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Exotic statistics for strings in 4d BF theory

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Abstract

After a review of exotic statistics for point particles in 3d BF theory, and especially 3d quantum gravity, we show that string-like defects in 4d BF theory obey exotic statistics governed by the "loop braid group". This group has a set of generators that switch two strings just as one would normally switch point particles, but also a set of generators that switch two strings by passing one through the other. The first set generates a copy of the symmetric group, while the second generates a copy of the braid group. Thanks to recent work of Xiao-Song Lin, we can give a presentation of the whole loop braid group, which turns out to be isomorphic to the "braid permutation group" of Fenn, Rimányi, and Rourke. In the context of 4d BF theory, this group naturally acts on the moduli space of flat G-bundles on the complement of a collection of unlinked unknotted circles in $\mathbb{R}^3$. When G is unimodular, this gives a unitary representation of the loop braid group. We also discuss "quandle field theory", in which the gauge group G is replaced by a quandle.

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1 Introduction

Physically speaking, the goal of this paper is to study the exotic statistics of loop-like defects in a 4d topological field theory called BF theory. We call these entities "closed strings" for short, though they behave differently from the closed strings familiar in string theory: the relevant Lagrangian is different. In fact, we postpone the study of their dynamics to another paper [3]. The considerations of this paper are purely topological, and accessible — we hope — to mathematicians with only a passing interest in physics.

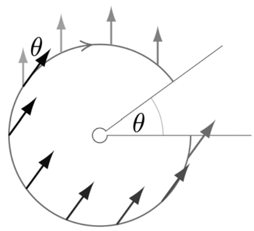
Mathematically speaking, the point of this paper is to study some representations of a higher-dimensional analogue of the braid group: the "loop braid group". Just as the braid group describes the topology of points moving in the plane, the loop braid group describes the topology of circles moving in $\mathbb{R}^3$. In the body of this paper, we describe this group and certain representations of it coming from the moduli space of flat bundles on $\mathbb{R}^3$ with these circles removed. But since everything we do has a more familiar analogue one dimension down, let us start by recalling that.

1.1 Exotic statistics in 3d BF theory

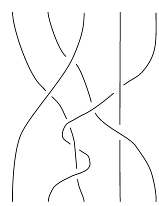
The behavior of a collection of identical particles when they are exchanged goes by the name of "statistics". Traditionally, statistics was described using representations of the symmetric group. However, it is well known that in 3d spacetime, "exotic" statistics are possible, in which the process of exchanging identical particles is described by a representation of the braid group. For example, exchanging two "abelian anyons" multiplies their wavefunction by a phase, which need not be 1 as it is for bosons, nor -1 as for fermions. This possibility has been investigated in experiments on the fractional quantum Hall effect [7]. Now researchers have begun the search for "non-abelian anyons", whose statistics are described by more complicated representations of the braid group [5]. Plans are already afoot to use these in quantum computers [12, 21].

Exotic statistics also arise naturally in the context of 3d quantum gravity. As we "turn on gravity", letting Newton's gravitational constant κ become non-zero, ordinary quantum field theory on 3d Minkowski spacetime deforms into a theory where the Poincaré group goes over to a quantum group called the κ-Poincaré group. Moreover, if we begin with a field theory of bosons, their statistics become exotic as we turn on gravity. For a thorough treatment of these fascinating phenomena, see the papers by Freidel and collaborators [13, 14], the paper by Krasnov [23], and many references therein.

In fact, the reason for exotic statistics in 3d quantum gravity is very simple. In 3d spacetime, Einstein's equations say that spacetime is *flat* except in regions where matter is present. A point particle at rest bends the nearby space into a cone. This cone is flat everywhere except at its tip, where there is a deficit angle proportional to the particle's mass. If we parallel transport a vector around the particle, it gets rotated by this angle θ.



More generally, if we have n particles, space will be flat except for conical singularities at n points. If we exchange these particles by moving them around the plane, they trace out a loop in the space of n-point subsets of the plane. Their energy-momenta will change in a way that depends on this loop — but only on the *homotopy class* of this loop, because they are being parallel transported with respect to a flat connection. A homotopy class of such loops is just an n-strand braid.



So, the group B_n of n-strand braids acts on the Hilbert space of states for n identical particles. In fact, this result holds classically as well: we get an action of B_n on the configuration space for n identical particles.

The above argument uses the fact that 3d gravity (with vanishing cosmological constant) can be described by BF theory with the Lorentz group $\mathrm{SO}(2,1)$ as gauge group. To understand this paper, the reader only needs to know one thing about BF theory: *it involves a flat connection on space*. For completeness, however, we recall that BF theory in n-dimensional spacetime with gauge group G involves two fields: a connection A and a $\mathfrak{g}$-valued $(n-2)$-form E. In the absence of matter, the Lagrangian is simply

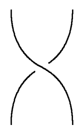
$$L = \frac{1}{\kappa}\mathrm{tr}\,(E \wedge F)$$

Here κ plays the role of Newton's constant in the case of 3d gravity, and $F = dA + A \wedge A$ is the curvature of A. The resulting equations of motion:

$$F = 0, \quad dE + [A, E] = 0,$$

imply that the connection A is flat.

In 3d BF theory, point particles can be included by considering spacetimes with curves removed: we think of these as the particles' worldlines. Away from these worldlines, the above equations still hold, while along the worldlines A becomes singular. The holonomy around a loop circling a worldline gives an element of the group G. A collection of n-particles in the plane thus gives rise to an n-tuple of elements of G. For simplicity, consider the case $n = 2$. As we exchange two particles by rotating them around each other counterclockwise, they trace out this braid.



As we recall in Section 3, this operation acts as the following map on G^2:

$$(g_1, g_2) \longmapsto (g_1 g_2 g_1^{-1}, g_1). \tag{1.1}$$

Applying this map twice does *not* give the identity, so we do not obtain an action of the symmetric group on G^2, but only an action of the braid group. In other words, the particles have exotic statistics!

In the case of 3d gravity, the singularity of the connection along a particle's worldline reflects the fact that the particle's mass creates a conical singularity in the metric. The holonomy around the worldline, an element of $G = \mathrm{SO}(2,1)$, describes the particle's *energy-momentum.* This may seem odd, since we are used to thinking of energy-momentum as a vector in Minkowski spacetime. However, in 3 dimensions, Minkowski spacetime is naturally isomorphic to the Lie algebra $\mathfrak{so}(2,1)$, and we can reinterpret Lie algebra elements as group elements via the map:

$$\begin{aligned} \mathfrak{so}(2,1) &\longrightarrow \mathrm{SO}(2,1) \\ p &\longmapsto \exp(\kappa p). \end{aligned}$$

So, we can encode the energy-momentum p of a particle in the holonomy $g = \exp(\kappa p)$ resulting from parallel transport around this particle's worldline.

Thanks to the factor of κ here, the group $\mathrm{SO}(2,1)$ effectively "flattens out" to $\mathfrak{so}(2,1)$ in the $\kappa \to 0$ limit. For example, multiplication in the group reduces to addition in the Lie algebra plus small corrections:

$$\exp(\kappa p_1)\exp(\kappa p_2) = \exp\left(\kappa(p_1 + p_2) + \frac{\kappa^2}{2}[p_1, p_2] + \cdots\right) \qquad (1.2)$$

This implies that in terms of $\mathfrak{so}(2,1)$-valued energy-momenta, the braiding in equation (1.1) is given by

$$(p_1, p_2) \longmapsto (p_2 + \kappa[p_1, p_2] + \cdots\ ,\ p_1)$$

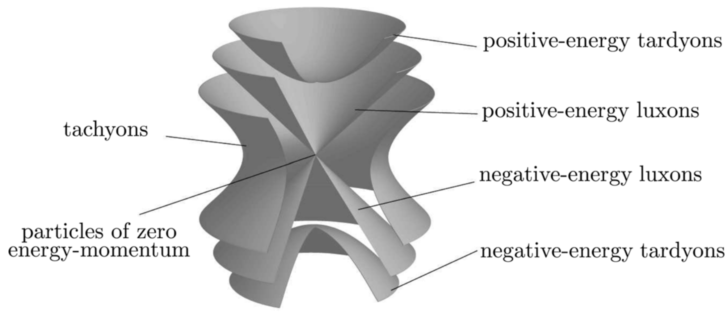
So, the exotic statistics reduce to ordinary bosonic statistics in the limit where Newton's constant goes to zero. They also reduce to bosonic statistics in the limit where the particles are at rest relative to each other, since then p_1 and p_2 become proportional and their commutator vanishes.

The corrections to the usual law for addition of energy-momenta implicit in equation (1.2) are interesting in themselves. Like the exotic statistics, these corrections become negligible in the limit $\kappa \to 0$. Under the name of "doubly special relativity", modified laws for adding energy-momentum have already been studied by many authors. The paper by Freidel, Kowalski-Glikman, and Smolin [14] gives a good account of doubly special relativity in the context of 3d quantum gravity; their paper also explains more of the history of this subject.

1.2 Quandle field theory

Besides exotic statistics and corrections to the usual rule for adding energy-momenta, there is yet another surprising consequence of the switch from vector-valued to group-valued energy-momentum as we turn on gravity in 3d physics. The classification of elementary particles changes!

In ordinary quantum field theory on Minkowski spacetime, the Lorentz group acts on the space of possible energy-momenta, and the orbits of this action correspond to different types of spin-zero particles. When spacetime is 3d, the space of energy-momenta is $\mathfrak{so}(2,1)$, and the orbits look like this:



If we write the energy-momentum as $p = (E, p_x, p_y)$ and let $p \cdot p = E^2 - p_x^2 - p_y^2$, we have six families of orbits, corresponding to six types of spin-zero particles:

1. positive-energy tardyons of mass $m > 0$: $\{p \cdot p = m^2,\ E > 0\}$,
2. negative-energy tardyons of mass $m > 0$: $\{p \cdot p = m^2,\ E < 0\}$,
3. positive-energy luxons: $\{p \cdot p = 0,\ E > 0\}$,
4. negative-energy luxons: $\{p \cdot p = 0,\ E < 0\}$,
5. tachyons of mass im for $m > 0$: $\{p \cdot p = -m^2\}$,
6. particles of vanishing energy-momentum: $\{p = 0\}$.

Given any orbit $Q \subseteq \mathfrak{so}(2,1)$, the Hilbert space for a single particle of type Q is just $L^2(Q)$.

The same philosophy applies when we turn on gravity, but now the space of energy-momenta is not the Lie algebra $\mathfrak{so}(2,1)$ but the Lorentz group itself. This acts on itself by conjugation, and the orbits are conjugacy classes. Types of spin-zero particles now correspond to conjugacy classes in the Lorentz group. Near the identity these conjugacy classes look just like orbits in the Lie algebra, so the classification of particles reduces to the above one in the limit of small energy-momenta. However, there are important differences, which show up for large energy-momenta.

Most notably, under the map

$$p \longmapsto \exp(\kappa p),$$

the Lie algebra element $p = (E, 0, 0)$ is mapped to a rotation by the angle κE in the xy-plane. So, the holonomy around a stationary particle of energy E is a rotation by the angle κE. This rotation does not change when we add $2\pi/\kappa$ to the particle's energy. Up to factors of order unity, this quantity $2\pi/\kappa$ is just the *Planck energy.* If we call it the Planck energy, then masses in 3d quantum gravity are defined only modulo the Planck mass.

This "periodicity of mass" affects the classification of tardyons — that is, the most familiar sort of particles, those with timelike energy-momentum. Instead of positive-energy tardyons of arbitrary mass $m > 0$ and negative-energy tardyons of arbitrary mass $m > 0$, we just have tardyons of arbitrary mass $m \in \mathbb{R}/(2\pi/\kappa)\mathbb{Z}$.

More generally, for any Lie group G, the various allowed types of spin-zero particles in 3d BF theory with gauge group G correspond to conjugacy classes $Q \subseteq G$. Any conjugacy class is closed under the operations

$$g \rhd h = ghg^{-1}, \quad h \lhd g = g^{-1}hg,$$

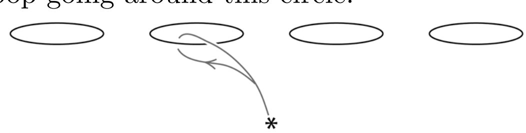
and these operations satisfy equations making Q into an algebraic structure called a "quandle" [19], whose definition we recall in Section 5. The Hilbert space for a single particle of type Q is just $L^2(Q)$, defined using a measure on Q that is invariant under these operations. In an easy generalization of 3d BF theory, we can study the exotic statistics of "particles of type Q" for any quandle Q equipped with an invariant measure. This takes advantage of the well-known relation between quandles and the braid group [11].

1.3 Exotic statistics in 4d BF theory

It would be wonderful to generalize all the above results to 4d gravity, but for now all we can handle is a simpler theory: 4d BF theory. This may eventually be relevant to gravity, since one can describe general relativity in 4d either as the result of *constraining* 4d BF theory with a certain gauge group, or *perturbing around* 4d BF theory with some other gauge group. The first approach goes back to Plebanski [34], and it underlies a great deal of work on spin foam models of quantum gravity [2, 30, 32], especially the Barrett–Crane model. The second approach goes back to MacDowell and Mansouri [25], and has recently been explored by Freidel and Starodubtsev [15]. However, we do not dwell on these possible applications here. They only focus our attention towards certain choices of gauge group:

$$\begin{aligned} \text{Plebanski gravity:} \quad & G = \mathrm{SO}(3,1) \\ \text{MacDowell–Mansouri gravity:} \quad & \begin{cases} G = \mathrm{SO}(4,1) & \Lambda > 0 \\ G = \mathrm{SO}(3,2) & \Lambda < 0 \end{cases} \end{aligned}$$

Our idea is simply to increase the dimension of everything in the previous section by 1. Thus, we consider BF theory on a 4d spacetime with the worldsheets of several "closed strings" removed. We focus on the case where the manifold representing space is $\mathbb{R}^3 - \Sigma$, where Σ is an "n-component unlink": a collection of n unknotted unlinked circles. A flat connection on $\mathbb{R}^3 - \Sigma$ gives us a group element for each circle, namely the holonomy of some standard loop going around this circle.

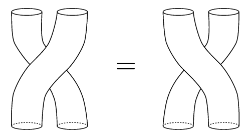


So, just as before, we obtain n-tuples of elements of G. Moreover, any way to exchange the circles in Σ gives a map from G^n to itself.

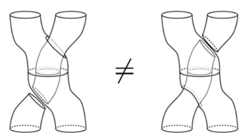
It is often said that exotic statistics are only possible when space has dimension 2 or less. However, this folklore only applies to point particles.

As pointed out by Balanchandran and others [1, 4, 29, 37, 38], exotic statistics are possible for closed strings in 3d space, since there are topologically non-trivial ways to exchange unknotted unlinked circles in $\mathbb{R}^3$. The statistics of such theories are governed not by the braid group B_n, but by a larger group: the "loop braid group" LB_n.

Using recent work of Lin [24], we show that this group is isomorphic to the "braid permutation group" of Fenn, Rimányi, and Rourke [10]. This is an apt name, because LB_n has a presentation with generators s_i that describe two strings trading places without passing through each other, just as if they were point particles:



but also generators σ_i that describe one string passing through another.



So, this group is a kind of "hybrid" of the symmetric group and the braid group. Indeed, the elements s_i generate a copy of the symmetric group S_n in LB_n, while the elements σ_i generate a copy of the braid group B_n.

In a 1d unitary representation of the loop braid group, the permutation generators s_i all act as ± 1, while the braid generators σ_i all act as an arbitrary phase $q \in \mathrm{U}(1)$. We could call particles that transform in this way "abelian bose-anyons" and "abelian fermi-anyons", respectively. They act like bosons or fermions when we switch them using the generators s_i, but like abelian anyons when we switch them using the generators σ_i.

BF theory gives us more interesting unitary representations of the loop braid group: whenever the group G is unimodular, we obtain a unitary representation of LB_n on $L^2(G^n)$. All the groups listed above are unimodular, so we get an interesting variety of exotic statistics for closed strings in 4d BF theory.

We can also restrict attention to a specific conjugacy class $Q \subseteq G$ and get a unitary representation of the loop braid group on $L^2(Q^n)$, as long as Q is equipped with a measure invariant under conjugation. As already mentioned, in the case of 3d gravity a choice of conjugacy class in $G = \mathrm{SO}(2,1)$ essentially amounts to choosing a specific *mass* for our point particles, which is a very natural thing to do. In the case of 4d BF theory with $G = \mathrm{SO}(3,1)$, choosing a conjugacy class essentially amounts to choosing a specific *mass density* for our closed strings.

2 The loop braid group

The loop braid group LB_n consists of all ways a collection of oriented, unknotted, unlinked circles can move around in $\mathbb{R}^3$ and come back to their original positions, perhaps trading places. More precisely, it consists of "isotopy classes" of such motions. This group thus plays the same role in describing the interchange of closed strings in $\mathbb{R}^3$ that the symmetric group S_n plays for point particles in $\mathbb{R}^3$, and the braid group plays for point particles in $\mathbb{R}^2$. In this section, we use the work of Lin [24] to obtain two presentations of the loop braid group. First, however, we explain the sense in which the loop braid group, the symmetric group and the braid group are all examples of "motion groups".

The general idea of a "motion group" goes back at least to Dahm's 1962 thesis [9], which unfortunately was never published. In the 1970s and 80s, some papers by Wattenberg [40] and Goldsmith [17, 18] clarified and expanded on Dahm's work. More recently, McCool [26] and Rubinsztein [35] have studied the motion group for unknotted and unlinked circles in $\mathbb{R}^3$. Surya has also given a description of the loop braid group as an iterated semidirect product [37]. Much of this work considers the motion of unoriented circles. Since we use oriented circles, we obtain a smaller motion group, which lacks the "circle-flipping" operations that reverse orientations.

Quite generally, suppose that S is a smooth oriented manifold and $\Sigma \subseteq S$ is a smooth oriented submanifold. Let $\mathrm{Diff}(S)$ be the group of orientation-preserving diffeomorphisms of S. Let $\mathrm{Diff}(S, \Sigma)$ be the subgroup of $\mathrm{Diff}(S)$ maps that restrict to give orientation-preserving diffeomorphisms of Σ.

We define a "motion" of Σ in S to be a smooth map $f\colon [0,1] \times S \to S$, which we write as $f_t\colon S \to S$ $(t \in [0,1])$, with the following properties:

- for all t, f_t lies in $\mathrm{Diff}(S)$;
- for all t sufficiently close to 0, f_t is the identity;
- for all t sufficiently close to 1, f_t is independent of t and lies in $\mathrm{Diff}(S, \Sigma)$.

Intuitively, a motion is a way of moving Σ through S so that it comes back to itself — not pointwise, but as a set — at $t = 1$. This suggests that one can "multiply" motions by doing one after the other, and indeed this is true. Given motions f and g, one can define a motion $f \cdot g$ called their "product" as follows:

$$(f \cdot g)_t = \begin{cases} f_{2t} & \text{for } 0 \le t \le \frac{1}{2}, \\ g_{2t-1} \circ f_1 & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Given a motion f we can also define a motion called its "reverse", denoted $\bar{f}$, by:

$$\bar{f}_t = f_{1-t} \circ f_1^{-1}.$$

We say two motions f and g are "equivalent" if $\bar{f} \cdot g$ is smoothly homotopic, as a path in $\mathrm{Diff}(S)$ with fixed endpoints, to a path that lies entirely in $\mathrm{Diff}(S, \Sigma)$. One can check that this is, indeed, an equivalence relation and that the operations of product and reverse make equivalence classes of motions into a group. This is called the "motion group" $\mathrm{Mo}(S, \Sigma)$.

Next we turn to examples:

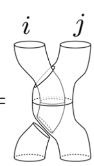
- When $\Sigma \subset \mathbb{R}^d$ is a collection of n points and $d > 2$, $\mathrm{Mo}(\mathbb{R}^d, \Sigma)$ is the symmetric group S_n.
- When $\Sigma \subset \mathbb{R}^2$ is a collection of n points, $\mathrm{Mo}(\mathbb{R}^2, \Sigma)$ is the braid group B_n.
- When $\Sigma \subset \mathbb{R}^3$ is a collection of n unknotted and unlinked oriented circles, we call $\mathrm{Mo}(\mathbb{R}^3, \Sigma)$ the "loop braid group" LB_n.

We shall use the work of Lin [24] to give two presentations of LB_n. First, note that there is a homomorphism

$$p \colon \mathrm{LB}_n \to S_n$$

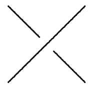
which simply forgets the details of the braiding, remembering only how the circles get permuted in the process. The image of p is all of S_n. We call the kernel of p the "pure loop braid group" PLB_n.

Suppose, just to be specific, that $\Sigma = \ell_1 \cup \cdots \cup \ell_n$ where $\ell_1, \ldots, \ell_n$ are disjoint unit circles in the xy-plane, lined up from left to right with their centers on the x-axis. Lin proves that PLB_n has a presentation with generators σ_{ij} for $i, j \in \{1, \ldots, n\}$ with $i \neq j$. The generator σ_{ij} describes a motion in which the ith circle floats up and over the jth circle, shrinks slightly and passes down through the jth circle, expands to its original size, and then moves straight back to its starting position. We draw this as follows:

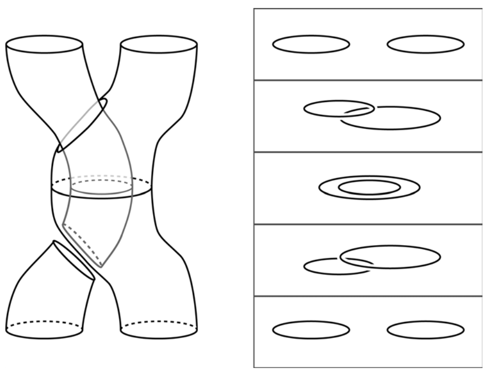
$\sigma_{ij} =$ 

where for purely artistic reasons we let the jth circle move a bit to the left in the process.

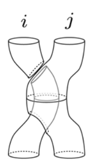
Here we are using a drawing style adapted from Carter and Saito's work on surfaces in 4d [8]. Crossings in a braid or knot are usually drawn with an artificial "break" in one of the strands to indicate that it lies under the other:



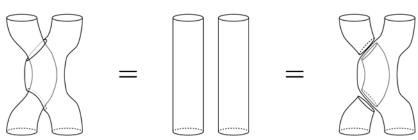
Similarly, Carter and Saito draw 3d projections of knotted surfaces in 4d, indicating by a broken surface which one passes "under" the other in the suppressed fourth dimension. In our context, we take this suppressed dimension to be one of the spatial dimensions, in order to make room for *time*, which we decree to flow downward in all our diagrams. The broken surfaces in σ_{ij} indicate whether one circle is above or below the other in the suppressed spatial dimension, so that the following diagram and "movie" illustrate the same process:



The inverse of σ_{ij} is of course obtained by running the movie backwards, which in diagrammatic notation becomes:

$\sigma_{ij}{}^{-1} =$ 

One advantage of this drawing style is that it immediately suggests Reidemeister-like moves for loop braids, such as this:



We shall study the loop braid group algebraically, relying on such diagrams for our intuition.

Given Lin's presentation of PLB_n, we can obtain a presentation of LB_n using the short exact sequence

$$1 \longrightarrow \mathrm{PLB}_n \xrightarrow{i} \mathrm{LB}_n \xrightarrow{p} S_n \longrightarrow 1.$$

First, note that there is a homomorphism

$$j\colon S_n \longrightarrow \mathrm{LB}_n$$

which takes a given permutation to what Lin calls a "permutation path" in the motion group: a loop braid in which circles trade places without any circle passing through another in a topologically non-trivial way. For example, we can have them trade places while remaining on the xy-plane. This map j is well defined since all such permutation paths are homotopic. Moreover, the composite $p \circ j\colon S_n \to S_n$ is the identity homomorphism on S_n, so j is a splitting of the short exact sequence above.

Since j is one-to-one, we may identify elements of S_n with their images in LB_n. Since PLB_n is a normal subgroup, elements of S_n act on PLB_n via conjugation. This allows us to define the semidirect product $S_n \ltimes \mathrm{PLB}_n$, and thanks to our split exact sequence, we get an isomorphism

$$\begin{aligned} f\colon \mathrm{LB}_n &\longrightarrow S_n \ltimes \mathrm{PLB}_n \\ g &\longmapsto (p(g), j(p(g))^{-1}g) \end{aligned}$$

with inverse

$$\begin{aligned} f^{-1}\colon S_n \ltimes \mathrm{PLB}_n &\longrightarrow \mathrm{LB}_n \\ (s,\sigma) &\longmapsto s\sigma. \end{aligned}$$

Writing the loop braid group as a semidirect product in this way, we easily obtain a presentation for it.

Theorem 2.1. *The loop braid group* LB_n *has a presentation with generators* s_i *for* $1 \le i \le n-1$ *and* σ_{ij} *for* $1 \le i, j \le n$ *with* $i \ne j$*, together with the following relations.*

(a) *The relations for the standard generators* s_i *of* S_n*:*

$$s_i s_j = s_j s_i \quad \text{for} \quad |i-j| > 1, \tag{2.1}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for} \quad 1 \le i \le n-2, \tag{2.2}$$

$$s_i^2 = 1 \quad \text{for} \quad 1 \le i \le n-1. \tag{2.3}$$

(b) *Lin's relations for the generators σ_{ij} of* PLB_n*:*

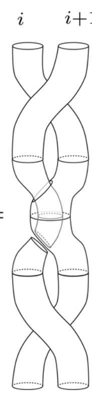
$$\sigma_{ij}\sigma_{k\ell} = \sigma_{k\ell}\sigma_{ij} \quad \text{for} \quad i,j,k,\ell \text{ distinct,} \tag{2.4}$$
$$\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik} \quad \text{for} \quad i,j,k \text{ distinct,} \tag{2.5}$$
$$\sigma_{ij}\sigma_{kj}\sigma_{ik} = \sigma_{ik}\sigma_{kj}\sigma_{ij} \quad \text{for} \quad i,j,k \text{ distinct.} \tag{2.6}$$

(c) *Relations expressing the action of S_n on* PLB_n*:*

$$s_i\sigma_{i(i+1)} = \sigma_{(i+1)i}s_i \quad \text{for} \quad 1 \le i \le n-1, \tag{2.7}$$
$$s_k\sigma_{ij} = \sigma_{ij}s_k \quad \text{for} \quad i,j,k,k+1 \text{ distinct,} \tag{2.8}$$
$$s_j\sigma_{ij} = \sigma_{i(j+1)}s_j \quad \text{for} \quad i,j,j+1 \text{ distinct,} \tag{2.9}$$
$$s_i\sigma_{ij} = \sigma_{(i+1)j}s_i \quad \text{for} \quad i,i+1,j \text{ distinct.} \tag{2.10}$$

Proof. Since the presentation (a) of S_n is well-known, and Lin [24] proved that PLB_n has the presentation (b), to present their semidirect product LB_n it suffices to add relations that express the result of conjugating any of Lin's generators σ_{ij} by the symmetric group generators s_k. For $1 \le i \le n-1$ we have:

$s_i\sigma_{i(i+1)}s_i^{-1}$ =



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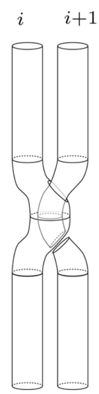
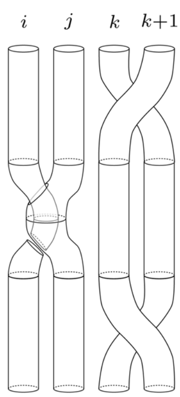


$= \sigma_{(i+1)i}$

For i, j, k and $k+1$ all distinct, we have:

$s_k\sigma_{ij}s_k^{-1}$ =



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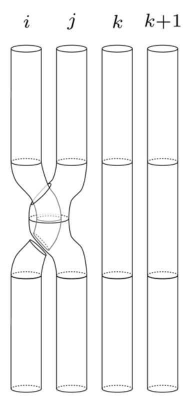
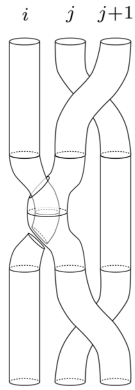


$= \sigma_{ij}$

For i, j and $j+1$ distinct, we have:

$s_j \sigma_{ij} {s_j}^{-1} =$ 

$= \sigma_{i(j+1)}$

and using a similar picture we see that for $i, i+1$ and j distinct, $s_i \sigma_{ij} {s_i}^{-1} = \sigma_{(i+1)j}$. The reader may notice that we have not included all possible conjugations of generators of PLB_n by generators of S_n — we would naively expect two additional such classes, yielding two more relations:

$$s_{j-1}\sigma_{ij} = \sigma_{i(j-1)} s_{j-1} \text{ for } i, j-1, j \text{ distinct} \tag{2.11}$$

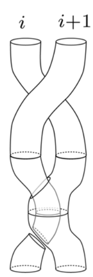
$$s_{i-1}\sigma_{ij} = \sigma_{(i-1)j} s_{i-1} \text{ for } i-1, i, j \text{ distinct} \tag{2.12}$$

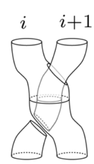
but these follow, respectively, from (2.9) and (2.10) combined with (2.3). So, we have precisely the relations in part (c), as desired. □

From this presentation of the loop braid group we now derive a presentation with fewer generators. We keep all the generators s_i, but replace the σ_{ij} with new generators defined as follows:

$$\sigma_i = s_i \sigma_{i(i+1)}$$

for $1 \le i \le n-1$. We can draw these as follows:

$\sigma_i =$ 

$=$ 

where we twist the picture a bit in the second step. To see that the generators s_i and σ_i indeed give a new presentation, note that we can express the old generators σ_{ij} in terms of these new ones as follows. First, repeatedly applying (2.9) we obtain:

$$\sigma_{ij} = s_{j-1}s_{j-2}\cdots s_{i+1}\sigma_{i(i+1)}s_{i+1}s_{i+2}\cdots s_{j-2}s_{j-1} \text{ for } i<j.$$

If instead of (2.9) we use its equivalent form (2.11), we obtain:

$$\sigma_{ij} = s_j s_{j+1}\cdots s_{i-2}\sigma_{i(i-1)}s_{i-2}\cdots s_{j+1}s_j \text{ for } i>j.$$

Rewriting these in terms of the new generators σ_i, and in the second case using relation (2.7), we obtain a way to write σ_{ij} in terms of the new generators:

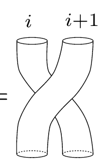
$$\sigma_{ij} = \begin{cases} s_{j-1}s_{j-2}\cdots s_i\sigma_i s_{i+1}s_{i+2}\cdots s_{j-2}s_{j-1} & \text{for } i<j \\ s_j s_{j+1}\cdots s_{i-2}\sigma_{i-1}s_{i-1}s_{i-2}\cdots s_{j+1}s_j & \text{for } i>j \end{cases} \tag{2.13}$$

Sometimes it is more convenient to use an alternate formula, obtained by applying (2.10), its equivalent form (2.12), and (2.7) again:

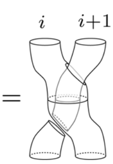
$$\sigma_{ij} = \begin{cases} s_i s_{i+1}\cdots s_{j-1}\sigma_{j-1}s_{j-2}\cdots s_{i+1}s_i & \text{for } i<j \\ s_{i-1}s_{i-2}\cdots s_{j+1}\sigma_j s_j s_{j+1}\cdots s_{i-2}s_{i-1} & \text{for } i>j. \end{cases} \tag{2.14}$$

What these formulas say is that when $j \neq i+1$ we can construct the loop braid σ_{ij} by permuting either the ith circle or the jth until they are adjacent, braiding one through the other, and then permuting the circles back to where they started.

The nice thing about using s_i and σ_i as generators of the loop braid group is that s_i describes how two neighboring circles can trade places by going around each other:

$s_i =$ 

while σ_i describes how two neighboring circles can trade places with the right one passing over and then down through the left one:

$\sigma_i =$ 

As a result, the generators s_i generate a subgroup of LB_n isomorphic to the symmetric group S_n, while the σ_i generate a subgroup isomorphic to the braid group B_n. There are also "mixed relations" involving generators of both kinds.

Theorem 2.2. *The loop braid group* LB_n *has a presentation with generators* s_i *and* σ_i *for* $1 \le i \le n-1$ *together with the following relations.*

(a) *Relations for the standard generators* s_i *of* S_n*:*

$$s_i s_j = s_j s_i \qquad \text{for } |i-j| > 1 \tag{2.15}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad \text{for } 1 \le i \le n-2 \tag{2.16}$$

$$s_i^2 = 1 \qquad \text{for } 1 \le i \le n-1 \tag{2.17}$$

(b′) *Relations for the standard generators* σ_i *of* B_n*:*

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{for } |i-j| > 1 \tag{2.18}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \text{for } 1 \le i \le n-2 \tag{2.19}$$

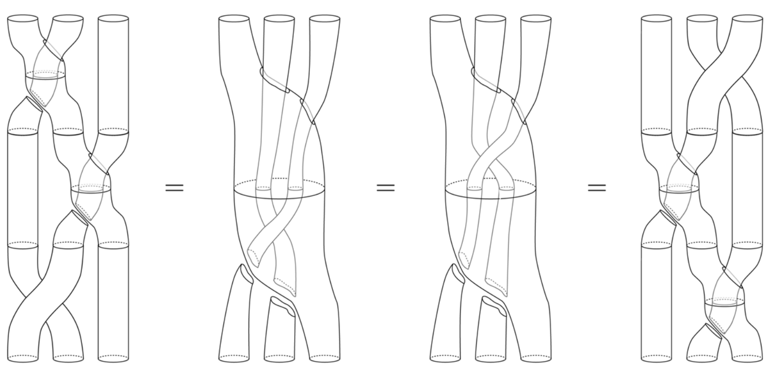
(c′) *The following mixed relations:*

$$s_i \sigma_j = \sigma_j s_i \qquad \text{for } |i-j| > 1 \tag{2.20}$$

$$s_i s_{i+1} \sigma_i = \sigma_{i+1} s_i s_{i+1} \qquad \text{for } 1 \le i \le n-2 \tag{2.21}$$

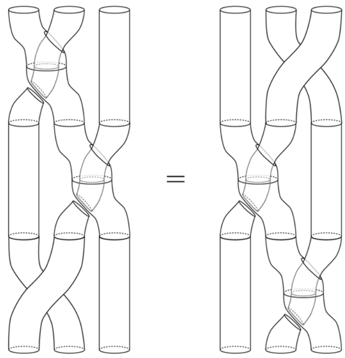
$$\sigma_i \sigma_{i+1} s_i = s_{i+1} \sigma_i \sigma_{i+1} \qquad \text{for } 1 \le i \le n-2 \tag{2.22}$$

Proof. The proof is somewhat lengthy, so we defer it to the Appendix. It is, however, simple to convince oneself using pictures that the given relations express topologically allowed moves for loop braids. Perhaps, the least obvious of these is (2.22), for which we supply a visual proof below.



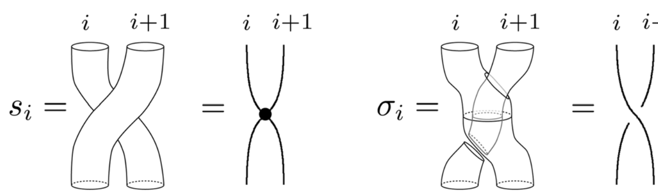
□

If we omit relations (2.22) we obtain the "virtual braid group" VB_n of Vershinin [39]. This plays a role in virtual knot theory analogous to that of the usual braid group in ordinary knot theory. If we include these relations, which say:

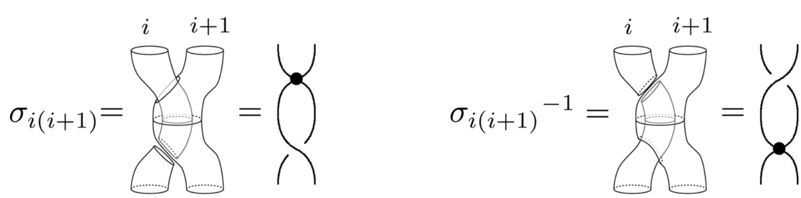


then we obtain precisely the "braid permutation group" BP_n of Fenn, Rimányi and Rourke [10]. So, the loop braid group is isomorphic to the braid permutation group.

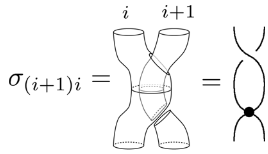
The isomorphism $\mathrm{LB}_n \cong \mathrm{BP}_n$ yields a simplified diagrammatic way of working with loop braids, which is in fact the method used by Fenn, Rimányi and Rourke in their original paper on BP_n. In the theory of "welded braids", the generators σ_i in BP_n correspond to the kind of crossings found in ordinary braids: , while the s_i describe "welded crossings", drawn like this: . These crossings are called "welded" because one imagines that the two strands have been "welded down" at the crossing. The point is that elements of the abstract group presented in Theorem 2.2 can be represented either as loop braid diagrams or as welded braid diagrams, as follows:



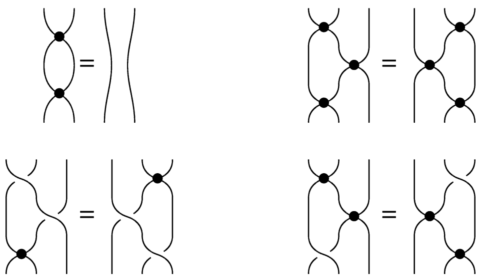
For the *pure* loop braid group PLB_n, the above correspondence implies the following welded braid pictures of the generators $\sigma_{i(i+1)}$ and their inverses.



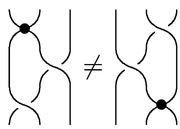
The other generators σ_{ij} can be obtained from these by conjugation, using (2.13) or (2.14). For example:



Diagrammatic calcuations with welded braids — and hence with loop braids — can be carried out by using the usual Reidemeister moves for real crossings, along with "welded Reidemeister moves":



which are of course simply graphical restatements of the relations in (a) and (c′). The non-existence of the following move:



is the rationale for the term "welded braid" — we are not allowed to pass a strand under the weld.

It is easy from the presentation in Theorem 2.2 to work out the 1-dimensional unitary representations of the loop braid group. If $\rho\colon \mathrm{LB}_n \to \mathrm{U}(1)$ is such a representation, we must have

$$\rho(s_i) = \pm 1$$

and

$$\rho(\sigma_i) = q$$

for all $1 \le i < n$, where $q \in \mathrm{U}(1)$ is a fixed phase. We call the representations with $\rho(s_i) = 1$ "bose-anyons", and the representations with $\rho(s_i) = -1$ "fermi-anyons". These have been studied in physics at least since the work of Balachandran [4], and recently Niemi has shown how they arise in the dynamics of vortices in a quantum fluid [29].

In Section 5, we describe more interesting unitary representations of the loop braid group, using some technology which we now develop. In related work, Szabo [38] has obtained a different class of representations using BF theory with abelian gauge group. Surya [37] has also studied representations of the loop braid group.

3 Motion groups and flat bundles

In this section, we recall Dahm's [9] action of the motion group $\mathrm{Mo}(S, \Sigma)$ on the fundamental group of $S - \Sigma$ and describe how this gives a unitary representation of the motion group on a certain Hilbert space of states for BF theory on $S - \Sigma$.

We consider BF theory in n-dimensional spacetime. So, we take "space" to be of the form $X = S - \Sigma$, where S is an oriented manifold of dimension $n-1$, and $\Sigma \subset S$ is an oriented submanifold. We let G be a Lie group and let $P \to X$ be a principal G-bundle. The "naive configuration space" of BF theory is $\mathcal{A}_0/\mathcal{G}$, where $\mathcal{A}_0$ is the space of flat connections on P and $\mathcal{G}$ is the group of gauge transformations. By "naive" we mean that we are ignoring boundary conditions; there are no boundary conditions to worry about when X is compact, but we shall mainly be interested in two examples where it is not:

1. X is $\mathbb{R}^2$ with a finite set of points removed (describing *point particles*):

$$X = S - \Sigma, \quad S = \mathbb{R}^2, \quad \Sigma = \{z_1, \ldots, z_n\}.$$

2. X is $\mathbb{R}^3$ with a finite set of unlinked unknotted circles removed (describing what one might call *closed strings*):

$$X = S - \Sigma, \quad S = \mathbb{R}^3, \quad \Sigma = \ell_1 \cup \cdots \cup \ell_n.$$

A rigorous study of BF theory may require that we impose boundary conditions at Σ. We ignore this issue now, leaving it for future research.

The space $\mathcal{A}_0/\mathcal{G}$ is a bit difficult to handle. It is often more convenient to start by fixing a basepoint $* \in X$ and working with $\mathcal{A}_0/\mathcal{G}_0$, where

$$\mathcal{G}_0 = \{g \in \mathcal{G}\colon\ g(*) = 1\}.$$

The group $\mathcal{G}/\mathcal{G}_0 \cong G$ acts on $\mathcal{A}_0/\mathcal{G}_0$ in a natural way. This lets us form $\mathcal{A}_0/\mathcal{G}$ as the quotient of the bigger space $\mathcal{A}_0/\mathcal{G}_0$ by this action of G.

The advantage of the space $\mathcal{A}_0/\mathcal{G}_0$ is that any point $[A]$ in this space gives a homomorphism

$$\mathrm{hol}([A])\colon \pi_1(X) \to G$$

which sends any homotopy class of loops $[\gamma]$ to the holonomy of A around γ. This gives a map

$$\mathrm{hol}\colon \mathcal{A}_0/\mathcal{G}_0 \to \hom(\pi_1(X), G)$$

which is known to be one-to-one. Note that G acts on $\hom(\pi_1(X), G)$ by conjugation:

$$(gf)(\gamma) = gf(\gamma)g^{-1}$$

where $f\colon \pi_1(X) \to G$ is any homomorphism. Moreover, the map hol is compatible with this group action:

$$\mathrm{hol}([gA]) = g\,\mathrm{hol}([A]).$$

So far we have fixed a principal G-bundle P. But, in gauge theory it is often better to treat this bundle as variable — part of the physical field along with the connection A. For example, path integrals in quantum chromodynamics involve a sum over bundles, which represent instantons. The mathematical advantage of treating P as variable is that all points of $\hom(\pi_1(X), G)$ are in the image of hol if we allow ourselves to vary P [22]. A point in this space represents a "G-bundle with flat connection over X, mod gauge transformations that equal the identity at the basepoint". Modding out by the rest of the gauge transformations we get a space known as the "moduli space of flat bundles", $\hom(\pi_1(X), G)/G$. This is the naive configuration space for BF theory where we treat the bundle P as variable.

Applying Schrödinger quantization to this configuration space, we obtain the (naive) Hilbert space for BF theory:

$$L^2(\hom(\pi_1(X), G)/G)$$

Of course, defining this L^2 space requires that we choose a measure on the moduli space of flat bundles. Alternatively, we can try to form a Hilbert space

$$L^2(\hom(\pi_1(X), G))$$

on which G acts as follows:

$$(g\psi)(f) = \psi(g^{-1}f).$$

Again, this requires choosing a measure on $\hom(\pi_1(X), G)$. Moreover, G will only have a unitary representation on $L^2(\hom(\pi_1(X), G)$ if this measure is G-invariant.

In Sections 4 and 6, we will show that for the two examples above, there is a "natural" choice of G-invariant measure on $\hom(\pi_1(X), G)$. In both these examples, the motion group $\mathrm{Mo}(S, \Sigma)$ acts on $\pi_1(X)$ and thus on $\hom(\pi_1(X), G)$. By saying a measure on $\hom(\pi_1(X), G)$ is "natural", we simply mean that it is preserved by this action.

Using such a natural measure to define the Hilbert space $L^2(\hom(\pi_1(X), G))$, we obtain a unitary representation of the motion group on this Hilbert space. This representation describes the statistics of point particles or closed strings in BF theory. As we have seen, in the first example the motion group is the braid group B_n, while in the 4d case it is the loop braid group LB_n. So, we obtain "exotic statistics" in both cases. This fact is somewhat familiar in 3d, but less so in 4d. So, in the following sections we first review the 3d case, and then move on to the 4d case after a brief digression on "quandle field theory".

Before doing this, however, let us see how the motion group acts on $\pi_1(X)$. The idea goes back to Dahm's original work on the motion group [9], and it has been nicely explained by Goldsmith [17]. The idea is simple: elements of the motion group $\mathrm{Mo}(S, \Sigma)$ give equivalence classes of diffeomorphisms of $X = S - \Sigma$, and these act on homotopy classes of loops in X. The only problem is that the fundamental group is defined using *based* loops, and the diffeomorphisms used in the definition of the motion group need not preserve the basepoint in X. Luckily, Wattenberg [40] has shown that we can use compactly supported diffeomorphisms in the definition of the motion group without changing this group. In the examples above, we can assume without loss of generality that these diffeomorphisms are supported in a fixed large ball containing Σ. So, if we choose a basepoint $* \in S$ that is sufficiently far from Σ, we can assume this basepoint is preserved by all the diffeomorphisms in the definition of the motion group. This makes it easy to check that $\mathrm{Mo}(S, \Sigma)$ acts as automorphisms of $\pi_1(X)$.

4 Point particles in 3d BF theory

Now let us apply the general ideas of the previous section to the case of a plane with n punctures:

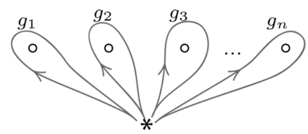
$$X = S - \Sigma, \quad S = \mathbb{R}^2, \quad \Sigma = \{z_1, \ldots, z_n\}$$

If we interpret these punctures as "particles", we shall see that a state of 3d BF theory on this space describes a collection of identical point particles with exotic statistics governed by the braid group.

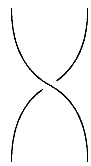
The fundamental group of X is the free group on n generators, so we have

$$\hom(\pi_1(X), G) = G^n$$

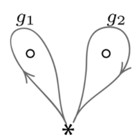
The n group elements here are nothing but the holonomies of a flat connection around based loops going clockwise around the particles:



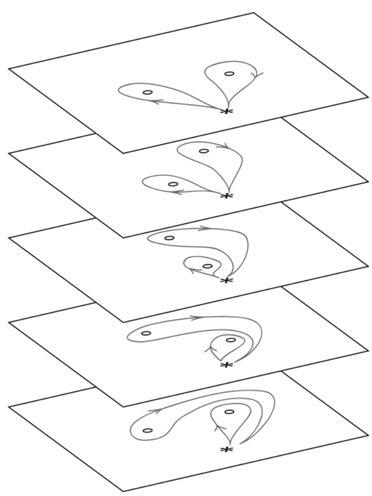
Having described particles as punctures in this theory, let us now consider what sort of statistics such particles obey. The previous section shows that the interchange of identical particles is described by an action of the n-strand braid group B_n on G^n, but we would like to work it out explicitly. For simplicity, consider the case $n = 2$ and consider what happens when the two particles switch places. As remarked earlier, there are infinitely many topologically distinct ways for the particles to move around each other, but they are all powers of the braid group generator σ_1:



If the holonomies around the two particles are g_1 and g_2:

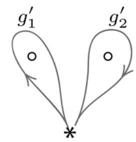


switching them via σ_1 induces a diffeomorphism of the plane which deforms the loops around which the holonomies are taken:



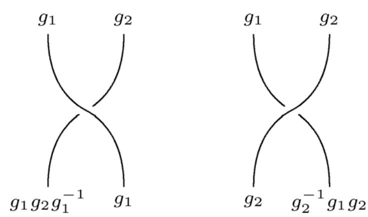
To see how the system changes in this process, compare the final frame in this "movie" to the first frame. Given that $(g_1, g_2) \in G^2$ describes the holonomies initially, a slight deformation of the loops in the final frame:

makes it clear that the corresponding holonomies around *these* loops in the final configuration:



are $(g_1', g_2') = (g_1 g_2 g_1{}^{-1}, g_1)$. Thus the effect of switching the two particles via σ_1 is to send (g_1, g_2) to $(g_1 g_2 g_1{}^{-1}, g_1)$.

We can work out the action of σ_1^{-1} in the same way, or simply derive it algebraically from the fact that it must undo the effect of σ_1. The easiest way to remember the results is with this picture:



More generally, we have a right action of the braid group B_n on G^n given as follows:

$$(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n)\sigma_i = (g_1, \ldots, g_i g_{i+1} g_i^{-1}, g_i, \ldots, g_n).$$

As mentioned in the previous section, we also have a left action of G on G^n via gauge transformations at the basepoint $*$. This works as follows:

$$g(g_1, \ldots, g_n) = (g g_1 g^{-1}, \ldots, g g_n g^{-1}).$$

We would like a measure on G^n that is invariant under both these group actions, so that the braid group and gauge transformations act as unitary operators on $L^2(G^n)$. Such a measure exists whenever G is "unimodular", meaning that its left-invariant Haar measure is also right-invariant. A Lie group is automatically unimodular if it is compact, or abelian, or semisimple. In particular, the groups $\mathrm{SO}(p, q)$ are all unimodular. Since these groups act on Minkowski spacetime in a way that preserves its Lebesgue measure, the

Poincaré groups $\mathrm{ISO}(p, q)$ are also unimodular. Also, the identity component of a unimodular group is unimodular, as is any covering space of a unimodular group.

From this, we see that the 3d Lorentz group $\mathrm{SO}(2, 1)$ is unimodular, as are its identity component $\mathrm{SO}_0(2, 1)$ and the double cover of its identity component, namely $\mathrm{SL}(2, \mathbb{R})$. All these are reasonable choices of gauge group when treating 3d — or more properly, $(2 + 1)$d — Lorentzian gravity as a BF theory.

Given a unimodular Lie group, Haar measure is typically not the only measure invariant under conjugation: we can multiply Haar measure by any function that only depends on the conjugacy class. As an extreme example, we can even try to multiply Haar measure by a "delta function" supported on one conjugacy class. More precisely, we can look for a conjugation-invariant measure supported on a single conjugacy class of G. In this case, we might as well be working not with G, but with just the conjugacy class. It turns out that in the case of 3d quantum gravity, this amounts to studying identical particles *of a specified mass.* This leads us to our next subject: quandle field theory.

5 Quandle field theory

In the previous section, we considered BF theory in 3d, and were led to a natural action of the braid group B_n on the space G^n for any group G. Notice that we did not actually need the multiplication in G to define this action; we only needed the operation of *conjugation.* This suggests that we can work more generally, replacing the group G by some algebraic structure that captures the properties of conjugation. Such a thing is called a "quandle".

More precisely, a "quandle" is a set Q equipped with two binary operations $\rhd : Q \times Q \to Q$ and $\lhd : Q \times Q \to Q$ called "left" and "right conjugation," which satisfy:

(i) left idempotence: $x \rhd x = x$,
(i′) right idempotence: $x \lhd x = x$,
(ii) left inverse law: $x \rhd (y \lhd x) = y$,
(ii′) right inverse law: $(x \rhd y) \lhd x = y$,
(iii) left distributive law: $x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z)$,
(iii′) right distributive law: $(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$,

for all $x, y, z \in Q$. In general, the operations of left and right conjugation in a quandle are neither associative nor commutative.

Quandles were first introduced as a source of knot invariants by David Joyce [19] in 1982. Many examples of quandles can be found in the work of Fenn and Rourke [11] and other authors [6, 19, 20]. For us, the most important examples come from taking a group G, letting Q be any union of conjugacy classes of G, and making Q into a quandle with

$$g \rhd h = ghg^{-1}, \quad h \lhd g = g^{-1}hg.$$

We are especially interested in the cases, where Q is either the whole group G or a single conjugacy class.

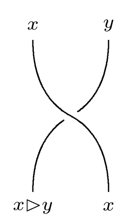
We can do some of the same things with quandles as with groups. For example, we can define a "topological quandle" to be a topological space that is also a quandle in such a way that the quandle operations $\rhd$ and $\lhd$ are continuous [36]. If G is a Lie group and $Q \subseteq G$ is a conjugacy class, Q becomes a topological quandle with the induced topology.

Given a topological quandle Q, we define an "invariant measure" on Q to be a Borel measure that is invariant under left conjugation by any element of Q — or equivalently, invariant under right conjugation by any element of Q. This implies that

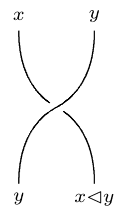
$$\begin{aligned}\int f(x)\, d\mu(x) &= \int f(q \rhd x)\, d\mu(x) \\ &= \int f(x \lhd q)\, d\mu(x)\end{aligned}$$

for any $q \in Q$ and any integrable function f on Q. As noted earlier, invariant measures on quandles are far from unique in general. In particular, we may multiply an invariant measure on a Lie group by any class function and obtain a new invariant measure.

In the previous section, we saw that the n-strand braid group B_n acts on G^n for any group G. But, since our argument relied only on properties of conjugation, it works just as well for a quandle. The idea is that we can braid two elements of a quandle past each other using left conjugation:



The inverse braiding uses right conjugation:



It is well known that with these rules, the braid group relations follow from the quandle axioms. So, generalizing our result from the previous section, we easily obtain:

Theorem 5.1. *Suppose Q is a topological quandle equipped with an invariant measure. Then there is a unitary representation ρ of the braid group B_n on $L^2(Q^n)$ given by*

$$(\rho(\sigma)\psi)(q_1, \ldots, q_n) = \psi((q_1, \ldots, q_n)\sigma)$$

for all $\sigma \in B_n$, where B_n has a right action on Q^n given by:

$$(q_1, \ldots, q_i, q_{i+1}, \ldots, q_n)\sigma_i = (q_1, \ldots, q_i \triangleright q_{i+1}, q_i, \ldots, q_n).$$

There is also a unitary operator $U(q)$ on $L^2(Q^n)$ for each element $q \in Q$, given by

$$(U(q)\psi)(q_1, \ldots, q_n) = \psi(q \triangleright q_1, \ldots, q \triangleright q_n).$$

The operators $U(q)$ represent gauge transformations when Q is a group, so we can think of them as representing some sort of "gauge transformation" even when Q is a quandle. Of course, if Q is a conjugacy class in a group G, there will be gauge transformations even for elements of G that do not lie in Q.

It is instructive to work out the details in the case of $(2+1)$-dimensional quantum gravity. This theory can be viewed as a BF theory with G being the connected Lorentz group $\mathrm{SO}_0(2,1)$, or perhaps better, its double cover $\mathrm{SL}(2,\mathbb{R})$. In either case, we shall see that different conjugacy classes Q describe different types of spinless particles. The Hilbert space for n particles of this type is $L^2(Q^n)$, and Theorem 3 describes the exotic statistics and gauge invariance of this n-particle system.

In quantum field theory without gravity on 3d Minkowski spacetime, we can describe the energy-momentum of a particle by an element $p \in \mathfrak{sl}(2,\mathbb{R})$:

$$p = \begin{pmatrix} p_x & p_y + E \\ p_y - E & -p_x \end{pmatrix}$$

Note that

$$\det p = E^2 - p_x^2 - p_y^2.$$

The adjoint action of $\mathrm{SL}(2,\mathbb{R})$ on its Lie algebra:

$$\begin{aligned} \mathrm{SL}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) &\to \mathfrak{sl}(2,\mathbb{R}) \\ (g,p) &\mapsto gpg^{-1} \end{aligned}$$

preserves the determinant of p. So, the adjoint action gives an action of $\mathrm{SL}(2,\mathbb{R})$ as Lorentz transformations on the space of energy-momenta. As explained in the Introduction, an orbit of this action is just a type of spin-zero particle.

When we turn on gravity, we must describe energy-momenta not by elements of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ but by elements of the group $\mathrm{SL}(2,\mathbb{R})$. Particle types are then described not by adjoint orbits but by conjugacy classes $Q \subseteq \mathrm{SL}(2,\mathbb{R})$. However, this new description is compatible with the old one, at least for energy-momenta that are small compared to the Planck energy $2\pi/\kappa$. The reason is that we can identify group elements near the identity with Lie algebra elements via the map

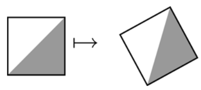
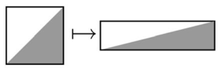
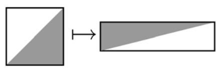
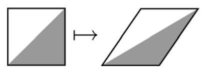
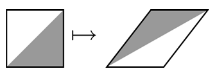
$$\begin{aligned} \mathfrak{sl}(2,\mathbb{R}) &\longrightarrow \mathrm{SL}(2,\mathbb{R}) \\ p &\longmapsto \exp(\kappa p) \end{aligned}$$

This maps any adjoint orbit of $\mathfrak{sl}(2,\mathbb{R})$ into a conjugacy class of $\mathrm{SL}(2,\mathbb{R})$. Indeed, it gives a one-to-one correspondence between the set of adjoint orbits close to $0 \in \mathfrak{sl}(2,\mathbb{R})$ and the set of conjugacy classes close to $1 \in \mathrm{SL}(2,\mathbb{R})$. But, as mentioned in the Introduction, important differences show up for large energy-momenta.

To understand the conjugacy classes in $\mathrm{SL}(2,\mathbb{R})$, it is handy to use the representation

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a+b & c+d \\ c-d & a-b \end{pmatrix} \; : \; a,b,c,d \in \mathbb{R},\; a^2 - b^2 - c^2 + d^2 = 1 \right\}$$

which says $\mathrm{SL}(2,\mathbb{R})$ is geometrically a "unit hyperboloid" in a space of signature $(+--+)$. Since conjugate matrices have the same eigenvalues, the *trace* and thus the number a is an invariant of conjugacy classes. It is not a complete invariant, but it is except for matrices with $\mathrm{tr}\, g = \pm 2$. Every matrix in $\mathrm{SL}(2,\mathbb{R})$ is conjugate to one of these five kinds:

		conjugate to...	trace
rotations		$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$	$-2 \le \mathrm{tr}\, g \le 2$
boosts		$\begin{pmatrix} \mathrm{e}^{\alpha} & 0 \\ 0 & \mathrm{e}^{-\alpha} \end{pmatrix}$	$\mathrm{tr}\, g \ge 2$
antiboosts		$\begin{pmatrix} -\mathrm{e}^{\alpha} & 0 \\ 0 & -\mathrm{e}^{-\alpha} \end{pmatrix}$	$\mathrm{tr}\, g \le -2$
shears		$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$	$\mathrm{tr}\, g = 2$
antishears		$\begin{pmatrix} -1 & \alpha \\ 0 & -1 \end{pmatrix}$	$\mathrm{tr}\, g = -2.$

Some explanation of this table is in order. Every "rotation" maps to a rotation in the connected Lorentz group $\mathrm{SO}_0(2,1)$: in other words, a transformation that preserves a timelike vector in 3d Minkowski spacetime. Similarly, every "boost" maps to a transformation that preserves a spacelike vector, and every "shear" maps to a transformation that preserves a lightlike vector. Since the two-to-one map from $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{SO}_0(2,1)$ maps the matrix -1 to the identity, "antiboosts" get mapped to the same elements as boosts, and "antishears" get mapped to the same elements as shears. (An "antirotation" would be just another rotation.)

The above chart counts certain conjugacy classes more than once. First of all, there is an overlap at $\mathrm{tr}\, g = 2$, since the identity rotation is also the identity shear and identity boost. Similarly, there is an overlap at $\mathrm{tr}\, g = -2$, since a rotation by π is also an antishear and an antiboost. Finally, all shears (resp. antishears) with $\alpha > 0$ are conjugate to each other, and all shears (resp. antishears) with $\alpha < 0$ are conjugate to each other. These are all the redundancies.

Knowing this, we can list all the conjugacy classes in $\mathrm{SL}(2,\mathbb{R})$ without any redundancies. However, it is less tiresome to list the conjugacy classes in $\mathrm{SO}_0(2,1)$, since the elements $\pm g \in \mathrm{SL}(2,\mathbb{R})$ get identified in $\mathrm{SO}_0(2,1)$, so we do not need to worry about "antiboosts" and "antishears".

Here are all the conjugacy classes in $\mathrm{SO}_0(2,1)$, and the corresponding five types of spin-zero particles.

1. For any $0 < m < 2\pi/\kappa$ there is a conjugacy class containing the image of
$$\begin{pmatrix} \cos\kappa m/2 & -\sin\kappa m/2 \\ \sin\kappa m/2 & \cos\kappa m/2 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$
This corresponds to a tardyon of mass m.
2. For any $0 < m < \infty$ there is a conjugacy class containing the image of
$$\begin{pmatrix} \mathrm{e}^{\kappa m/2} & 0 \\ 0 & \mathrm{e}^{-\kappa m/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$
This corresponds to a tachyon of mass im.
3. There is a conjugacy class containing the image of
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$
This corresponds to a positive-energy luxon.
4. There is a conjugacy class containing the image of
$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$
This corresponds to a negative-energy luxon.
5. There is a conjugacy class containing the image of
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$
This corresponds to a particle of vanishing energy-momentum.

The factors of 1/2 here arise from the double cover $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}_0(2,1)$. As explained in the Introduction, masses of tardyons really take values in the circle $\mathbb{R}/(2\pi/\kappa)\mathbb{Z}$.

Each conjugacy class $Q \subseteq \mathrm{SO}_0(2,1)$ admits an invariant measure which is unique up to an overall scale. So, Theorem 5.1 applies: we can form a Hilbert space $L^2(Q)$ for particles of type Q, and more generally an n-particle Hilbert space $L^2(Q^n)$, on which the braid group and $\mathrm{SO}_0(2,1)$ gauge transformations act as unitary transformations.

6 Strings in 4d BF theory

All the work in the previous two sections generalizes nicely from 3 to 4 dimensions, using the loop braid group as a substitute for the braid group.

Let space be $\mathbb{R}^3$ with n unknotted and unlinked circles removed:

$$X = S - \Sigma, \quad S = \mathbb{R}^3, \quad \Sigma = \ell_1 \cup \cdots \cup \ell_n.$$

The fundamental group of X is the free group on n generators, so for any Lie group G we have

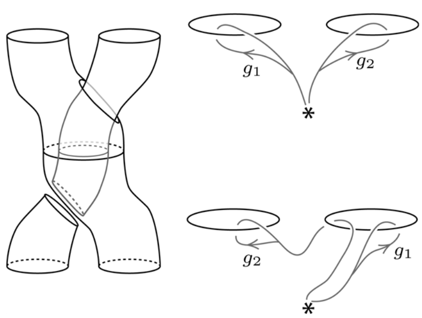
$$\hom(\pi_1(X), G) = G^n.$$

As explained in Section 3, a point in this space represents a G-bundle with flat connection over X, mod gauge transformations that equal the identity at a chosen basepoint. The n elements of G describing this point are just the holonomies around the circles $\ell_1, \ldots, \ell_n$. Physically, we think of these circles as string-like "topological defects" where the flat connection on space becomes singular.

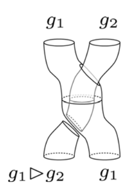
We explained quite generally in Section 3 how the motion group $\mathrm{Mo}(S, \Sigma)$ acts on $\hom(\pi_1(X), G)$. In the present case, the motion group is just the loop braid group LB_n, and its generators act on $\hom(\pi_1(X), G) = G^n$ as follows:

$$(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n)s_i = (g_1, \ldots, g_{i+1}, g_i, \ldots, g_n),$$
$$(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n)\sigma_i = (g_1, \ldots, g_i g_{i+1} g_i^{-1}, g_i, \ldots, g_n).$$

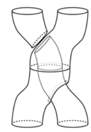
This is easy to see using pictures. For example, the generator σ_1 has the following effect:



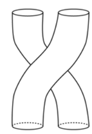
By an argument like the one we made in Section 3 for the ordinary braid group action in 3d BF theory, it follows that σ_1 acts on the holonomies g_1, g_2 by switching them while left conjugating g_2 by g_1:



Similarly, the inverse of σ_1 acts to switch the group elements while right conjugating g_1 by g_2:



The generator s_1 simply switches the holonomies g_1 and g_2:



It is easy to see that if G is unimodular, this action of the loop braid group on G^n gives rise to a unitary representation of the loop braid group on $L^2(G^n)$. And, just as in 3d, we can generalize this result to the case of a quandle:

Theorem 6.1. *Suppose Q is a topological quandle equipped with an invariant measure. Then there is a unitary representation ρ of the loop braid group LB_n on $L^2(Q^n)$ given by*

$$(\rho(\sigma)\psi)(q_1, \ldots, q_n) = \psi((q_1, \ldots, q_n)\sigma)$$

for all $\sigma \in \mathrm{LB}_n$, where LB_n has a right action on Q^n given by:

$$(q_1, \ldots, q_i, q_{i+1}, \ldots, q_n)s_i = (q_1, \ldots, q_{i+1}, q_i, \ldots, q_n)$$
$$(q_1, \ldots, q_i, q_{i+1}, \ldots, q_n)\sigma_i = (q_1, \ldots, q_i \triangleright q_{i+1}, q_i, \ldots, q_n)$$

There is also a unitary operator $U(q)$ on $L^2(Q^n)$ for each element $q \in Q$, given by

$$(U(q)\psi)(q_1, \ldots, q_n) = \psi(q \triangleright q_1, \ldots, q \triangleright q_n)$$

Proof. While the proof is straightforward, it is worth comparing Theorem 5.1 of Fenn, Rimányi and Rourke [10]. This says that the braid permutation group BP_n is the group of automorphisms of the free quandle on n generators. Since BP_n is isomorphic to the loop braid group LB_n, it follows that LB_n acts on Q^n for any quandle Q. The action is precisely as above. □

Let us illustrate these ideas in the case where the gauge group is the connected Lorentz group $\mathrm{SO}_0(3, 1)$ or its double cover $\mathrm{SL}(2, \mathbb{C})$. With either of these gauge groups, BF theory in 4d is sometimes called "topological gravity".

In Section 5, we recalled the classification of conjugacy classes in $\mathrm{SO}_0(2,1)$ and its double cover $\mathrm{SL}(2,\mathbb{R})$. The classification for $\mathrm{SO}_0(3,1)$ and its double cover $\mathrm{SL}(2,\mathbb{C})$ is very similar, but simpler, because every complex number has a square root. It is also more familiar, since any element of

$$\mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;:\; a,b,c,d \in \mathbb{C},\; ad - bc = 1 \right\}$$

gives a fractional linear transformation

$$z \longmapsto \frac{az+b}{cz+d}.$$

Such transformations are precisely the conformal transformations of the Riemann sphere. Note that both 1 and -1 in $\mathrm{SL}(2,\mathbb{C})$ map to the identity fractional linear transformation, so the conformal group of the Riemann sphere is

$$\mathrm{SL}(2,\mathbb{C})/\{\pm 1\} \cong \mathrm{SO}_0(3,1).$$

Indeed, Lorentz transformations can be thought of as conformal transformations of the "celestial sphere": the set of light rays through an observer at the origin [31]. A list of conjugacy classes in $\mathrm{SO}_0(3,1)$ can thus be read off from the well-known classification of conformal transformations of the Riemann sphere [28]. But in fact, it is easy enough to construct this list from first principles.

Every element of $\mathrm{SO}_0(3,1)$ is either conjugate to the image of the shear

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$

or conjugate to the image of

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$

for some $\lambda \neq 0$. The conjugacy class of the latter element is unchanged if we make the replacement $\lambda \mapsto 1/\lambda$, and its image in $\mathrm{SO}_0(3,1)$ is unchanged if we make the replacement $\lambda \mapsto -\lambda$. These replacements (and their composite) are the only ways we can change λ without changing the conjugacy class of the corresponding element of $\mathrm{SO}_0(3,1)$. Using this, we can see there are five types of conjugacy classes in $\mathrm{SO}_0(3,1)$.

1. For any real m with $0 < m \le \pi/\kappa$ there is a conjugacy class containing the image of

$$\begin{pmatrix} e^{i\kappa m/2} & 0 \\ 0 & e^{-i\kappa m/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$

An element conjugate to one of this form is called "elliptic".

2. For any purely imaginary m with $0 < \mathrm{Im}(m) < \infty$ there is a conjugacy class containing the image of
$$\begin{pmatrix} \mathrm{e}^{i\kappa m/2} & 0 \\ 0 & \mathrm{e}^{-i\kappa m/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$
An element conjugate to one of this form is called "hyperbolic".
3. For any $m \in \mathbb{C}$ with $0 < \mathrm{Re}(m) < 2\pi/\kappa$ and $0 < \mathrm{Im}(m) < \infty$ there is a conjugacy class containing the image of
$$\begin{pmatrix} \mathrm{e}^{i\kappa m/2} & 0 \\ 0 & \mathrm{e}^{-i\kappa m/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$
An element conjugate to one of this form is called "loxodromic".
4. There is a conjugacy class containing the image of
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$
An element conjugate to one of this form is called "parabolic".
5. There is a conjugacy class containing the image of
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$
This class contains only the identity element.

Now let us return to BF theory with gauge group $\mathrm{SO}_0(3,1)$, taking space to be $\mathbb{R}^3$ with a collection of unknotted unlinked circles $\ell_1, \ldots, \ell_n$ removed. For brevity, let us call these circles "closed strings". A flat connection on space will have some holonomy $g_i \in \mathrm{SO}_0(3,1)$ around the ith string. The above list of conjugacy classes lets us list possible "types" of strings, just as we used conjugacy classes in $\mathrm{SO}_0(2,1)$ to list types of point particles in 3d gravity:

1. If g_i is elliptic, it acts on Minkowski spacetime as a spatial rotation in some reference frame. In this reference frame, parallel transport around the string ℓ_i is a spatial rotation by some angle $0 < \theta \le \pi$ about some axis. (A rotation by an angle $\theta > \pi$ is a rotation by $\theta - \pi$ about the opposite axis.) This angle θ is proportional to the real number m which appears in item 1 of the above list, as follows:
$$\theta = \kappa m.$$
By analogy to 3d gravity, we could call the string a "tardyon" in this case, and call the number m its "mass density". The number m is real and takes values $0 < m \le \pi/\kappa$.
2. If g_i is hyperbolic, it acts on Minkowski spacetime as a boost in some reference frame. In this reference frame, parallel transport around the string ℓ_i is a boost with rapidity $0 < \beta < \infty$ along some axis. The

rapidity β is proportional to the imaginary number m which appears in item 2 of the above list, as follows:

$$\beta = \kappa \,\mathrm{Im}(m).$$

By analogy to 3d gravity, we could call the string a "tachyon" in this case, and call the number m its "mass density". The number m is purely imaginary and takes values in the upper half of the imaginary axis: $0 < \mathrm{Im}(m) < \infty$.

3. If g_i is loxodromic, it acts on Minkowski spacetime as a combined rotation and boost about the same axis in some reference frame. In this reference frame, parallel transport around the string ℓ_i is a combination of a rotation by an angle $0 < \theta < 2\pi$ and a boost with rapidity $0 < \beta < \infty$ about the same axis, where

$$\theta = \kappa \,\mathrm{Re}(m), \quad \beta = \kappa \,\mathrm{Im}(m).$$

This case has no analogue in 3d gravity. We can still think of m as some sort of mass density, but it is complex, with $0 < \mathrm{Re}(m) < 2\pi/\kappa$ and $0 < \mathrm{Im}(m) < \infty$.
4. If g_i is parabolic, it acts on Minkowski spacetime as a Lorentz transformation fixing a single null vector. By analogy to 3d gravity, we could call the string a "luxon" in this case, and say $m = 0$.
5. If g_i is the identity, we can say the string carries no energy-momentum, and again say $m = 0$.

Each of these conjugacy classes $Q \subseteq \mathrm{SO}_0(3,1)$ is a quandle. The question then arises which of these quandles admits an invariant measure, and whether this measure is unique up to scale. One can work this out on a case-by-case basis.

One important case is when Q is the conjugacy class containing all rotations by some fixed angle $0 < \theta < \pi$. This conjugacy class corresponds to a "tardyonic" closed string with a given mass density $0 < m \leq \pi/\kappa$. It is easy to see that this conjugacy class Q indeed admits an invariant measure. To see this, note that to specify a rotation by the angle θ one must first pick a future-pointing unit timelike vector $u \in \mathbb{R}^4$, to split Minkowski spacetime into space and time, and then pick a unit spacelike vector v orthogonal to u, to serve as the axis of rotation. The allowed choices of u lie in the hyperboloid

$$H = \{(t,x,y,z)\colon \; t^2 - x^2 - y^2 - z^2 = 1, \; t > 0\}.$$

This hyperboloid H is a Riemannian submanifold of $\mathbb{R}^4$. An allowed choice of u together with v amounts to a point in SH, the unit sphere bundle of H. So, we have $Q \cong SH$. Since the unit sphere bundle of a Riemannian manifold is itself a Riemannian manifold in a natural way, we get a well-defined Lebesgue measure on SH and thus Q, which is invariant under $\mathrm{SO}_0(3,1)$, since our construction respected the Lorentz group symmetry.

Given an invariant measure on Q, we obtain a Hilbert space $L^2(Q^n)$ for n strings of type Q. Note that we do not try to "symmetrize" the states in this Hilbert space. Instead, we describe the statistics using a representation of the loop braid group, following Theorem 6.1. Of course, one should work out the details explicitly, but we leave this for future research.

7 Conclusions

Much more needs to be done to ferret out the physical significance of the theory we have been considering here. First, there are some nice projects for the mathematician. One should determine for various Lie groups G which conjugacy classes $Q \subseteq G$ admit invariant measures, and when these invariant measures are unique up to an overall scale. We have only done this for $G = \mathrm{SO}_0(2,1)$, but for applications to 4d physics other groups are more relevant — especially the Lorentz, Poincaré, deSitter and anti-deSitter groups. Then, given a conjugacy class $Q \subseteq G$ with an invariant measure, one should work out explicitly the representation of the loop braid group LB_n on the Hilbert space $L^2(Q^n)$, if possible decomposing this representation into irreducibles, so as to understand in detail the workings of the exotic statistics. It would also be interesting to study how, in the $\kappa \to 0$ limit, the exotic statistics approach ordinary bosonic statistics.

For the physicist, one interesting project would be to study the *dynamics* and *interactions* of the "closed strings" discussed at the purely kinematical level here. In a paper with Perez [3], we describe a Lagrangian whereby these objects can couple to the fields in BF theory. We work out the equations of motion and propose a strategy for quantizing the resulting theory, analogous to the known quantization of point particles coupled to 3d gravity [33].

A more ambitious project would be to generalize all our results from collections of unlinked unknotted circles to arbitrary embedded graphs. Finally, a still more ambitious project would be to use these ideas as part of a perturbative expansion of MacDowell–Mansouri gravity about 4d BF theory, as proposed by Freidel and Starodubtsev [15].

Appendix A

Here we present a proof of Theorem 2.2 on p. 15.

Proof. We begin by demonstrating that the relations in the statement of Theorem 2.2 follow from those given in Theorem 2.1. It clearly suffices to show that the relations in (b′) and (c′) follow from the relations in (a), (b) and (c).

In what follows, we make frequent use of the correspondence between generators σ_{ij} of PLB_n and generators σ_i of LB_n as given in (2.13) and (2.14). In fact, since these follow from different relations in the presentation of Theorem 2.1, it suffices for our purposes to take one expression from each of these, say

$$\sigma_{ij} = \begin{cases} s_i s_{i+1} \cdots s_{j-1} \sigma_{j-1} s_{j-2} \cdots s_{i+1} s_i & \text{for } i < j, \\ s_j s_{j+1} \cdots s_{i-2} \sigma_{i-1} s_{i-1} \cdots s_{j+1} s_j & \text{for } i > j. \end{cases} \tag{A.1}$$

These representations of σ_{ij} follow directly from the definition of σ_i along with the relations (2.7), (2.9), and (2.10).

- Relation (2.20): We wish to show that $s_j \sigma_i = \sigma_i s_j$ for $|i-j| > 1$. To check this, we begin with relation (2.8) in the form:

  $$s_j \sigma_{i(i+1)} = \sigma_{i(i+1)} s_j,$$

  where $|i-j| > 1$. Using (A.1) above, this becomes:

  $$s_j s_i \sigma_i = s_i \sigma_i s_j.$$

  Applying relation (2.1) of to the left-hand side and then cancelling s_i from each side gives $s_j \sigma_i = \sigma_i s_j$ when $|i-j| > 1$, which is (2.20).
- Relation (2.21): We wish to show that $s_i s_{i+1} \sigma_i = \sigma_{i+1} s_i s_{i+1}$ for $1 \leq i \leq n-2$. Beginning with relation (2.9) with $j = i+1$, we obtain:

  $$s_{i+1} \sigma_{i(i+1)} = \sigma_{i(i+2)} s_{i+1}.$$

  By (A.1) this gives:

  $$s_{i+1} s_i \sigma_i = s_i s_{i+1} \sigma_{i+1} s_i s_{i+1}.$$

  Multiplying on the right by $s_{i+1} s_i$ and on the left by $s_i s_{i+1}$, we have:

  $$\begin{aligned} \sigma_i s_{i+1} s_i &= s_i s_{i+1} s_i s_{i+1} \sigma_{i+1} \\ &= s_i s_i s_{i+1} s_i \sigma_{i+1} && \text{by (2.2)} \\ &= s_{i+1} s_i \sigma_{i+1} && \text{by (2.3)} \end{aligned}$$

  This can be rewritten as $s_i s_{i+1} \sigma_i = \sigma_{i+1} s_i s_{i+1}$, which is (2.21).
- Relation (2.22): We wish to show that $\sigma_i \sigma_{i+1} s_i = s_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n-2$. To verify this we use relation (2.5) with $i, i+1$ and $i+2$,

which gives:

$$\sigma_{i(i+2)}\sigma_{(i+1)(i+2)} = \sigma_{(i+1)(i+2)}\sigma_{i(i+2)}.$$

By (A.1) this becomes:

$$(s_i s_{i+1}\sigma_{i+1}s_i)(s_{i+1}\sigma_{i+1}) = (s_{i+1}\sigma_{i+1})(s_i s_{i+1}\sigma_{i+1}s_i).$$

Applying relation (2.21) on the left hand side gives:

$$s_i s_{i+1} s_i s_{i+1}\sigma_i\sigma_{i+1} = (s_{i+1}\sigma_{i+1})(s_i s_{i+1}\sigma_{i+1}s_i).$$

Multiplying by $s_i s_{i+1} s_i$ on the left produces:

$$\begin{aligned} s_{i+1}\sigma_i\sigma_{i+1} &= s_i s_{i+1} s_i s_{i+1}\sigma_{i+1}s_i s_{i+1}\sigma_{i+1}s_i \\ &= s_{i+1}s_i\sigma_{i+1}s_i s_{i+1}\sigma_{i+1}s_i && \text{by (2.16)} \\ &= \sigma_i\sigma_{i+1}s_i && \text{by (2.21)} \end{aligned}$$

which is (2.22).

- Relation (2.18): We wish to show that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$. To do so, we use relation (2.4) with $i, i+1, j, j+1$, which are clearly all distinct for $|i-j| > 1$. We therefore have:

$$\sigma_{i(i+1)}\sigma_{j(j+1)} = \sigma_{j(j+1)}\sigma_{i(i+1)},$$

which, by (A.1), becomes:

$$s_i\sigma_i s_j\sigma_j = s_j\sigma_j s_i\sigma_i.$$

Applying (2.20) to both sides of this equation, followed by relation (2.1), we obtain:

$$\sigma_i\sigma_j = \sigma_j\sigma_i$$

with $|i-j| > 1$, which is (2.18).

- Relation (2.19): We wish to show that $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-2$. To check this we start with relation (2.6) with $i, i+1$, and $i+2$, which are clearly all distinct. Thus, we have:

$$\sigma_{i(i+1)}\sigma_{(i+2)(i+1)}\sigma_{i(i+2)} = \sigma_{i(i+2)}\sigma_{(i+2)(i+1)}\sigma_{i(i+1)}.$$

Using the correspondence given in (A.1) and cancelling s_i from both sides, we obtain:

$$\begin{aligned} s_{i+1}\sigma_{i+1}s_i\sigma_{i+1}s_{i+1}s_i\sigma_i &= \sigma_i\sigma_{i+1}s_{i+1}s_i s_{i+1}\sigma_{i+1}s_i \\ &= \sigma_i\sigma_{i+1}s_i s_{i+1}s_i\sigma_{i+1}s_i && \text{by (2.2)} \\ &= \sigma_i\sigma_{i+1}s_i\sigma_i s_{i+1} && \text{by (2.21), (2.3)} \\ &= s_{i+1}\sigma_i\sigma_{i+1}s_{i+1} && \text{by (2.22).} \end{aligned}$$

Cancelling s_{i+1} on the left and multiplying by s_{i+1} on the right produces:

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}s_i\sigma_{i+1}s_{i+1}s_i\sigma_i s_{i+1} \\ &= \sigma_{i+1}\sigma_i\sigma_{i+1} \end{aligned}$$

where in the last step we used (2.5) in the form $s_i\sigma_i\sigma_{i+1}s_{i+1} = \sigma_{i+1}s_{i+1}s_i\sigma_i$. This is (2.19).

The loop braid group thus has generators that satisfy all of the relations of the braid permutation group. It remains to show that these relations are sufficient, which we do by demonstrating that the relations in the statement of Theorem 2.1 follow from those given in Theorem 2.2. In this direction of the proof it is convenient to use both of the equivalent expressions (2.13) and (2.14) as the correspondence between generators σ_i and σ_{ij}.

- Relation (2.7): This relation simply says $s_i\sigma_{i(i+1)} = \sigma_{(i+1)i}s_i$, which is immediate from (A.1) since both sides are equal to σ_i.
- Relation (2.8): We wish to show $s_k\sigma_{ij} = \sigma_{ij}s_k$, whenever $i, j, k, k+1$ are distinct. When either $k+1 < i < j$ or $i < j < k$, s_k commutes with each of the factors in the expansion

$$\sigma_{ij} = s_i s_{i+1} \cdots s_{j-1}\sigma_{j-1}s_{j-2}\cdots s_{i+1}s_i$$

by (2.15) and (2.20). Similarly, when $k+1 < j < i$ or $j < i < k$, s_k commutes with each factor in

$$\sigma_{ij} = s_j s_{j+1} \cdots s_{i-2}\sigma_{i-1}s_{i-1}\cdots s_{j+1}s_j.$$

When $i < k < k+1 < j$ we also need two applications of (2.16):

$$\begin{aligned}
s_k\sigma_{ij} &= s_k s_i \cdots s_{j-1}\sigma_{j-1}s_{j-2}\cdots s_i \\
&= s_i \cdots s_{k-2}s_k s_{k-1}s_k s_{k+1}\cdots s_{j-1}\sigma_{j-1}s_{j-2}\cdots s_i && \text{by (2.15)} \\
&= s_i \cdots s_{k-2}s_{k-1}s_k s_{k-1}s_{k+1}\cdots s_{j-1}\sigma_{j-1}s_{j-2}\cdots s_i && \text{by (2.16)} \\
&= s_i \cdots s_{k-2}s_{k-1}s_k s_{k+1}\cdots s_{k-1}s_{j-1}\sigma_{j-1}s_{j-2}\cdots s_i && \text{by (2.15)} \\
&= s_i \cdots s_{j-1}\sigma_{j-1}s_{j-2}\cdots s_{k+1}s_{k-1}s_k s_{k-1}s_{k-2}\cdots s_i && \text{by (2.15), (2.20)} \\
&= s_i \cdots s_{j-1}\sigma_{j-1}s_{j-2}\cdots s_{k+1}s_k s_{k-1}s_k s_{k-2}\cdots s_i && \text{by (2.16)} \\
&= \sigma_{ij}s_k && \text{by (2.15)}
\end{aligned}$$

The only remaining case is $j < k < k+1 < i$, which is handled similarly.

- Relation (2.9): We wish to show that $s_j\sigma_{ij} = \sigma_{i(j+1)}s_j$ whenever $i \neq j+1$. When $i < j$ we have:

$$\begin{aligned}
s_j\sigma_{ij} &= s_i s_{i+1}\cdots s_j s_{j-1}\sigma_{j-1}s_{j-2}\cdots s_{i+1}s_i && \text{by (2.15)} \\
&= s_i s_{i+1}\cdots s_{j-1}s_j s_{j-1}s_j\sigma_{j-1}s_{j-2}\cdots s_{i+1}s_i && \text{by (2.16)} \\
&= s_i s_{i+1}\cdots s_{j-1}s_j\sigma_j s_{j-1}s_j s_{j-2}\cdots s_{i+1}s_i && \text{by (2.20)} \\
&= \sigma_{i(j+1)}s_j && \text{by (2.15), (A.1)}
\end{aligned}$$

and the case $i > j+1$ is similar.
- Relation (2.10): The proof that $s_i\sigma_{ij} = \sigma_{(i+1)j}s_i$ is essentially the same as the proof of (2.9) above.

- Relation (2.4): We wish to show $\sigma_{ij}\sigma_{k\ell} = \sigma_{k\ell}\sigma_{ij}$, whenever i, j, k, and ℓ are distinct. Naively there are 4! orderings of i, j, k, ℓ to consider, but symmetry of the relation implies only eight are independent. All cases are proved similarly; we demonstrate only the case $i < j < k < \ell$:

$$\begin{aligned}
\sigma_{ij}\sigma_{k\ell} &= (s_i \cdots s_{j-1}\sigma_{j-1}s_{j-2} \cdots s_i)(s_k \cdots s_{\ell-1}\sigma_{\ell-1}s_{\ell-2} \cdots s_k) \\
&= s_k \cdots s_{\ell-1}(s_i \cdots s_{j-1}\sigma_{j-1}s_{j-2} \cdots s_i) \\
&\quad (\sigma_{\ell-1}s_{\ell-2} \cdots s_k) \text{ by (2.15), (2.20)} \\
&= s_k \cdots s_{\ell-1}\sigma_{\ell-1}(s_i \cdots s_{j-1}\sigma_{j-1}s_{j-2} \cdots s_i) \\
&\quad (s_{\ell-2} \cdots s_k) \text{ by (2.20), (2.18)} \\
&= (s_k \cdots s_{\ell-1}\sigma_{\ell-1}s_{\ell-2} \cdots s_k) \\
&\quad (s_i \cdots s_{j-1}\sigma_{j-1}s_{j-2} \cdots s_i) \text{ by (2.15), (2.20)} \\
&= \sigma_{k\ell}\sigma_{ij}.
\end{aligned}$$

- Relation (2.5): We wish to show that $\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}$ when i, j, k are distinct. We have three independent cases: $i < j < k$, $i < k < j$, and $k < i < j$. In the case $i < j < k$, we first note that if $j \neq i + 1$, then by (2.8) and (2.9) we have:

$$\begin{aligned}
\sigma_{ik}\sigma_{jk} &= s_{j-1}(\sigma_{ik}\sigma_{(j-1)k})s_{j-1} \\
\text{and } \sigma_{jk}\sigma_{ik} &= s_{j-1}(\sigma_{(j-1)k}\sigma_{ik})s_{j-1}.
\end{aligned}$$

  By repeated application of these facts, it suffices to consider the subcase where $j = i + 1$. Similarly, if $k \neq j + 1$, we can use (2.8) and (2.10) to reduce to the case where $k = j + 1$. Thus it suffices to consider only the cases where i, j, k are consecutive:

$$\begin{aligned}
\sigma_{i(i+2)}\sigma_{(i+1)(i+2)} &= (s_i s_{i+1}\sigma_{i+1}s_i)(s_{i+1}\sigma_{i+1}) \\
&= s_i s_{i+1} s_i s_{i+1}\sigma_i\sigma_{i+1} && \text{by (2.21)} \\
&= s_{i+1} s_i \sigma_i \sigma_{i+1} && \text{by (2.16)} \\
&= s_{i+1} s_i s_{i+1}\sigma_i\sigma_{i+1}s_i && \text{by (2.22)} \\
&= s_{i+1}\sigma_{i+1}s_i s_{i+1}\sigma_{i+1}s_i && \text{by (2.21)} \\
&= \sigma_{(i+1)(i+2)}\sigma_{i(i+2)}.
\end{aligned}$$

  This proves the case $i < j < k$. The remaining two cases are similar.
- Relation (2.6): We wish to show that $\sigma_{ij}\sigma_{kj}\sigma_{ik} = \sigma_{ik}\sigma_{kj}\sigma_{ij}$ when i, j, k are distinct. In light of (2.5) this equation is symmetric under the interchange of i and k, and this symmetry reduces the number of independent cases to 3: $i < j < k$, $i < k < j$, and $j < i < k$. In the case $i < j < k$, we first note that if $j \neq i + 1$, then by (2.8) and (2.9)

we have

$$\begin{aligned}
\sigma_{ij}\sigma_{kj}\sigma_{ik} &= s_{j-1}(\sigma_{i(j-1)}\sigma_{k(j-1)}\sigma_{ik})s_{j-1} \\
\text{and} \quad \sigma_{ik}\sigma_{kj}\sigma_{ij} &= s_{j-1}(\sigma_{ik}\sigma_{k(j-1)}\sigma_{i(j-1)})s_{j-1}
\end{aligned}$$

By repeated application of these facts, it suffices to consider the subcase where $j = i+1$. Similarly, if $k \neq j+1$, we can use (2.8) and (2.10) to reduce to the case where $k = j+1$. Thus it suffices to consider only the cases where i, j, k are consecutive:

$$\begin{aligned}
\sigma_{i(i+1)}\sigma_{(i+2)(i+1)}\sigma_{i(i+2)} &= (s_i\sigma_i)(\sigma_{i+1}s_{i+1})(s_is_{i+1}\sigma_{i+1}s_i) \\
&= s_i\sigma_i\sigma_{i+1}s_is_{i+1}s_i\sigma_{i+1}s_i && \text{by (2.16)} \\
&= s_i\sigma_i\sigma_{i+1}s_i\sigma_is_{i+1} && \text{by (2.21)} \\
&= s_is_{i+1}\sigma_i\sigma_{i+1}\sigma_is_{i+1} && \text{by (2.22)} \\
&= s_is_{i+1}\sigma_{i+1}\sigma_i\sigma_{i+1}s_{i+1} && \text{by (2.19)} \\
&= \sigma_{i(i+2)}\sigma_{i(i+1)}\sigma_{(i+2)(i+1)} \\
&= \sigma_{i(i+2)}\sigma_{(i+2)(i+1)}\sigma_{i(i+1)} && \text{by (2.5)}
\end{aligned}$$

This proves the case of $i < j < k$. The other two independent cases are similar.

Thus, the relations of Theorem 2.2 imply those of Theorem 2.1. □

As pointed out by Blake Winter, one can also prove Theorem 2.2 as follows. Fenn, Rimányi, and Rourke [10] show that the braid permutation group BP_n is isomorphic to the subgroup of $\mathrm{Aut}(F_n)$ generated by all permutations of basis elements, together with all operations of conjugating one basis element by another. Let X be $\mathbb{R}^3$ with unlinked unknotted circles $\ell_1, \ldots, \ell_n$ removed. As we have seen, $\pi_1(X) = F_n$, the free group on n generators, so by the work of Dahm, the loop braid group acts as automorphisms of F_n. Let $D\colon LB_n \to \mathrm{Aut}(F_n)$ be the resulting homomorphism. Goldsmith [17] shows that the image of D is precisely the above subgroup of $\mathrm{Aut}(F_n)$ and that, moreover, D is one-to-one. It follows that LB_n and BP_n are isomorphic. Since Fenn, Rimányi and Rourke prove that BP_n has the presentation given in Theorem 2.2, it follows that LB_n also has this presentation.

After this paper appeared on the arXiv, Sumati Surya pointed out that Theorem 2.2 can also be proved using results of Fuks-Rabinowitz [16] and McCullough and Miller [27].

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