Quantization of strings and branes coupled to BF theory

John C. Baez and Alejandro Perez

Department of Mathematics, University of California, Riverside,
CA 92521, USA
Centre de Physique Théorique, Unité Mixte de Recherche (UMR 6207) du
CNRS et des Universités Aix-Marseille I, Aix-Marseille II, et du Sud
Toulon-Var; laboratoire afilié à la FRUMAM (FR 2291), Campus de
Luminy, 13288 Marseille, France
baez@math.ucr.edu and perez@cpt.univ-mrs.fr

Abstract

BF theory is a topological theory that can be seen as a natural generalization of three-dimensional gravity to arbitrary dimensions. Here, we show that the coupling to point particles that is natural in three dimensions generalizes in a direct way to BF theory in d dimensions coupled to $(d-3)$-branes. In the resulting model, the connection is flat except along the membrane world-sheet, where it has a conical singularity whose strength is proportional to the membrane tension. As a step towards canonically quantizing these models, we show that a basis of kinematical states is given by "membrane spin networks", which are spin networks equipped with extra data where their edges end on a brane.

Interest in the quantization of $2 + 1$ gravity coupled to point particles has been revived in the context of the spin foam [1] and loop quantum

gravity [2] approaches to the nonperturbative and background-independent quantization of gravity. On the one hand this simple system provides a non-trivial example where the strict relation between the covariant and canonical approaches can be demonstrated [3]. On the other hand, intriguing relation-ships with field theories with infinitely many degrees of freedom have been obtained [4].

The idea of generalizing this construction to higher dimensions is very appealing. We will argue that in $3 + 1$ dimensions, the natural objects replacing point particles are strings. This idea has already been studied in a companion paper [5], which treated these strings merely as defects in the gauge field, i.e., places where it has a conical singularity. Here, we propose a specific dynamics for the theory and a strategy for quantizing it. More generally, in d-dimensional spacetime we describe a way to couple $(d - 3)$-branes to BF theory.

To understand this, first recall that in three dimensions, Einstein's equa-tions force the curvature to vanish at every point of spacetime. Therefore, except for global topological excitations, three-dimensional pure gravity does not have local degrees of freedom. However, it is precisely this local rigidity of Einstein's gravity in three dimensions that makes it easy to couple the theory to point particles. The presence of massive point particles in three-dimensional gravity modifies the classical solutions by producing conical cur-vature singularities along the particles' world-lines. With this idea in mind, one can write an action for a single particle coupled to gravity by introducing a source term to the standard action in the first order formalism, namely:

$$S(A, e) = \int_{\mathscr{M}} \text{tr}[e \wedge F(A)] + m \int_{\gamma} \text{tr}[e \, v], \qquad (1)$$

where m is the mass of the particle, v is a fixed unit vector in the Lie algebra $\mathfrak{su}(2)$, and γ is the world-line of the particle. It is easy to see that the previ-ous action leads to equations of motion whose solutions are flat everywhere except for a conical singularity along γ, as desired.

Unfortunately, this action suffers two drawbacks. First, it is no longer invariant under the standard gauge symmetries of pure gravity. Second, there is no explicit dependence on the particle degrees of freedom: one is describing the particle simply as a gauge defect along γ. One can solve both problems in one stroke by adding degrees of freedom for the particles, and choosing an action invariant under an appropriate extension of the gauge group of the system. The result is the Sousa Gerbert action [6] for a spinless point particle of mass m coupled to three-dimensional Riemannian gravity:

$$S(A, e, q, \lambda) = \int_{\mathscr{M}} \text{tr}[e \wedge F(A)] + m \int_{\gamma} \text{tr}[(e + d_A q) \, \lambda v \lambda^{-1}]. \qquad (2)$$

Here v is a fixed unit vector in $\mathfrak{su}(2)$ as before, while the particle's degrees of freedom are described by an $\mathfrak{su}(2)$-valued function q and an SU(2)-valued function λ defined on the world-line γ. The physical interpretation of q is a bit obscure, but we can think of it as 'position in an internal space'. In a similar way, $p = m\lambda v\lambda^{-1}$ represents the particle's momentum, which is treated as an independent variable in this first-order formulation.

This action is invariant under the gauge transformations

$$
\begin{aligned}
e &\longmapsto geg^{-1} \\
A &\longmapsto gAg^{-1} + gdg^{-1} \\
q &\longmapsto gqg^{-1} \\
\lambda &\longmapsto g\lambda,
\end{aligned}
\tag{3}
$$

where $g \in C^{\infty}(\mathscr{M}, \mathrm{SU}(2))$ and

$$
\begin{aligned}
e &\longmapsto e + d_A\eta \\
q &\longmapsto q - \eta,
\end{aligned}
\tag{4}
$$

where $\eta \in C^{\infty}(\mathscr{M}, \mathfrak{su}(2))$. In addition to these gauge symmetries, the action is invariant under $\lambda \mapsto \lambda h$ where $h \in C^{\infty}(\gamma, H)$ and $H \subset \mathrm{SU}(2)$ is the subgroup consisting of elements $g \in \mathrm{SU}(2)$ that stabilize the vector v, meaning that $gvg^{-1} = v$. The action is also invariant under reparametrization of the world-line γ.

A generalization of the naive action (1) to arbitrary dimensions can be constructed as follows. Let G be a Lie group such that its Lie algebra $\mathfrak{g}$ is equipped with an inner product invariant under the adjoint action of G. Let $\mathscr{M}$ be a d-dimensional smooth-oriented manifold equipped with an oriented $(d-2)$-dimensional submanifold $\mathscr{W}$, which we call the "membrane world-sheet". Let P be a principal G-bundle over M; to simplify the discussion we shall assume P is trivial, but this is not essential. One can define the action

$$
S(A, B) = \int_{\mathscr{M}} \mathrm{tr}[B \wedge F(A)] + \tau \int_{\mathscr{W}} \mathrm{tr}[B\, v]
\tag{5}
$$

where τ is the membrane tension, B is a $\mathfrak{g}$-valued $(d-2)$-form, A is a connection on P, v is a fixed but arbitrary unit vector in $\mathfrak{g}$, and "tr" stands for the invariant inner product in $\mathfrak{g}$. The first term is the standard BF theory action, whereas the second is a source term that couples BF theory to the membrane world-sheet.

As with the action in equation (1), the above action is only gauge-invariant if we restrict gauge transformations to be trivial on the membrane world-sheet. We can relax this condition by introducing appropriate

degrees of freedom for the $(d-3)$-brane whose world-sheet is $\mathscr{W}$. The resulting action is

$$S(A, B, q, \lambda) = \int_{\mathscr{M}} \mathrm{tr}[B \wedge F(A)] + \tau \int_{\mathscr{W}} \mathrm{tr}[(B + d_A q)\, \lambda v \lambda^{-1}], \qquad (6)$$

where q is a $\mathfrak{g}$-valued $(d-3)$-form on $\mathscr{W}$ and λ is a G-valued function on $\mathscr{W}$.

This action is invariant under the gauge transformations:

$$
\begin{aligned}
B &\longmapsto gBg^{-1} \\
A &\longmapsto gAg^{-1} + gdg^{-1}, \\
q &\longmapsto gqg^{-1}, \\
\lambda &\longmapsto g\lambda,
\end{aligned}
\qquad (7)
$$

where $g \in C^\infty(\mathscr{M}, G)$ and

$$
\begin{aligned}
B &\longmapsto B + d_A \eta, \\
q &\longmapsto q - \eta,
\end{aligned}
\qquad (8)
$$

where η is any $\mathfrak{g}$-valued $(d-3)$-form. As in the particle case, the action is also invariant under $\lambda \mapsto \lambda h$, where $h \in C^\infty(\mathscr{W}, H)$ and $H \subseteq G$ is the subgroup stabilizing v, and under reparametrization of the membrane world-sheet.

Perhaps the most intuitive equation of motion comes from varying the B field. This says that the connection A is flat except at $\mathscr{W}$:

$$F = -p\delta_{\mathscr{W}}, \qquad (9)$$

where $p = \tau \lambda v \lambda^{-1}$ and $\delta_{\mathscr{W}}$ is the distributional 2-form (current) associated to the membrane world-sheet. So, the membrane causes a conical singularity in the otherwise flat connection A. The strength of this singularity is determined by the field p, which plays the role of a "momentum density" for the brane. Note that while the connection A is singular in the directions transverse to $\mathscr{W}$, it is smooth and indeed flat when restricted to $\mathscr{W}$. Thus the equation of motion obtained from varying q makes sense:

$$d_A p = 0. \qquad (10)$$

This expresses conservation of momentum density.

1   The canonical analysis for $d = 4$

In this section, we work out the other equations of motion as part of a canonical analysis of the action (6). But, in order to simplify the presentation, we restrict for the moment to the case $d = 4$ — that is, the coupling of a string

to four-dimensional BF theory. In Section 3, we generalize the calculations to arbitrary dimensions.

For this canonical analysis, we assume the spacetime manifold is of the form $\mathscr{M} = \Sigma \times \mathbb{R}$. We choose local coordinates (t, x^a) for which Σ is given as the hypersurface $\{t = 0\}$. By definition, x^a with $a = 1, 2, 3$ are local coordinates on Σ. We also choose local coordinates (t, s) on the two-dimensional world-sheet $\mathscr{W}$, where $s \in [0, 2\pi]$ is a coordinate along the one-dimensional string formed by the intersection of $\mathscr{W}$ with Σ. We pick a basis, e_i, of the Lie algebra $\mathfrak{g}$, raise and lower Lie algebra indices using the inner product, and define structure constants by $[e_i, e_j] = c_{ij}^k e_k$.

Performing the standard Legendre transformation one obtains $E_i^a = \epsilon^{abc} B_{ibc}$ as the momentum canonically conjugate to A_a^i. Similarly, $\pi_i^a = \tau(\partial x^a)/(\partial s)\mathrm{tr}[e_i \lambda v \lambda^{-1}]$ is the momentum canonically conjugate to q_a^i. This is a version of the p field mentioned in the previous section. There are also certain fields σ_i defined on the string, which are essentially[1] the momenta conjugate to λ. These phase space variables satisfy the following primary constraints:

$$\sigma_i = 0 \tag{11}$$

$$\pi_i^a = \tau \frac{\partial x^a}{\partial s} \, \mathrm{tr}[e_i \lambda v \lambda^{-1}] \tag{12}$$

$$D_a \pi_i^a = 0 \tag{13}$$

$$D_a E_i^a = \int_{\mathscr{S}} c_{ij}^k q_a^j \pi_k^a \, \delta^{(3)}(x - x_{\mathscr{S}}(s)) \tag{14}$$

$$\epsilon^{abc} F_{ibc}(x) = -\int_{\mathscr{S}} \pi_i^a \, \delta^{(3)}(x - x_{\mathscr{S}}(s)). \tag{15}$$

Here, $\mathscr{S} \subset \Sigma$ denotes the one-dimensional curve representing the string, parametrized by $x_{\mathscr{S}}(s)$. Equation (11) expresses the fact that no time derivatives of λ appear in the action. Equation (12) relates the conjugate momentum π to the field λ. The constraint (13) implies that the momentum π_i^a

---

[1]The field λ takes values in the group G, so if we think of it as a kind of "position" variable, position-momentum pairs lie in T^*G. Each basis element e_i of $\mathfrak{g}$ gives a left-invariant vector field on G and thus a function σ_i on T^*G, which describes one component of the "momentum". The usual symplectic structure on T^*G gives

$$\{\sigma_i, \sigma_j\} = c_{ij}^k \sigma_k,$$

but recalling that λ and thus its conjugate momentum is actually a function of the coordinate s on the string world-sheet, we expect

$$\{\sigma_i(s), \sigma_j(s')\} = c_{ij}^k \sigma_k(s)\delta^{(1)}(s - s')$$

and indeed this holds, in analogy to Sousa Gerbert's [6] calculation for the three-dimensional case.

is covariantly constant along the string. This constraint is redundant, since it could be obtained by taking the covariant derivative of (15) and applying the Bianchi identity. However, this argument requires some regularization due to the presence of the δ distribution on the right. The constraint (14) is the modified Gauss law of BF theory due to the presence of the string.

Finally, (15) is the modified curvature constraint containing the dynamical information of the theory. This constraint implies that the connection A is flat away from the string $\mathscr{S}$. Some care must be taken to correctly intepret the constraint for points on $\mathscr{S}$. By analogy with the case of three-dimensional gravity, the correct interpretation is that the holonomy of an infinitesimal loop circling the string at some point $x \in \mathscr{S}$ is $\exp(-p(x)) \in G$, where $p = \tau \lambda v \lambda^{-1}$ as before. This describes the conical singularity of the connection at the string world-sheet.

The BF phase space variables satisfy the standard commutation relations:

$$\{E_i^a(x), A_b^j(y)\} = \delta_b^a \delta_i^j \, \delta^{(3)}(x-y) \tag{16}$$

$$\{E_i^a(x), E_j^b(y)\} = \{A_a^i(x), A_b^j(y)\} = 0. \tag{17}$$

Concerning the string canonical variables, there are second class constraints (this can be seen from the consistency conditions which say that the time derivatives of (11) and (12) vanish).

They can be solved in a way analogous to the point particle case [6, 7]. As in the latter, this leads to a convenient parametrization of the phase space of the string in terms of the momentum π_i^a and the "total angular momentum" $J_i = c_{ij}^k q_a^j \pi_k^a + \sigma_i$. The Poisson brackets of these variables are given by

$$\{\pi_i^a(s), J_j(s')\} = c_{ij}^k \pi_k^a(s)\delta^{(1)}(s-s') \tag{18}$$

$$\{J_i(s), J_j(s')\} = c_{ij}^k J_k(s)\delta^{(1)}(s-s'). \tag{19}$$

It is important to calculate the Poisson bracket[2]

$$\{J_i(s), \lambda(s')\} = -e_i \lambda(s)\delta^{(1)}(s-s'). \tag{20}$$

The string variables are still subject to the following first class constraints:

$$\mathrm{tr}[e_i \lambda z \lambda^{-1}]J^i = 0 \tag{21}$$

$$\mathrm{tr}[\pi^a \lambda z \lambda^{-1}] = \tau \frac{\partial x^a}{\partial s}\mathrm{tr}[vz], \tag{22}$$

---

[2]The presence of second class constraints in the phase space of the string means that instead of the standard Poisson bracket one should use the appropriate Dirac bracket. However, due to the fact that both π_i^a and J_i commute with the constraints, the Dirac bracket and the standard Poisson bracket coincide for the previous two equations as well as for the following one.

where $z \in \mathfrak{g}$ is such that $[z, v] = 0$. The last constraint is the generalization of the mass shell condition for point particles in three-dimensional gravity.

The Poisson bracket of the string variables with the BF variables is trivial, as well as the Poisson brackets among the π_i^a. In the next section, we shall find a representation of the previous variables as self-adjoint operators acting on an auxiliary Hilbert space $\mathscr{H}_{\mathrm{aux}}$. The constraints above will also be quantized and imposed on $\mathscr{H}_{\mathrm{aux}}$ in order to construct the physical Hilbert space $\mathscr{H}_{\mathrm{phys}}$.

2 Quantization

The auxiliary Hilbert space has the tensor product structure

$$\mathscr{H}_{\mathrm{aux}} = \mathscr{H}_{\mathrm{BF}} \otimes \mathscr{H}_{\mathrm{ST}},$$

where $\mathscr{H}_{\mathrm{BF}}$ and $\mathscr{H}_{\mathrm{ST}}$ are the BF theory and string auxiliary Hilbert spaces, respectively. In the following two subsections, we describe the construction of such Hilbert spaces; in the third we define the so-called kinematical Hilbert space $\mathscr{H}_{\mathrm{kin}}$ by quantizing and imposing all the constraints but the curvature constraint (15). In the last subsection, we sketch the definition of the physical Hilbert space.

2.1 The BF auxiliary Hilbert space

When the group G is compact, we may quantize the BF theory degrees of freedom just as in standard loop quantum gravity. For this reason, we only provide a quick review of how to construct the relevant Hilbert space. A detailed description of this construction can be found in [8].

Briefly, the auxiliary Hilbert space for BF theory, $\mathscr{H}_{\mathrm{BF}}$, is given by $L^2(\bar{\mathcal{A}}, \mu)$ where $\bar{\mathcal{A}}$ is a certain completion of the space $\mathcal{A}$ of smooth connections on P, and μ is the standard gauge- and diffeomorphism-invariant measure on $\bar{\mathcal{A}}$. A bit more precisely, the construction goes as follows.

One starts from a certain algebra Cyl_{BF} of so-called "cylinder functions" of the connection A. The basic building blocks of this algebra are the holonomies $h_\gamma(A) \in G$ of A along paths γ in the manifold Σ representing space:

$$h_\gamma(A) = P \exp\left(-\int_\gamma A\right) \tag{23}$$

where P stands for the path-ordered exponential. An element of Cyl_{BF} is a function

$$\Psi_{\gamma,f} \colon \mathcal{A} \longrightarrow \mathbb{C},$$

where γ is a finite directed graph embedded in Σ and $f \colon G^m \to \mathbb{C}$ is any continuous function, m being the number of edges of γ. This function $\Psi_{\gamma,f}$ is given by

$$\Psi_{\gamma,f}(A) = f(h_1(A), \ldots, h_m(A)) \tag{24}$$

where $h_i(A)$ is the holonomy along the ith edge of the graph γ and m is the number of edges.

Given any larger graph γ' formed by adding vertices and edges to γ, the function $\Psi_{\gamma,f}$ equals $\Psi_{\gamma',f'}$ for some continuous function $f' \colon G^{m'} \to \mathbb{C}$, where m' is the number of edges of γ'. Using this fact, we can define an inner product on cylinder functions. Given any two elements of Cyl_{BF}, we can write them as $\Psi_{\gamma,f}$ and $\Psi_{\gamma,g}$ where γ is a sufficiently large graph. Their inner product is then defined by:

$$\langle \Psi_{\gamma,f}, \Psi_{\gamma,g} \rangle = \int_{G^m} \overline{f(h_1, \ldots, h_m)}\, g(h_1, \ldots, h_m)\, dh_1 \cdots dh_m \tag{25}$$

where dh_i is the normalized Haar measure on G.

The auxiliary Hilbert space $\mathscr{H}_{\mathrm{BF}}$ is defined as the Cauchy completion of Cyl_{BF} under the inner product in (25). Using projective techniques it has been shown [8] that $\mathscr{H}_{\mathrm{BF}}$ is also the space of square-integrable functions on a certain space $\bar{\mathcal{A}}$ containing the space $\mathcal{A}$ of smooth connections on Σ. Elements of $\bar{\mathcal{A}}$ are called "generalized connections". The measure μ in equation (25) is actually a measure on $\bar{\mathcal{A}}$, and we have $\mathscr{H}_{\mathrm{BF}} = L^2(\bar{\mathcal{A}}, \mu)$. In other words, we have

$$\langle \Psi_{\gamma,f}, \Psi_{\gamma,g} \rangle = \int_{\bar{\mathcal{A}}} \overline{\Psi_{\gamma,f}(A)}\, \Psi_{\gamma,g}(A)\, d\mu(A). \tag{26}$$

The (generalized) connection is quantized by promoting the holonomy (23) to an operator acting by multiplication on cylinder functions as follows:

$$\widehat{h_\gamma(A)}\Psi(A) \;=\; h_\gamma(A)\Psi(A) \,. \tag{27}$$

It is easy to check that this defines a self-adjoint operator on $\mathscr{H}_{\mathrm{BF}}$. Similarly, the conjugate momentum E_j^a is promoted to a self-adjoint operator valued distribution that acts by differentiation on smooth cylinder functions, namely:

$$\hat{E}_j^a = -i\frac{\delta}{\delta A_a^j}. \tag{28}$$

Next, one can introduce an orthonormal basis of states in $\mathscr{H}_{\mathrm{BF}}$ using harmonic analysis on the compact group G. Thanks to the Peter–Weyl theorem, any continuous function $f\colon G \to \mathbb{C}$ can be expanded as follows:

$$f(g) = \sum_{\rho\in\mathrm{Irrep}(G)} \langle f_\rho, \rho(g)\rangle. \tag{29}$$

Here, $\mathrm{Irrep}(G)$ is a set of unitary irreducible representations of G containing one from each equivalence class. For any $g \in G$, a representation $\rho \in \mathrm{Irrep}(G)$ gives a linear transformation $\rho(g)\colon H_\rho \to H_\rho$ for some finite-dimensional Hilbert space H_ρ. We may think of $\rho(g)$ as an element of the Hilbert space $H_\rho \otimes H_\rho^*$. The "Fourier component" f_ρ is another element of $H \otimes H^*$, and $\langle f_\rho, \rho(g)\rangle$ is their inner product.

The straightforward generalization of this decomposition to functions $f\colon G^m \to \mathbb{C}$ allows us to write any cylindrical function (24) as:

$$\Psi_{\gamma,f}(A) = \sum_{\rho_1,\ldots,\rho_m\in\mathrm{Irrep}(G)} \prod_{i=1}^{m} \langle f_{\rho_i},\, \rho_i(h_i(A))\rangle, \tag{30}$$

where the "Fourier component" f_{ρ_i} associated to the ith edge of the graph γ is an element of $H_{\rho_i} \otimes H_{\rho_i}^*$. We call the functions appearing in this sum *open spin networks*. A general open spin network is of the form

$$\Psi_{\gamma,\vec{\rho},\vec{f}}(A) = \prod_{i=1}^{m} \langle f_{\rho_i}, \rho_i(h_i(A))\rangle. \tag{31}$$

Here, $\vec{\rho}$ is an abbreviation for the list of representations $(\rho_1, \ldots, \rho_m)$ labelling the edges of the graph, and $\vec{f}$ is an abbreviation for the tensor product $f_{\rho_1} \otimes \cdots \otimes f_{\rho_m}$. Note that $\Psi_{\gamma,\vec{\rho},\vec{f}}$ depends in a multilinear way on the vectors f_{ρ_i}, so it indeed depends only on their tensor product $\vec{f}$.

2.2 The string auxiliary Hilbert space

The auxiliary Hilbert space for the string degrees of freedom, $\mathscr{H}_{\mathrm{ST}}$, is obtained in an analogous fashion. Just as we built the auxiliary Hilbert space for BF theory starting from continuous functions of the connection's holonomies along edges in space, we build the space $\mathscr{H}_{\mathrm{ST}}$ starting from continuous functions of the λ field's values at points on the string. This space $\mathscr{H}_{\mathrm{ST}}$ can be described as $L^2(\bar{\Lambda}, \mu_{\mathrm{ST}})$, where $\bar{\Lambda}$ is a certain completion of the space of G-valued functions on the string $\mathscr{S}$, and μ_{ST} is the natural measure on this space (see for instance [9]).

To achieve this, we first define an algebra Cyl_{ST} of "cylinder functions" on the space of λ fields, $\Lambda = C^\infty(\mathscr{S}, G)$. An element of Cyl_{ST} is a function

$$\Phi_{p,f} \colon \Lambda \longrightarrow \mathbb{C},$$

where $p = \{p_1, \ldots, p_n\}$ is a finite set of points in $\mathscr{S}$ and $f \colon G^n \to \mathbb{C}$ is any continuous function. This function $\Phi_{p,f}$ is given by

$$\Phi_{p,f}(\lambda) = f(\lambda(p_1), \ldots, \lambda(p_n)). \tag{32}$$

As in the previous section, if p' is a finite set of points in $\mathscr{S}$ with $p \subset p'$, then the function $\Phi_{p,f}$ is equal to $\Phi_{p',f'}$ for some continuous function $f' \colon G^{n'} \to \mathbb{C}$. This lets us define an inner product on Cyl_{ST}. Given any two cylinder functions, we can write them as $\Phi_{p,f}$ and $\Phi_{p,g}$, where p is a sufficiently large finite set of points in $\mathscr{S}$. We define their inner product by

$$\langle \Phi_{p,f}, \Phi_{p,g} \rangle = \int_{G^n} \overline{f(h_1, \ldots, h_n)} \, g(h_1, \ldots, h_n) \, dh_1 \cdots dh_n \tag{33}$$

where dh_i is the normalized Haar measure on G. One can check that this is independent of the choices involved.

The auxiliary Hilbert space $\mathscr{H}_{\mathrm{ST}}$ is then defined to be the Cauchy completion of Cyl_{ST} under this inner product. Using projective techniques [9] it has been shown that $\mathscr{H}_{\mathrm{ST}}$ is $L^2(\bar{\Lambda}, \mu_{\mathrm{ST}})$ for some measure μ_{ST} on a certain space $\bar{\Lambda}$ containing the space Λ:

$$\langle \Phi_{p,f}, \Phi_{p,g} \rangle = \int_{\bar{\Lambda}} \overline{\Phi_{\gamma,f}(\lambda)} \, \Phi_{\gamma,g}(\lambda) \, d\mu_{\mathrm{ST}}(\lambda). \tag{34}$$

In fact, $\bar{\Lambda}$ is just the space of *all* functions $\lambda \colon \mathscr{S} \to G$. Though very large, this is actually a compact topological group by Tychonoff's theorem, and μ_{ST} is the Haar measure on this group.

The field λ is quantized in terms of operators acting by multiplication in $\mathscr{H}_{\mathrm{ST}}$. Therefore, the wave functional $\Phi(\lambda)$ gives the momentum representation of the quantum state of the string. More precisely, in this representation the momentum operator $\pi_i^a = \tau(\partial x^a)/(\partial s)\mathrm{tr}[e_i \lambda v \lambda^{-1}]$ acts by multiplication, namely:

$$\widehat{\pi_i^a(\lambda)} \Phi(\lambda) = \tau \frac{\partial x^a}{\partial s} \mathrm{tr}[e_i \lambda v \lambda^{-1}] \Phi(\lambda). \tag{35}$$

It is easy to check that the momentum operator is self-adjoint on $\mathscr{H}_{\mathrm{BF}}$. According to (20), the "total angular momentum" $J_i \equiv c_{ij}^k q_a^j \pi_k^a + \sigma_i$ is

promoted to a self-adjoint operator-valued distribution that acts as a derivation, namely

$$J^j = -i\frac{\delta}{\delta\lambda^j}. \tag{36}$$

An application of harmonic analysis on the group G, analogous to what was done in the previous section, lets us write any cylinder function (32) as

$$\Phi_{p,f}(\lambda) = \sum_{\rho_1,\ldots,\rho_n\in\mathrm{Irrep}(G)} \prod_{i=1}^{n} \langle f_{\rho_i}, \rho_i(\lambda(p_i))\rangle, \tag{37}$$

where ρ_i runs over irreducible unitary representations of G on finite-dimensional Hilbert spaces H_{ρ_i}, and the "Fourier component" f_{ρ_i} is an element of $H_{\rho_i} \otimes H_{\rho_i}^*$. We call the functions appearing in the sum n-point spin states. A typical n-point spin state is of the form

$$\Phi_{p,\vec{\rho},\vec{f}}(\lambda) = \prod_{i=1}^{n} \langle f_{\rho_i}, \rho_i(\lambda(p_i))\rangle. \tag{38}$$

Here, $\vec{\rho}$ is an abbreviation for the list of representations $(\rho_1, \ldots, \rho_n)$ labelling the points in p, and $\vec{f}$ is an abbreviation for the tensor product $f_{\rho_1} \otimes \cdots \otimes f_{\rho_n}$.

We hope the strong similarity between the BF and string auxiliary Hilbert spaces is clear. The only real difference is that the A field assigns group elements to edges, while the λ field assigns group elements to points. So, we need one-dimensional spin networks to describes states of BF theory, but their zero-dimensional analogues for the λ field.

2.3 The kinematical Hilbert space

The next step in the Dirac program is to implement the first class constraints found above as operator equations in order to define the physical Hilbert space. Here, we implement the constraints (14), (21), and (22). The states in the kernel of these quantum constraints define a proper subspace of $\mathscr{H}_{\mathrm{aux}}$ that we call the kinematical Hilbert space

$$\mathscr{H}_{\mathrm{kin}} \subset \mathscr{H}_{\mathrm{aux}} = \mathscr{H}_{\mathrm{BF}} \otimes \mathscr{H}_{\mathrm{ST}}.$$

The implementation of the remaining curvature constraint (15) (which also implies (13)) will be discussed in the next subsection.

The constraint (22) is automatically satisfied. This can be easily checked using the fact that one is working in the momentum representation where equation (35) holds.

The Gauss constraint (14) acts on the connection A generating gauge transformations $g \in C^\infty(\Sigma, G)$ whose action transforms the holonomies along edges of any graph as follows:

$$h_e(A) \longmapsto g(s(e))h_e(A)g(t(e))^{-1} \tag{39}$$

where $s(e), t(e) \in \Sigma$ are the source and target vertices of the edge e, respectively. As a result, such gauge transformations act on open spin networks in $\mathscr{H}_{\mathrm{BF}}$ as follows:

$$\prod_{i=1}^{n} \langle f_{\rho_i}, \rho_i(h_i(A)) \rangle \longmapsto \prod_{i=1}^{n} \langle f_{\rho_i}, \rho_i(g(s(e_i))h_i(A)g(t(e_i))^{-1}) \rangle. \tag{40}$$

Such gauge transformations also act on the λ field:

$$\lambda \longmapsto g\lambda, \tag{41}$$

so they act on n-point spin states in $\mathscr{H}_{\mathrm{ST}}$ as follows:

$$\prod_{i=1}^{n} \langle f_{\rho_i}, \rho_i(\lambda(p_i)) \rangle \longmapsto \prod_{i=1}^{n} \langle f_{\rho_i}, \rho_i(g(p_i)\lambda(p_i)) \rangle. \tag{42}$$

Combining these representations, we obtain a unitary representation of the group $C^\infty(\Sigma, G)$ on $\mathscr{H}_{\mathrm{aux}} = \mathscr{H}_{\mathrm{BF}} \otimes \mathscr{H}_{\mathrm{ST}}$. Gauge-invariant states are those invariant under this action.

A spanning set of gauge-invariant states can then be constructed in analogy with the known construction for three-dimensional quantum gravity coupled to point particles [3]. We form such states by taking the tensor product of an open spin network $\Psi_{\gamma, \vec{\rho}, \vec{f}}$ and an n-point spin state $\Phi_{p, \vec{\rho'}, \vec{f'}}$. Such a tensor product state will be invariant under the action of $C^\infty(\Sigma, G)$ if we:

(1) Require the graph γ for the open spin network to have vertices that include the points $\{p_1, \ldots, p_n\}$ forming the set p.

(2) Associate an intertwining operator to each vertex v of the graph γ as follows:

   (a) If the vertex v is not on the string, then choose an intertwining operator

   $$\iota_v \colon \rho_{i_1} \otimes \cdots \otimes \rho_{i_t} \longrightarrow \rho_{j_1} \otimes \cdots \otimes \rho_{j_s},$$

   where $i_1, \ldots, i_t$ are the edges of γ whose target is v, and $j_1, \ldots, j_s$ are the edges of γ whose source is v.

(b) If the vertex v is on the string, it coincides with some point $p_k \in p$. Then choose an intertwining operator

$$\iota_v \colon \rho_{i_1} \otimes \cdots \otimes \rho_{i_t} \longrightarrow (\rho_{j_1} \otimes \cdots \otimes \rho_{j_s}) \otimes \rho'_k,$$

where ρ'_k is the representation labelling the point p_k in the n-point spin state $\Phi_{p,\vec{\rho}',\vec{f}'}$.

(3) Tensor all the intertwining operators ι_v. The result is an element of

$$\bigotimes_{i=1}^{m}(H_{\rho_i} \otimes H_{\rho_i}^*) \otimes \bigotimes_{i=1}^{n}(H_{\rho'_i} \otimes H_{\rho'_i}^*).$$

Demand that this equals $\vec{f} \otimes \vec{f}'$. This fixes our choice of $\vec{f}$ for the open spin network and $\vec{f}'$ for the n-point spin state.

One can check that such states actually span the space of states in $\mathscr{H}$ that are invariant under gauge transformations in $C^\infty(\Sigma, G)$. So, we have solved the Gauss constraint.

Finally, constraint (21) generates gauge transformations

$$\lambda \longmapsto \lambda h \tag{43}$$

for any $h \in C^\infty(\mathscr{S}, H)$, where $H \subseteq G$ is the subgroup stabilizing the vector v. These transformations are unitarily represented on $\mathscr{H}_{\mathrm{ST}}$. The gauge transformation h acts on n-point spin functions as follows:

$$\prod_{i=1}^{n}\langle f_{\rho_i}, \rho_i(\lambda(p_i))\rangle \longmapsto \prod_{i=1}^{n}\langle f_{\rho_i}, \rho_i(\lambda(p_i)h(p_i))\rangle. \tag{44}$$

We can find n-point spin functions $\Phi_{p,\vec{\rho}',\vec{f}'}$ that are invariant under these transformations by choosing the vectors $\vec{f}'$ in such a way that each vector $f'_{\rho'_j}$ is invariant under the action of the group H.

We call the resulting states $\Psi_{\gamma,\vec{\rho},\vec{f}} \otimes \Phi_{p,\vec{\rho}',\vec{f}'}$ string spin networks. They span $\mathscr{H}_{\mathrm{kin}}$. A typical string spin network state appears in figure 1. The interplay between the quantum degrees of freedom in the "bulk" and those on the string (or membrane, in the general setting of the next section) is reminiscent of that appearing in the loop quantization of the degrees of freedom of an isolated horizon in loop quantum gravity [10].

2.4   The physical Hilbert space

In order to construct the physical Hilbert space $\mathscr{H}_{\mathrm{phys}}$, we have to impose the remaining curvature constraint (15). This can be achieved by an application of the techniques developed in [3]. The physical inner product can be
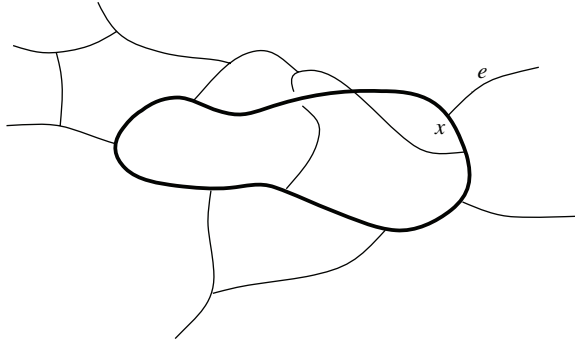
Figure 1: A typical string spin network. The Gauss law implies that if a single spin network edge e ends at some point x on the string, the representation ρ_e is evaluated on the product of the associated holonomy $h_e(A)$ and the value of the λ field at x.

represented as a sum over spin foam amplitudes, which are a simple generalization of the amplitudes in three dimensions. The associated state sum invariants can be directly derived from the canonical perspective presented here. The details of the construction will be provided elsewhere.

3   The general case: membranes coupled to *BF* theory

Let us now describe the phase space of the general case in detail. Recall that G is a general Lie group with Lie algebra $\mathfrak{g}$ equipped with an invariant inner product. Performing the canonical analysis along the same lines as in Section 1 one obtains $E_i^a = \epsilon^{aa_1\cdots a_{d-2}} B_{ia_1\cdots a_{d-2}}$ as the momentum canonically conjugate to A_a^i, where as before i labels a basis e_i of $\mathfrak{g}$. The momentum canonically conjugate to q_a^i is given by

$$\pi_i^{a_1\cdots a_{d-3}} = \tau \frac{\partial x^{[a_1}}{\partial s_1} \frac{\partial x^{a_2}}{\partial s_2} \cdots \frac{\partial x^{a_{d-3}]}}{\partial s_{d-3}} \mathrm{tr}[e_i \lambda v \lambda^{-1}],$$

where $t, s_1, \ldots, s_{d-3}$ are local coordinates on the membrane world-sheet. The Gauss law now becomes

$$D_a E_i^a = \int_{\mathscr{B}} c_{ij}^k q_{a_1\cdots a_{d-3}}^j \pi_k^{a_1\cdots a_{d-3}} \delta^{(d-1)}(x - x_{\mathscr{B}}), \qquad (45)$$

where $\mathscr{B}$ denotes the brane, i.e., the intersection of the membrane world-sheet $\mathscr{W}$ with Σ. The curvature constraint becomes

$$\epsilon^{a_1\cdots a_{d-3}bc} F_{ibc} = -\int_{\mathscr{B}} \pi_i^{a_1\cdots a_{d-3}} \delta^{(d-1)}(x - x_{\mathscr{B}}). \qquad (46)$$

We also have

$$D_a \pi_i^{aa_1\cdots a_{d-4}} = 0. \tag{47}$$

There are additional constraints for the degrees of freedom of the $(d-3)$-branes, namely

$$\mathrm{tr}[e_i \lambda z \lambda^{-1}] J^i = 0 \quad \text{where} \quad J_i \equiv c_{ij}^k q_{a_1\cdots a_{d-3}}^j \pi_k^{a_1\cdots a_{d-3}} + \sigma_i \tag{48}$$

and

$$\mathrm{tr}[\pi^{a_1\cdots a_{d-3}} \lambda z \lambda^{-1}] = \tau \frac{\partial x^{[a_1}}{\partial s_1} \frac{\partial x^{a_2}}{\partial s_2} \cdots \frac{\partial x^{a_{d-3}]}}{\partial s_{d-3}} \mathrm{tr}[vz], \tag{49}$$

for $[z, v] = 0$.

The quantization of the general d-dimensional BF theory coupled to $(d-3)$-branes can be achieved by following an essentially analogous path as the one described in detail for four-dimensional BF theory coupled to strings. As long as the gauge group G is compact, the techniques used in the construction of the auxiliary Hilbert spaces as well as the definition of the kinematical Hilbert space and finally the physical Hilbert space can be directly generalized. In particular, the kinematical Hilbert space is spanned by membrane spin networks, which generalize the string spin networks of the four-dimensional case.

4 Conclusions

There are formulations of gravity in four dimensions that are closely related to BF theory. The results presented here could lead to natural candidates for the introduction of matter in those models. Examples of interest are the MacDowell–Mansouri formulation of gravity [13], which is a perturbed version of BF theory with gauge group SO(3,2), SO(4,1) or SO(5) depending on the signature of the metric and sign of the cosmological constant. Another interesting approach to gravity is the Plebanski formulation, obtained by imposing extra constraints on BF theory with gauge group SO(3,1) or SO(4). The well-known Barrett–Crane model [14] is a tentative quantization of this theory. At least classically, the BF theories associated to all these theories can be coupled to strings using the techniques developed here.

When the gauge group G is compact, we can also quantize these theories. However, for Lorentzian models G is typically not compact. In the noncompact case, it seems that there is no good measure on the space of generalized connections, which precludes the construction of the auxiliary Hilbert spaces used above. The main obstacle is the non-normalizability of

the Haar measure. As long as G is "unimodular" i.e., as long as it admits a measure invariant under both right and left translations, as in all the examples mentioned above — formulas (25) and (33) can still be given a meaning on a fixed graph [11]. However, it is no longer possible to promote this inner product to an inner product on cylindrical functions [12]. One can still attempt to deal with the theory in a more restricted setting by defining it on a fixed cellular decomposition of spacetime and then showing that physical amplitudes are independent of this choice. This is expected for topological theories such as the ones defined here, but the study of these models still presents interesting challenges.

Another subtlety of the noncompact case is that while the Lie algebra $\mathfrak{g}$ may still admit an invariant nondegenerate inner product, this inner product typically fails to be positive definite. Indeed, this happens for all noncompact semisimple groups, such as $SO(p, q)$ for $p + q > 2$. This affects the interpretation of the action (6) for our theory. Recall that we imposed the normalization condition $v \cdot v = 1$ for the vector $v \in \mathfrak{g}$. We used this condition to give a meaning to the tension parameter τ, but the action only depends on the combination $p = \tau \lambda v \lambda^{-1}$. As we have seen in the four-dimensional case, the field p has a simple meaning: the holonomy of the connection A around any small loop encircling the membrane world-sheet is $\exp(-p) \in G$. The same is true in any dimension.

This suggests a simpler action:

$$S(A, B, q, p) = \int_{\mathscr{M}} \mathrm{tr}[B \wedge F(A)] + \int_{\mathscr{W}} \mathrm{tr}[(B + d_A q)\, p], \qquad (50)$$

where p is a $\mathfrak{g}$-valued function on the world-sheet $\mathscr{W}$ which under the gauge transformations (7) transforms in the adjoint representation: $p \mapsto gpg^{-1}$. One can check that the equations of motion still imply A is flat except at points on $\mathscr{W}$. If $\mathscr{W}$ is connected, this implies that the holonomy around any small loop encircling the world-sheet is in the same conjugacy class. As before, the holonomy around an infinitesimal loop around some point $x \in \mathscr{W}$ is $\exp(-p(x))$. It follows that p remains in the same adjoint orbit over the whole world-sheet. So, we can write p as $\tau \lambda v \lambda^{-1}$ for some fixed vector $v \in \mathfrak{g}$ and some G-valued field λ on the world-sheet.

When the inner product on $\mathfrak{g}$ is positive definite, we can then fix τ by normalizing v to have $v \cdot v = 1$. However, when the inner product is not positive definite, the new action (50) is more general than the old one, even for a connected world-sheet, since it allows the momentum density of the membrane to be space-like ($p \cdot p > 0$) or null ($p \cdot p = 0$), as well as time-like ($p \cdot p < 0$). One can check that with this new action, the canonical

analysis of Section 1 requires only mild modifications, and the kinematical construction of the quantum theory presented in Section 2 can still be used, with the precautions described above for noncompact Lie groups.

It will be interesting to carry out the study of four-dimensional BF theory coupled to strings in analogy to what has already been done for three-dimensional gravity coupled to point particles. For example, point particles in three-dimensional gravity are known to obey exotic statistics governed by the braid group. Similarly, we have argued in the companion to this paper that strings coupled to four-dimensional BF theory obey exotic statistics governed by the "loop braid group" [5]. In that paper, we studied these statistics in detail for the case $G = \mathrm{SO}(3,1)$, but we treated the strings merely as gauge defects. It would be good to study this issue more carefully with the help of the framework developed here.

Acknowledgments

References

[1] A. Perez, *Spin foam models for quantum gravity,* Class. Quant. Grav. **20**, (2003), R43, arXiv:gr-qc/0301113; D. Oriti, *Spacetime geometry from algebra: spin foam models for non-perturbative quantum gravity,* Rept. Prog. Phys. **64**, (2001), 1489, arXiv:gr-qc/0106091. J.C. Baez, *An introduction to spin foam models of BF theory and quantum gravity,* Lect. Notes Phys. **543**, (2000), 25, arXiv:gr-qc/9905087; J.C. Baez, *Spin foam models,* Class. Quant. Grav. **15**, (1998), 1827, arXiv: gr-qc/9709052.

[2] T. Thiemann, *Modern canonical quantum general relativity*, Cambridge Monographs on Mathematical Physics, University press Cambridge, UK, 2007; C. Rovelli, *Quantum gravity,* University Press, Cambridge, UK: 2004, 455; A. Ashtekar and J. Lewandowski, *Background independent quantum gravity: a status report,* Class. Quant. Grav. **21** (2004), R53, arXiv:gr-qc/0404018. A. Perez, *Introduction to loop quantum gravity and spin foams,* Proceedings of the International Conference on Fundamental Interactions, Domingos Martins, Brazil, 2004, arXiv:gr-qc/0409061.

[3] K. Noui and A. Perez, *Three dimensional loop quantum gravity: coupling to point particles,* Class. Quant. Grav. **22** (2005), 4489, arXiv:gr-qc/0402111; K. Noui and A. Perez, *Three dimensional loop quantum gravity: physical scalar product and spin foam models,* Class. Quant. Grav. **22**, (2005), 1739, arXiv:gr-qc/0402110; L. Freidel and D. Louapre, *Ponzano-Regge model revisited. I: Gauge fixing, observables and interacting spinning particles,* Class. Quant. Grav. **21**, (2004), 5685, arXiv:hep-th/0401076.

[4] L. Freidel and E.R. Livine, *Ponzano-Regge model revisited. III: Feynman diagrams and effective field theory,* Class. Quant. Grav. **23**, (2006), 2021, arXiv:hep-th/0502106; J.W. Barrett, *Feynman diagams coupled to three-dimensional quantum gravity,* Class. Quant. Grav. **23**, (2006), 137, arXiv:gr-qc/0502048; J.W. Barrett, *Feynman loops and three-dimensional quantum gravity,* Mod. Phys. Lett. A **20** (2005), 1271, arXiv:gr-qc/0412107.

[5] J.C. Baez, D.K. Wise and A.S. Crans, *Exotic statistics for strings in 4d BF theory,* arXiv:gr-qc/0603085.

[6] P. de Sousa Gerbert, *On Spin And (Quantum) Gravity In (2 + 1)-Dimensions,* Nucl. Phys. B **346** (1990), 440.

[7] E. Buffenoir and K. Noui, *Unfashionable observations about 3 dimensional gravity,* arXiv:gr-qc/0305079.

[8] J.C. Baez, *Diffeomorphism-invariant generalized measures on the space of connections modulo gauge transformations,* Proceedings of the Conference on Quantum Topology, D.N. Yetter, Ed., World Scientific Press, Singapore, 1994, arXiv:hep-th/9305045; A. Ashtekar and J. Lewandowski, *Projective techniques and functional integration for gauge theories,* J. Math. Phys. **36** (1995), 2170, arXiv:gr-qc/9411046; A. Ashtekar and J. Lewandowski, *Differential geometry on the space of connections via graphs and projective limits,* J. Geom. Phys. **17**, (1995), 191, arXiv:hep-th/9412073.

[9] J.C. Baez, *Generalized measures in gauge theory,* Lett. Math. Phys. **31** (1994), 213, arXiv:/hep-th/9310201; T. Thiemann, *Kinematical Hilbert spaces for fermionic and Higgs quantum field theories,* Class. Quant. Grav. **15** (1998), 1487, arXiv:gr-qc/9705021. A. Ashtekar, J. Lewandowski and H. Sahlmann, *Polymer and Fock representations for a scalar field,* Class. Quant. Grav. **20** (2003), L11, arXiv:gr-qc/0211012. W. Kaminski, J. Lewandowski and M. Bobienski, *Background independent quantizations: the scalar field I*, Class. Quant. Grav. **23** (2006), 2761, arXiv:gr-qc/0508091; W. Kaminski, J. Lewandowski and A. Okolow, *Background independent quantizations: the scalar field II* arXiv:gr-qc/0604112.

[10] A. Ashtekar, J. Baez, A. Corichi and K. Krasnov, *Quantum geometry and black hole entropy,* Phys. Rev. Lett. **80**, (1998), 904, arXiv:gr-qc/9710007; A. Ashtekar, J.C. Baez and K. Krasnov, *Quantum geometry of isolated horizons and black hole entropy,* Adv. Theor. Math. Phys. **4** (2000), 1, arXiv:gr-qc/0005126.

[11] L. Freidel and E.R. Livine, *Spin networks for non-compact groups*, J. Math. Phys. **44** (2003), 1322, arXiv:hep-th/0205268.

[12] J.L. Willis, *On the low-energy ramifications and a mathematical extension of loop quantum gravity,* UMI-31-48692.

[13] L. Freidel and A. Starodubtsev, *Quantum gravity in terms of topological observables*, arXiv:hep-th/0501191.

[14] J.W. Barrett and L. Crane, *Relativistic spin networks and quantum gravity,* J. Math. Phys. **39** (1998), 3296, arXiv:gr-qc/9709028.