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Minimal AdS_3

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Abstract

We show that type IIB string theory on $AdS_3 \times S^3 \times M_4$ with p units of NS flux contains an integrable subsector, isomorphic to the minimal $(p, 1)$ bosonic string. To this end, we construct a topological string theory with target space Euclidean $AdS_3 \times S^3$. We use a variant of Hamiltonian reduction to prove its equivalence to the minimal $(p, 1)$ string. The topological theory is then embedded in the physical 10-dimensional IIB string theory. Correlators of tachyons in the minimal string are mapped to correlators of spacetime chiral primaries in the IIB theory, in the presence of background 5-form RR flux. We also uncover a ground ring structure in $AdS_3 \times S^3$ analogous to the well-known ground ring of the minimal string. This tractable model provides a literal incarnation of the idea that the holographic direction of AdS space is the Liouville field. We discuss a few generalizations; in particular, we show that the $N = 4$ topological string on an A_{p-1} ALE singularity also reduces to the $(p, 1)$ minimal string.

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1 Introduction and summary

Exactly solvable models have played an important historical role in physics. In string theory, topological strings and strings in low dimensions are two related classes of models often amenable to exact treatment. Apart from their intrinsic value as tractable toy models, topological strings can also be embedded in physical superstring theories; the topological observables correspond to a subsector of BPS quantities of the physical theory. The best known examples are type II compactifications on Calabi–Yau 3-folds, where the topological string of the Calabi–Yau computes certain F-terms of the four-dimensional effective action [1, 2].

It is of great interest to extend these ideas to backgrounds involving an Anti-de Sitter factor, with the prospect of obtaining new insights into the AdS/CFT correspondence and the workings of open/closed duality. Recent results [3–5] about 1/2 BPS configurations in AdS spaces and their dual field theories are strongly suggestive of a topological string structure. In fact there is a natural class of exactly solvable models that seem tailor-made to be embedded into backgrounds of the form AdS $\times X$, for X a compact manifold: the string theories defined by coupling (minimal) $c \leq 1$ matter to gravity. Computations of special BPS quantities should be captured by the topological string on AdS $\times X$. Moreover — one is tempted to speculate — the AdS factor of the topological sigma model may be replaced by Liouville CFT, whereas the X factor may reduce to $c \leq 1$ matter.

In this paper, we show that this general guess is precisely realized in what is, in some technical sense, the simplest AdS background: IIB string theory on $\mathrm{AdS}_3 \times S^3 \times M_4$, with p units of NS flux. This well-known background arises as the near horizon geometry of Q_1 fundamental strings and $Q_5 \equiv p$ NS5 branes wrapping the four-manifold M_4, which can be either K_3 or T^4. In this concrete example, the worldsheet description is under complete control [6] and we can carry out a very explicit analysis. We are going to prove that the B-model with target space (Euclidean) $\mathrm{AdS}_3 \times S^3$ is equivalent to the $(p, 1)$ bosonic string, to all orders in the topological genus expansion. This equivalence could be phrased as "taking seriously" the $SL(2)$ current algebra in the KPZ description [7] of worldsheet gravity for the minimal string: the $SL(2)$ is given a literal spacetime interpretation as the AdS_3 factor of the sigma model. Similarly, the minimal matter is carved out of the $SU(2)$ current algebra for the S^3 factor of the sigma model. As in the Calabi–Yau case, the topological theory can be viewed as a BPS subsector of the physical theory. Special correlators of chiral primaries in the IIB theory, in the presence of background 5-form RR flux, are thus reduced to computations in the minimal $(p, 1)$ string. We hasten to add that the minimal $(p, 1)$ theory considered in this paper is not quite the same as the purely topological theory discussed, e.g., in [8–10] and which can be realized as an $N = 2$ minimal model coupled to topological gravity. This latter theory only captures the "resonant" Liouville amplitudes of the $(p, 1)$ minimal string.

We are actually going to base our technical analysis on the Euclidean version of AdS_3, the hyperbolic space H_3^+, since the topological theory seems more naturally defined on an Euclidean manifold. The supersymmetric sigma model on $H_3^+ \times S^3$ consists of an $SU(2)_{p-2}$ WZW model for the sphere, of an $SU(2)_{-p-2}$ WZW model for the hyperbolic space, and of six real free fermions [6]. We define the topological string theory by the usual procedure of twisting the $(2, 2)$ worldsheet supersymmetry. The A-model turns out to be trivial, so we focus on the B-model. Because of the special symmetries of this theory, four of the six coordinates are redundant — the two "transverse" coordinates along the boundary of H_3^+ and two of the S^3 coordinates. Indeed, the structure of the BRST operator of the topological theory is such that these coordinates and four of the free fermions fill two "$bc\beta\gamma$ quartets", which can be argued to decouple. The reduced theory is identical to the $(p,1)$ bosonic string: The remaining S^3 coordinate provides a Coulomb gas description of the $(p,1)$ matter; the "holographic" coordinate of H_3^+ becomes the Liouville field; and the two remaining free fermions transmute into the diffeomorphism ghosts of the bosonic string. We demonstrate the equivalence at the full quantum level, showing that the rules for computation of string amplitudes inherited from the B-model are the expected ones.

A precise dictionary describes the embedding of the minimal string into IIB string theory on $H_3^+ \times S^3 \times M_4$:

- The tachyons of the minimal string map to universal 1/2 BPS operators in $H_3^+ \times S^3$, which are (chiral, chiral) both on the worldsheet and in spacetime. By "universal" we mean independent of the details of M_4. The condition of being chiral in spacetime says that their angular momentum j around a preferred axis of the S^3 equals the spacetime conformal dimension h. These states can be organized in a sequence with increasing $h = j$ quantum number, with $h = (n-1)/2$, one state for each integer $n \geq 1$, $n \neq 0 \mod p$ (see figure 2 in Section 4.4). The first $p-1$ of them are constructed in $H_3^+ \times S^3$ by combining conventional primaries of the two current algebras, and map to the "small phase space" of the minimal theory. The higher states are constructed using the operation of "spectral flow" – they are long string states that are supported on worldsheets that wrap the boundary of H_3^+ multiple times — and map to the "gravitational descendants" of the minimal theory. Every pth state is missing both in the minimal string and in the IIB theory. This is natural from the viewpoint of the p-KdV hierarchy, where every pth flow parameter is redundant.
- Besides the tachyons, which carry ghost number one, the minimal string has non-trivial cohomology elements at ghost number zero. These states form the so-called ground ring and are the hallmark of integrability of the model. The ground ring lifts to an analogous ring structure in the IIB theory. We find non-trivial cohomology classes in $H_3^+ \times S^3$, carrying zero ghost number with respect to the ghosts of the 10d superstring. The appearance of the ground ring structure follows from the fact that the $\widehat{\mathfrak{su}}(2)_{p-2}$ current algebra representations are degenerate: imitating the construction of [11–14], each primitive null over an $\widehat{\mathfrak{su}}(2)_{p-2}$ primary gives rise to a ground ring state; the H_3^+

sector provides the "gravitational dressing". In principle, it should be possible to organize calculations of 1/2 BPS amplitudes in the IIB theory in terms of ring multiplication rules, in analogy with the calculations in the minimal string [15]. We expect ground ring structures to be ubiquitous in supersymmetric backgrounds of the form AdS $\times X$, though demonstrating their existence from a worldsheet viewpoint as we do here may be possible only in special cases.

- Finally, D-branes of the minimal string must correspond to supersymmetric B-branes in $H_3^+ \times S^3$. We believe that the FZZT brane of the minimal string lifts to the supersymmetric $H_2^+ \times S^2$ brane discussed in [16, 17], but we leave a detailed comparison for future work.

Our work should have implications for the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. The dual boundary CFT is a deformation of the symmetric product sigma model $\mathrm{Sym}^{Q_1 Q_5}(M_4)$, and is not very well understood. It should be viewed in some sense as the theory of the long strings that make up the geometry before taking the near horizon limit. Thus the holographic correspondence is not an instance of open/closed duality (that would be the case for the S-dual D1/D5 background), and is rather more similar in spirit to matrix string theory [18, 19]. For amplitudes of chiral primaries, we are entitled to expect simplifications. Many details remain to be worked out, but we describe the main idea that the computation of such amplitudes can be reduced to a counting problem familiar from Hurwitz theory.

Our results are connected with several ideas of current interest. In this paper, we only begin to explore some of these relations:

- An obvious connection is with the "bubbling" AdS_3 geometries of [5, 20–22]. As in these works, we are focusing on 1/2 BPS states. Moving in the small phase space of the minimal string corresponds to exploring (a subset of) 1/2 BPS configurations in IIB that preserve the $H_3^+ \times S^3$ asymptotic boundary conditions. It would be interesting to understand in detail the six-dimensional geometric interpretation of the small phase space and of its W_p integrable structure.
- The background $H_3^+ \times S^3$ provides a "critical" ($\hat{c} = 3$) topological realization of the $(p, 1)$ theories. Recently another critical realization of the $(p, 1)$ models has been proposed, as the B-model on a target Calabi–Yau related to the ground ring curve [23]. We conjecture that the $H_3^+ \times S^3$ sigma model is T-dual to this Calabi–Yau, in analogy with the T-duality between NS5 brane geometries and ALE spaces [24].

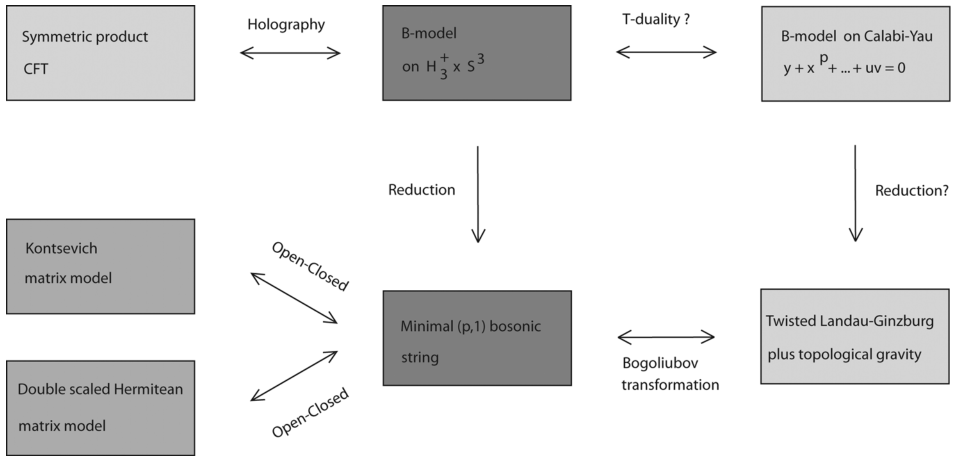

Figure 1: Web of relations centering on the $(p, 1)$ model coupled to gravity. In this paper, we focus on the two central boxes.

- The B-model on $H_3^+ \times S^3$ is arguably the simplest critical topological theory with torsion. It should provide a good testing ground for the abstract framework of twisted generalized complex geometry [25, 26].
- Besides the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, we may also discuss open/closed duality. To this end, we need to introduce D-branes in the $H_3^+ \times S^3$ background. The open string field theory on the FZZT branes of the $(p, 1)$ model reduces to a Kontsevich matrix model [27–31], while the open string field theory on infinitely many decayed ZZ branes corresponds to the doubled scaled matrix model [32, 33]. It would be interesting to lift these statements to the superstring theory on $H_3^+ \times S^3 \times M_4$ (figure 1).

One can envision several generalizations of this work. In this paper, we focus on the purely Neveu–Schwarz $H_3^+ \times S^3$ background (apart from the RR "graviphoton" 5-form flux needed to twist the theory). It is possible to turn on additional self-dual 3-form RR flux in six dimensions while preserving supersymmetry. An obvious question is whether this more general background can be reduced to a minimal string. A simple speculation is that the 3-form RR deformation corresponds to deforming the KdV hierarchy into the mKdV hierarchy of complex matrix models. This amounts to embedding the minimal bosonic theory into the 0A theory [34–36]. If this guess is correct, the self-dual 3-form RR flux in six dimensions would map to the RR background flux of the 0A theory. The natural framework to investigate this issue is the hybrid formalism of [37]. Other interesting generalizations are to supersymmetric backgrounds of the form $H_3^+ \times \mathcal{N}$, where $\mathcal{N}$ is a coset model. We briefly study the simple example $H_3^+ \times S^1$ and show that it reduces to the $c = 1$ bosonic string at the self-dual radius. In the final section, we consider the $N = 4$ topological string on an A_{p-1} ALE singularity, and show that it can also be mapped to the $(p, 1)$ minimal string by the same reasoning used for $\mathrm{AdS}_3 \times S^3$.

The organization of the paper is as follows. In Section 2, the topological theory on $H_3^+ \times S^3$ is defined. In Section 3, we prove that this theory reduces to the minimal string. In Section 4, we describe its embedding into the physical theory. Section 5 contains some remarks on the holographic correspondence. In Section 6, we comment on the relation to Calabi–Yau spaces and discuss some generalizations. Appendix A contains some mathematical details needed in the reduction of Section 3. In Appendix B, we review the (p, q) minimal theories and spell out our viewpoint about the $(p, 1)$ models, which require special consideration.

Topological strings on $\mathrm{AdS}_3 \times \mathcal{N}$ have previously been considered in [38].

2 Topological string theory on $H_3^+ \times S^3$

In this section, we construct a topological string theory with target space $H_3^+ \times S^3$. We include a review of the worldsheet CFT and of its supersymmetry structure.[1]

2.1 RNS description

We start with a brief summary of the RNS worldsheet description of superstring theory with target space $H_3^+ \times S^3$ [6]. We will fix the amount of NS flux H for the remainder as

$$\int_{S^3} H = p \, . \tag{2.1}$$

The bosonic sector of the CFT consists of two WZW models, one for the sphere and one for the hyperbolic space. The S^3 factor is described by an $\widehat{\mathfrak{su}}(2)$ current algebra at level $k = p - 2$, with central charge

[1] The Mathematica package OPEdefs.m [39] was used to check some OPEs in this paper.

$3k/(k+2) = 3 - 6/p$. We denote the left-moving $\widehat{\mathfrak{su}}(2)_{p-2}$ generators by $j_\pm = j_1 \pm i j_2$ and j_3, with OPEs

$$\begin{aligned} j_+(z)\cdot j_-(w) &\sim \frac{p-2}{(z-w)^2} + \frac{2\,j_3(w)}{z-w} \\ j_3(z)\cdot j_\pm(w) &\sim \pm\frac{j_\pm(w)}{z-w} \\ j_3(z)\cdot j_3(w) &\sim \frac{p-2}{2(z-w)^2}. \end{aligned} \tag{2.2}$$

Similarly, we have right-moving generators $\tilde{j}_\pm(\bar{z})$, $\tilde{j}_3(\bar{z})$. The hyperbolic space H_3^+ is described by an $SL(2,\mathbf{C})/SU(2)$ coset model at level $p+2$, equivalently by an $\widehat{\mathfrak{su}}(2)$ current algebra at level $k' = -p-2$, with central charge $3k'/(k'+2) = 3 + 6/p$. The total bosonic central charge of $S^3 \times H_3^+$ is then six, as expected. We denote the left-moving $\widehat{\mathfrak{su}}(2)_{-p-2}$ generators by $k_\pm = k_1 \pm i k_2$ and k_3, with OPEs

$$\begin{aligned} k_+(z)\cdot k_-(w) &\sim -\frac{p+2}{(z-w)^2} + \frac{2\,k_3(w)}{z-w} \\ k_3(z)\cdot k_\pm(w) &\sim \pm\frac{k_\pm(w)}{z-w} \\ k_3(z)\cdot k_3(w) &\sim -\frac{p+2}{2(z-w)^2}. \end{aligned} \tag{2.3}$$

Similarly, for the right-moving generators $\tilde{k}_\pm(\bar{z})$, $\tilde{k}_3(\bar{z})$. It is convenient to adopt a six-dimensional notation and introduce the symbol J_a with $a = 1,\ldots,6$ to denote all the (left-moving) currents:

$$\begin{aligned} J_a &\equiv j_a \quad \text{for } a = 1,2,3 \\ J_a &\equiv k_{a-3} \quad \text{for } a = 4,5,6. \end{aligned} \tag{2.4}$$

The fermionic sector of the CFT consists of the superpartners of the currents, six free fermions $\lambda^a(z)$, $a = 1,\ldots,6$. (Similarly, we have $\tilde{J}_a(\bar{z})$ and $\tilde{\lambda}^a(\bar{z})$. In the following we shall usually avoid explicit mention of the right movers.) Notice that the currents have lower indices and the fermions have upper indices. This is a natural notation since the (zero modes of the) currents correspond to tangent vectors, while the (zero modes of the) fermions correspond to one-forms.

Let us introduce a metric h_{ab} in this six-dimensional vector space:

$$h_{ab} = \frac{1}{2}\begin{pmatrix} p\,\mathbf{1}_{3\times 3} & 0 \\ 0 & -p\,\mathbf{1}_{3\times 3} \end{pmatrix}_{ab} \tag{2.5}$$

The h_{ab} are the residues of the double poles of the currents, up to a shift by δ_{ab} (= half the dual Coxeter number of $\mathfrak{su}(2)$). The fermions are normalized to satisfy the following OPEs,

$$\lambda^a(z) \cdot \lambda^b(w) \sim \frac{h^{ab}}{z-w}. \tag{2.6}$$

In these notations, the stress tensor and supercurrent are given by the simple standard expressions

$$\begin{aligned} \mathsf{G} &= J_a \lambda^a - \frac{1}{6} h_{cd} f^d_{ab} \lambda^a \lambda^b \lambda^c, \\ \mathsf{T} &= \frac{1}{2}(h^{ab} J_a J_b + h_{ab} \partial \lambda^a \lambda^b). \end{aligned} \tag{2.7}$$

Here f^c_{ab} are the structure constants of $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$. It is easy to check that

$$\mathsf{G}(z) \cdot \mathsf{G}(w) \sim \frac{6}{(z-w)^3} + \frac{2\mathsf{T}}{z-w}. \tag{2.8}$$

In particular, the central charge is $\hat{c} = c/3 = 3$.

2.2 (2,2) Structure

In order to define a topological string theory, we need to identify an extended $(2,2)$ worldsheet supersymmetry. For the case at hand, there is a standard construction that we now briefly review ([40], see also [41, 42]).[2]

The conditions for extended susy for a general σ-model with torsion H are well known [43]. The target space manifold must admit two integrable complex structures $\mathcal{J}_\pm$, the metric must be Hermitian with respect to both complex structures, and $\mathcal{J}_\pm$ must be covariantly constant, each with respect to a different affine connection,

$$\begin{aligned} \Gamma^{\pm\mu}_{\rho\nu} &\equiv \Gamma^\mu_{\rho\nu} \pm g^{\mu\sigma} H_{\sigma\rho\nu} \\ \nabla^\pm_\rho \mathcal{J}^\mu_{\pm\nu} &= 0. \end{aligned} \tag{2.9}$$

In the case of group manifolds, there is a canonical way to construct globally defined almost complex structures $\mathcal{J}^\mu_{\pm\nu}$: start with a constant matrix $\mathcal{J}^a_b$ acting on the Lie algebra, and use the left and right group action to obtain, respectively, $\mathcal{J}^\mu_{-\nu}$ and $\mathcal{J}^\mu_{+\nu}$. The various conditions on $\mathcal{J}^\mu_{\pm\nu}$ translate into conditions for $\mathcal{J}^a_b$.

Let us see in more detail how this works. The left- and right-invariant global vector fields J_a and $\tilde{J}_a$ on the group manifold are covariantly constant

[2]Readers willing to take on faith the basic definitions (2.13) can skip directly there.

with respect to the two connections,[3]

$$\nabla^- J_a = 0, \quad \nabla^+ \tilde{J}_a = 0. \tag{2.10}$$

Given a constant matrix $\mathcal{J}^a_b$ acting on the Lie algebra and squaring to minus one, we parallel-transport it in two different ways to give the two candidate complex structures $\mathcal{J}^\mu_{\pm\nu}$, globally defined on the manifold,

$$\mathcal{J}^\mu_{-\nu} = J^\mu_a \mathcal{J}^a_b J^b_\nu, \quad \mathcal{J}^\mu_{+\nu} = \tilde{J}^\mu_a \mathcal{J}^a_b \tilde{J}^b_\nu. \tag{2.11}$$

The point of this construction is that the resulting $\mathcal{J}^\mu_\nu$ are automatically covariantly constant with respect to $\Gamma^\pm$. The hermiticity condition for $g_{\mu\nu}$ with respect to $\mathcal{J}^\mu_{\pm\nu}$ translates into the hermiticity condition of the metric h_{ab} on the Lie algebra with respect to $\mathcal{J}^a_b$. To express the integrability condition, it is convenient to use $\mathcal{J}^a_b$ to split the real indices $a, b, \ldots$ into holomorphic indices $i, j, \ldots$ and antiholomorphic indices $\bar{i}, \bar{j}, \ldots$. Then the integrability of $\mathcal{J}^\mu_{\pm\nu}$ translates into the condition $f^{\bar{k}}_{ij} = f^{k}_{\bar{i}\bar{j}} = 0$; that is, holomorphic and antiholomorphic generators form closed subalgebras. In summary, the existence of $(2,2)$ supersymmetry on an even-dimensional group manifold is associated to a decomposition of the complexified Lie algebra $\mathfrak{g}^c = \mathfrak{g}_- \oplus \mathfrak{g}_+$, such that $\mathfrak{g}_\pm$ (the $\pm\sqrt{-1}$ eigenspaces of $\mathcal{J}^a_b$) are subalgebras and are maximally isotropic with respect to the metric h_{ab} (this means $h_{ij} = h_{\bar{i}\bar{j}} = 0$). The structure $(\mathfrak{g}^c, \mathfrak{g}_-, \mathfrak{g}_+)$ is called a Manin triple.

Finally, we are in the position to quote the general expressions for the $N = 2$ supercurrents:

$$\begin{aligned} \mathsf{G}^+ &= J_i \lambda^i - \frac{1}{6} f^k_{ij} \lambda^i \lambda^j \lambda_k, \\ \mathsf{G}^- &= J^i \lambda_i - \frac{1}{6} f^{ij}_k \lambda_i \lambda_j \lambda^k. \end{aligned} \tag{2.12}$$

Here we have used the Hermitian metric $h_{i\bar{j}}$ to get rid of all antiholomorphic indices by raising or lowering them. The decomposition of $\mathsf{G} = \mathsf{G}^+ + \mathsf{G}^-$ hinges on the condition $f_{ijk} = f_{\bar{i}\bar{j}\bar{k}} = 0$, that is, on the integrability of the complex structures.

Let us now specify the choice of complex structure in our concrete example. We just need to indicate how the the real Lie algebra indices $a, b = 1, \ldots, 6$ split into holomorphic indices $i, j = 1, 2, 3$ antiholomorphic

[3]By a slight abuse of notation, here we use the symbols J_a, $\tilde{J}_a$ to denote the zero modes of the current $J_a(z)$, $\tilde{J}_a(\bar{z})$.

$\bar{i}, \bar{j} = 1, 2, 3$. We choose

$$\begin{aligned} &J_{z_1} = j_- \quad J_{z_2} = +k_+ \quad J_{z_3} = j_3 + k_3 \\ &J_{\bar{z}_1} = j_+ \quad J_{\bar{z}_2} = -k_- \quad J_{\bar{z}_3} = j_3 - k_3. \end{aligned} \tag{2.13}$$

It is easy to check that this defines a Manin triple. This choice of complex structure is essentially unique up to symmetries. For example, we could apply the automorphisms $k_+ \leftrightarrow k_-, k_3 \leftrightarrow -k_3$, or $j_+ \leftrightarrow j_-, j_3 \leftrightarrow -j_3$, etc., which correspond to spacetime isometries. We make the choice (2.13) for future convenience, since the $(p, 1)$ string will arise with a minimum amount of field redefinitions. The fermions with holomorphic and antiholomorphic indices are defined as

$$\begin{aligned} &\lambda^{z_1} = \frac{1}{2}(\lambda^1 + i\lambda^2) \quad \lambda^{z_2} = +\frac{1}{2}(\lambda^4 - i\lambda^5) \quad \lambda^{z_3} = \frac{1}{2}(\lambda^3 + \lambda^6), \\ &\lambda^{\bar{z}_1} = \frac{1}{2}(\lambda^1 - i\lambda^2) \quad \lambda^{\bar{z}_2} = -\frac{1}{2}(\lambda^4 + i\lambda^5) \quad \lambda^{\bar{z}_3} = \frac{1}{2}(\lambda^3 - \lambda^6). \end{aligned} \tag{2.14}$$

The metric on the Lie algebra takes the form

$$h_{i\bar{j}} = p\,\delta_{i\bar{j}}, \quad h_{ij} = h_{\bar{i}\bar{j}} = 0, \tag{2.15}$$

which is manifestly Hermitian. Specializing (2.12) to our case, we obtain the following $N = 2$ structure

$$\begin{aligned} \mathsf{T} &= \frac{1}{2p}(2j_3j_3 + j_+j_- + j_-j_+ - 2k_3k_3 - k_-k_+ - k_+k_-) \\ &\quad + \frac{1}{2}\left(\partial c^i b_i - c^i \partial b_i\right) \\ \mathsf{G}^+ &= c^1(j_-) + c^2(k_+) + c^3(j_3 + k_3 + c^1b_1 - c^2b_2) \\ \mathsf{G}^- &= \frac{1}{p}\left(b_1j_+ - b_2k_- + b_3(j_3 - k_3 + c^2b_2 + c^1b_1)\right) \\ \mathsf{J} &= c^ib_i - \frac{2}{p}J_3^{\text{tot}} \end{aligned} \tag{2.16}$$

with

$$J_3^{\text{tot}} \equiv j_3 + k_3 + c^1b_1 - c^2b_2 = \{\mathsf{G}_0^+, b_3\}. \tag{2.17}$$

We have renamed the fermions as $c^i \equiv \lambda^i$ and $b_i \equiv \lambda_i$, which will be a useful notation. From now on, we shall never lower or raise indices: b_i (c^i) will consistently have lower (upper) indices. They satisfy the OPEs

$$b_i(z)\, c^j(w) \sim \frac{\delta_i^j}{z-w}. \tag{2.18}$$

In fact there is a whole family of $N = 2$ structures generalizing (2.16),

$$\begin{aligned}
\mathsf{T} &= \frac{1}{2p}(2j_3j_3 + j_+j_- + j_-j_+ - 2k_3k_3 - k_-k_+ - k_+k_-) \\
&\quad + \frac{1}{2}\left(\partial c^i b_i - c^i \partial b_i\right) + \frac{\kappa}{2p}\partial J_3^{\text{tot}} \\
\mathsf{G}^+ &= c^1(j_-) + c^2(k_+) + c^3(j_3 + k_3 + c^1b_1 - c^2b_2) \\
\mathsf{G}^- &= \frac{1}{p}\left(b_1j_+ - b_2k_- + b_3(j_3 - k_3 + c^2b_2 + c^1b_1) + \kappa\partial b_3\right) \\
\mathsf{J} &= c^ib_i + \frac{\kappa-2}{p}J_3^{\text{tot}}.
\end{aligned} \tag{2.19}$$

A preferred value of the parameter κ will emerge when we discuss the reduction to the minimal string, but it will be useful to maintain the explicit κ dependence.

2.3 Twisting

The next step is to twist the $(2,2)$ structure [44]. There are four choices. We can twist T either to $\mathsf{T} + \frac{1}{2}\partial\mathsf{J}$ or to $\mathsf{T} - \frac{1}{2}\partial\mathsf{J}$, and similarly for the right movers. We will focus on the B-model with $(+,+)$ twist, which will be shown to be equivalent to the minimal string. There is no obstruction in defining the B-model because the worldsheet fermions are free, and the axial current is manifestly non-anomalous. The A-model will turn out to be trivial. We hasten to add that because of the torsion, one's usual geometric intuition about observables in the A- and B-models does not readily apply. In particular B-model amplitudes generically receive quantum corrections [45].

Using this topological sigma model, we define a string theory by "coupling it to topological gravity". In practice, one can ignore the topological gravity multiplet and compute amplitudes using an analogy with the bosonic string, as we now schematically review. The analogy is

$$\begin{aligned}
\mathsf{T} + \frac{1}{2}\partial\mathsf{J} &\longrightarrow T \\
\mathsf{G}^+ &\longrightarrow J_{\text{BRST}} \\
\mathsf{G}^- &\longrightarrow B \\
\mathsf{J} &\longrightarrow J_{\text{ghost}}.
\end{aligned} \tag{2.20}$$

Here T, $J_{\rm BRST}$, B, and $J_{\rm ghost}$ are the usual stress tensor, BRST current, antighost, and ghost number current of the bosonic string. It is possible to add improvement terms to $J_{\rm BRST}$ and $J_{\rm ghost}$ such that T, $J_{\rm BRST}$, B, and $J_{\rm ghost}$ generate *precisely* a twisted $N = 2$ algebra [46].

The physical states of the topological theory are cohomology classes of G_0^+, that is, chiral primaries of the worldsheet $N = 2$ algebra. If the $N = 2$ algebra is unitary, chiral primaries are automatically annihilated by G_0^-. In our case, the $N = 2$ algebra is not unitary. As in the bosonic string, proper care must be taken in combining left and right movers to form closed string states: the closed-string Hilbert space is defined as the semi-relative cohomology of of $\mathsf{Q} \equiv \mathsf{G}_0^+ + \bar{\mathsf{G}}_0^+$, which is the Q-cohomology on the complex of states annihilated by $\mathsf{G}_0^- - \bar{\mathsf{G}}_0^-$,

$$\mathcal{H}_{\rm Hilbert} = H_{\mathsf{Q}}({\rm Ker}_{\mathsf{G}_0^- - \bar{\mathsf{G}}_0^-}). \tag{2.21}$$

The measure on the moduli space of Riemann surfaces is given by insertions of G^- contracted with quadratic differentials,

$$\mathsf{G}^-(\mu_k) = \int d^2z\, \mathsf{G}^-_{zz}\, \mu_{k\,\bar{z}}^{z}. \tag{2.22}$$

Using the substitution (2.20), one defines string amplitudes for genus $g > 1$ through the formula

$$F_g(t) = \left\langle \prod_{j=1}^{3g-3} |\mathsf{G}^-(\mu_j)|^2 \quad e^{\sum_i t_i \int d^2z\, \mathsf{G}^-_{-1}\bar{\mathsf{G}}^-_{-1}\phi^i} \right\rangle. \tag{2.23}$$

Here $\{\phi_i\}$ are (unintegrated) physical vertex operators, and absolute values squared denote contributions from left and right movers. For genus zero, there is a similar formula, except that we must absorb Killing vectors fixing the position of three vertex operators,

$$\partial_i\partial_j\partial_k F_0(t) = \left\langle \phi^i\phi^j\phi^k e^{\sum_l t_l \int d^2z\, \mathsf{G}^-_{-1}\bar{\mathsf{G}}^-_{-1}\phi^l} \right\rangle. \tag{2.24}$$

Finally, the genus one amplitude is often presented in the operator formalism,

$$F_1 = \frac{1}{2}\int \frac{d^2\tau}{{\rm Im}(\tau)} {\rm Tr}\left[(-1)^{f_L+f_R} f_L f_R\, q^{\mathsf{L}_0 - \frac{1}{2}\mathsf{J}_0} \bar{q}^{\bar{\mathsf{L}}_0 - \frac{1}{2}\bar{\mathsf{J}}_0}\right], \tag{2.25}$$

where $q = e^{2\pi i\tau}$ and f is the worldsheet fermion number.

Analogy (2.20) will be an important guiding principle in relating our topological theory to the minimal string.

3 Reduction to the minimal string

We are now going to argue that the topological string theory on $H_3^+ \times S^3$ is equivalent to the minimal $(p, 1)$ string theory. To this end, it is useful to use a free field representation for $H_3^+ \times S^3$.

3.1 Wakimoto representation

There is a well-known free-field realization of the $\widehat{\mathfrak{su}}(2)$ current algebra in terms of a linear dilaton and of a first-order (β, γ) system of dimensions $(1, 0)$. All in all, we introduce two (β, γ) systems and two linear dilaton fields φ and x with central charges $c_\varphi = 1 + 6/p$ and $c_x = 1 - 6/p$. The OPEs read (notice that we take $\alpha' = 1$)

$$\begin{aligned} \beta_L(z)\gamma_L(w) &\sim -\frac{1}{z-w}, \quad \varphi(z)\varphi(w) \sim -\frac{1}{2}\log(z-w), \\ \beta_M(z)\gamma_M(w) &\sim -\frac{1}{z-w}, \quad x(z)x(w) \sim -\frac{1}{2}\log(z-w). \end{aligned} \tag{3.1}$$

For S^3, we represent the $\widehat{\mathfrak{su}}(2)_{p-2}$ current algebra as

$$\begin{aligned} j_+ &= \beta_M \\ j_3 &= +\gamma_M\beta_M + i\sqrt{p}\partial x \\ j_- &= -\gamma_M^2\beta_M - i2\sqrt{p}\gamma_M\partial x - (p-2)\partial\gamma_M. \end{aligned} \tag{3.2}$$

Similarly for H_3^+, we write the $\widehat{\mathfrak{su}}(2)_{-p-2}$ algebra as

$$\begin{aligned} k_+ &= \beta_L \\ k_3 &= \gamma_L\beta_L - \sqrt{p}\partial\varphi \\ k_- &= -\gamma_L^2\beta_L + 2\sqrt{p}\gamma_L\partial\varphi + (p+2)\partial\gamma_L. \end{aligned} \tag{3.3}$$

As we will see in more detail in Section 4, the variables $\{e^\varphi, \gamma_L, \bar{\gamma}_L\}$ are closely related to the Poincaré coordinates for H_3^+. Substituting in the (untwisted) stress tensor (2.16) yields

$$\begin{aligned} \mathsf{T_j} + \mathsf{T_k} &= \frac{1}{2p}(2j_3j_3 + j_+j_- + j_-j_+ - 2k_3k_3 - k_-k_+ - k_+k_-) \\ &= -(\partial x)^2 - \frac{i}{\sqrt{p}}\partial^2 x - (\partial\varphi)^2 - \frac{1}{\sqrt{p}}\partial^2\varphi \\ &\quad - \beta_M\partial\gamma_M - \beta_L\partial\gamma_L. \end{aligned} \tag{3.4}$$

The free-field representation must be supplemented with appropriate screening charges. For S^3, the screeners are

$$\begin{aligned} \mathsf{Q}_-^M &= \oint dz\, \beta_M\, e^{-2ix/\sqrt{p}} \\ \mathsf{Q}_+^M &= \oint dz\, \beta_M^{-p}\, e^{+2i\sqrt{p}x}, \end{aligned} \tag{3.5}$$

while for H_3^+,

$$\begin{aligned} \mathsf{Q}_-^L &= \oint dz\, \beta_L\, e^{-2\varphi/\sqrt{p}} \\ \mathsf{Q}_+^L &= \oint dz\, \beta_L^{p}\, e^{-2\sqrt{p}\varphi}. \end{aligned} \tag{3.6}$$

Let us briefly review the free-field resolution [47] of the irreducible $\widehat{\mathfrak{su}}(2)$ modules, focusing on the S^3 factor. We introduce the Fock spaces $\mathcal{F}_{m,n}$, obtained by acting with oscillators on the vacuum of x-momentum $p_x = \frac{m-1}{2\sqrt{p}} + \frac{(1-n)\sqrt{p}}{2}$,

$$\begin{aligned} \mathcal{F}_{m,n} \equiv \mathrm{Span}\{&\beta_{-i_1}\cdots\beta_{-i_\alpha}\,\gamma_{-j_1}\cdots\gamma_{-j_\beta}\,a_{-k_1}\cdots \\ &a_{-k_\gamma}\, e^{((2i(m-1)+2i(1-n)p)/2\sqrt{p}x)}|0\rangle\,\}. \end{aligned} \tag{3.7}$$

Here the a_n's are the usual oscillators for the field x, $i\partial x(z) = \sum_n a_n z^{-n+1}$, and $|0\rangle$ is the $SL(2)$ invariant vacuum. Irreducible representations of $\widehat{\mathfrak{su}}(2)_{p-2}$ are labeled by a semi-integer spin j, with $0 \le j \le p/2-1$. For a given j in this range, consider the sequence

$$\cdots \xrightarrow{\mathsf{Q}_-^{2j+1}} \mathcal{F}_{2p-2j-1,1} \xrightarrow{\mathsf{Q}_-^{p-2j-1}} \mathcal{F}_{2j+1,1} \xrightarrow{\mathsf{Q}_-^{2j+1}} \mathcal{F}_{-2j-1,1} \xrightarrow{\mathsf{Q}_-^{p-2j-1}} \cdots \tag{3.8}$$

This defines a complex, i.e., $(\mathsf{Q}_F^M)^2 = 0$, where the symbol Q_F^M denotes $(\mathsf{Q}_-^M)^{2j+1}$ or $(\mathsf{Q}_-^M)^{p-2j-1}$ according to which space it acts on. Moreover, the sequence is exact except at the middle Fock space $\mathcal{F}_{2j+1,1}$, where the cohomology $H_{\mathsf{Q}_F^M}(\mathcal{F}_{2j+1})$ is isomorphic to the irreducible $\widehat{\mathfrak{su}}(2)_{p-2}$ module of spin j [47]. To obtain each spin j representation with $0 \le j \le p/2-1$

once, we consider the cohomology $H_{\mathsf{Q}_F^M}(\mathcal{F})$, where $\mathcal{F}$ is the direct sum of Fock spaces

$$\mathcal{F} = \bigoplus_{j=0}^{p/2-1} \mathcal{F}_{[j]} \equiv \bigoplus_{j=0}^{p/2-1} \bigoplus_{k\in\mathbb{Z}} \mathcal{F}_{\pm(2j+1)+2kp,1} = \bigoplus_{m\in\mathbb{Z}}^{m\neq 0 \bmod p} \mathcal{F}_{m,1}. \tag{3.9}$$

The absence of the Fock spaces $\mathcal{F}_{m=kp,1}$ will play a role in the following.

There is a similar contruction for $\widehat{\mathfrak{su}}(2)_{-p-2}$, with the analogous Felder BRST charge Q_F^L built from powers of Q_-^L. The representations of $\widehat{\mathfrak{su}}(2)_{-p-2}$ that will be relevant for us are the principal discrete representations $\mathcal{D}^\pm_{j'}$, also labeled by semi-integer spins j'. For the purposes of this section, it will be sufficient to keep track of the representations for S^3.

3.2 Preview

Consider now the *twisted* stress tensor

$$\begin{aligned} \mathsf{T} + \frac{1}{2}\partial\mathsf{J} = &-(\partial x)^2 + i\frac{\kappa-2}{\sqrt{p}}\partial^2 x - (\partial\varphi)^2 - \frac{\kappa}{\sqrt{p}}\partial^2\varphi \\ &- b_1\partial c^1 + \frac{\kappa-1}{p}\partial(c^1 b_1) - b_2\partial c^2 - \frac{\kappa-1}{p}\partial(c^2 b_2) - b_3\partial c^3 \\ &- \beta_M\partial\gamma_M + \frac{\kappa-1}{p}\partial(\gamma_M\beta_M) - \beta_L\partial\gamma_L + \frac{\kappa-1}{p}\partial(\gamma_L\beta_L). \end{aligned} \tag{3.10}$$

We summarize in Table 1 the conformal dimensions of the fields and vertex operators. For the special value $\kappa = p+1$, the additional twist by J_3^{tot} is precisely of one unit,

$$\mathsf{T}_{\text{twisted}} = \mathsf{T}_{\text{phys}} + \frac{1}{2}\partial(c^i b_i) + \partial J_3^{\text{tot}}, \tag{3.11}$$

and we see from (3.10) that the central charges of x and φ are the expected ones for the $(p,1)$ model coupled to gravity:

$$c_x = 1 - 6\frac{(p-1)^2}{p}, \quad c_\varphi = 1 + 6\frac{(p+1)^2}{p}. \tag{3.12}$$

Table 1: Conformal dimensions in the twisted theory, for arbitrary κ.

	β_M	γ_M	β_L	γ_L	c^1	c^2	c^3	b_1	b_2	b_3	$e^{2i\alpha x}$	$e^{2\beta\varphi}$
Δ	$\frac{p+1-\kappa}{p}$	$\frac{\kappa-1}{p}$	$\frac{p+1-\kappa}{p}$	$\frac{\kappa-1}{p}$	$\frac{1-\kappa}{p}$	$\frac{\kappa-1}{p}$	0	$\frac{\kappa+p-1}{p}$	$\frac{p+1-\kappa}{p}$	1	$\alpha\left(\alpha-\frac{\kappa-2}{\sqrt{p}}\right)$	$-\beta\left(\beta+\frac{\kappa}{\sqrt{p}}\right)$

Moreover, the (b_1, c^1) system has dimensions $(2, -1)$ and can be identified with the diffeomorphism ghosts of the minimal string. The remaining degrees of freedom are two pairs of $\beta\gamma$, bc systems: (β_L, γ_L) (b_2, c^2), and (β_M, γ_M) (c^3, b_3), of dimensions $(0,1)(0,1)$. We expect them to decouple by the quartet mechanism (bosonic and fermionic degrees of freedom with the same quantum numbers canceling pairwise in the path integral). In fact there are well-known procedures to obtain minimal matter and Liouville CFT from $SL(2)$ WZW models, known, respectively, as Hamiltonian [48] and KPZ [7] reduction, which also exploit the idea of quartet confinement.

While this is promising, we should not dictate any additional rules; the equivalence with the minimal string should arise from the established rules for perturbative (topological) string theory that we reviewed above. To prove this, we will decompose the BRST operator of the topological theory as $\mathsf{G}_0^+ = \mathsf{Q}_1 + \mathsf{Q}_R$, and show that taking the cohomology with respect to Q_R implements a version of Hamiltonian + KPZ reduction, reducing the field content to that of the minimal string. Then we will find a similarity transformation that maps the $N = 2$ generators of the topological string to the corresponding generators of the bosonic string (2.20), up to Q_R-trivial terms. In particular, the operator Q_1 is mapped to the usual BRST operator $\mathsf{Q}_{\mathrm{Vir}}$ of the bosonic string. Schematically, we are going to show the following equivalences of cohomologies

$$H_{\mathsf{G}_0^+}(H_3^+ \times S^3) \cong H_{\mathsf{Q}_1}(H_{\mathsf{Q}_R}(H_3^+ \times S^3)) \cong H_{\mathsf{Q}_{\mathrm{Vir}}}((\mathrm{p}, 1) + \mathrm{gravity}). \quad (3.13)$$

Finally, through the use of the similarity transformation, the computations of topological amplitudes are exactly mapped to the corresponding computations in the $(p, 1)$ bosonic string.

We should mention at the outset that there is at least a superficial resemblance of this story to computations done for G/G WZW models with $G = SU(2)_{p-2}$, where the spectrum also exactly reproduces that of the $(p, 1)$ minimal bosonic string [49–51]. Our BRST operator differs from the one used in the G/G context. There is also the crucial conceptual difference that to compute correlation functions in the G/G model one would naturally integrate over the moduli space of flat $SU(2)$ connections, whereas in our case we want to have a string theory and so we integrate over the moduli space of Riemann surfaces. We do not exclude however that a deeper connection may be found. Although both the interpretation and several details are different, we found it useful to borrow some technical aspects of the work by Sadov [52], who performed a cohomological analysis in the context of G/G models (see also Frenkel, Appendix in [53]).

3.3 Setup for the reduction

Our task is to study the cohomology of G_0^+ acting on the state space of the $H_3^+ \times S^3$ sigma model. Using the free-field representation, we are instructed to first evaluate the cohomology of the Felder charges on the free-field state space, and then the G^+ cohomology,

$$H_{\mathsf{G}^+}(H_3^+ \times S^3) \cong H_{\mathsf{G}^+}(H_{\mathsf{Q}_F^M + \mathsf{Q}_F^L}(\mathcal{F})). \tag{3.14}$$

Here $\mathcal{F}$ is the total space state of the *free* fields

$$\mathcal{F} = \mathcal{F}_x \otimes \mathcal{F}_{\beta^M \gamma^M} \otimes \mathcal{F}_\varphi \otimes \mathcal{F}_{\beta^L \gamma^L} \otimes \mathcal{F}_{b_i c^i}. \tag{3.15}$$

The main idea is to split up the BRST differential as

$$\mathsf{G}_0^+ = \mathsf{Q}_1 + \mathsf{Q}_2 + \mathsf{Q}_3, \quad \{\mathsf{Q}_i, \mathsf{Q}_j\} = 0. \tag{3.16}$$

where we choose

$$\begin{aligned} \mathsf{Q}_1 &= \oint c^1 j_- \qquad (3.17)\\ \mathsf{Q}_2 &= \oint c^2 \beta_L \\ \mathsf{Q}_3 &= \oint c^3(-\sqrt{p}\partial\varphi + i\sqrt{p}\partial x + \gamma_M \beta_M + c^1 b_1) + \oint c^3(\gamma_L \beta_L - c^2 b_2). \end{aligned}$$

In relation to the previous section, $\mathsf{Q}_R \equiv \mathsf{Q}_2 + \mathsf{Q}_3$ is the operator that implements the reduction, while Q_1 will turn out to be equivalent to $\mathsf{Q}_{\mathrm{Vir}}$ of the bosonic string. We can assign gradings to the fields by defining

$$q_1 = \oint c^1 b_1, \quad q_2 = \oint c^2 b_2, \quad q_3 = \oint c^3 b_3. \tag{3.18}$$

Then Q_i has degree 1 with respect to q_i and degree zero with respect to the other gradings. These three pieces of G_0^+ mutually commute and are separately nilpotent, but in general this does not guarantee that the cohomology of G_0^+ is obtained by computing the cohomologies of the individual Q_i's successively. In the case at hand, it turns out that we can compute the cohomology (3.14) as:

$$\begin{aligned} &H_{\mathsf{G}^+}(H_{\mathsf{Q}_F^M + \mathsf{Q}_F^L}(\mathcal{F})) \cong H_{\mathsf{G}^+ + \mathsf{Q}_F^M + \mathsf{Q}_F^L}(\mathcal{F}) \\ &\quad \cong H_{\mathsf{Q}_1 + \mathsf{Q}_F^M + \mathsf{Q}_F^L}(H_{\mathsf{Q}_R}(\mathcal{F})) \cong H_{\mathsf{Q}_1 + \mathsf{Q}_F^M + \mathsf{Q}_F^L}(H_{\mathsf{Q}_3}(H_{\mathsf{Q}_2}(\mathcal{F}))). \end{aligned} \tag{3.19}$$

The justification of this claim is based on a simple property of double complexes and can be found in Appendix A. We see that we can evaluate the cohomology in the order Q_2, Q_3, $\mathsf{Q}_1 + \mathsf{Q}_F$. Here is a schematic outline of the calculation:

Q_2 *reduction*: The BRST operator Q_2 is associated to a positive root of $\widehat{\mathfrak{su}}(2)$, and computing its cohomology is similar to Hamiltonian reduction, except that here we are setting $\beta_L \to 0$ rather than to a constant. This step gets rid of the quartet $\{b_2, c^2, \beta_L, \gamma_L\}$.

Q_3 *reduction*: The operator Q_3 is associated to a $\hat{\mathfrak{u}}(1)$ generator of $\widehat{\mathfrak{su}}(2)_{p-2} \oplus \widehat{\mathfrak{su}}(2)_{-p-2}$. Taking its cohomology amounts to restricting to the states which are invariant under this current. This is similar in spirit the reduction from $SL(2) \times U(1)$ to $SL(2)/U(1)$ described in [54, 55]. This steps essentially kills the second quartet $\{b_3, c^3, \beta_M, \gamma_M\}$.

We now turn to a more detailed analysis.

3.4 Q_2 reduction

The operator $\mathsf{Q}_2 = \oint c^2\beta_L$ is just the supercharge for the quartet $\{b_2 c^2 \beta_L \gamma_L\}$. The standard Kugo–Ojima quartet mechanism applies: in Q_2 cohomology, only the vacuum state survives. Actually because of the usual phenomenon of picture degeneracy for a $\beta\gamma$ system, there are infinitely many vacua, one for each choice of picture. Vacua in different pictures must be considered physically identical. One can move between different pictures with the help of the picture raising and lowering operators,

$$Y_L = b_2\delta(\beta_L), \quad Z_L = c^2\delta(\gamma_L). \tag{3.20}$$

The precise statement of Q_2 reduction is that once we commit ourselves to a choice of picture — for example, by restricting to the Fock space built on the $SL(2)$ vacuum — only the vacuum in that picture survives. In particular for a given picture, H_{Q_2} is non-trivial for only one value of the grading q_2.

3.5 Q_3 reduction

The operator Q_3 be viewed as the BRST charge for the $U(1)$ symmetry generated by the current

$$\mathcal{J} = -\sqrt{p}\partial\varphi + i\sqrt{p}\partial x + \gamma_M\beta_M + c^1 b_1, \tag{3.21}$$

with c^3, b_3 the corresponding ghost and antighost fields.[4] The Q_3 cohomology consists of the space of gauge-invariant states. In particular, since $\mathcal{J}_0 = \{\mathsf{Q}_3, (b_3)_0\}$, all states in H_{Q_3} must be singlets with respect to the $U(1)$ current.

A familiar way to organize the calculation is to separate out the ghost zero mode. To find H_{Q_3}, we can first calculate the relative cohomology $H^R_{\mathsf{Q}_3}$

[4] We have left out the term $\beta_L\gamma_L - c^2 b_2 = \{\mathsf{Q}_2, b_2\gamma_L\}$ because it acts trivially on H_{Q_2}.

in the complex annihilated by $(b_3)_0$. By standard arguments, the relative cohomology is non-trivial only for degree $q_3 = 0$. Then the absolute cohomology H_{Q_3} is simply given by $H_{\mathsf{Q}_3} \cong H^R_{\mathsf{Q}_3} \oplus (c^3)_0 H^R_{\mathsf{Q}_3}$. It remains to find a useful representation for $H^R_{\mathsf{Q}_3}$. We have seen that for $\kappa = p+1$, the fields φ, x, b_1, and c^1 are very reminiscent of the fields of the minimal string and we would like to use them as generators of the H_{Q_3}, but they are not $U(1)$ invariant. There is a simple remedy for this — we are going to dress them up with appropriate powers of β_M or β_M^{-1}:

$$\begin{aligned} e^{-2\Phi/\sqrt{p}} &= \beta_M e^{-2\varphi/\sqrt{p}} \\ e^{-2iX/\sqrt{p}} &= \beta_M e^{-2ix/\sqrt{p}} \\ B &= b_1 \beta_M \\ C &= c^1 \beta_M^{-1}. \end{aligned} \tag{3.22}$$

By a simple counting we see that the fields Φ, X, B, C in fact generate all gauge-invariant combinations. These fields have the remarkable property that their dimensions are independent of κ. The (B, C) fields are fermionic ghosts of dimensions and $(2, -1)$, and the background charges of X and Φ are the correct ones for the matter and Liouville fields of the $(p, 1)$ model. So we seem to have obtained simple expressions for the generators that can be identified exactly with the fields of the minimal bosonic string.

This is essentially correct; however, these generators contain some redundancy with respect to the original degrees of freedom, introduced in the step of taking the formal inverse power of β_M. To make sense of β_M^{-1}, we can bosonize the $\beta_M\gamma_M$-system. To this end we introduce two scalar fields ρ, σ with stress tensor

$$\mathsf{T}_{\rho\sigma} = -(\partial\rho)^2 + \frac{1-2\lambda}{\sqrt{2}}\partial^2\rho - (\partial\sigma)^2 + \frac{i}{\sqrt{2}}\partial^2\sigma, \tag{3.23}$$

and write[5]

$$\beta_M = \eta e^{-\sqrt{2}\rho} \equiv e^{-i\sqrt{2}\sigma - \sqrt{2}\rho}, \quad \gamma_M = -\partial\xi e^{\sqrt{2}\rho} \equiv -i\sqrt{2}\partial\sigma e^{i\sqrt{2}\sigma + \sqrt{2}\rho}. \tag{3.24}$$

The $\eta\xi$ fields are fermionic ghosts of dimensions $(1, 0)$. We have quoted the general formula for $\Delta(\beta_M, \gamma_M) = (\lambda, 1-\lambda)$, but in the following we take $\lambda = \frac{p+1-\kappa}{p}$. Now we can set

$$\beta_M^{-1} = \xi e^{\sqrt{2}\rho}. \tag{3.25}$$

As is familiar, bosonization introduces one additional zero mode, the zero mode of $\xi = e^{i\sqrt{2}\sigma}$. Moreover we encounter the usual picture degeneracy:

[5]The unconventional factors of $\sqrt{2}$ arise from our choice $\alpha' = 1$, which is awkward here but quite useful elsewhere.

there are infinitely many copies of the Hilbert space of the $\beta_M \gamma_M$ system labeled by the picture $\oint(\beta_M\gamma_M + \eta\xi) = p_\rho - p_\sigma$, where $p_\rho \equiv \sqrt{2}\oint \partial\rho$, $p_\sigma \equiv \sqrt{2}\oint i\partial\sigma$. Since

$$\{\mathsf{Q}_R, b_3 - b_2\gamma_L\} = -\sqrt{p}\partial\varphi + i\sqrt{p}\partial x + \gamma_M\beta_M + c^1 b_1, \tag{3.26}$$

we can define the picture changing operators

$$Z_M = e^{\sqrt{p}\varphi - i\sqrt{p}x - \sqrt{2}\rho - \int c^1 b_1}, \quad Y_M = e^{-\sqrt{p}\varphi + i\sqrt{p}x + \sqrt{2}\rho + \int c^1 b_1}. \tag{3.27}$$

These operators are Q_R-closed and their derivatives are Q_R-exact.

The fields Φ, X, B, C are all automatically at zero picture. We still need to restrict to the "small Hilbert space" of states that do not contain ξ_0, or equivalently to the kernel of η_0. It is convenient to use the zero-picture version of η_0, the nilpotent operator

$$\begin{aligned} \mathsf{F} \equiv \oint \eta\, Z_M &= \oint \beta_M e^{\sqrt{p}\varphi - i\sqrt{p}x - \int c^1 b_1} \\ &= \oint B e^{\sqrt{p}(\Phi - iX)}. \end{aligned} \tag{3.28}$$

We can now state the final result for $H^R_{\mathsf{Q}_3}$: it consists of the states generated by Φ, X, B, C and in the kernel of F.[6]

3.6 Comparison with the cohomology of the minimal string

Let us now examine the Felder charges. The $\widehat{\mathfrak{su}}(2)_{p-2}$ screeners are

$$\beta_M e^{-2ix/\sqrt{p}} = e^{-2iX/\sqrt{p}}, \quad \beta_M^{-p} e^{2i\sqrt{p}x} = e^{2i\sqrt{p}X}, \tag{3.29}$$

which are just the usual matter screening operators for the $(p,1)$ model (see (B.9) in Appendix B). The H_3^+ screeners on the other hand turn out to be exact:

$$\beta_L e^{-2\varphi/\sqrt{p}} = \{\mathsf{Q}_R, b_2 e^{-2\varphi/\sqrt{p}}\}, \tag{3.30}$$

[6]It is worth noticing that F is precisely the fermionic screening operator $\tilde{Q}$ encountered in [46] in the analysis of the underlying $N=2$ structure of the minimal bosonic string. It is also equivalent to the operator usually denoted by Q_S in the context of topological gravity [9, 56]. We suspect that a deeper understanding of this structure may involve an $N=4$ topological algebra [57], with Q_{Vir} and F as the $N=4$ G^+ and $\tilde{G}^+$ generators.

and similarly for the + screener. Thus the H_3^+ screener does not reduce to the cosmological constant operator, which instead descends from

$$\beta_M e^{-2\varphi/\sqrt{p}} = e^{-2\Phi/\sqrt{p}}. \tag{3.31}$$

The last piece Q_1 of the original topological BRST operator G_0^+ will be shown below to be equivalent to Q_{Vir} of the minimal string up to Q_R exact terms. All in all, we can summarize our findings as[7]

$$H_{\mathsf{G}_0^+}(H_3^+ \times S^3) \cong H_{\mathsf{Q}_{\text{Vir}}+\mathsf{Q}_F^X}(\text{Ker}_{\mathsf{F}}(\mathcal{F}_X \otimes \mathcal{F}_\Phi \otimes \mathcal{F}_{BC})). \tag{3.32}$$

Here Q_F^X denotes the Felder charge for the matter $(p,1)$ model. We should also recall that the field X inherits from x the restriction to the lattice of momenta $p_X = \frac{m-1}{2\sqrt{p}}$, $m \in \mathbb{Z}$, $m \neq 0 \bmod p$ (see Equation (3.9)).

We claim that (3.32) contains all the expected "small phase space" states of the minimal $(p,1)$ string. As we review in detail in Appendix B, the cohomology of the $(p,1)$ string[8] consists of infinitely many tachyon states $T_{r,s}$ and infinitely many ground ring states $G_{r,s}$, with $1 \leq r \leq p-1$, $s \geq 1$. By definition, the small phase space is spanned by the states with $s = 1$. Using the explicit expressions for $T_{r,1}$ and $G_{r,1}$, it is straightforward to check that these states are elements of (3.32): they are annihilated by F and define non-trivial representatives of the cohomology classes. The restriction of the complex to the kernel of F is crucial. Without this restriction, the cohomology (3.32) would be empty, since for $(p,1)$ matter the Felder complex is exact. This implies that, for example, $T_{r,1} = (\mathsf{Q}_{\text{Vir}} + \mathsf{Q}_F^X)(\Lambda_{r,s})$ for some state $\Lambda_{r,1}$; but one finds $\Lambda_{r,1} \notin \text{Ker}_{\mathsf{F}}$.

To obtain the "gravitational descendants" of the minimal string, which are the states with $s > 1$, we will need to enlarge the spectrum of the theory on $H_3^+ \times S^3$ by allowing for "spectral flowed" states (long strings). Including such states is actually a necessity for consistency of string theory on $H_3^+ \times S^3$ [58]. The relation between gravitational descendants and long strings will be discussed in Section 4.4. Ultimately we will find that all the states are correctly matched.

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[7]For clarity, we omit here the doubling of the cohomology due to $(c^3)_0$.

[8]We focus in the following on the chiral cohomology relative to B_0, since if that is found to agree, the construction of closed string states (in semi-relative cohomology) poses no problem. We also restrict to states obeying the Seiberg bound.

3.7 Similarity transformation

We will now establish a correspondence between the twisted $N = 2$ structure of the topological string and the underlying twisted $N = 2$ structure of the minimal bosonic string. It will follow from this map that the rules for computing $N = 2$ amplitudes on $H_3^+ \times S^3$ coincide with the usual rules for minimal bosonic string amplitudes, up to some additional insertions of picture changing operators.

The strategy will be to perform a similarity transformation on the $N = 2$ algebra given in (2.19), repeated here for reference:

$$\begin{aligned}
\mathsf{T} &= \frac{1}{2p}(2j_3j_3 + j_+j_- + j_-j_+ - 2k_3k_3 - k_-k_+ - k_+k_-) \\
&\quad + \frac{1}{2}\left(\partial c^i b_i - c^i \partial b_i\right) + \frac{\kappa}{2p}\partial J_3^{\rm tot} \\
\mathsf{G}^+ &= c^1(j_-) + c^2(k_+) + c^3(j_3 + k_3 - c^2b_2 + c^1b_1) \\
\mathsf{G}^- &= \frac{1}{p}\left(b_1j_+ - b_2k_- + b_3(j_3 - k_3 + c^2b_2 + c^1b_1) + \kappa\partial b_3\right) \\
\mathsf{J} &= c^ib_i + \frac{\kappa - 2}{p}J_3^{\rm tot}.
\end{aligned} \tag{3.33}$$

We take as generator of the similarity transformation the operator

$$\mathsf{L} = -\oint (c^1 j_+^{-1})\left(-b_2k_- + b_3(j_3 - k_3 + c^2b_2 + c^1b_1) + \kappa\partial b_3\right). \tag{3.34}$$

Then we find[9]

$$\exp \mathrm{Ad}(\mathsf{L})\mathsf{G}_0^+ = \mathsf{Q}_R + p\,\mathsf{Q}_{\rm Vir} + \{\mathsf{Q}_R, \bullet\}, \tag{3.35}$$

where $\mathsf{Q}_{\rm Vir}$ is just the usual Virasoro BRST operator of the minimal string theory,

$$\mathsf{Q}_{\rm Vir} = \oint C\left(T_\Phi + T_X + \frac{1}{2}T_{BC}\right). \tag{3.36}$$

Next we consider G^-. This generator plays the analog of the B ghost in bosonic string theory. A short calculation shows that it is in fact exactly equivalent to B,

$$\exp \mathrm{Ad}(\mathsf{L})\mathsf{G}^- = \frac{1}{p}(b_1j_+) = \frac{1}{p}B. \tag{3.37}$$

This has the important consequence that the integration over the moduli space of Riemann surfaces and the definition of semi-relative cohomology

[9] We have checked this expression up to derivatives in β_M (which arise through normal ordering) and passed it through some consistency checks.

are the same both in the topological string and in the minimal string. Of course, applying the similarity transformation to the commutator of G_0^+ and G^-, we find the expected stress tensor of the minimal string up to Q_R-exact terms.

Finally the $U(1)_R$ current is found to be invariant,

$$\exp \mathrm{Ad}(\mathsf{L})\mathsf{J} = \mathsf{J}. \tag{3.38}$$

The current J differs from the ghost number current CB of the minimal string. This is actually expected, since the $U(1)_R$ anomaly prescribes how we should saturate the background charges. Even though two $\{b, c, \beta, \gamma\}$ quartets effectively decouple, we still must insert picture changing operators to saturate their fermionic zero modes. Taking $\kappa \equiv p + 1$, at genus g, we need $(g-1)$ insertions of $Z_L \bar{Z}_L$ operators and $(g-1)$ insertions of $Z_3 \bar{Z}_3 \equiv \int d^2z\, b_3\, \bar{b}_3$.

3.8 General (p,q) models?

At a formal level, the manipulations of this section work equally well with the replacement $p \to p/q$. Then the Q_R reduction yields precisely the field content of the (p, q) minimal model coupled to gravity. All the expected (r, s) matter representations of the minimal string arise provided start in the "upstairs" theory with a direct sum of $\mathcal{F}_{m,n}$ Fock spaces with $m \in \mathbb{Z}$, $m \neq 0 \bmod p$, $1 \leq n \leq q$ (see Equation (3.9)). So the question arises whether we can also give minimal (p, q) strings an interpretation as a string theory on a Euclidean space with H-flux. For H_3^+ the H-flux need not be quantized, so fractional level already makes sense. Of course the issue is how to interpret $SU(2)$ at fractional level.[10]

4 What does the topological string compute?

In this section, we turn our attention to IIB string theory on $H_3^+ \times S^3 \times M_4$. In close analogy with the Calabi–Yau case, the topological string theory on $H_3^+ \times S^3$ computes a set of special amplitudes of spacetime chiral primaries in the IIB theory, in the presence of background RR 5-form flux. The additional RR insertions are responsible for twisting the worldsheet theory. These special amplitudes are at string tree level from the viewpoint of the

[10] As we are going to see shortly, there is a sense in which the topological theory discussed so far computes physical amplitudes in the sector of one long string. It is tempting to speculate that the (p, q) models should correspond to the sector of q long strings. This suggestion is due to Seiberg.

physical theory. The perturbative expansion of the topological theory corresponds to an expansion in powers of the 5-form RR flux. A difference with respect to the Calabi–Yau case is the extra twist by $\partial J_3^{\rm tot}$. This gives additional insertions, which turn out to be insertions of "long string" vertex operators. The topological theory computes amplitudes in the sector of one long string.

In the remainder of this section, we work out the correspondence between spacetime chiral primaries on $H_3^+ \times S^3$ and the physical operators in the minimal string theory, both for the tachyons and the ground ring elements. Finally we discuss a family of exact deformations of the $H_3^+ \times S^3$ sigma model.

4.1 Amplitudes

We would like to understand exactly which physical amplitudes can be reduced to the topological string. The analysis is almost identical to [1] and [2] and so we will be brief. We can focus on the partition function, since the addition of external operators is straightforward. The physical partition function is given by the path-integral over the full set of 10-dimensional fields: the matter fields of the supersymmetric sigma model on $H_3^+ \times S^3 \times M_4$ plus the diffeomorphism ghosts and superghosts $\{b, c, \beta, \gamma\}$. The topological partition function is given instead by the path-integral over the six-dimensional fields only, with the fermions having twisted dimensions. Finally the measure on the moduli space of Riemann surfaces is different in the two theories. Nevertheless, in analogy with the Calabi–Yau case, one can show that the topological partition function equals the physical partition function at the only cost of introducing some extra insertions. Because of the relation (3.11) between the physical and the twisted stress tensors, we need to make insertions of two kinds: "graviphoton" vertex operators implementing the usual twist by $\frac{1}{2}\partial(c^i b_i)$; "spectral flow" operators implementing the twist by $\partial J_3^{\rm tot}$.

The operators analogous to the graviphotons of the Calabi–Yau case are

$$T^{\pm} = e^{-\phi/2-\tilde{\phi}/2} e^{1/2\int c^i b_i + 1/2\int \tilde{c}^i \tilde{b}_i} \Sigma^{\pm} \bar{\Sigma}^{\pm}. \tag{4.1}$$

Here ϕ is the boson arising from the usual bosonization of the $\beta\gamma$ superghosts [59]. The operators $\Sigma^{\pm}$ are the spectral flow operators for the fermions on M_4. In the case of $M_4 = T^4$, denoting the four free fermions by ψ_i, we can write

$$\Sigma^{\pm} = e^{\pm 1/2\int \psi_1\psi_2 + \psi_3\psi_4}. \tag{4.2}$$

One can can check that $c\bar{c}T^{\pm}$ are physical vertex operators for the superstring. Indeed we have $\Delta(T^{+}) = 1$ and $T^{\pm}$ has no $z^{-3/2}$ pole in its OPE with the $N = 1$ worldsheet supercurrent G. Just as in the Calabi–Yau case, $T^{\pm}$ correspond to the 5-form RR flux, with three indices along the "internal" six-dimensional manifold $H_3^+ \times S^3$ and two indices along the M_4. They are also closely related to *spacetime* supercurrents. For $H_3^+ \times S^3$, the left-moving spacetime supercurrents are [6]

$$\mathcal{S}_{\vec{\epsilon}} = e^{-\phi/2} e^{1/2 \int \sum_i^3 \epsilon_i c^i b_i + \epsilon_4 \psi_1 \psi_2 + \epsilon_5 \psi_3 \psi_4}, \tag{4.3}$$

where $\epsilon_i = \pm 1$ with the constraints $\prod_{i=1}^{3} \epsilon_i = \prod_{i=1}^{5} \epsilon_i = 1$. So we see that $T^{\pm}$ are products of a left-moving and a right-moving spacetime supercurrent.

The additional "spectral flow" operator is defined as

$$U \equiv \exp\left(-\int J_3^{\mathrm{tot}}\right), \tag{4.4}$$

and its interpretation will be discussed shortly. With this notation in place, we can formulate the basic claim: the genus g topological partition function F_g corresponds to the physical amplitude

$$F_g = \int_{\mathcal{M}_g} \left\langle \Theta_g\, R^2 \prod^{g-1} \int d^2 z\, T^+ \prod^{g-1} \int d^2 z\, T^- \right.$$

$$\left. \prod^{3g-3} |\,\delta(\beta)\mathsf{G}|^2 \prod^{3g-3} |b(\mu_i)|^2 \right\rangle_g^{\mathrm{physical}}, \tag{4.5}$$

with

$$\Theta_g \equiv (U\bar{U})^{2-2g}\, (Z_L \bar{Z}_L)^{g-1} (Z_3 \bar{Z}_3)^{g-1}. \tag{4.6}$$

As in [1], R^2 denotes the extra insertion of a ψ^8 expression needed to account properly for the fermion zero modes. The proof of this claim imitates the arguments of [1] and [2]. Here is a very brief sketch: the $T^{\pm}$ insertions are responsible for twisting the fermions and changing the background charge of the $\beta\gamma$ system, in such a way that the path-integral over the M_4 fields and the $bc\beta\gamma$ system can be trivially performed. Similarly, the insertions of U implement the twist by $\partial J_3^{\mathrm{tot}}$. The insertions of Z_L and Z_3 are needed to saturate fermion zero modes of $c^2 b_2$ and $c^3 b_3$, as discussed in Section 3.7. Writing $\mathsf{G} = \mathsf{G}^+ + \mathsf{G}^-_{\kappa=0} + \mathsf{G}_{M_4}$, we see that only G^-

can contribute due to the anomalous background charge for J. So finally we have

$$F_g = \int_{\mathcal{M}_g} \left\langle \prod^{3g-3} |\mathsf{G}^-_{\kappa=0}(\mu_i)|^2 \, (Z_L \bar{Z}_L)^{g-1} (Z_3 \bar{Z}_3)^{g-1} \right\rangle_g^{\text{topological}} . \qquad (4.7)$$

Apparently there is a discrepancy between the value of $\kappa = p + 1$ in the twisted stress tensor and the value of $\kappa = 0$ in G^-. This is not a problem: the saturation of fermionic zero modes is such that we can replace $\mathsf{G}^-_{\kappa=0}$ with $\mathsf{G}^-_{\kappa=p+1}$ because the κ-dependent term does not contribute. Equivalently, under the similarity transformation, $\mathsf{G}^-_{\kappa=0}$ gets mapped to $(B - (p + 1)\partial b_3)/p$, and again the term proportional to ∂b_3 does not contribute.

Finally we turn to the interpretation of the operator U. This is exactly the supersymmetric spectral flow operator constructed in [60] following [58]. The intuitive picture is that an insertion of U^w creates a "long string" — a worldsheet that wraps w times the boundary of H_3^+; it also adds angular momentum around the S^3, in such a way that spacetime supersymmetry is preserved.

4.2 Vertex operators for $H_3^+ \times S^3$

Our task is now to identify the physical states of the topological theory, defined as cohomology classes of G_0^+, with physical states of the 10-dimensional theory, defined as cohomology classes of the usual superstring BRST charge $\mathsf{Q}_{10\text{d}}$. To this end, we review in this section some general properties of vertex operators in $H_3^+ \times S^3$ [6, 61], emphasizing their relation with bulk-to-boundary propagators. In the following section, we will specialize to vertex operators for spacetime chiral primaries [60, 62], which are the natural candidates to match with the topological observables. This section and the next are almost entirely review of well-known material.

The WZW model on H_3^+ with a pure imaginary B-field is described by the following action

$$S = \int d^2 z \, \left(\partial\varphi \bar{\partial}\varphi + e^{2\varphi} \bar{\partial}\gamma \partial\bar{\gamma} \right) . \qquad (4.8)$$

The coordinates $\gamma, \bar{\gamma}$ and φ satisfy $\gamma^* = \bar{\gamma}, \varphi^* = \varphi$. This action is real and positive definite on a Euclidean worldsheet. The zero modes of the currents are given by the left- and right-invariant vector fields

$$\oint k_+ = -\partial_\gamma, \quad \oint k_3 = -\gamma\partial_\gamma + \frac{1}{2}\partial_\phi, \quad \oint k_- = \gamma^2\partial_\gamma - \gamma\partial_\phi - e^{-2\phi}\partial_{\bar\gamma}$$
$$\oint \bar k_+ = -\partial_{\bar\gamma}, \quad \oint \bar k_3 = -\bar\gamma\partial_{\bar\gamma} + \frac{1}{2}\partial_\phi, \quad \oint \bar k_- = \bar\gamma^2\partial_\gamma - \bar\gamma\partial_\phi - e^{-2\phi}\partial_\gamma. \tag{4.9}$$

Here we have put a bar on the currents to denote right movers on the worldsheet. Let us also define the total currents

$$\begin{aligned} &\mathsf{K}_3 = k_3 - c^2 b_2 \quad \mathsf{J}_3 = j_3 + c^1 b_1 \\ &\mathsf{K}_+ = k_+ + \left(c^3 - \frac{1}{p}b_3\right) b_2 \quad \mathsf{J}_+ = j_+ - c^1(pc^3 + b_3) \\ &\mathsf{K}_- = k_- + (pc^3 - b_3)c^2 \quad \mathsf{J}_- = j_- - b_1\left(c^3 + \frac{1}{p}b_3\right) \end{aligned} \tag{4.10}$$

They form an $\widehat{\mathfrak{su}}(2)$ current algebra of level $-p$, and level p respectively. The zero modes of the currents (4.10) are conserved charges in the spacetime theory. It is convenient to use these charges to organize the spectrum of string theory on $H_3^+ \times S^3$. In particular, according to Brown and Henneaux [63], the spacetime theory is itself a conformal theory and comes with its own set of Virasoro generators $\mathcal{L}_n$ which are distinct form the worldsheet Virasoro generators. As described in [6], the zero modes of $\mathsf{K}_+, \mathsf{K}_-, \mathsf{K}_3$ map to the spacetime Virasoro generators, and the zero mode of J_3 to the space time R-charge. The spacetime Virasoro algebra is given by

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{c^{st}}{12}(n^3 - n)\delta_{n+m}\,, \quad c^{st} = 6Q_1Q_5. \tag{4.11}$$

In order for this to agree with our commutation relations for $\mathsf{K}_+, \mathsf{K}_-, \mathsf{K}_3$, we need either $\{\mathcal{L}_1, \mathcal{L}_{-1}, \mathcal{L}_0\} \longrightarrow \left\{\oint \mathsf{K}_+, -\oint \mathsf{K}_-, -\oint \mathsf{K}_3\right\}$ or to $\{-\oint \mathsf{K}_-, \oint \mathsf{K}_+, \oint \mathsf{K}_3\}$. Since we wish to identify $\mathcal{L}_{-1}$ with the generator ∂_γ of translations on the boundary of H_3^+, we pick the second possibility,

$$\{\mathcal{L}_1, \mathcal{L}_{-1}, \mathcal{L}_0\} \longrightarrow \left\{-\oint \mathsf{K}_-, \oint \mathsf{K}_+, \oint \mathsf{K}_3\right\}. \tag{4.12}$$

In particular, the (left-moving) spacetime weight is measured by $\mathcal{L}_0 = +\oint \mathsf{K}_3$. We will denote the eigenvalues of $\mathcal{L}_0$ by h. Similarly the spacetime R-charge is measured by

$$q \equiv -\oint \mathsf{J}_3. \tag{4.13}$$

The vertex operators for the physical string states are built from the free fermions and primaries of the two $\mathfrak{su}(2)$ current algebras (and 10d ghosts and superghosts b, c, β, γ). The primaries $\Phi_{jm}(z)$ for H_3^+ fall in certain representations of $\mathfrak{su}(2)$ at negative level. The relevant ones for us are the principal discrete representations:

$$\begin{aligned}\mathcal{D}_j^-&: m = -j, -j-1, \dots\\ \mathcal{D}_j^+&: m = j, j+1, \dots\end{aligned} \tag{4.14}$$

These representations are labeled by the worldsheet dimension (which coincides with the Casimir)

$$\Delta = -\frac{j(j-1)}{p}, \tag{4.15}$$

and states are labeled by their eigenvalue m under k_3.

To discuss strings on H_3^+, it is convenient to exchange the label m for the isotopic spin coordinate x. We can interpret x, $\bar{x}$ as parametrizing the boundary of H_3^+. The vertex operator $\Phi_j(z|x)$ is then required to satisfy the OPEs

$$k_a(z)\Phi_j(w,\bar{w}|x,\bar{x}) \sim \frac{D_a}{z-w}\Phi_j(w,\bar{w}|x,\bar{x}) \tag{4.16}$$

with

$$D_+ = \frac{\partial}{\partial x}, \quad D_3 = x\frac{\partial}{\partial x} + j, \quad D_- = x^2\frac{\partial}{\partial x} + 2jx. \tag{4.17}$$

This implies the following expression:

$$\Phi_j(x) = \frac{1-2j}{\pi}\left(\frac{1}{|\gamma - x|^2 e^{\varphi} + e^{-\varphi}}\right)^{2j}. \tag{4.18}$$

This is exactly the standard bulk-boundary propagator written in Poincaré coordinates, for a field of spacetime dimension $h = j$ inserted at $(x, \bar{x})$ on the boundary. It admits the following expansion:

$$\Phi_j(x) = e^{2(j-1)\varphi}\delta^2(\gamma - x) + \cdots + \frac{e^{-2j\varphi}}{|\gamma - x|^{4j}} + \cdots \tag{4.19}$$

Now we turn to a free-field description. We introduce two new variables $\beta_L, \bar{\beta}_L$ and rewrite the sigma model action as

$$S = \int d^2z\left(\partial\varphi\bar{\partial}\varphi + \beta_L\bar{\partial}\gamma_L + \bar{\beta}_L\partial\bar{\gamma}_L - e^{-2\varphi}\beta_L\bar{\beta}_L\right). \tag{4.20}$$

By classically integrating out β_L and $\bar{\beta}_L$, we recover the original action. Taking into account the quantum measure introduces a linear dilaton and renormalizes some of the terms. The correct confomally invariant action is

$$S = \int d^2z\left(\partial\varphi\bar{\partial}\varphi - \frac{2}{\sqrt{p}}R\varphi + \beta_L\bar{\partial}\gamma_L + \bar{\beta}_L\partial\bar{\gamma}_L - e^{-2\varphi/\sqrt{p}}\beta_L\bar{\beta}_L\right). \tag{4.21}$$

Now we can treat $\beta_L\bar{\beta}_L e^{-2\varphi/\sqrt{p}}$ as a screening operator and use $\varphi, \beta_L, \gamma_L$ as free fields. Writing out the currents yields the Wakimoto representation used in Section 3. Calculations with this free-field representation are valid when the correlation functions are supported near the boundary $\varphi \to \infty$, and merely use the leading term in the bulk-boundary propagator:

$$\Phi_j(0) \longrightarrow V^{H_3^+}_{j,j} \equiv \delta(\gamma_L) e^{2(j-1)\varphi/\sqrt{p}}. \tag{4.22}$$

This operator is annihilated by $\mathcal{L}_{n\geq 1}$, and it is therefore a spacetime Virasoro primary inserted at $x = 0$. Acting on it with k_+ maps out a $\mathcal{D}^+_j$ representation.

There is another set of vertex operators which use the $\mathcal{D}^-_j$ representation:

$$V^{H_3^+}_{j,-j} \equiv e^{-2j\varphi/\sqrt{p}}. \tag{4.23}$$

These operators have regular OPE with k_+; therefore, they are annihilated by $\mathcal{L}_{-1}$, in fact by all $\mathcal{L}_{n\leq -1}$. Acting on $V^{H_3^+}_{j,-j}$ with k_- maps out the $\mathcal{D}^-_j$ representation. These operators correspond to normalizable modes in spacetime – they carry the "Δ_- dressing" in the usual AdS/CFT language. Their insertion in correlation functions should be interpreted as infinitesimal changes of the *state* of the theory, specified by the vacuum expectation values of the spacetime operators.

Finally we need to consider Wakimoto vertex operators for S^3. The primary vertex operators for the spin j representation of $\widehat{\mathfrak{su}}(2)_{p-2}$ are given by

$$V^{S^3}_{j,m} = \gamma_M^{j-m} e^{2ijx/\sqrt{p}}, \quad 0 \leq j \leq \frac{p}{2} - 1, \quad -j \leq m \leq j, \tag{4.24}$$

with worldsheet dimensions

$$\Delta(V^{S^3}_{j,m}) = \frac{j(j+1)}{p}. \tag{4.25}$$

They satisfy the following OPEs:

$$\begin{aligned} j_+(z) \cdot V^{S^3}_{j,m}(w) &= \frac{-j+m}{z-w} V^{S^3}_{j,m+1}(w) \\ j_3(z) \cdot V^{S^3}_{j,m}(w) &= \frac{2m}{z-w} V^{S^3}_{j,m}(w) \\ j_-(z) \cdot V^{S^3}_{j,m}(w) &= \frac{-j-m}{z-w} V^{S^3}_{j,m-1}(w) \end{aligned} \tag{4.26}$$

In particular, for a highest weight state, we have the simple expression

$$V^{S^3}_{j,j} = e^{2ijx/\sqrt{p}}. \tag{4.27}$$

For lowest weight states, we could take $m = -j$ in (4.24), but it turns out that a different (equivalent) representation makes contact more directly with the minimal string. By analogy with (4.22), we represent lowest weight states for S^3 as

$$\tilde{V}^{S^3}_{j,-j} \equiv \delta(\gamma_M) e^{-2i(j+1)x/\sqrt{p}}. \tag{4.28}$$

4.3 Chiral primaries

The spacetime theory has $(4,4)$ supersymmetry. Here we are defining chiral operators with respect to a given subgroup $U(1)_R \subset SU(2)_R$ of the R-symmetry. Each short multiplet of $N = 4$ contains precisely one chiral and one anti-chiral primary with respect to this $U(1)_R$. In defining the topological theory, we picked a preferred $U(1)_R$ (generated by J_3) in the choice of the complex structure.

Since $\mathsf{J}_3^{\mathrm{tot}} = \mathsf{K}_3 + \mathsf{J}_3$ is G_0^+-exact, a cohomology element of the topological string necessarily obeys $h = q$. There are two possibilities. If $h = q > 0$, the vertex operator is a (spacetime) chiral primary inserted at $x = 0$; it is the lowest weight state of a $\mathcal{D}_j^+$ representation for H_3^+, and the lowest weight state for a finite representation of S^3. If $h = q < 0$, it is the highest weight state of a $\mathcal{D}_j^-$ representation for H_3^+, and the highest weight state for a finite representation of S^3; it corresponds to turning on a vev.

Vertex operators for spacetime chiral primaries have been classified, so we can check the known list of such primaries for candidates that can descend to the minimal string. For the purpose of the topological string, we are interested in operators in the Neveu–Schwarz sector. Then one finds the following two series [60, 62][11]

$$\text{NS:} \quad \mathcal{W}_j, \ \mathcal{X}_j \tag{4.29}$$

These are the left movers only. The operator $\mathcal{W}_j$ has an index along H_3^+, and $\mathcal{X}_j$ has an index along S^3. By combining left and right movers, one obtains various modes of $g_{\mu\nu} + b_{\mu\nu}$ on $H_3^+ \times S^3$. The index j comes in half-integer steps and labels the spacetime dimension, or equivalently the R-charge, as follows:[12]

$$\begin{aligned} \mathcal{W}_j &= c^2\, V^{H_3^+}_{j+1,j+1}\, V^{S^3}_{j,-j}\colon h = q = j, \qquad & l = 0, \ldots, \frac{p-2}{2} \\ \mathcal{X}_j &= b_1\, V^{H_3^+}_{j+1,j+1}\, V^{S^3}_{j,-j}\colon h = q = j+1, \qquad & l = 0, \ldots, \frac{p-2}{2}. \end{aligned} \tag{4.30}$$

[11] In the cited papers, they are denoted by $\mathcal{W}_j^-$, $\mathcal{X}_j^+$. The superscripts have no meaning for us, so we will leave them out.

[12] We write only the matter part of the operators. The full operators are obtained by adding the usual ghost and superghost factor $ce^{-\phi}$.

We are also interested in the operators with $h = q < 0$. These are given by

$$\widetilde{\mathcal{W}}_j = b_2\, V^{H_3^+}_{j+1,-j-1} V^{S^3}_{j,j} : h = q = -j, \qquad l = 0, \ldots, \frac{p-2}{2}$$
$$\widetilde{\mathcal{X}}_j = c^1\, V^{H_3^+}_{j+1,-j-1} V^{S^3}_{j,j} : h = q = -j-1, \qquad l = 0, \ldots, \frac{p-2}{2}. \tag{4.31}$$

This is however not the full story. It has been demonstrated in [58] that apart from the usual vertex operators based on $\mathcal{D}^-_j$ and $\mathcal{D}^+_j$, string theory on H_3^+ has additional physical vertex operators that can be obtained by spectral flow. Spectral-flowed vertex operators correspond to "long strings" wrapping the boundary of H_3^+ multiple times. The resulting "winding number" w need not be conserved in correlation functions because H_3^+ is a contractible space. In the context of superstring theory on $H_3^+ \times S^3$, it is natural to consider spectral flow operators which are local with respect to the spacetime supercharges. The operator relevant for us is

$$U = \exp\left(-\int J_3^{\text{tot}}\right). \tag{4.32}$$

This operator increases h and q by $\delta h = \delta q = p/2$. By repeated application of U, one finds finds the following towers of chiral primaries:

$$\mathcal{W}^w_l l = 0, \ldots, \frac{p-2}{2}, \quad w \geq 0$$
$$\mathcal{X}^w_l l = 0, \ldots, \frac{p-2}{2}, \quad w \geq 0. \tag{4.33}$$

We have restricted ourselves to $w \geq 0$. Applying spectral flow in the opposite direction yields $\mathcal{D}^{-,w}_j$, because $\mathcal{D}^-_j$ and $\mathcal{D}^+_j$ are related by one unit of spectral flow:

$$\mathcal{D}^{+,w=-1}_j = \mathcal{D}^-_{p/2-j}. \tag{4.34}$$

4.4 Matching with the minimal string

We have discussed a list of vertex operators that can potentially survive in the topological theory. Since these are ghost number one states (with respect to the cb ghosts of the 10d string theory), they should correspond to ghost number one states of the minimal string (with respect to the CB ghosts of the minimal string). This is necessary if the counting of Riemann surface moduli in correlation functions is to work out correctly.

The spectrum of the minimal string is discussed in detail in Appendix B. At ghost number one (in the relative, chiral cohomology), one has the $p-1$

tachyons and their gravitational descendants,

$$T_{n=2j+w+1} = \hat{y}^w T_{2j+1}, \;\; j = 0, \ldots, \frac{p-2}{2}, \quad w \geq 0. \tag{4.35}$$

The T_1 tachyon is also known as the puncture operator, and the T_{p-1} tachyon is also known as the cosmological constant operator. As we have mentioned, one needs to consider both chiral primaries with $h > 0$ and $h < 0$. Let us start with $\widetilde{\mathcal{X}}_j$. In the Wakimoto representation, it can be written as

$$\begin{aligned}\widetilde{\mathcal{X}}_j &= c^1 \, V^{H_3^+}_{j+1,-j-1} \, V^{S^3}_{j,j} \\ &= c^1 e^{-2j(\varphi - ix)/\sqrt{p} - 2\varphi/\sqrt{p}} \\ &= C e^{-2j(\Phi - iX)/\sqrt{p} - 2\Phi/\sqrt{p}}.\end{aligned} \tag{4.36}$$

Remarkably these are precisely the expressions for the tachyons T_{p-1-2j} of the minimal string theory. As we discussed, these operators are interpreted as normalizable modes in H_3^+. They are of course non-normalizable in Liouville theory. This difference arises because of the shift in the background charge of φ due to the $\partial J_3^{\text{tot}}$ twist.

One can also check that $\widetilde{\mathcal{W}}_j$ and $\mathcal{X}_j$ are *not* in the cohomology of G_0^+.

Next we consider $\mathcal{W}_j$. We can write them as

$$\begin{aligned}\mathcal{W}_j &= c^2 \, V^{H_3^+}_{j+1,j+1} \, V^{S^3}_{j,-j} \\ &= c^2 \delta(\gamma_L) \delta(\gamma_M) e^{2(j+1)(\varphi - ix)/\sqrt{p}} e^{-2\varphi/\sqrt{p}}\end{aligned} \tag{4.37}$$

Comparing the Liouville momenta of these operators with the spectrum for the minimal string suggests that the $\mathcal{W}_j$ should be identified with the first group of descendants $\hat{y} T_{2j+1}$.

To understand how this comes about, let us first look more closely at the spectral flow operator [60]:

$$U(z) = \exp\left(-\int^z J_3^{\text{tot}}\right) = c^2 b_1 \delta(\gamma_L) \delta(\gamma_M) \exp(\sqrt{p}(\varphi - ix)). \tag{4.38}$$

It is precisely the picture changing operator $Z_M Z_L$ that we have encountered before. Essentially U plays the role of the ground ring generator $\hat{y}$ — it has the correct x and φ dependence — but being a picture changing operator, formally it does not create any new states. A similar situation was encountered in [53]. The resolution is to consider the closely related operator

$$\tilde{U} = U e^{-i\sqrt{2}\sigma} = c^2 \delta(\gamma_L) B \exp(\sqrt{p}(\Phi - iX)), \tag{4.39}$$

where we replaced $\delta(\gamma_M) \to \beta_M$ by combining with $e^{-i\sqrt{2}\sigma}$. This operator is still not quite the one we need since while its total ghost number is zero, it has $q_1 = -1$ and $q_2 = +1$. We should then apply a descent procedure to obtain an operator with $q_1 = q_2 = 0$. It is not hard to check that

$$\{\mathsf{Q}_{\rm Vir}, \tilde{U}\} = \{\mathsf{Q}_R, \theta(\gamma_L)\hat{y}\} - \partial(c^2\delta(\gamma_L)\hat{y}). \tag{4.40}$$

Therefore $\tilde{U}$ is responsible for creating the descendants.

Now it is easy to check that

$$U\widetilde{\mathcal{X}}_j = \mathcal{W}_j \tag{4.41}$$

and so by dressing up $\delta(\gamma_M) \to \beta_M$ and applying the descent procedure, the operators $\mathcal{W}_j$ should indeed be regarded as the first descendants of the tachyons. Further acting with $\tilde{U}$ yields the remaining descendants. These correspond to honest long string states in H_3^+ which cannot be seen in supergravity.

Figure 2 summarizes the correspondence between the ghost number one states of the minimal string and the chiral primaries in $H_3^+ \times S^3$. A striking feature is the presence of gaps in the spectrum: every p steps, a state is missing. This is very natural from the viewpoint of the p-KdV integrable hierarchy, where every p-th flow parameter is redundant. We see that this emerges naturally in the reduction from $H_3^+ \times S^3$. Physically, the absence of these states has been a bit of a mystery for string theory on $H_3^+ \times S^3$. The holographic CFT on the boundary of H_3^+, a deformation of the symmetric product $\mathrm{Sym}^{Q_1Q_5}(M_4)$ superficially appears to contain such states. It has been suggested that their absence may be related to the singular behavior of this CFT [64]. Heuristically, at the point in moduli space at which we are working, there is no cost in energy for the system to emit a long string. This leads to a continuum of states above a certain threshold and it has been proposed that the missing states may be related to this continuum. Does the correspondence with the minimal string give any insight into this issue? In the minimal string, some of the gaps in the closed string spectrum are believed to be filled by open string states [65]. For example, the first missing state has precisely the Liouville dressing to be identified with the operator

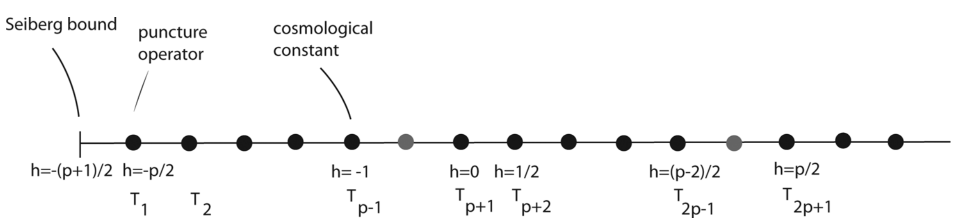

Figure 2: The spectrum is organized according to increasing Liouville momentum, or equivalently increasing spacetime dimension. The lighter dots indicate the missing states.

that couples to the *boundary* length — the boundary cosmological constant that can be turned on on an FZZT brane [65]. An analogous interpretation becomes viable for string theory in $H_3^+ \times S^3$: the first missing state has the correct φ dressing to be identified with the boundary screener of an H_2^+ brane.

4.5 Ground ring

Besides the tachyons, which carry ghost number one, the minimal string has physical states at ghost number zero, the ground ring elements $G_{n=r+ps} = \hat{x}^{r-1}\hat{y}^{s-1}$, $1 \leq r \leq p-1$, $s \geq 1$. These states are clearly in the G_0^+ cohomology of the topological theory. The question is what is their intepretation in the physical theory. It is natural to expect that they get lifted to ground ring elements of the IIB theory, that is, cohomology classes of the BRST operator $\mathsf{Q}_{\rm 10d}$ carrying zero ghost number with respect to 10d *cb* ghosts. The explicit expressions of $\hat{x}$ and $\hat{y}$ fail to be annihilated by $\mathsf{Q}_{\rm 10d}$, but it is possible to add improvement terms involving the 10d $bc\beta\gamma$ ghosts such that they become elements of the $\mathsf{Q}_{\rm 10d}$ cohomology. One could have anticipated the existence of a ground ring structure in $H_3^+ \times S^3$ just from the representation theory of the $\widehat{\mathfrak{su}}(2)_{p-2}$ current algebra: a ground ring element must exist for each primitive null over a primary, by a generalization to current algebras of the mechanism [11, 12, 14] reviewed in Appendix B.

It is clear that the ground ring states can be constructed by a descent procedure entirely analogous to (B.23),

$$\mathsf{Q}_{\rm 10d}\,|\mathcal{G}_n\rangle = ce^{-\phi}\,\mathsf{Q}_F^M|T_n\rangle. \tag{4.42}$$

The operators $\mathcal{G}_n$ are the "improved" versions of G_n that we are after. They generate a W_p symmetry which encodes the exact solutions of the (topological) theory. More general ground ring elements can be obtained acting on $\mathcal{G}_n$ with isometries of $H_3^+ \times S^3$, though they will not be in the G_0^+ cohomology.

4.6 Small phase space deformations

As we have seen, deformations of the minimal string obtained by turning on the first $p-1$ tachyons (the so-called "small phase space") lift to deformations of the *state* of string theory on $H_3^+ \times S^3$. Each point in the small phase space maps to a certain 1/2 BPS configuration in $H_3^+ \times S^3$. These configurations are exact solutions of tree-level string theory, for finite

α'. We can write down the corresponding sigma models in the Wakimoto representation,

$$S_{H_3^+\times S^3} \to S_{H_3^+\times S^3} + \sum_{n=1}^{p-1} t_n \int d^2z\ \beta_M \bar{\beta}_M e^{2i((p-1-n)/2\sqrt{p})x} e^{2\frac{n-p-1}{2\sqrt{p}}\phi}. \quad (4.43)$$

We see that they are deformations of the S^3, "gravitationally dressed" by the warp factor of H_3^+. The underlying integrable structure guarantees that they are exactly marginal. They correspond to states since they preserve the $H_3^+ \times S^3$ asymptotics as $\varphi \to \infty$.

In the limit $p \to \infty$, these sigma models must correspond to 1/2 BPS supergravity solutions. Such supergravity solutions for the NS5/F1 system have been classified [20–22, 66, 67], and the sigma models (4.43) provide a generalization to finite α' for a subclass of them. A natural guess is that this class corresponds to (a subsector of) the Coulomb branch of the near horizon geometry of the NS5/F1 system. It would be nice to understand the geometric interpretation of (4.43) in more detail.

5 Holography and symmetric products

The main interest of string theory on $H_3^+ \times S^3$ is in the context of the AdS/CFT correspondence. This background arises as the near horizon geometry of Q_5 NS5 branes wrapping M_4 and Q_1 parallel fundamental strings. The dual spacetime CFT on the boundary of H_3^+ is believed to be the low-energy limit of the worldvolume theory on the NS5–F1 system: a sigma model with target space a certain deformation of $\mathrm{Sym}^{Q_1Q_5}(M_4)$. The string coupling constant g_s is fixed by the relation [6]

$$Q_1 \sim \frac{\mathrm{Vol}_{M_4}}{\alpha'^2 g_s^2 \sqrt{Q_5}}. \quad (5.1)$$

For a fixed level $p \equiv Q_5$ of the worldsheet sigma model, string perturbation theory corresponds to $Q_1 \to \infty$. All the topological amplitudes that we have considered in this paper arise at string tree level and map to boundary correlators at leading order in the large Q_1 expansion.

While the details of the boundary theory are not very well understood, the spectrum of spacetime chiral primaries of the string theory has been matched with the spectrum of chiral twist operators of the symmetric product CFT. It has been suggested [60] that it is more natural to phrase the correspondence in terms of $\mathrm{Sym}^{Q_1}(\mathrm{Sym}^{Q_5}(M_4))$, which has the same chiral spectrum of $\mathrm{Sym}^{Q_1Q_5}(M_4)$ and is believed to be in the same moduli space. Here one thinks of $\mathrm{Sym}^{Q_5}(M_4)$ as the theory of a single long string. Twist operators of

$\mathrm{Sym}^{Q_5}(M_4)$ in the untwisted sector of Sym^{Q_1} correspond to ordinary vertex operators for $H_3^+ \times S^3$, with zero amount of spectral flow. The operation of taking w units of spectral flow in the bulk maps to considering the Z_w twisted sector of Sym^{Q_1}.

Let us first consider the chiral twist operators for $\mathrm{Sym}^{Q_5}(M_4)$. We restrict to universal operators that do not depend on the structure of M_4. They were constructed for instance in [68, 69]:

$$\begin{aligned} \sigma_n^-\colon h &= \frac{n-1}{2} \quad n = 1, \dots, Q_5 \\ \sigma_n^+\colon h &= \frac{n+1}{2} \quad n = 1, \dots, Q_5. \end{aligned} \tag{5.2}$$

We have written the left movers only, and by left movers, we now mean left moving in the boundary CFT. The operator $\sigma_n^{\pm}$ contains a twist operator of length n. The correspondence between the bulk and boundary theories is

$$\begin{aligned} \mathcal{W}_l &\longrightarrow \sigma_{2l+1}^-, \quad l = 0, \dots, \frac{p-2}{2} \\ \mathcal{X}_l &\longrightarrow \sigma_{2l+1}^+, \quad l = 0, \dots, \frac{p-2}{2}. \end{aligned} \tag{5.3}$$

By including the Z_w twisted sector for Sym_1^Q, one finds [60] chiral operators with the correct dimensions to be identified with the spectral flowed bulk vertex operators.

From the bulk viewpoint, the spectrum of dimensions appears to increase indefinitely, while from the boundary viewpoint it is cut-off. The reason for this is that in the bulk we work in perturbation theory, so we have implicitly taken $g_s \to 0$. If we want to recover the same results int the boundary theory, we should take $Q_1 \sim 1/g_s^2 \to \infty$, which removes the cut-off. If we want to consider Q_1 finite, then there must be non-perturbative effects in the bulk to cut-off the spectrum. This is the well-known “stringy exclusion principle” [70].

The missing states in the bulk spectrum correspond to the twist operator of length Q_5 in $\mathrm{Sym}^{Q_5}(M_4)$ and its images in Sym^{Q_1}. In particular, they are naively present in the boundary theory, and one must invoke subtle effects [64] to argue them away.

Next we would briefly like to discuss some properties of amplitudes in the symmetric product theory. The maps from the boundary of H_3^+ to the symmetric product can be described in terms of a covering surface Σ_g of degree $Q_1 Q_5$ of the boundary, together with a map from this covering surface to M_4. Since our topological string theory was completely defined in terms of $H_3^+ \times S^3$, the amplitudes we are looking for in the boundary theory can only depend on the constant modes of M_4. So we can effectively ignore the map $\Sigma_g \to M_4$ and focus on the structure of the covering $\Sigma_g \to \mathbf{P}^1$.

Consider an amplitude of various tachyons and descendants in the bulk. This translates to computing an amplitude in the boundary theory with certain twist operator insertions. We further need to turn on interactions in the symmetric product theory. This is done by inserting DVV [18] twist operators on the boundary and integrating over their positions. Finally to recover the perturbative answer one should take the limit $Q_1 \to \infty$.

There are many details that need to be worked out in order to compute such amplitudes. However since we are dealing with a relatively simple topological theory in the bulk, our guess is that the amplitudes on the boundary also end up being those of a well known topological theory, namely Hurwitz theory. That is, we suspect that up to a numerical factor the amplitudes simply count the number of covers with the branching we have just described.

The Hurwitz problem has been completely solved. Some interesting works that might be relevant here are [71, 72]. The Hurwitz problem is known to be related to topological strings [73, 74].

6 Relation with Calabi–Yau spaces and generalizations

6.1 Calabi–Yau description

We have seen that $H_3^+ \times S^3$ gives a $\hat{c} = 3$ topological string realisation of the minimal $(p, 1)$ string theories. Another such realisation was recently proposed in [23]. These authors considered the non-compact Calabi-Yau manifold defined by the equation

$$y + x^p + \cdots + uv = 0, \tag{6.1}$$

and argued that an appropriately defined B-model topological string on this Calabi–Yau is equivalent to the $(p, 1)$ minimal string.[13] Since this threefold is closely related to the ground ring of the minimal $(p, 1)$ string, one can view this as "quantization of the deformations of the ground ring". This Calabi–Yau has no Kähler moduli and therefore the associated A-model is trivial.

[13] A certain amount of interpretation is needed, since we expect the minimal string (which has a linear dilaton) to describe the leading behaviour near a Calabi-Yau singularity, whereas if we regard this equation as an equation in $\mathbf{C}^4$ we get no singularities. We will not discuss this issue here.

Our topological theory has the same B-model, namely the $(p, 1)$ minimal string, and in fact it has also a trivial A-model. This can be seen easily by recalling that G^- is equivalent to the B antighost of the minimal string, which clearly has trivial cohomology. Thus we have two $\hat{c} = 3$ $(2, 2)$ SCFTs for which both the A and B-models agree. In such a situation it is expected that superconformal invariance also forces the D-terms to agree. So we conclude that the two theories must be identical. The question then arises if there is a natural explanation for this equivalence. We conjecture that $H_3^+ \times S^3$ and the Calabi-Yau sigma-model are related by T-duality. An example of such a duality for SCFTs that is very similar in spirit is the equivalence between the NS5-brane geometry (which is analogous to $H_3^+ \times S^3$) and the ALE sigma model (which is analogous to the Calabi-Yau threefold). These two CFTs are related by T-duality [24].

6.2 $H_3^+ \times S^1$

It is also interesting to consider more general backgrounds of the form $H_3^+ \times \mathcal{N}$ with NS flux, where $\mathcal{N}$ is some coset model. So long as the worldsheet theory has $N = 2$ supersymmetry, one can write down a topological string theory and study its properties. We will now take a look at the simplest of these examples, namely $H_3^+ \times S^1$. We will take H_3^+ at fractional level $t + 2 = p/q + 2$ for now and restrict ourselves later on.

The generators are given as follows:

$$\begin{aligned}
\mathsf{T} &= -(\partial x)^2 - \frac{1}{2t}\left(2k_3k_3 + k_+k_- + k_-k_+\right) \\
&\quad + \frac{1}{2}(\partial c^2 b_2 - c^2\partial b_2) + \frac{1}{2}(\partial c^1 b_1 - c^1\partial b_1) + \frac{\kappa}{2t}\,\partial \mathsf{J}_3^{\rm tot} \\
\mathsf{G}^+ &= c^1 k_- + c^2(i\sqrt{t}\,\partial x + k_3 + c^1 b_1) \\
\mathsf{G}^- &= \frac{1}{t}\Big(-b_1 k_+ + b_2(i\sqrt{t}\,\partial x - k_3 - c^1 b_1) + \kappa\,\partial b_2\Big) \\
\mathsf{J} &= c^1 b_1 + c^2 b_2 + \frac{2}{t}(k_3 + c^1 b_1) + \frac{\kappa}{t}\mathsf{J}_3^{\rm tot}
\end{aligned} \tag{6.2}$$

with

$$\mathsf{J}_3^{\rm tot} = k_3 + i\sqrt{t}\,\partial x + c^1 b_1 = \{\mathsf{G}_0^+, b_2\}. \tag{6.3}$$

Here we used the same Wakimoto representation and OPEs as in Sections 2 and 3, and we used the current $j = i\sqrt{t}\,\partial x$ to describe the S^1 factor. From the OPEs, one finds that the central charge is given by $\hat{c} = (2t + 2)/t$. When we expand the twisted stress tensor in the Wakimoto representation with $\kappa = t-1$ we find

$$\begin{aligned} \mathsf{T} + \frac{1}{2}\partial \mathsf{J} = -\partial x^2 + i\frac{t-1}{\sqrt{t}}\partial^2 x - \partial\varphi^2 - \frac{t+1}{\sqrt{t}}\partial^2\varphi \\ + c^1\partial b_1 + 2\partial c^1 b_1 + \partial c^2 b_2 + \partial\beta\gamma, \end{aligned} \tag{6.4}$$

which for $t = p/q$ yields exactly the stress tensor for the (p,q) minimal model coupled to gravity plus an additional $\{b_2, c^2, \beta, \gamma\}$ multiplet.

In this realization, t is no longer restricted to be an integer since there is no flux quantization. On the other hand, in the untwisted theory, the scalar x does not have a background charge and we do not have any matter screening operators, but in a minimal theory we do need a screening operator, so we need to deal with non-minimal theories. Fortunately there is one special case, namely $t = 1$ (and $\hat{c} = 4$), where we do not have any background charge for x either before or after twisting. In this case, by arguments very similar to those for $H_3^+ \times S^3$, we recover the $c = 1$ string at self-dual radius. Namely, we can split up the BRST charge as

$$\mathsf{G}_0^+ = \mathsf{Q}_1 + \mathsf{Q}_2 \tag{6.5}$$

with

$$\begin{aligned} \mathsf{Q}_1 &= \oint c^1 k_- \\ \mathsf{Q}_2 &= \oint c^2(i\sqrt{t}\,\partial x + k_3 + c^1 b_1) \end{aligned} \tag{6.6}$$

which satisfy $\{\mathsf{Q}_i, \mathsf{Q}_j\} = 0$. The cohomology of Q_2 is generated by the familiar expressions

$$\begin{aligned} e^{-2\Phi/\sqrt{t}} &= \beta e^{-2\varphi/\sqrt{t}} \\ e^{-2iX/\sqrt{t}} &= \beta e^{-2ix/\sqrt{t}} \\ B &= b_1\beta \\ C &= c^1\beta^{-1}. \end{aligned} \tag{6.7}$$

The similarity transformation is now defined using the operator

$$\mathsf{L} = -\oint (c^1 k_+^{-1}) \left(b_2(i\sqrt{t}\,\partial x - k_3 - c^1 b_1) + \kappa\,\partial b_2 \right) \tag{6.8}$$

and we find that

$$\exp \mathrm{Ad}(\mathsf{L})\mathsf{G}^- = \frac{1}{t}(-b_1 k_+) = -\frac{1}{t}B. \tag{6.9}$$

and[14]

$$\exp \mathrm{Ad}(\mathsf{L})\mathsf{G}_0^+ = \mathsf{Q}_2 - t \oint C \left(T_X + T_\Phi + \frac{1}{2} T_{BC} \right) + \{\mathsf{Q}_2, \bullet\}. \tag{6.10}$$

Moreover, J is invariant under the similarity transformation. Therefore this topological theory is equivalent to the $c = 1$ string.

The fact that the spectrum of topological string theory on $H_3^+ \times S^1$ coincides with the $c = 1$ string is perhaps not too surprising. Indeed imposing invariance under Q_2 reduces $H_3^+ \times S^1$ to the Kazama–Suzuki coset $SL(2)/U(1)$ at level 3 and Q_1 coincides with the BRST operator for topological string theory on the coset, which is of course known to be equivalent to the $c = 1$ string [53]. On the other hand, the coupling to gravity naively appears to be different for $H_3^+ \times S^1$ and $SL(2)/U(1)$. The similarity transformation shows that they are in fact equivalent.

Since $\hat{c} = 4$, one can ask if there is a Calabi–Yau four-fold realization of this conformal field theory. By analogy with (6.1) there is a natural candidate, namely the Calabi–Yau defined by the equation

$$y + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0. \tag{6.11}$$

The topological string theories on more general spaces of the form $H_3^+ \times \mathcal{N}$ with NS flux should be very similar. The horizon manifold $\mathcal{N}$ will get reduced to some matter theory, and H_3^+ should yield the gravitational dressing by the Liouville field. One may also speculate about a Calabi–Yau dual of such conformal field theories. A natural guess is that if $SL(2)/U(1) \times \mathcal{N}/U(1)$ (modulo GSO projection) is described by a Calabi–Yau n-fold of the form $P(x) = \mu$, then $H_3^+ \times \mathcal{N}$ is described by the $n+1$-fold $y^q + P(x) = 0$, with $q = 1$ if the level of H_3^+ is an integer.

6.3 Minimal NS5 branes

Another application of the ideas used in this paper is to the $N = 4$ topological string theory [57] on ALE spaces, or equivalently [53] near horizon NS5-brane geometries. This topological string theory computes certain BPS correlators in six dimensions. The (4,4) superconformal field theory for the four transverse directions to a stack of p NS5-branes consists of an $SU(2)_{p-2}$ WZW model, a linear dilaton, and four free fermions. We will now show

[14] With the same qualifications as in a previous footnote.

that this topological string theory also reduces to the $(p, 1)$ bosonic string.[15] A relation of this kind was anticipated in [24]. However, the disclaimer mentioned in the introduction applies: by the $(p, 1)$ bosonic string, we do not mean the purely topological model realized, e.g., as the $N = 2$ A_{p-1} minimal model coupled to topological gravity ([10] and references therein).

The main point is that if we throw out the quartet $\{c^2, b_2, \beta_L, \gamma_L\}$ of $H_3^+ \times S^3$, we end up with the CHS geometry. Since this quartet did not give any contribution in topological correlators except for zero modes, we again end up with the minimal string. By T-duality, this also describes the topological string on the A_{p-1} ALE spaces.

The generators of the left-moving $N = 4$ algebra of the CHS background can be taken to be[16]

$$
\begin{aligned}
\mathsf{T} &= -\partial\varphi^2 - \frac{1}{\sqrt{p}}\partial^2\varphi + \frac{1}{2p}(2j_3j_3 + j_+j_- + j_-j_+) \\
&\quad + \frac{1}{2}(\partial c^3 b_3 - c^3\partial b_3) + \frac{1}{2}(\partial c^1 b_1 - c^1\partial b_1) \\
\mathsf{G}^+ &= c^1 j_- + c^3(-\sqrt{p}\,\partial\varphi + j_3 + c^1 b_1) - \partial c^3 \\
\mathsf{G}^- &= \frac{1}{p}\left(b_1 j_+ + b_3(\sqrt{p}\,\partial\varphi + j_3 + c^1 b_1) + \partial b_3\right) \\
\tilde{\mathsf{G}}^+ &= \frac{1}{p}\left(c^3 j_+ - c^1(\sqrt{p}\,\partial\varphi + j_3 - c^3 b_3) - \partial c^1\right) \\
\tilde{\mathsf{G}}^- &= b_3 j_- + b_1(\sqrt{p}\,\partial\varphi - j_3 + c^3 b_3) + \partial b_1 \\
\mathsf{J} &= c^3 b_3 + c^1 b_1 \\
\mathsf{J}^{++} &= c^3 c^1 \\
\mathsf{J}^{--} &= b_3 b_1.
\end{aligned}
\tag{6.12}
$$

We are going to use the same Wakimoto representation as before, i.e., $j_+ = \beta_M$, etc. Following our previous arguments, we split up the BRST operator $\mathsf{G}_0^+ = \mathsf{Q}_1 + \mathsf{Q}_R$ with

$$
\begin{aligned}
\mathsf{Q}_1 &= \oint c^1 j_- \\
\mathsf{Q}_R &= \oint c^3(-\sqrt{p}\,\partial\varphi + j_3 + c^1 b_1).
\end{aligned}
\tag{6.13}
$$

[15] This result was announced at a talk at Stony Brook in August 2005 [75]. The recent papers [76–78] make a similar claim and check the correspondence for certain tree-level amplitudes. Our approach proves the correspondence for any scattering amplitude at any genus.

[16] For simple comparison with the rest of the paper, we have labeled the fermions by c^1, b_1 and c^3, b_3.

We can perform a similarity transformation by

$$\mathsf{L} = \oint (c^1 \beta_M^{-1}) \left(b_3(\sqrt{p}\, \partial\varphi + j_3 + c^1 b_1) + \partial b_3 \right) . \tag{6.14}$$

As usual this yields the expected operators for the $(p, 1)$ string:

$$\begin{aligned} \exp(\mathrm{Ad}\,\mathsf{L})\mathsf{G}_0^+ &= \mathsf{Q}_R + p\mathsf{Q}_{\mathrm{Vir}} + \{\mathsf{Q}_R, \bullet\} \\ \exp(\mathrm{Ad}\,\mathsf{L})\mathsf{G}^- &= \frac{1}{p} B. \end{aligned} \tag{6.15}$$

We also find

$$\exp(\mathrm{Ad}\,\mathsf{L})\tilde{\mathsf{G}}^+ = \frac{1}{p} c^3 j_+ = \frac{1}{p} c^3 \beta_M . \tag{6.16}$$

The cohomology of G_0^+ yields the physical states of the $(p, 1)$ minimal string, as before. It is easy to check that these cohomology elements are also in the kernel of $\tilde{\mathsf{G}}_0^+$, as required for observables of the $N = 4$ topological string.[17] Mimicking our previous arguments, one finds that the correlation functions also agree with the minimal string. We will elaborate on aspects of this correspondence in a future paper [80].

The relation to the minimal string leads to a lot of new insight. For instance, the $(2, 1)$ minimal string, or equivalently $c = -2$ matter coupled to gravity is solved by a supermatrix model [81–83]. The authors of [83] find the following all genus formula for the free energy as a function of the cosmological constant μ,

$$\mathcal{F} = -\frac{1}{6}\mu^3 \log \mu + \frac{1}{12} \log\, \mu - \sum_{g=2}^{\infty} \frac{(3g-4)!}{12^g g!} \mu^{3-3g} . \tag{6.17}$$

Notice that this answer is very different from pure topological gravity, which has $\mathcal{F}_{\mathrm{top}} = -\left(\frac{1}{6}\right)\mu^3 + \left(\frac{1}{12}\right)$. Expression (6.17) can be checked for $g = 0$ and $g = 1$ in the continuum worldsheet formulation of the $(2, 1)$ minimal string. The logarithmic terms are seen to arise from integration over the Liouville zero mode. In particular, this implies that there is a non-vanishing 4-point function at tree level, scaling like $1/\mu$.

An interesting observation, pointed out to us by Marcos Mariño, is that for each genus, the coefficients (6.17) are precisely the leading singular terms

[17] It is conceivable that by using free-field representations, we miss some discrete states of the full theory. Such extra states appear in the analysis of [79]. However, our free-field representation should capture all the asymptotic states, and hence the minimal string should describe all the scattering amplitudes in the $N = 4$ topological string.

in a $\mu = 0$ expansion of a simple polylog expression:

$$\mathcal{F}_g(\mu) = -\frac{1}{12^g g!}\mathrm{Li}_{4-3g}(e^{-\mu}) = -\frac{1}{12^g g!}\sum_{d=0}^{\infty} d^{3g-4}e^{-d\mu}. \qquad (6.18)$$

This has the form of a worldsheet instanton expansion, and so we get a prediction for counting genus g curves of degree d on an A_1 ALE space! Similarly, we expect a matrix model with multiple supermatrices to solve the $(p, 1)$ minimal string. This issue is currently under investigation [80].

The genus zero term in the free energy (6.17) can be understood as follows [84]. Suppose we consider type IIB on the CHS geometry or equivalently type IIA on the A_1 ALE. Then there are light 1/2 BPS states (W-bosons) coming from wrapped branes with a mass of order $M^2 \sim \mu$. Now the coefficient of the F^4 term in the effective six-dimensional theory is given by $\partial^4\mathcal{F}_0/\partial\mu^4$, where F is the field strength of a $U(1)$ vector multiplet and $\mathcal{F}_0$ is the genus zero free energy of the topological string. This coefficient can also be calculated by a one-loop computation with only charged light 1/2 BPS states running around the loop. Therefore this coefficient should scale like $M^{-2} \sim \mu^{-1}$ and one deduces that $\mathcal{F}_0 \sim \mu^3 \log \mu$ [84].

For higher genus, the topological string computes the coefficient of certain R^4F^{4g-4} terms in the six-dimensional effective theory [57], where R is the curvature tensor. These terms are chiral and one might hope that the answer is fixed by a one-loop Schwinger calculation involving the 1/2 BPS states.[18] But it turns out that the Schwinger computation yields an answer for $\mathcal{F}_1$ that scales as μ^{-1} instead of $\log \mu$.[19] On closer inspection, this is actually not a problem: the string worldsheet computation which reduces to the minimal string is valid for large μ, and the Schwinger computation is valid when the W-bosons are very light, which is for small μ. In our setting, there is no non-renormalization theorem which protects these terms. It is quite likely that there are additional instanton corrections to the R^4F^{4g-4} terms. One might hope that reasoning similar to [88–91] fixes these terms completely.

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[18] See [85–87] for similar computations for the case of the conifold.

[19] This discrepancy has been noticed independently by O. Aharony and D. Kutasov (unpublished).

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Appendix A: double complexes

In this appendix, we justify Equation (3.19). We need a simple lemma about double complexes [92].

Consider the double complex generated by the differentials Q_H and Q_V, with $\mathsf{Q}_H^2 = \mathsf{Q}_V^2 = \{\mathsf{Q}_H, \mathsf{Q}_V\} = 0$. The differentials act on the vector space

$$K = \bigoplus_{h,v\in\mathbb{Z}} K^{h,v},$$

double-graded by the ghost numbers h and v:

$$Q_H \colon K^{h,v} \longrightarrow K^{h+1,v}$$
$$Q_V \colon K^{h,v} \longrightarrow K^{h,v+1}.$$

The total differential $\mathsf{Q} = \mathsf{Q}_H + \mathsf{Q}_V$ acts on the complex graded by the total ghost number, $K = \bigoplus_t K^t \equiv \bigoplus_{h+v=t} K^{h,v}$. With these definitions, we have: if the horizontal sequence is exact except at most two consecutive gradings, say $h = 0, 1$, i.e., $H_{\mathsf{Q}_H}(K^{h,v}) = 0$ for $h \neq 0, 1$; then $H_{\mathsf{Q}_H+\mathsf{Q}_V}(K) \cong H_{\mathsf{Q}_V}(H_{\mathsf{Q}_H}(K))$. This statement follows immediately by considering the associated spectral sequence, which converges after one step [92].

Claim (3.19) is easily proved by repeated application of the lemma. The first equality follows if we take $\mathsf{Q}_V = \mathsf{G}_0^+$ and $\mathsf{Q}_H = \mathsf{Q}_F^M + \mathsf{Q}_F^L$. The exactness of the horizontal sequence except at zero grading is a famous fact of the Felder construction, as we reviewed in the text. The second equality follows if we take $\mathsf{Q}_H = \mathsf{Q}_2$, $\mathsf{Q}_V = \mathsf{Q}_1 + \mathsf{Q}_F^M + \mathsf{Q}_F^L$. The hypothesis holds since for a given choice of $\beta_L\gamma_L$ picture, the cohomology at Q_2 is non-trivial only for one value of the q_2 grading. Finally the last equality follows with $\mathsf{Q}_H = \mathsf{Q}_3$, $\mathsf{Q}_V = \mathsf{Q}_1 + \mathsf{Q}_F^M + \mathsf{Q}_F^L$. Here the lemma can be applied because as explained in Section 3.5 the Q_3 cohomology is non-trivial only for $q_3 = 0$ and $q_3 = 1$.

Appendix B: facts about minimal strings

In this appendix, we review some standard facts about the (p, q) models coupled to gravity, emphasizing their free-field description. The main goal is to point out the special features of the $(p, 1)$ models. Useful reviews with a different perspective include [9, 10, 93].

B.1 The (p,q) minimal models and their Felder description

The Virasoro minimal models are labeled by a pair or relatively prime integers (p, q), $p, q > 1$. They have central charge

$$c_{p,q} = 1 - 6Q_{p,q}^2, \quad Q_{p,q} \equiv \sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}. \tag{B.1}$$

There are $(p-1)(q-1)/2$ primary fields labeled by two integers (r, s) satisfying $1 \le r \le p-1$, $1 \le s \le q-1$, with the identification $(r, s) \sim (p-r, q-s)$. Their conformal dimensions are

$$\Delta_{r,s} = \frac{(qr-ps)^2 - (p-q)^2}{4pq}. \tag{B.2}$$

The Coulomb gas description of these models is in terms of the theory of free boson X with background charge, supplemented by appropriate screening operators. The stress tensor reads ($\alpha' = 1$)

$$\mathsf{T}_X = -\partial X \partial X + iQ_{p,q}\partial^2 X. \tag{B.3}$$

Vertex operators are given by

$$V_\alpha = e^{2i\alpha X} \tag{B.4}$$

and have conformal dimension

$$\Delta_\alpha = \alpha(\alpha - Q_{p,q}). \tag{B.5}$$

It is useful to introduce the lattice of momenta

$$\alpha_{m,n} = \frac{1}{2}(1-m)\alpha_- + \frac{1}{2}(1-n)\alpha_+ \quad \text{with} \quad \alpha_+ = \sqrt{\frac{p}{q}}, \quad \alpha_- = -\sqrt{\frac{q}{p}}. \tag{B.6}$$

The conformal dimensions of the corresponding operators $V_{m,n} = e^{2i\alpha_{m,n}X}$ can be expressed as

$$\Delta_{\alpha_{m,n}} = \frac{1}{4}(m\alpha_- + n\alpha_+)^2 - \frac{1}{4}Q_{p,q}^2. \tag{B.7}$$

Note the invariance of $\Delta_{m,n}$ under $(m, n) \to (-m, -n)$ and $(m, n) \to (m + p, n + q)$. The spaces $\mathcal{F}_{m,n}$ are defined as the Fock spaces built on the

(m, n) vacua,

$$\mathcal{F}_{m,n} \equiv \text{Span}\{a_{-n_1} \cdots a_{-n_k} V_{m,n}(0)|0\rangle\}, \tag{B.8}$$

where the a_n's denote the usual oscillators, $i\partial X(z) = \sum_n a_n z^{-n+1}$, and $|0\rangle$ is the $SL(2)$ invariant vacuum. Finally we introduce the conformally invariant screening charges

$$\mathsf{Q}_\pm = \oint e^{2i\alpha_\pm X}. \tag{B.9}$$

With all the ingredients in place we can now present the free-field resolution of the irreducible Virasoro module $\mathcal{L}_{r,s}$. The sequence

$$\cdots \xrightarrow{\mathsf{Q}_-^r} \mathcal{F}_{2p-r,s} \xrightarrow{\mathsf{Q}_-^{p-r}} \mathcal{F}_{r,s} \xrightarrow{\mathsf{Q}_-^r} \mathcal{F}_{-r,s} \xrightarrow{\mathsf{Q}_-^{p-r}} \cdots \tag{B.10}$$

is a complex, i.e., $\mathsf{Q}_F^2 = 0$, where the Felder BRST charge Q_F is defined as Q_-^r or Q_-^{p-r} according to which space it acts on. Felder proved that the sequence is exact except on the middle Fock space $\mathcal{F}_{r,s}$, where the cohomology $H_{\mathsf{Q}_F}(\mathcal{F}_{r,s})$ is isomorphic to the irreducible Virasoro module $\mathcal{L}_{r,s}$ [94]. This construction has the following rationale: The reducible representation $\mathcal{F}_{r,s}$ contains two primitive submodules, one built on the null at level rs and the other built on the null at level $(p-r)(q-s)$. Restricting to $\text{Ker}_{\mathsf{Q}_F}(\mathcal{F}_{r,s})$ factors out the null at level rs, while modding out by $\text{Im}_{\mathsf{Q}_F}(\mathcal{F}_{r,s})$ factors out the null at level $(p-r)(q-s)$.

An equivalent resolution is obtained by considering the dual Fock space, $(r, s) \to (p-r, q-s)$.

B.2 Liouville

The Liouville field Φ has stress tensor

$$\mathsf{T}_\Phi = -\partial\Phi\partial\Phi - \tilde{Q}_{p,q}\partial^2\Phi \tag{B.11}$$

and central charge

$$c_\Phi = 1 + 6\,\tilde{Q}_{p,q}^2, \quad \tilde{Q} = \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}. \tag{B.12}$$

Vertex operators of the form

$$W_\beta = e^{2\beta\Phi} \tag{B.13}$$

have conformal dimensions

$$\tilde{\Delta}_\beta = \beta(-\beta - \tilde{Q}_{p,q}). \tag{B.14}$$

We are adopting almost the same conventions as in the recent Liouville literature (e.g., [15, 95, 96] with the exception that for us the weak coupling region is at $\Phi = \infty$. The Seiberg bound is

$$\beta \geq -\frac{\tilde{Q}_{p,q}}{2}. \tag{B.15}$$

For special values of β, one encounters degenerate representations:

$$\beta_{m,n} = -\frac{1}{2}(1-m)\beta_- - \frac{1}{2}(1-n)\beta_+ \text{ with } \beta_+ = \sqrt{\frac{p}{q}}, \quad \beta_- = \sqrt{\frac{q}{p}} \tag{B.16}$$

with

$$\tilde{\Delta}_{\beta_{mn}} = -\frac{1}{4}(m\beta_- + n\beta_+)^2 + \frac{1}{4}\tilde{Q}^2. \tag{B.17}$$

Next we would like to introduce vertex operators $e^{2\gamma_{mn}\Phi}$ with

$$\gamma_{m,n} = -\frac{1}{2}(1-m)\beta_- - \frac{1}{2}(1+n)\beta_+. \tag{B.18}$$

and dimension

$$\tilde{\Delta}_{\gamma_{m,n}} = \tilde{\Delta}_{\beta_{m,n}} + mn = -\Delta_{\alpha_{m,n}} + 1 \tag{B.19}$$

Note the invariances of $\tilde{\Delta}_{\gamma_{m,n}}$ under $(m,n) \to (-m,-n)$, $(m,n) \to (m+p, n+q)$. Finally define

$$\gamma^+_{mn} = \max(\gamma_{m,n}, \gamma_{-m,-n}), \quad \gamma^-_{mn} = \min(\gamma_{m,n}, \gamma_{-m,-n}). \tag{B.20}$$

The Liouville momentum $\gamma^+_{r,s}$, which satisfies the Seiberg bound, is used to dress the (r,s) matter primary to obtain a tachyon vertex operator,

$$T_{r,s} = e^{2i\alpha_{r,s}X} e^{2\gamma^+_{r,s}\Phi} C\bar{C}. \tag{B.21}$$

B.3 Cohomology of the (p,q) models

Physical states of minimal string theory correspond to cohomology classes of the BRST operator $\mathsf{Q}_{\rm Vir}$. Let us briefly review the situation for the ordinary (p,q) models with $p,q > 1$. Denote by gh the BC ghost number, in conventions where the $SL(2)$ vacuum $|0\rangle$ has gh $= 0$, and gh$(C) = +1$, gh$(B) = -1$. Consider first the chiral (left-moving or right-moving) cohomology relative to B_0, that is, evaluated in the complex of states annihilated by B_0. The states that obey the Seiberg bound are:

gh $= 1$: *Tachyon states.* These are the $(p-1)(q-1)/2$ states of the form

$$T_{r,s} = e^{2i\alpha_{r,s}X(0)} e^{2\gamma^+_{r,s}\Phi(0)} C. \tag{B.22}$$

gh ≤ 0: *Lian-Zuckerman states.* In correspondence to each matter null, there is an LZ state of zero or negative ghost number, with Liouville dressing equal to that of the corresponding null. In the free-field realization, LZ states can be found by a descent procedure, starting with a state $|T_{r+(k+1)p,s}\rangle$,

$1 \leq r \leq p-1$, $1 \leq s \leq q-1$, $k \geq 0$, that is, a "tachyon" built with a matter primary outside the minimal Kac table. The descent is [11–14]

$$\begin{aligned} \mathsf{Q}_{\text{Vir}}|LZ\rangle^{(-k)} &= \mathsf{Q}_F|I_1\rangle^{(-k+1)} \\ &\cdots\cdots \\ \mathsf{Q}_{\text{Vir}}|I_k\rangle^{(0)} &= (-1)^k \mathsf{Q}_F|T_{m,n}\rangle^{(1)}, \end{aligned} \tag{B.23}$$

where the superscripts indicate the ghost number. Notice that here we are *not* imposing the restriction to half of the Kac table, in other terms we have $(p-1)(q-1)$ states for each gh ≤ 0. This doubling is due to the fact that over each matter primary there are *two* primitive null vectors, each of which generates a LZ state upon descent. (Applying literally the above descent procedure, half of the LZ states will land in the dual matter representation—they can be dualized back to the usual representation if desired.)

The LZ states with gh $= 0$ form the *ground ring*, generated by [97]

$$\begin{aligned} \hat{x} &= \left(BC + \sqrt{\frac{p}{q}}(\partial\Phi - i\partial X)\right) e^{\sqrt{\frac{q}{p}}(\Phi + iX)} \\ \hat{y} &= \left[BC + \sqrt{\frac{q}{p}}(\partial\Phi + i\partial X)\right] e^{\sqrt{\frac{p}{q}}(\Phi - iX)}, \end{aligned} \tag{B.24}$$

subject to the relations (taking zero cosmological constant for simplicity)

$$\hat{x}^{p-1} = 0, \quad \hat{y}^{q-1} = 0. \tag{B.25}$$

Cohomology spaces at given ghost number form representations of the ground ring. The representation is faithful for the states with gh ≤ 0, but not for the tachyons. When acting on the tachyon module, the ground ring generators obey the additional relation $\hat{x}^{p-2} = \hat{y}^{q-2}$.

The states listed so far represent only half of the relative chiral cohomology. Because of the pairing induced by the bpz inner product, $\langle\Psi|C_0|\Psi'\rangle$, to each of the above states corresponds a dual state with ghost number $2 -$ gh and dual matter and Liouville momenta. The dual states violate the Seiberg bound.

In building closed string states, we need to combine left and right movers. We are instructed to work in the semi-relative cohomology H_S, the cohomology of $\mathsf{Q}_{\text{Vir}} + \bar{\mathsf{Q}}_{\text{Vir}}$ relative to $B_0 - \bar{B}_0$. Consider first the closed-string cohomology H_R, relative to *both* B_0 and $\bar{B}_0$. These are the traditional closed string states obtained by gluing left and right movers belonging to the relative chiral cohomology discussed above. Then one can show [98] that $H_S^{(n)} \cong H_R^{(n)} \oplus H_R^{(n-1)}$. Each representative $\psi^{(n)}$ in the relative cohomology $H_R^{(n)}$ of ghost number n gives rise to two representatives in semi-relative cohomology: one in $H_S^{(n)}$ immediately given by $\psi^{(n)}$, and one in $H_S^{(n+1)}$ of the form $(C_0 + \bar{C}_0)\psi^{(n)} + \cdots$.

The discussion of semi-relative cohomology is more than a technical nuisance; in fact, it is important for the construction of symmetry currents. For the $c = 1$ string, one can obtain symmetry currents by descent from physical states of (left, right) ghost number $(1, 0)$ made by gluing a left-moving tachyon with a right-moving ground ring state. For the (p, q) models, such physical states are not allowed since the left and right Liouville momenta do not match. Fortunately there are states of total ghost number one in $H_S^{(1)}$ of the form $(C_0 + \bar{C}_0)\psi^{(0)} + \cdots$, where $\psi^{(0)}$ is a ground ring state. As explained in [98], these more general states of ghost number one still lead to symmetry currents.

B.4 The $(p,1)$ models

We are mainly interested in the $(p, 1)$ models, which are outside the range of definition of the "minimal" Virasoro series, since the fundamental domain of the Kac table is empty. Nevertheless it is possible to construct consistent CFTs with central charge $c_{p,1}$ [99–101] These models share with the minimal series the property of containing only degenerate Virasoro representations — but infinitely many of them, of course. They are rational with respect to an extended W-algebra [99, 101]. Modular invariant partition functions have also been constructed [101].

In the $(p, 1)$ model, the structure of the degenerate Virasoro representations is somewhat different. Without loss of generality, we can label the degenerate Virasoro representations of a theory with central charge $c_{p,1}$ by pairs of integers (r, s) with restrictions $1 \leq r \leq p$ and $s \geq 1$. Consider again the complex (B.10), for a theory of central charge $c_{p,1}$, and with the understanding that now (r, s) obey the new restrictions. One can prove that the sequence is exact and that the irreducible Virasoro module $\mathcal{L}_{r,s}$ is isomorphic to $\mathrm{Ker}_{\mathsf{Q}_F}(\mathcal{F}_{r,s})$. This is because the reducible representation $\mathcal{F}_{r,s}$ has only *one* primitive submodule, built on the null at level rs; the other putative null is absent since it would appear at level $(p - r)(1 - s) \leq 0$.

We are going to define the (chiral sector of the) $(p, 1)$ model as containing each irreducible representation $\mathcal{L}_{r,s}$ with $1 \leq r \leq p - 1$, $s \geq 1$ precisely once. Notice that representations with $r = 0 \bmod p$ are excluded. This is the natural definition that makes contact with the $N = 2$ topological minimal models and the p-KdV integrable hierarchy. It is also the definition that emerges in our reduction from $H_3^+ \times S^3$.

The existence of infinitely many matter primaries has the effect of changing the structure of the cohomology. In essence, the putative LZ states with gh < 0 are replaced by states with gh $= 0$ and gh $= 1$. The states in the (chiral, relative) cohomology obeying the Seiberg bound are:

gh $= 1$: *Tachyon states,*

$$T_{r,s} = e^{2i\alpha_{r,s}X} e^{2\gamma^+_{r,s}\Phi} C, \;\; 1 \le r \le p-1, \quad s \ge 1, \tag{B.26}$$

one for each matter primary. For convenience, we may re-label them using a single index $n \equiv ps - r$,

$$T_n = e^{2i((p-1-n)/2\sqrt{p})X} e^{2((n-p-1)/2\sqrt{p})\Phi} C, \;\; n \ge 1, \quad n \ne 0 \bmod p. \tag{B.27}$$

gh $= 0$: *Ground ring states,*

$$G_{r,s} = \hat{x}^{r-1}\hat{y}^{s-1}, \;\; 1 \le r \le p-1, \quad s \ge 1, \tag{B.28}$$

where $\hat{x}$ and $\hat{y}$ are given by the same expression (B.24) with $q = 1$. The $\hat{x}$ generator obeys the usual relation $\hat{x}^{p-1} = 0$ but there is no restrictions on the power of $\hat{y}$. Since each matter primary has a single primitive null, there is precisely one ground ring state for each tachyon. States with gh < 0 are absent: the descent procedure of the previous subsection always terminates after one step, on a ground ring state.

As in the (p, q) case, each of the above states has a dual (of ghost number $2 - $ gh) violating the Seiberg bound.

In the $(p, 1)$ models, the tachyons and the ground ring states form two isomorphic modules of the ground ring. In the formulation of the $(p, 1)$ model as twisted $N = 2$ minimal matter coupled to topological gravity, these two modules should be viewed as equivalent copies of the (gravitationally extended) chiral ring. In that language, the states with $s = 1$ span the so-called "small phase space", while the states with $s > 1$ are interpreted as gravitational descendants.

The discussion of the closed string semi-relative cohomology exactly parallels that for general (p, q).

Finally, we would like to express the view that the most natural way to treat the usual (p, q) minimal strings with $q > 1$ should parallel the above analysis of the $(p, 1)$ models—one should define the matter theory so that it contains an infinite number of primaries. Indeed by coupling to gravity, the fusion rules of the (p, q) matter minimal model get erased—the zeroes of the matter correlators are offset by infinities in the Liouville sector [102]—and it seems necessary to go outside the minimal Kac table. This treatment of the (p, q) models is also natural from the viewpoint of obtaining them by gravitational RG flow starting from the $c = 1$ theory.

References

[1] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes,* Commun. Math. Phys. **165** (1994), 311 arXiv:hep-th/9309140.

[2] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, *Topological amplitudes in string theory,* Nucl. Phys. B **413** (1994), 162 arXiv:hep-th/9307158.

[3] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, Adv. Theor. Math. Phys. **5** (2002), 809 arXiv:hep-th/0111222.

[4] D. Berenstein, *A toy model for the AdS/CFT correspondence,* JHEP **0407** (2004), 018, arXiv:hep-th/0403110.

[5] H. Lin, O. Lunin and J. Maldacena, *Bubbling AdS space and 1/2 BPS geometries,* JHEP **0410** (2004), 025 arXiv:hep-th/0409174.

[6] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on AdS(3),* Adv. Theor. Math. Phys. **2** (1998), 733, arXiv:hep-th/9806194.

[7] V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, *Fractal structure Of 2d-quantum gravity,* Mod. Phys. Lett. A **3** (1988), 819.

[8] E. Witten, *The N matrix model and gauged WZW models*, Nucl. Phys. B **371** (1992), 191.

[9] R. Dijkgraaf, H. Verlinde and E. Verlinde, *Notes on topological string theory and 2-D quantum gravity,* PUPT-1217 Based on lectures given at Spring School on Strings and Quantum Gravity, Trieste, Italy, Apr 24–May 2, 1990 and at Cargese Workshop on Random Surfaces, Quantum Gravity and Strings, Cargese, France, May 28–Jun 1, 1990.

[10] R. Dijkgraaf, *Intersection theory, integrable hierarchies and topological field theory,* arXiv:hep-th/9201003.

[11] B. H. Lian and G. J. Zuckerman, *New selection rules and physical states in 2-D gravity: conformal gauge,* Phys. Lett. B **254** (1991), 417.

[12] P. Bouwknegt, J. G. McCarthy and K. Pilch, *BRST analysis of physical states for 2-D gravity coupled to $c \leq 1$ matter,* Commun. Math. Phys. **145** (1992), 541.

[13] C. Imbimbo, S. Mahapatra and S. Mukhi, *Construction of physical states of nontrivial ghost number in $c < 1$ string theory,* Nucl. Phys. B **375** (1992), 399.

[14] S. Govindarajan, T. Jayaraman and V. John, *Chiral rings and physical states in $c < 1$ string theory,* Nucl. Phys. B **402** (1993), 118, arXiv:`hep-th/9207109`; S. Govindarajan, T. Jayaraman, V. John and P. Majumdar, *States of nonzero ghost number in $c < 1$ matter coupled to 2-D gravity,* Mod. Phys. Lett. A **7** (1992), 1063, arXiv:`hep-th/9112033`.

[15] N. Seiberg and D. Shih, *Branes, rings and matrix models in minimal (super)string theory,* JHEP **0402** (2004), 021, arXiv:`hep-th/0312170`.

[16] C. Bachas and M. Petropoulos, *Anti-de-Sitter D-branes,* JHEP **0102** (2001), 025, arXiv:`hep-th/0012234`.

[17] B. Ponsot, V. Schomerus and J. Teschner, *Branes in the Euclidean AdS(3),* JHEP **0202** (2002), 016, arXiv:`hep-th/0112198`.

[18] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Matrix string theory,* Nucl. Phys. B **500** (1997), 43, arXiv:`hep-th/9703030`.

[19] K. Hosomichi and Y. Sugawara, *Multi-strings on AdS(3) x S(3) from matrix string theory,* JHEP **9907** (1999), 027, arXiv:`hep-th/9905004`; K. Hosomichi and Y. Sugawara, *Hilbert space of space-time SCFT in AdS(3) superstring and T**(4kp)/S(kp) SCFT,* JHEP **9901** (1999), 013, arXiv:`hep-th/9812100`.

[20] O. Lunin, J. Maldacena and L. Maoz, *Gravity solutions for the D1–D5 system with angular momentum,* arXiv:`hep-th/0212210`.

[21] D. Martelli and J. F. Morales, *Bubbling AdS(3),* JHEP **0502** (2005), 048, arXiv:`hep-th/041213`.

[22] J. T. Liu and D. Vaman, *Bubbling 1/2 BPS solutions of minimal six-dimensional supergravity,* arXiv:`hep-th/0412242`.

[23] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino and C. Vafa, *Topological strings and integrable hierarchies,* arXiv:`hep-th/0312085`.

[24] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds,* Nucl. Phys. B **463** (1996), 55, arXiv:`hep-th/9511164`.

[25] N. Hitchin, *Generalized Calabi–Yau manifolds,* Q. J. Math. (2003), **54**(**3**), 281-308, arXiv:`math.DG/0209099`.

[26] M. Gualtieri, *Generalized complex geometry,* Oxford University DPhil thesis, arXiv:`math.DG/0401221`.

[27] D. Gaiotto and L. Rastelli, *A paradigm of open/closed duality: Liouville D-branes and the Kontsevich model,* arXiv:`hep-th/0312196`.

[28] J. Maldacena, G. W. Moore, N. Seiberg and D. Shih, *Exact vs. semiclassical target space of the minimal string,* JHEP **0410** (2004), 020, arXiv:hep-th/0408039.

[29] A. Hashimoto, M. x. Huang, A. Klemm and D. Shih, *Open/closed string duality for topological gravity with matter,* JHEP **0505** (2005), 007, arXiv:hep-th/0501141.

[30] D. Ghoshal, S. Mukhi and S. Murthy, *Liouville D-branes in two-dimensional strings and open string field theory,* JHEP **0411** (2004), 027, arXiv:hep-th/0406106.

[31] S. Giusto and C. Imbimbo, *The Kontsevich connection on the moduli space of FZZT Liouville branes,* Nucl. Phys. B **704** (2005), 181, arXiv:hep-th/0408216.

[32] J. McGreevy and H. L. Verlinde, *Strings from tachyons: the $c = 1$ matrix reloaded,* JHEP **0312** (2003), 054, arXiv:hep-th/0304224.

[33] I. R. Klebanov, J. Maldacena and N. Seiberg, *D-brane decay in two-dimensional string theory,* JHEP **0307** (2003), 045, arXiv:hep-th/0305159.

[34] I. R. Klebanov, J. Maldacena and N. Seiberg, *Unitary and complex matrix models as 1-d type 0 strings,* Commun. Math. Phys. **252** (2004), 275, arXiv:hep-th/0309168.

[35] C. V. Johnson, *Non-perturbative string equations for type 0A,* JHEP **0403** (2004), 041, arXiv:hep-th/0311129.

[36] N. Seiberg and D. Shih, *Flux vacua and branes of the minimal superstring,* JHEP **0501** (2005), 055, arXiv:hep-th/0412315.

[37] N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux,* JHEP **9903** (1999), 018, arXiv:hep-th/9902098.

[38] Y. Sugawara, *Topological string on AdS(3) x N,* Nucl. Phys. B **576** (2000), 265, arXiv:hep-th/9909146.

[39] K. Thielemans, *A Mathematica package for computing operator product expansions,* Int. J. Mod. Phys. C **2** (1991), 787.

[40] P. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, *Extended supersymmetric sigma models on group manifolds. 1. The complex structures,* Nucl. Phys. B **308** (1988), 662; A. Sevrin, W. Troost, A. Van Proeyen and P. Spindel, *Extended supersymmetric sigma models on group manifolds. 2. Current algebras,* Nucl. Phys. B **311** (1988), 465.

[41] S. E. Parkhomenko, *Extended superconformal current algebras and finite dimensional Manin triples,* Sov. Phys. JETP **75** (1992), 1 [Zh. Eksp. Teor. Fiz. **102** (1992), 3].

[42] E. Getzler, *Manin triples and $N = 2$ superconformal field theory,* arXiv:hep-th/9307041; E. Getzler, *Manin pairs and topological field theory,* Annals Phys. **237** (1995), 161, arXiv:hep-th/9309057.

[43] S. J. Gates, C. M. Hull and M. Rocek, *Twisted multiplets and new supersymmetric nonlinear sigma Models,* Nucl. Phys. B **248** (1984), 157.

[44] E. Witten, *Mirror manifolds and topological field theory,* arXiv: hep-th/9112056.

[45] A. Kapustin and Y. Li, *Topological sigma-models with H-flux and twisted generalized complex manifolds,* arXiv:hep-th/0407249.

[46] M. Bershadsky, W. Lerche, D. Nemeschansky and N. P. Warner, *Extended $N = 2$ superconformal structure of gravity and W gravity coupled to matter,* Nucl. Phys. B **401** (1993), 304, arXiv:hep-th/9211040.

[47] D. Bernard and G. Felder, *Fock representations and Brst cohomology in Sl(2) current algebra,* Commun. Math. Phys. **127** (1990), 145.

[48] M. Bershadsky and H. Ooguri, *Hidden Sl(N) symmetry in conformal field theories,* Commun. Math. Phys. **126** (1989), 49.

[49] H. L. Hu and M. Yu, *On BRST cohomology of SL(2,R)p/q-2/SL (2,R)p/q-2 gauged WZNW models,* Nucl. Phys. B **391** (1993), 389.

[50] O. Aharony, O. Ganor, J. Sonnenschein and S. Yankielowicz, *$c = 1$ string theory as a topological G/G model,* Phys. Lett. B **305** (1993), 35, arXiv:hep-th/9302027; O. Aharony, O. Ganor, J. Sonnenschein and S. Yankielowicz, *On the twisted G/H topological models,* Nucl. Phys. B **399** (1993), 560, arXiv:hep-th/9208040; O. Aharony, J. Sonnenschein and S. Yankielowicz, *G/G models and W(N) strings,* Phys. Lett. B **289** (1992), 309, arXiv:hep-th/9206063; O. Aharony, O. Ganor, J. Sonnenschein, S. Yankielowicz and N. Sochen, *Physical states in G/G models and 2-d gravity,* Nucl. Phys. B **399** (1993), 527, arXiv:hep-th/9204095.

[51] V. Sadov, *On the spectra of sl(N)-K / sl(N)-k cosets and W(N) gravities. 1,* Int. J. Mod. Phys. A **8** (1993), 5115, arXiv:hep-th/9302060.

[52] V. Sadov, *The Hamiltonian reduction of the BRST complex and $N = 2$ SUSY,* arXiv:hep-th/9304049.

[53] S. Mukhi and C. Vafa, *Two-dimensional black hole as a topological coset model of $c = 1$ string theory,* Nucl. Phys. B **407** (1993), 667, arXiv:hep-th/9301083.

[54] K. Gawedzki and A. Kupiainen, *G/H conformal field theory from gauged WZW model,* Phys. Lett. B **215** (1988), 119; K. Gawedzki and A. Kupiainen, *Coset construction from functional integrals,* Nucl.

Phys. B **320** (1989), 625; K. Gawedzki, *Noncompact WZW conformal field theories,* arXiv:hep-th/9110076.

[55] D. Karabali and H. J. Schnitzer, *Brst quantization of the gauged WZW action and coset conformal field theories,* Nucl. Phys. B **329** (1990), 649.

[56] K. Li, *Recursion relations in topological gravity with minimal matter,* Nucl. Phys. B **354** (1991), 725; K. Li, *Topological gravity with minimal matter,* Nucl. Phys. B **354** (1991), 711.

[57] N. Berkovits and C. Vafa, $N = 4$ *topological strings,* Nucl. Phys. B **433** (1995), 123, arXiv:hep-th/9407190.

[58] J. M. Maldacena and H. Ooguri, *Strings in AdS(3) and SL(2,R) WZW model. I,* J. Math. Phys. **42** (2001), 2929, arXiv:hep-th/0001053.

[59] D. Friedan, E. J. Martinec and S. H. Shenker, *Conformal invariance, supersymmetry and string theory,* Nucl. Phys. B **271** (1986), 93.

[60] R. Argurio, A. Giveon and A. Shomer, *Superstrings on AdS(3) and symmetric products,* JHEP **0012** (2000), 003, arXiv:hep-th/0009242.

[61] D. Kutasov and N. Seiberg, *More comments on string theory on AdS(3),* JHEP **9904** (1999), 008, arXiv:hep-th/9903219.

[62] D. Kutasov, F. Larsen and R. G. Leigh, *String theory in magnetic monopole backgrounds,* Nucl. Phys. B **550** (1999), 183, arXiv:hep-th/9812027.

[63] J. D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity,* Commun. Math. Phys. **104** (1986), 207.

[64] N. Seiberg and E. Witten, *The D1/D5 system and singular CFT,* JHEP **9904** (1999), 017, arXiv:hep-th/9903224.

[65] E. J. Martinec, G. W. Moore and N. Seiberg, *Boundary operators in 2-D gravity,* Phys. Lett. B **263** (1991), 190.

[66] O. Lunin and S. D. Mathur, *Metric of the multiply wound rotating string,* Nucl. Phys. B **610** (2001), 49, arXiv:hep-th/0105136.

[67] O. Lunin, S. D. Mathur and A. Saxena, *What is the gravity dual of a chiral primary?,* Nucl. Phys. B **655** (2003), 185, arXiv:hep-th/0211292.

[68] C. Vafa and E. Witten, *A strong coupling test of S duality,* Nucl. Phys. B **431** (1994), 3, arXiv:hep-th/9408074.

[69] O. Lunin and S. D. Mathur, *Three-point functions for M(N)/S(N) orbifolds with* $N = 4$ *supersymmetry,* Commun. Math. Phys. **227** (2002), 385, arXiv:hep-th/0103169.

[70] J. M. Maldacena and A. Strominger, *AdS(3) black holes and a stringy exclusion principle,* JHEP **9812** (1998), 005, arXiv:hep-th/9804085.

[71] I. K. Kostov, M. Staudacher and T. Wynter, *Complex matrix models and statistics of branched coverings of 2D surfaces,* Commun. Math. Phys. **191** (1998), 283, arXiv:hep-th/9703189.

[72] A. Okounkov, *Toda equations for Hurwitz numbers,* arXiv:math.AG/0004128.

[73] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves,* Invent. Math. **146**(**2**) (2001), 297327.

[74] A. Okounkov and R. Pandharipande, *Gromov–Witten theory, Hurwitz numbers, and matrix models. I,* arXiv:math.ag/0101147.

[75] L. Rastelli, M. Wijnholt, *Minimal superstrings, topological strings and NS5 branes,* Talk by L. Rastelli at the Simons Workshop, Stony Brook, August 2005, available online at http://insti.physics.sunysb.edu/itp/conf/simonswork3/talks.html.

[76] T. Takayanagi, *Notes on S-matrix of non-critical $N = 2$ string,* JHEP **0509** (2005), 001, arXiv:hep-th/0507065.

[77] V. Niarchos, *On minimal $N = 4$ topological strings and the (1,k) minimal bosonic string,* arXiv:hep-th/0512222.

[78] D. A. Sahakyan and T. Takayanagi, *On the connection between $N = 2$ minimal string and (1,n) bosonic minimal string,* arXiv:hep-th/0512112.

[79] A. Konechny, A. Parnachev and D. A. Sahakyan, *The ground ring of $N = 2$ minimal string theory,* Nucl. Phys. B **729** (2005), 419, arXiv:hep-th/0507002.

[80] L. Rastelli, M. Wijnholt, in progress.

[81] F. David, *Randomly triangulated surfaces in two-dimensions,* Phys. Lett. B **159** (1985), 303.

[82] I. K. Kostov and M. L. Mehta, *Random surfaces of arbitrary genus: exact results for $D = 0$ and -2 dimensions,* Phys. Lett. B **189** (1987), 118.

[83] I. R. Klebanov and R. B. Wilkinson, *Critical potentials and correlation functions in the minus two-dimensional matrix model,* Nucl. Phys. B **354** (1991), 475.

[84] O. Aharony, B. Fiol, D. Kutasov and D. A. Sahakyan, *Little string theory and heterotic/type II duality,* Nucl. Phys. B **679** (2004), 3, arXiv:hep-th/0310197.

[85] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, *$N = 2$ type II heterotic duality and higher derivative F terms,* Nucl. Phys. B **455** (1995), 109, arXiv:hep-th/9507115.

[86] C. Vafa, *A stringy test of the fate of the conifold,* Nucl. Phys. B **447** (1995), 252, arXiv:hep-th/9505023.

[87] R. Gopakumar and C. Vafa, *M-theory and topological strings. I, II* arXiv:hep-th/9809187, arXiv:hep-th/9812127.

[88] N. Berkovits and C. Vafa, *Type IIB R**4 H**(4g-4) conjectures,* Nucl. Phys. B **533** (1998), 181, arXiv:hep-th/9803145.

[89] M. B. Green and S. Sethi, *Supersymmetry constraints on type IIB supergravity,* Phys. Rev. D **59** (1999), 046006, arXiv:hep-th/9808061.

[90] M. B. Green and M. Gutperle, *Effects of D-instantons,* Nucl. Phys. B **498** (1997), 195, arXiv:hep-th/9701093.

[91] E. Kiritsis and B. Pioline, *On R**4 threshold corrections in type IIB string theory and (p,q) string instantons,* Nucl. Phys. B **508** (1997), 509, arXiv:hep-th/9707018.

[92] R. Bott and L. Tu, *Differential forms in algebraic topology,* Springer Verlag, New York.

[93] W. Lerche, *Chiral rings and integrable systems for models of topological gravity,* arXiv:hep-th/9401121.

[94] G. Felder, *Brst approach to minimal methods,* Nucl. Phys. B **317** (1989), 215, [Erratum: Nucl. Phys. B **324** (1989), 548].

[95] J. Teschner, *Liouville theory revisited,* Class. Quant. Grav. **18** (2001), R153, arXiv:hep-th/0104158.

[96] I. K. Kostov, B. Ponsot and D. Serban, *Boundary Liouville theory and 2D quantum gravity,* Nucl. Phys. B **683** (2004), 309, arXiv:hep-th/0307189.

[97] D. Kutasov, E. J. Martinec and N. Seiberg, *Ground rings and their modules in 2-D gravity with $c \leq 1$ matter,* Phys. Lett. B **276** (1992), 437, arXiv:hep-th/9111048.

[98] E. Witten and B. Zwiebach, *Algebraic structures and differential geometry in 2-D string theory,* Nucl. Phys. B **377** (1992), 55, arXiv:hep-th/9201056.

[99] H. G. Kausch, *Extended conformal algebras generated by a multiplet of primary fields,* Phys. Lett. B **259** (1991), 448.

[100] H. G. Kausch, *Curiosities at $c = -2$,* arXiv:hep-th/9510149.

[101] M. A. I. Flohr, *On modular invariant partition functions of conformal field theories with logarithmic operators,* Int. J. Mod. Phys. A **11** (1996), 4147, arXiv:hep-th/9509166.

[102] V. S. Dotsenko, *Three point correlation functions of the minimal conformal theories coupled to 2-D gravity,* Mod. Phys. Lett. A **6** (1991), 3601.