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Topological Landau–Ginzburg models on the world-sheet foam

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Abstract

We define topological Landau–Ginzburg models on a world-sheet foam, that is, on a collection of 2-dimensional surfaces whose boundaries are sewn together along the edges of a graph. We use the matrix factorizations in order to formulate the boundary conditions at these edges and then produce a formula for the correlators. Finally, we present the gluing formulas, which correspond to various ways in which the pieces of a world-sheet foam can be joined together.

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Contents

1 Introduction

It is always easier to define a quantum field theory on a closed manifold: there is no need to formulate the boundary conditions for the fields in the path integral. If the boundary exists, then one might limit oneself to the easiest case of the Neumann boundary conditions. The last decade showed, however, that the world of boundary conditions may be even more interesting and diverse than the world of the "bulk" QFTs. A bulk 2-dimensional QFT yields an algebra, boundary conditions are objects of a category, and all morphisms between two objects form a module over the bulk algebra. Now it turns out that the existence of a rather general class of boundary conditions may change the very nature of the world-sheet manifold: instead of being just a surface with boundary, it may become a "foam," that is, a version of a 2-dimensional CW-complex endowed with a complex structure, if needed.

Although the QFTs themselves do not require the presence of a world-sheet foam, a foam appeared in the paper [4] as a necessary element in the categorification of the SU(3) HOMFLY polynomial. The paper [6] interpreted that part of the categorification as a 2-dimensional topological A-model defined on a world-sheet foam. Each 2-dimensional connected component Σ_i of the foam carries its own topological σ-model, whose target space is the complex Grassmannian $\mathrm{Gr}_{m_i,n}$, while the boundary condition at an edge of the seam graph is specified by selecting a Lagrangian submanifold in the cross-product of the Grassmannians assigned to the components Σ_i bounding the edge.

The paper [6] described the general setup of a QFT on a world-sheet foam, using the topological A-models as an illustration. Topological A-models, however, are notorious for their complexity even on the usual smooth surfaces, and the paper [6] presented neither an accurate description of the Hilbert spaces corresponding to the seam graph vertices, nor a complete formula for the partition function.

In this paper, we give a detailed description of a topological Landau–Ginzburg model on a world-sheet foam. Each component Σ_i of the foam carries its own fields $\boldsymbol{\phi}_i = \phi_{i,1}, \ldots, \phi_{i,m_i}$ and its own potential $W_i(\boldsymbol{\phi}_i)$, while each edge of the seam graph carries a matrix factorization of the sum of potentials of the bounding components Σ_i, in the spirit of [3]. We describe the Hilbert spaces of the vertices of the seam graph and also provide a formula for the correlators, which generalizes the formulas of Vafa [7] and Kapustin–Li [3]. Finally, we present the gluing formulas for joining "space-like" and even "time-like" boundary components of the world-sheet foam.

This paper is closely related to our categorification [5] of the SU(N) HOMFLY polynomial. Although we do not use world-sheet foams explicitly in [5], the construction of the graded vector spaces associated to 3-valent graphs in [5] is a particular case of the definition of an operator space H_γ related to a decorated local graph γ, as described in Section 5.2. We refer the reader to [5] for a detailed discussion of matrix factorizations.

2 A topological LG theory on a world-sheet with a boundary

According to [6], any QFT defined on a 2-dimensional surface with boundary can be transferred to a world-sheet foam. Hence we begin by reviewing the topological LG theory $\mathcal{T}$ on a surface with boundary, as presented in [2] (see also references therein). We assume for simplicity that the target space of $\mathcal{T}$ is a flat $\mathbb{C}^m$ and there are no gauge fields. Then the bulk theory is characterized by a (polynomial) super-potential $W \in \mathbb{C}[\boldsymbol{\phi}]$, $\boldsymbol{\phi} = \phi_1, \ldots, \phi_m$, and we denote this Landau–Ginzburg theory by $\mathcal{T} = (\boldsymbol{\phi}; W)$.

2.1 The bulk Lagrangian

The topological LG theory $\mathcal{T}$ contains bosonic fields $\boldsymbol{\phi}$ and $\overline{\boldsymbol{\phi}}$ as well as the fermionic fields $\eta^{\bar{i}}$, θ_i, ρ^i_z, and $\rho^i_{\bar{z}}$. The bulk Lagrangian of the theory is

$$\begin{aligned} L_{\mathcal{T}} = {} & \frac{1}{2}\Big(\partial_z\phi^i\,\partial_{\bar z}\phi^{\bar i} + \partial_z\phi^{\bar i}\,\partial_{\bar z}\phi^i - \rho^i_z\,\partial_{\bar z}\eta^{\bar i} - \rho^i_{\bar z}\,\partial_z\eta^{\bar i}\Big) \\ & -2i\Big(\theta_i\left(\partial_z\rho^i_{\bar z} - \partial_{\bar z}\rho^i_z\right) + \partial_i\partial_j W\,\rho^i_z\rho^j_{\bar z}\Big) \\ & +\frac{1}{4}\Big(\partial_{\bar i}\partial_{\bar j}\bar W\,\theta_i\eta^{\bar j} - \partial_i W\,\partial_{\bar i}\bar W\Big). \end{aligned} \tag{2.1}$$

Each line in this expression is invariant under the topological BRST transformation Q

$$\begin{aligned} &\delta_Q \phi^i = 0 \quad \delta_Q \phi^{\bar{i}} = \eta^{\bar{i}} \\ &\delta_Q \eta^{\bar{i}} = 0 \quad \delta_Q \theta_i = \partial_i W \\ &\delta_Q \rho^i_z = \partial_z \phi^i \quad \delta_Q \rho^i_{\bar{z}} = \partial_{\bar{i}} \phi^i, \end{aligned} \tag{2.2}$$

except that the second line generates the boundary Warner term: for a world-sheet Σ with a boundary $\partial\Sigma$ the BRST variation of the action is

$$\delta_Q \int_\Sigma L_{\mathcal{T}} = \int_{\partial\Sigma} \partial_i W (\rho^i_z dz + \rho^i_{\bar{z}} d\bar{z}). \tag{2.3}$$

Note that if we treat $(\rho^i_z, \rho^i_{\bar{z}})$ as a 1-form on Σ, then the Lagrangian (2.1) can be written without a reference to the complex structure of the world-sheet. The only remnant of that complex structure would be the orientation that it induces on Σ. Let $L_{\bar{\mathcal{T}}}$ denote the Lagrangian (2.1) in which we conjugated the complex structure or, equivalently, reversed the orientation of the world-sheet. It is easy to see that this change can be compensated by switching two signs: the sign of the field θ_i (that is, θ_i is a "pseudo-scalar") and the sign of the super-potential W. Thus,

$$L_{\overline{(\phi;W)}} = L_{(\phi;-W)}. \tag{2.4}$$

2.2 The boundary Wilson line

Kontsevich suggested that the Warner term (2.3) could be compensated by putting the appropriate Wilson lines at the boundary components of Σ. This idea was implemented in papers [1–3]. We will follow the approach of Lazaroiu [2] as the most suitable for our purposes.

The linear space for a LG Wilson line is provided by a matrix factorization of the super-potential W. According to [3], a *matrix factorization* of W is a triple (M, D, W), where M is a finite-dimensional $\mathbb{Z}_2$-graded free $\mathbb{C}[\boldsymbol{\phi}]$ module, $M = M^0 \oplus M^1$ ($\operatorname{rank} M^0 = \operatorname{rank} M^1$), while the *twisted differential* D is an operator $D \in \operatorname{End}(M)$, such that $\deg D = 1$ and

$$D^2 = W \,\mathrm{Id}. \tag{2.5}$$

Simply saying, a matrix factorization is described by a $2n \times 2n$-dimensional matrix with polynomial entries

$$D = \left(\begin{array}{c|c} 0 & F \\ \hline G & 0 \end{array}\right), \quad \text{such that} \quad FG = GF = W\,\mathrm{Id}. \tag{2.6}$$

Lazaroiu introduces a connection

$$A_{\mathcal{T}} = \left(\begin{array}{c|c} W\,\mathrm{Id} & (\rho^i_z dz + \rho^i_{\bar{z}} d\bar{z})\,\partial_i F \\ \hline (\rho^i_z dz + \rho^i_{\bar{z}} d\bar{z})\,\partial_i G & W\,\mathrm{Id} \end{array}\right) \tag{2.7}$$

acting on M. Suppose that $\partial\Sigma$ is a union of disjoint circles:

$$\partial\Sigma = \bigsqcup_k C_k. \tag{2.8}$$

To each circle C_k, we assign a matrix factorization (M_k, D_k, W) of W. Then Lazaroiu proves that the path integrand

$$\exp\left(\int_\Sigma L_{\mathcal{T}}\right) \prod_k \mathcal{W}_{C_k}, \quad \mathcal{W}_{C_k} = \mathrm{S\,Tr}_{M_k}\ \mathrm{Pexp} \oint_{C_k} A, \tag{2.9}$$

with the orientation of C_k induced by the orientation of Σ, is invariant under the topological BRST transformation (2.2).

An orientation of a boundary component C_k can be reversed without affecting its super-trace, if we replace the associated matrix factorization (M_k, D_k, W) with its dual. First, observe that the matrix factorizations can be tensored:

$$(M_1, D_1, W_1) \otimes (M_2, D_2, W_2) = (M_1 \otimes M_2, D_1 + D_2, W_1 + W_2)\,. \tag{2.10}$$

Then we define the dual matrix factorization as

$$(M, D, W)^* = (M^*, D^*, -W)\,, \tag{2.11}$$

where the module M^* is the dual of M over $\mathbb{C}[\boldsymbol{\phi}]$ and

$$D^* = \left(\begin{array}{c|c} 0 & G^* \\ \hline -F^* & 0 \end{array}\right), \tag{2.12}$$

where F^* and G^* are the dual maps (transposed matrices) of F and G. Choice (2.12) guarantees that the natural pairing map $M^* \otimes M \xrightarrow{f} \mathbb{C}[\boldsymbol{\phi}]$ satisfies the property $fD = 0$ and thus "commutes" with D.

2.3 Boundary operators

In order to simplify our notations, whenever we work with multiple matrix factorizations, we will denote all their twisted differentials by D, if it is clear on which particular module that D is acting. Also, let $[\cdot,\cdot]_{\mathrm{s}}$ denote the super-commutator:

$$[A, B]_{\mathrm{s}} = AB - (-1)^{\deg A \deg B} BA. \tag{2.13}$$

The super-traces of (2.9) have an obvious generalization. Let $t \in [0,1]$ parameterize a boundary circle C. The values $0 = t_0 \le t_1 \le \cdots \le t_n = 1$

split C into n segments. We can assign any matrix factorizations (M_j, D_j, W) to the segments $[t_{j-1}, t_j]$. To each value t_j we assign an operator $O_j \in \mathrm{Hom}(M_j, M_{j+1})$, which super-commutes with the twisted differential D: $[D, O_j]_{\mathrm{s}} = 0$. Then the Wilson line contribution $\mathcal{W}_C$ can be replaced in the integrand (2.9) by

$$\mathcal{W}_C = \mathrm{STr}_{M_1} \left(\mathrm{Pexp} \int_{[t_0,t_1]} A \right) O_1 \cdots \left(\mathrm{Pexp} \int_{[t_{n-1},t_n]} A \right) O_n, \qquad (2.14)$$

while still maintaining the BRST invariance.

Following [3], let us give a more precise description of the space of the operators O_j. For two matrix factorizations (M_0, D_0, W) and (M_1, D_1, W) consider the $\mathbb{Z}_2$-graded module $\mathrm{Hom}(M_0, M_1)$ of $\mathbb{C}[\boldsymbol{\phi}]$-linear maps between the modules. Define a differential d on this module by

$$dO = [D, O]_{\mathrm{s}}\,, \qquad (2.15)$$

where $O \in \mathrm{Hom}(M_0, M_1)$. It turns out that $d^2 = 0$, and d describes the BRST action on $\mathrm{Hom}(M_0, M_1)$. The space H_P of operators at the junction P of two segments carrying the modules M_0 and M_1 can be presented as

$$H_P = \mathrm{Ext}(M_0, M_1) = \frac{\ker\, d}{\mathrm{im}\, d}. \qquad (2.16)$$

An equivalent presentation of the operator space comes from the dual matrix factorization. Namely, consider the tensor product of matrix factorizations

$$\begin{aligned} (M_1, D_1, W) \otimes (M_0, D_0, W)^* &= (M_1 \otimes {M_0}^*, D, 0)\,, \\ D &= D_1 + {D_0}^*. \end{aligned} \qquad (2.17)$$

Since $D^2 = 0$, we can take

$$H_P = \frac{\ker\, D}{\mathrm{im}\, D}, \qquad (2.18)$$

because the cohomology of D is canonically isomorphic to $\mathrm{Ext}(M_1, M_0)$ in view of the canonical isomorphism

$$\mathrm{Hom}(M_0, M_1) = M_1 \otimes {M_0}^*, \qquad (2.19)$$

and the fact that d corresponds to D.

3 A topological LG theory on a world-sheet foam

3.1 The world-sheet foam

Let us recall the definition of a world-sheet foam given in [6]. Let Γ be a graph such that every vertex has adjacent edges. Γ is allowed to contain disjoint circles. A cycle on Γ is defined to be either a disjoint circle or a cyclicly ordered finite sequence of edges, such that the beginning of the next edge corresponds to the end of the previous edge. Let Σ be an orientable and possibly disconnected smooth 2-dimensional surface, its boundary $\partial\Sigma$ being a union of disjoint circles. A *world-sheet foam* (Σ, Γ) is a union $\Gamma \cup \Sigma$, in which the boundary circles of Σ are glued to some cycles on Γ in such a way that every edge of Γ is glued to at least one circle of $\partial\Sigma$.

Defining a topological LG theory on a world-sheet foam (Σ, Γ) involves three steps: first, we assign bulk theories to oriented connected components of Σ; second, we assign appropriate boundary condition to the oriented seam edges; third, we choose the operators at the seam vertices. The first two steps comprise a *decoration* of the world-sheet foam.

3.2 Bulk theories on the 2-dimensional connected components

The first step is simple. Suppose that the 2-dimensional surface Σ is a union of N_Σ disjoint components $\Sigma = \bigsqcup_{i=1}^{N_\Sigma} \Sigma_i$. Then to every oriented component Σ_i, we assign its own topological LG theory $\mathcal{T}_i = (\boldsymbol{\phi}_i; W_i)$ with its own target space $\mathbb{C}^{m_i}$, bosonic fields $\boldsymbol{\phi}_i$, fermionic fields, and a super-potential W_i in such a way that if Σ_i and $\bar{\Sigma}_i$ represent the same component of Σ with opposite orientations, then $\bar{\Sigma}_i$ should be assigned the conjugated theory $\bar{\mathcal{T}}_i$.

3.3 Boundary conditions at the seam graph edges

Next, we have to formulate the boundary conditions at the seam edges, which would be compatible with the BRST-invariance of the actions of Landau–Ginzburg theories sitting on the adjacent surfaces Σ_i. According to [6], these boundary conditions are just particular cases of an ordinary boundary condition for a Landau–Ginzburg theory defined on a smooth surface with a boundary.

Let us orient an edge e of Γ and let $\mathcal{I}_e$ be the set of indices i, such that e bounds Σ_i. We orient all Σ_i $(i \in \mathcal{I}_e)$ in such a way that their orientations are compatible with the orientation of e. If P is an inner point of e, then a small neighborhood of P looks like a union of upper half-planes $\mathbb{H}_i^+ \subset \mathbb{C}$ glued along the common real line, the point P being the origin of that real line. If $\mathbb{H}^+$ is a "standard" upper half-plane, then we can identify all $\mathbb{H}_i^+$ analytically (preserving orientation) with it. Thus, if every $\mathbb{H}_i^+$ carries a QFT $\mathcal{T}_i$ with the Lagrangian $L_{\mathcal{T}_i}$, then formulating a boundary condition for them at e is equivalent to formulating it for the combined theory $\mathcal{T}_e$ with the Lagrangian $L_{\mathcal{T}_e} = \sum_{i \in \mathcal{I}_e} L_{\mathcal{T}_i}$. In case of the topological LG theories, this means that to every seam edge e we assign the theory

$$\mathcal{T}_e = (\boldsymbol{\phi}_e; W_e)\,, \quad \text{where } \boldsymbol{\phi}_e = (\boldsymbol{\phi} \,|\, i \in \mathcal{I}_e), \quad W_e = \sum_{i \in \mathcal{I}_e} W_i. \tag{3.1}$$

Then to every oriented edge e of the seam graph Γ, we associate a matrix factorization (M_e, D_e, W_e) in such a way that if two oriented edges e and e^* represent the same edge with opposite orientations, then $(M_{e^*}, D_{e^*}, W_{e^*}) = (M_e, D_e, W_e)^*$. Also, we assign to e the Lazaroiu connection $A_e = A_{\mathcal{T}_e}$. If the matrix factorization (M_e, D_e, W_e) does not factor into a tensor product of matrix factorizations of individual super-potentials W_i, then the boundaries of the components Σ_i cannot be "unglued" at the edge e.

It is important to note that the construction of a matrix factorization associated to a seam edge must be local. It may happen that because of the global structure of the world-sheet foam (Σ, Γ), some of the strips that bound an edge e come from the same world-sheet component Σ_i. In this case, we first treat their theories $\mathcal{T}_i$ as different, that is, they share the same dimension of the target space m_i and the same super-potential W_i, yet their target spaces and fields are considered distinct. After we pick a matrix factorization (M_e, D_e, W_e), we impose a condition that the fields coming from the different strips of the same world-sheet component are the same.

3.4 Operators at the seam graph vertices

Let v be a seam graph vertex and let $\mathcal{I}_v$ be the set of seam edges, which are adjacent to v. We orient these edges away from v and then consider the factorization

$$(M_v, D_v, W_v) = \bigotimes\nolimits_{e \in \mathcal{I}_v} (M_e, D_e, W_e)\,. \tag{3.2}$$

Obviously,

$$M_v = \bigotimes\nolimits_{e \in \mathcal{I}_v} M_e, \quad W_v = 0, \tag{3.3}$$

the latter equation following from the fact that for every component Σ_i attached to v there are two (or, more generally, an even number of) bounding edges, which are attached to v in such a way that Σ_i contributes an equal number of W_i and $-W_i$ to W_v.

Since $W_v = 0$, then $D_v^2 = 0$ and we can consider its cohomology

$$H_v = \frac{\ker\, D_v}{\mathrm{im}\; D_v}. \tag{3.4}$$

The space H_v is $\mathbb{Z}_2$-graded: $H_v = H_v^0 \oplus H_v^1$. D_v plays the role of the BRST operator at v, so we take H_v as the space of operators at the vertex v. In other words, to every vertex v, we associate an element

$$O_v \in \ker\, D_v, \tag{3.5}$$

and the BRST-invariance of the path integral will guarantee that the correlators depend on O_v only modulo $\mathrm{im}\, D_v$.

3.5 Wilson network

For a world-sheet foam (Σ, Γ), the analog of the Wilson lines at the boundary components of the world-sheet Σ is the Wilson network formed by the seam graph Γ. Its contribution is a generalization of the super-trace (2.14) and it is expressed through multiple contractions between the pairs of dual modules in a big tensor product

$$\left(\bigotimes_{v\in\mathcal{V}} M_v\right) \otimes \left(\bigotimes_{e\in\mathcal{E}} (M_e \otimes {M_e}^*)\right), \tag{3.6}$$

where $\mathcal{V}$ and $\mathcal{E}$ are the sets of all vertices and of all edges of Γ. If we substitute the formulas (3.3) for M_v, then all the elementary modules M_e and ${M_e}^*$ can be grouped in pairs of mutually dual modules: M_e (or ${M_e}^*$) coming from $M_e \otimes {M_e}^*$, while the dual module ${M_e}^*$ (or M_e) coming from the tensor product expression (3.3) for M_v, where v is the beginning (or the end point) of the oriented edge e. Performing contractions within each pair, we get the map

$$\left(\bigotimes_{v\in\mathcal{V}} M_v\right) \otimes \left(\bigotimes_{e\in\mathcal{E}} M_e \otimes {M_e}^*\right) \xrightarrow{f_\Gamma} \mathbb{C}[\boldsymbol{\phi}_\Sigma], \tag{3.7}$$

where

$$\boldsymbol{\phi}_\Sigma = \boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{N_\Sigma}, \tag{3.8}$$

and N_Σ is the number of connected components Σ_i of Σ. If we consider the Lazaroiu connection holonomy along an edge e to be the element of $M_e \otimes {M_e}^*$, then the Wilson network contribution $\mathcal{W}_\Gamma$ can be expressed as the contraction map (3.7) applied to the tensor product of all the Lazaroiu holonomies and all the vertex operators O_v:

$$\mathcal{W}_\Gamma = f_\Gamma\left(\left(\bigotimes_{v\in\mathcal{V}} O_v\right)\otimes\left(\bigotimes_{e\in\mathcal{E}} \mathrm{Pexp}\int_e A_e\right)\right) \tag{3.9}$$

Clearly, this expression is a generalization of the super-trace (2.14), if we assume that the seam graph Γ in the latter case consists of a disjoint union of cycles formed by the seam edges $[t_{j-1}, t_j]$.

3.6 Operators at the internal points of the seam edges

The network structure of the boundary Wilson lines of the world-sheet foam forces us to always include the seam vertex operators in any correlator on (Σ, Γ). Moreover, note that the operator spaces H_v generally do not contain canonical elements, so there is no special choice for O_v.

In addition to the required operators at the seam vertices, the correlators may also include the optional operators at the internal points of the seam edges e and at the internal points of the world-sheet components Σ_i.

Let P be an internal point of a seam edge e. In order to describe its space of local operators, we can simply declare it to be a new seam vertex, thus breaking the edge e into two consecutive edges. Then the space of the operators can be presented either in the form (2.16) or in the form (2.18), in which we substitute $M_0 = M_1 = M_e$. Note that the space $\mathrm{Ext}(M_e, M_e)$ has a canonical element, which is the identity map.

3.7 Jacobi algebra and the local operators of the bulk

The description of the operator spaces for the internal points of the components Σ_i comes from the topological LG theories on closed surfaces. Namely, for a topological LG theory $(\boldsymbol{\phi}; W)$

$$H_P = \frac{\mathbb{C}[\boldsymbol{\phi}]}{\boldsymbol{\partial} W}, \tag{3.10}$$

where $\boldsymbol{\partial} W$ is the ideal of $\mathbb{C}[\boldsymbol{\phi}]$ generated by the first partial derivatives $\partial_i W$ (indeed, H_P must be the cohomology of the BRST operator acting according to (2.2) on the algebra of the fields). The $\mathbb{C}[\boldsymbol{\phi}]$-module H_P has an algebra structure and is called the Jacobi algebra of W, so we will also denote it as J_W.

The operator product expansion arguments show that for any world-sheet foam component Σ_i, whose boundary passes through a seam vertex v, the operator space H_v must be a module over the Jacobi algebras J_{W_i}. The space H_v by its definition (3.4) is already a module over $\mathbb{C}[\boldsymbol{\phi}_i]$, so we have to check that if $O_v \in \ker D_v$, then

$$\left(\frac{\partial W_i}{\partial \phi_{i,j}}\right) O_v \in \text{im } D_v. \tag{3.11}$$

The proof is based on a general relation

$$\{\partial_j D, D\} = \partial_j W, \tag{3.12}$$

which follows from Equation (2.5) by taking the derivative ∂_j of both sides. Now let e_0 be an adjacent edge of v, which is a part of the boundary $\partial\Sigma_i$. Suppose that all edges, which are adjacent to v, go out of v. Then $D_v = \sum_{e\in\mathcal{I}_v} D_e$. Since $\{D_e, D_{e_0}\} = 0$ for all $e \neq e_0$, then Equation (3.12) implies that

$$\{\partial_j D_e, D_v\} = \{\partial_j D_e, D_e\} = \partial_j W_e = \partial_j W_i, \tag{3.13}$$

the latter equality following from the last Equation of (3.1). Thus

$$\partial_j W_e\, O_v = \{\partial_j D_e, D_v\}\, O_v = (\partial_j D_e)\, D_v\, O_v + D_v\, (\partial_j D_e)\, O_v. \tag{3.14}$$

Since $O_v \in \ker\, D_v$, then the first term in the r.h.s. of this formula is zero. The second term belongs to im D_v, and this proves our assertion.

3.8 Topological Landau–Ginzburg path integral

Finally, we combine all the data into the path integral, which represents the correlator of the operators O_v at the seam graph vertices and the operators O_P at the internal points of Σ. The exponential part of the integrand is simply $\exp\left(\sum_i L_{\mathcal{T}_i}\right)$, whereas the operators O_P and the Wilson network contribution $\mathcal{W}_\Gamma$ provide the pre-exponential factors:

$$\left\langle \prod_{P\in\mathcal{P}} O_P \prod_{v\in\mathcal{V}} O_v \right\rangle_{(\Sigma,\Gamma)} = \int \exp\left(\sum_i L_{\mathcal{T}_i}\right) \left(\prod_P O_P\right) \mathcal{W}_\Gamma\, \mathcal{D}\boldsymbol{\phi}_\Sigma \mathcal{D}\boldsymbol{\eta}_\Sigma \mathcal{D}\boldsymbol{\theta}_\Sigma \mathcal{D}\boldsymbol{\rho}_\Sigma, \tag{3.15}$$

where (field)$_\Sigma$ means all fields of the given type from all the components Σ_i.

3.9 An example of a topological LG theory on a world-sheet foam

Let us now consider a specific example of a topological LG theory which can be put on a world-sheet foam. In other words, we are going to present a set of topological LG theories $\mathcal{T}_i = (\boldsymbol{\phi}_i; W_i)$ and some matrix factorizations of the sums of their super-potentials which do not factor into the tensor products of matrix factorizations of the individual super-potentials W_i. This example is inspired by the construction of [4], and following [8] we suggest that it is the mirror image of the world-sheet foam theory presented in [6]. Also the constructions of our paper [5] are based on a particular case of the matrix factorizations described here.

Let us fix a positive integer N and a complex parameter a. Following [8], for $1 \leq i \leq N-1$ we consider the polynomial

$$c_m(\boldsymbol{\phi}_m; t) = 1 + \sum_{j=1}^{m} \phi_{m,j}\, t^j, \quad \boldsymbol{\phi}_m = (\phi_{m,j} \,|\, 1 \leq j \leq m) \tag{3.16}$$

and the expansion of its logarithm in power series of t:

$$\ln c_m(\boldsymbol{\phi}_m; t) = \sum_{j=1}^{\infty} c_{m,j}(\boldsymbol{\phi}_m)\, t^j. \tag{3.17}$$

Then we set

$$W_m(\boldsymbol{\phi}_m) = (-1)^{N+1}\, c_{m,N+1}(\boldsymbol{\phi}_m) - a\, \phi_{m,1}, \tag{3.18}$$

thus defining the topological LG theory $\mathcal{T}_m = (\boldsymbol{\phi}_m; W_m)$ with the target space $\mathbb{C}^m$. Its Jacobi algebra coincides with the quantum cohomology algebra of the complex Grassmannian $\mathrm{Gr}_{m,N}$.

The matrix factorizations that we are going to use belong to a special class sometimes called the "Koszul factorizations". Here is the general construction. Suppose that a super-potential $W(\boldsymbol{\phi})$ factors over $\mathbb{C}[\boldsymbol{\phi}]$

$$W = p\, q, \quad p, q \in \mathbb{C}[\boldsymbol{\phi}]. \tag{3.19}$$

Then there exists a $(1|1)$-dimensional matrix factorization of W with the twisted differential

$$D = \left(\begin{array}{c|c} 0 & q \\ \hline p & 0 \end{array}\right). \tag{3.20}$$

We denote this matrix factorization by $(p; q)$. If $W(\boldsymbol{\phi})$ can be presented as a sum of products

$$W = \sum_{j=1}^{n} p_j\, q_j, \quad \mathbf{p}, \mathbf{q} \in \mathbb{C}[\boldsymbol{\phi}], \tag{3.21}$$

then there is a $(2^{n-1}|2^{n-1})$-dimensional factorization of W

$$(\mathbf{p};\mathbf{q}) = \bigotimes_{j=1}^{n} (p_j;q_j). \tag{3.22}$$

Now we come back to the potentials (3.18). For a list of integer numbers $\mathbf{m}$ such that

$$\sum_i m_i = N, \tag{3.23}$$

we are going to construct a Koszul matrix factorization of the super-potential

$$W_{\mathbf{m}} = \sum_i W_{m_i} \tag{3.24}$$

by presenting it in the form (3.21). Consider the polynomial $W_N(\tilde{\mathbf{p}})$ as defined by Equations (3.16) to (3.18), in which the variables $\boldsymbol{\phi}$ are replaced by the variables $\tilde{\mathbf{p}} = (p_j \,|\, 1 \le j \le N)$. The equation

$$\prod_i c_{m_i}(\boldsymbol{\phi}_{m_i};t) = 1 + \sum_{j=1}^{N} p_j(\boldsymbol{\phi})\, t^j \tag{3.25}$$

defines $\tilde{\mathbf{p}}$ as polynomial functions of all variables $\boldsymbol{\phi} = (\boldsymbol{\phi}_{m_i} \,|\, i)$. In particular,

$$p_1 = \sum_i \phi_{m_i,1}, \tag{3.26}$$

and it is easy to verify that

$$W_{\mathbf{m}}(\boldsymbol{\phi}) = W_N(\tilde{\mathbf{p}}(\boldsymbol{\phi})). \tag{3.27}$$

Consider again the polynomial $W_N(\tilde{\mathbf{p}})$. If we assign degrees to the variables $\tilde{\mathbf{p}}$ as $\deg p_j = j$, then W_N is a homogeneous polynomial of degree $N+1$. Therefore each monomial of $W_N(\mathbf{p})$ is proportional to at least one variable $\mathbf{p} = (p_j \,|\, 1 \le j \le (N+1)/2)$ and we can present W_N as a sum of products

$$W_N(\tilde{\mathbf{p}}) = \sum_{1 \le j \le (N+1)/2} p_j\, r_j(\tilde{\mathbf{p}}). \tag{3.28}$$

If we recall that Equation (3.25) turns $\tilde{\mathbf{p}}$ into the polynomials of $\boldsymbol{\phi}$, and if we define the new polynomials $\mathbf{q}(\boldsymbol{\phi})$ as

$$q_j(\boldsymbol{\phi}) = r_j(\tilde{\mathbf{p}}(\boldsymbol{\phi})), \tag{3.29}$$

then, according to Equation (3.27),

$$W_{\mathbf{m}}(\boldsymbol{\phi}) = \sum_{1 \le j \le (N+1)/2} p_j(\boldsymbol{\phi})\, q_j(\boldsymbol{\phi}), \tag{3.30}$$

and the super-potential $W_{\mathbf{m}}$ has a matrix factorization $(\mathbf{p};\mathbf{q})$. Although the presentation of W_N as a sum of products (3.28) is not unique, all these presentations lead to isomorphic Koszul matrix factorizations. We expect them to be the mirror images of the special Lagrangian submanifolds introduced in [4] and [6]: a Lagrangian submanifold of the cross-product of the complex Grassmannians $\mathrm{Gr}_{m_i,N}$ is defined by the condition that the subspaces $\mathbb{C}^{m_i} \subset \mathbb{C}^N$ provide an orthogonal decomposition of $\mathbb{C}^N$.

It is interesting to note a resemblance between these topological LG theories and the representation theory of $\mathrm{SU}(N)$. If V denotes the fundamental N-dimensional representation of $\mathrm{SU}(N)$, then the critical points of a super-potential W_m correspond to the weights of the fundamental representation $\bigwedge^m V$, a matrix factorization $(\mathbf{p};\mathbf{q})$ corresponds to the invariant element in the tensor product $\bigotimes_i \bigwedge^{m_i} V$ and for a local graph γ the dimension of the space H_γ equals the result of the contraction of the Clebsch–Gordan tensors placed at its vertices. This correspondence is at the heart of the categorification construction of [5].

4 Formulas for the correlators

4.1 Correlators on a closed surface

The formula for the correlator of a topological LG theory on a closed surface was derived by Vafa in [7]. Let us define a "Frobenius trace" map $\mathbb{C}[\boldsymbol{\phi}] \xrightarrow{\mathrm{Tr}_W} \mathbb{C}$ by the formula

$$\mathrm{Tr}_W(O) = \frac{1}{(2\pi i)^m} \oint \frac{O(\boldsymbol{\phi})\, d\phi_1 \cdots d\phi_m}{\partial_1 W \cdots \partial_m W}, \tag{4.1}$$

where the variables $\boldsymbol{\phi}$ are integrated over the contours which encircle all critical points of W. Let $\mathcal{P}$ be a finite set of *punctures* (marked points) on a closed surface Σ of genus g. Then, according to [7], the correlator of the operators $O_P \in \mathbb{C}[\boldsymbol{\phi}]$ placed at the punctures $P \in \mathcal{P}$, is

$$\left\langle \prod_{P\in\mathcal{P}} O_P \right\rangle_\Sigma = \mathrm{Tr}_W \left((\det \partial_i\partial_j W)^g \prod_{P\in\mathcal{P}} O_P(\boldsymbol{\phi}) \right). \tag{4.2}$$

This formula indicates that the Frobenius trace (4.1) computes the correlator of the operators on a sphere S^2, while the factors $\det \partial_i\partial_j W$ represent the "effective contributions" of the handles: if we imagine that Σ is a sphere with g tori attached to it by thin tubes, then these tori can be equivalently replaced by the operators $\det \partial_i\partial_j W$ placed at the points where the tubes join the sphere.

An important property of the trace (4.1) is that it annihilates the ideal $\boldsymbol{\partial} W$: for any $O \in \mathbb{C}[\boldsymbol{\phi}]$

$$\mathrm{Tr}_W(\partial_i W\, O) = 0. \tag{4.3}$$

This is consistent with the fact that the space of local operators is the quotient (3.10).

4.2 Correlators on a surface with a boundary

Suppose that a boundary of the world-sheet Σ of genus g is a union of disjoint circles (2.8), each circle C_k is split into n_k segments $e_{k,l}$, a matrix factorization $(M_{k,l}, D_{k,l}, W)$ is assigned to each segment and an operator $O_{k,l}$ is placed at the junction of the segments $e_{k,l}$ and $e_{k,l+1}$. We also place some operators O_P, $P \in \mathcal{P}$ at the internal points of Σ. In order to write a simple expression for the resulting correlator, we have to introduce a general notation. For a matrix factorization (M, D, W), Kapustin and Li [3] define an operator $\partial D^{\wedge} \in \mathrm{End}(M)$ by the formula

$$\partial D^{\wedge} = \frac{1}{m!} \sum_{\sigma \in \mathrm{S}_m} (-1)^{\mathrm{sign}(\sigma)} \partial_{\sigma(1)} D \cdots \partial_{\sigma(m)} D, \tag{4.4}$$

where S_m is the symmetric group of m elements, m being the dimension of the target space of the topological LG theory $(\boldsymbol{\phi}; W)$. Now to a circle C_k, which is a part of the boundary $\partial\Sigma$, we associate an element $O_{C_k} \in \mathbb{C}[\boldsymbol{\phi}]$ defined by the formula

$$O_{C_k} = \mathrm{STr}_{M_1}\, \partial D^{\wedge}_{k,1}\, O_{k,1} \cdots O_{k,n_k}. \tag{4.5}$$

Note the similarity between the expressions (4.5) and (2.14): the former is obtained from the latter by replacing all holonomies with the identity operators except the first one, which is replaced by $\partial D^{\wedge}_{k,1}$.

Kapustin and Li [3] derived the formula for the correlator of the boundary and bulk operators:

$$\left\langle \prod_{P \in \mathcal{P}} O_P \prod_{k,l} O_{k,l} \right\rangle_{\Sigma} = \mathrm{Tr}_W \left((\det \partial_i \partial_j W)^g \prod_{P \in \mathcal{P}} O_P(\boldsymbol{\phi}) \prod_k O_{C_k} \right). \tag{4.6}$$

By comparing this formula with Equation (4.2), we see that the factors O_{C_k} represent the boundary state operators corresponding to the boundary components C_k: the correlator does not change if C_k is contracted to a point and the operator O_{C_k} is placed at that point.

Following [3], let us verify directly (without using the path integral arguments) that the correlator formula (4.6) satisfies two properties related to the BRST invariance. First of all, if one of the boundary operators $O_{k,l}$ is BRST-exact ($O_{k,l} = \left[D, O'_{k,l}\right]_{\mathrm{s}}$), then the correlator (4.6) is zero, since we can move the operator D around the super-trace expression (4.5): all the operators O commute with D, while $\{\partial_j D, D\} = \partial_j W$ and a term proportional to a derivative $\partial_j W$ is annihilated by the Frobenius trace.

Second, we could insert the operator $\partial D^{\wedge}$ at any place in the product of the operators $O_{k,1} \cdots O_{k,n_k}$ in Equation (4.5): the r.h.s. of Equation (4.6) would not change. Indeed, if for some value of l

$$O_{k,l} = \left[\partial_j D, O'_{k,l}\right]_{\mathrm{s}}, \text{ while } \left[D, O'_{k,l}\right]_{\mathrm{s}} = 0, \tag{4.7}$$

then the r.h.s. of Equation (4.6) is zero (take the derivative ∂_j of the second Equation of (4.7) and use the already established fact that D-commutators annihilate the r.h.s. of Equation (4.6)).

4.3 Correlators on a world-sheet foam

Now we consider a correlator on a world-sheet foam (Σ, Γ). First, we present a formula and then comment on its derivation.

We define an operator $O_\Gamma \in \mathbb{C}[\boldsymbol{\phi}_\Sigma]$ which is the analog of O_{C_k} from Equation (4.5) and which represents the boundary state operator contribution of the Wilson network. For each connected component $C_{i,j}$ of the boundary $\partial\Sigma_i$, we choose a seam edge among the edges to which $C_{i,j}$ is glued, and we denote that edge as $e_{i,j}$. Then we introduce the operators (4.4)

$$\partial D^{\wedge}_{i,j} \in \mathrm{End}(M_{e_{i,j}}), \quad \partial D^{\wedge}_{i,j} = \sum_{\sigma \in \mathrm{S}_{m_i}} (-1)^{\mathrm{sign}(\sigma)} \partial_{\phi_{i,\sigma(1)}} D_{e_{i,j}} \cdots \partial_{\phi_{i,\sigma(m)}} D_{e_{i,j}}, \tag{4.8}$$

where $\phi_{i,1}, \ldots, \phi_{i,m_i}$ are the bosonic fields of the topological LG theory $(\boldsymbol{\phi}_i; W_i)$ assigned to the connected component Σ_i of Σ.

To every seam edge e we assign an operator

$$O_e = \prod_{(i,j)\colon e = e_{i,j}} \partial D^{\wedge}_{i,j} \tag{4.9}$$

(if $e = e_{i,j}$ for more than one combination (i, j), then we choose any order of the operators in the product (4.9); if e never appears as i, j, then the corresponding operator O_e is the identity). Then, similar to Equation (3.9), we define

$$O_\Gamma = f_\Gamma\left(\Big(\bigotimes_{v\in\mathcal{V}} O_v\Big)\otimes\Big(\bigotimes_{e\in\mathcal{E}} O_e\Big)\right) \tag{4.10}$$

and the correlator (3.15) is expressed as

$$\left\langle\prod_{P\in\mathcal{P}} O_P \prod_{v\in\mathcal{V}} O_v\right\rangle_{(\Sigma,\Gamma)} = \mathrm{Tr}_\Sigma\left(O_\Gamma\prod_{P\in\mathcal{P}} O_P\prod_{i=1}^{N_\Sigma}(\det\partial_j\partial_k W_i)^{g(\Sigma_i)}\right), \tag{4.11}$$

where

$$\mathrm{Tr}_\Sigma\colon \mathbb{C}[\boldsymbol{\phi}_\Sigma]\longrightarrow\mathbb{C},\qquad \mathrm{Tr}_\Sigma=\mathrm{Tr}_{W_1}\cdots\mathrm{Tr}_{W_{N_\Sigma}} \tag{4.12}$$

(cf. Equation (4.6)).

Now let us comment on the derivation of this correlator formula. Note that the original Vafa's formula (4.2) was derived in the assumption that the critical points of the super-potential W are non-degenerate. Then the BRST-invariance of the theory guarantees that the correlator is a sum of the contributions of the individual critical points and at each point the super-potential W can be replaced by its quadratic part.

Kapustin and Li derived their formula under the same assumption, although it is harder to justify in their case: W can be perturbed in order to make its critical points non-degenerate, but it is not clear, whether its matrix factorizations can be deformed together with it.

Kapustin and Li derived the correlator for the disk world-sheet. This is sufficient in order to establish the boundary state operator corresponding to the boundary. Then their general formula would follow from Vafa's formula (4.2). We use the same approach. Namely, it would be sufficient to derive Equation (4.11) under the assumption that the surface Σ is a union of disjoint disks. Then the computation of the path integral (3.15) that leads to Equation (4.11) is exactly the same as in [3]. The only minor novelty is that it may happen that a seam edge e is assigned to two (or more) different disks Σ_i and Σ_j $(i\neq j)$. Then it bears their operators $\partial D_i^\wedge$ and $\partial D_j^\wedge$ (we left only one index in their notation, since Σ_i and Σ_j have only one boundary component). The path integration over the fermionic fields leaves the derivatives $\partial_{\phi_{i,k}} D_e$ $(1\leq k\leq m_i)$ and $\partial_{\phi_{j,l}} D_e$ $(1\leq l\leq m_j)$, which enter in expressions (4.8), intermixed. However, they can still be pulled apart into the operators $\partial D_i^\wedge$ and $\partial D_j^\wedge$, because $\partial_{\phi_{i,k}} D_e$ and $\partial_{\phi_{j,l}} D_e$ anticommute up to a BRST-closed operator. Indeed, since the super-potential W_e is a sum (3.1) of the individual super-potentials, each depending on its own set of fields, then $\partial_{\phi_{i,k}}\partial_{\phi_{j,l}}W_e = 0$. Hence, if we apply $\partial_{\phi_{i,k}}\partial_{\phi_{j,l}}$ to both sides of the relation $D_e^2 = W_e$, then we find that

$$\{\partial_{\phi_{i,k}}D_e, \partial_{\phi_{j,l}}D_e\} = -\{D_e, \partial_{\phi_{i,k}}\partial_{\phi_{j,l}}D_e\}. \tag{4.13}$$

5 Gluing formulas

5.1 Gluing of a 2-dimensional world-sheet

The gluing property of the correlators is an important feature of general QFTs and of topological theories, in particular. Let us quickly review the gluing rules of a 2-dimensional topological QFT.

For a point $P \in \Sigma$, let γ_P denote the intersection between Σ and a small sphere centered at P. We will call γ_P the *local space section* of P. In the context of a string theory, γ_P is simply called a string. If P is an internal point of Σ, then γ_P is a circle (closed string), and if P is a point at the boundary $\partial\Sigma$, then γ_P is a segment (half-circle, or open string). The endpoints of the segment come from the intersection of the small sphere and the boundary $\partial\Sigma$, so they are "decorated" with the TQFT boundary conditions at $\partial\Sigma$. If P is a point at the junction of two different boundary conditions, then the decorations at its endpoints are also different. The segment is oriented, its orientation being induced by the orientation of Σ. For a decorated local space section γ, we define its dual γ^* to be the same as γ but with the opposite orientation. Obviously, a circle is self-dual.

The space of the TQFT states corresponding to γ_P coincides with the space H_P of the local operators that can be inserted at P.

Suppose that for two points $P_1, P_2 \in \Sigma$, their decorated local space section are dual to each other: $\gamma_1^* = \gamma_2$. Then the spaces H_1 and H_2 are also dual. In order to define the duality pairing between them, we consider the world-sheet $S_{(1,2)} = U_1 \# U_2$ constructed by gluing together the small neighborhoods U_1 and U_2 of P_1 and P_2 over the boundaries $\gamma_1 \sim \gamma_2$ identified with opposite orientations. If γ_1 is a circle, then $S_{(1,2)}$ is a 2-sphere, and if γ_1 is a segment, then $S_{(1,2)}$ is a disk. The pairing is defined by the correlator on $S_{(1,2)}$:

$$(O_1, O_2) = \langle O_1\, O_2 \rangle_{S_{(1,2)}}, \quad O_1 \in H_1, \quad O_2 \in H_2. \tag{5.1}$$

As a result, there is a canonical dual element

$$I_{1,2} \in H_1^* \otimes H_2^*, \tag{5.2}$$

defined by the relation

$$(I_{1,2}, O_1 \otimes O_2) = (O_1, O_2). \tag{5.3}$$

We will need the inverse element

$$I^{-1}_{1,2} \in H_1 \otimes H_2. \tag{5.4}$$

Let us cut the small neighborhoods U_1 and U_2 from the world-sheet Σ and glue (that is, identify) the boundaries γ_1 and γ_2 of the cuts in such a way that their orientations are opposite. Denote the resulting oriented manifold as Σ', then according to the gluing property of a TQFT,

$$\left\langle \prod_{P\in\mathcal{P}} O_P \right\rangle_{\Sigma'} = \left\langle I^{-1}_{1,2} \prod_{P\in\mathcal{P}} O_P \right\rangle_{\Sigma}. \tag{5.5}$$

A more "pedestrian" way to formulate the same property is to introduce a basis of operators $O_j \in H_1$ and a dual basis $O^*_j \in H_2$ so that $(O_j, O^*_{j'}) = \delta_{j,j'}$. Then

$$\left\langle \prod_{P\in\mathcal{P}} O_P \right\rangle_{\Sigma'} = \sum_j \left\langle O_j\, O^*_j \prod_{P\in\mathcal{P}} O_P \right\rangle_{\Sigma}. \tag{5.6}$$

5.2 Complete space gluing

Let us check how the general gluing formula (5.5) works for a topological LG theory on a world-sheet foam.

Let P be a point on a world-sheet foam (Σ, Γ). We call its local space section a *local graph* and denote it as γ_P. If P is an internal point of Σ, then γ_P is a circle. If P is an internal point of a seam edge e, then γ_P is a graph with two vertices connected by multiple edges, each edge corresponding to a strip of a component Σ_i attached to e. If P is a seam vertex v, then γ_P is a graph, whose vertices correspond to the seam edges adjacent to v and whose edges correspond to the strips of the components Σ_i, which pass through v. In fact, the vertex-edge and edge-surface correspondence between γ_P and (Σ, Γ) works for all three types of points P. The orientation of the components Σ_i induces the orientation of the corresponding edges of γ_P.

A small neighborhood U_P of P in (Σ, Γ) can be restored from its local graph γ_P, because U_P is the cone of γ_P:

$$U_P = \mathrm{C}\gamma_P. \tag{5.7}$$

Generally speaking, a local graph γ is just a graph. A *decorated* local graph means the following. To an oriented edge ϵ of γ, we assign a topological LG theory $(\boldsymbol{\phi}_\epsilon; W_\epsilon)$ in such a way that if ϵ and ϵ^* represent the same edge with opposite orientations, then they are assigned the conjugated theories. To a vertex ν of γ we associate a matrix factorization (M_ν, D_ν, W_ν), such that

$$W_\nu = \sum_{\epsilon \in \Upsilon_\nu} W_\epsilon, \tag{5.8}$$

where Υ_ν is the set of edges of γ, which are attached to ν (we assume that they are oriented away from ν).

For a decorated local graph γ consider the matrix factorization $(M_\gamma, D_\gamma, W_\gamma)$, which is the tensor product of all the matrix factorizations of its vertices

$$(M_\gamma, D_\gamma, W_\gamma) = \bigotimes_\nu (M_\nu, D_\nu, W_\nu)\,. \tag{5.9}$$

Obviously

$$W_\gamma = 0, \tag{5.10}$$

so $D_\gamma^2 = 0$ and we denote its cohomology as

$$H_\gamma = \frac{\ker\, D_\gamma}{\mathrm{im}\, D_\gamma}. \tag{5.11}$$

If a world-sheet foam (Σ, Γ) is decorated, then for any $P \in (\Sigma, \Gamma)$ its local graph γ_P is also decorated: a topological LG theory of an edge ϵ is the theory of the corresponding component Σ_i and a matrix factorization of a vertex ν is the matrix factorization of the corresponding seam edge, if it is oriented away from P, or the conjugated matrix factorization otherwise. Then it is easy to see that

$$H_{\gamma_P} = H_P, \tag{5.12}$$

that is, the space of operators at a point P is determined by its decorated local graph γ_P.

For a decorated local graph γ, we define its dual graph γ^* to be the same graph as γ, except that it is decorated with the conjugate topological LG theories and with the dual matrix factorizations.

Consider a suspension $\Sigma\gamma$ of a local graph γ: by definition, it is constructed by gluing together the cones $\mathrm{C}\gamma$ and $\mathrm{C}\gamma^*$ along their common boundary γ. $\Sigma\gamma$ has a structure of a world-sheet foam: its seam graph consists of two vertices v_1 and v_2, which are the vertices of the cones, the suspensions of the vertices of γ form the edges that connect v_1 and v_2, and the 2-dimensional components Σ_i are the suspensions of the edges of γ. Obviously, $\gamma_{v_1} = \gamma$ and $\gamma_{v_2} = \gamma^*$.

If a local graph γ is decorated, then a topological LG theory is defined on its suspension $\Sigma\gamma$. The correlators of this theory provide the paring between the spaces H_γ and H_{γ^*}

$$(O, O') = \langle O\, O' \rangle_{\Sigma\gamma}\,. \tag{5.13}$$

This pairing determines an inverse canonical element $I^{-1}_{\gamma,\gamma^*} \in H_\gamma \otimes H_{\gamma^*}$.

Suppose that for two points $P_1, P_2 \in (\Sigma, \Gamma)$ their local graphs are dual: $\gamma_1^* = \gamma_2$. Then we can cut their small neighborhoods from the world-sheet foam (Σ, Γ) and glue the cut borders together, thus forming a new world-sheet foam (Σ', Γ'). If a topological LG theory is defined on (Σ', Γ'), then it induces a topological LG theory on (Σ, Γ) and the correlators of both theories are related by the gluing formula

$$\left\langle \prod_{P \in \mathcal{P}'} O_P \prod_{v \in \mathcal{V}'} O_v \right\rangle_{(\Sigma',\Gamma')} = \left\langle I^{-1}_{1,2} \prod_{P \in \mathcal{P}'} O_P \prod_{v \in \mathcal{V}'} O_v \right\rangle_{(\Sigma,\Gamma)}, \tag{5.14}$$

where $\mathcal{P}'$ and $\mathcal{V}'$ are the punctures and seam vertices of (Σ', Γ'). Thus the gluing property of a topological LG theory on a world-sheet foam is very similar to the gluing property (5.5) on a usual world-sheet.

5.3 A topological LG theory on a world-sheet foam as a 2-category

The graph structure of world-sheet foam local space sections permits more complicated types of gluing than those described in the previous section, when two dual local space sections are glued together. These new types of gluing can be arranged into the mathematical structure known as a 2-category.

The usual 1-category structure of a TQFT on a 2-dimensional world-sheet comes from the composition property of transition amplitudes. Consider a world-sheet Σ with a finite puncture set $\mathcal{P}$ and two special punctures P_1 and P_2 with local graphs γ_1, γ_2. If we choose the operators at the punctures of $\mathcal{P}$, then the correlator on Σ defines a transition amplitude

$$H_{\gamma_1} \xrightarrow{A[\gamma_1,\gamma_2^*]} H_{\gamma_2^*} \tag{5.15}$$

by the formula

$$I_{\gamma_2^*,\gamma_2}(A[\gamma_1,\gamma_2^*](O_1),O_2) \;=\; \left\langle O_1 O_2 \prod_{P\in\mathcal{P}} O_P \right\rangle_\Sigma \quad \text{for any}$$
$$O_1 \;\in\; H_{\gamma_1}, \quad O_2 \in H_{\gamma_2}. \tag{5.16}$$

Let Σ_{12} denote the result of cutting small neighborhoods of P_1 and P_2 from Σ_{12}. If we have another surface Σ' with special punctures P_3 and P_4 such that $\gamma_2^* = \gamma_3$, then we can glue the boundary components γ_2 and γ_3 of Σ_{12} and Σ'_{34} together to form a new world-sheet with boundary Σ''_{14}. The transition amplitude of Σ'' is given by the composition of transition amplitudes

$$A[\gamma_1,\gamma_4^*] = A[\gamma_3,\gamma_4^*]\,A[\gamma_1,\gamma_2^*], \tag{5.17}$$

and this formula corresponds to the gluing formula (5.5) formulated for P_2 and P_3.

The composition property (5.17) extends verbatim to the world-sheet foams. In the foam case, however, there is an important generalization. Namely, the surface or the world-sheet foam Σ_{12} presented a cobordism between two closed space section (be it 1-manifolds or graphs). Now we are going to consider cobordisms between the spaces that have boundaries.

Let us take a local graph γ and make cuts across some of its edges, so that γ splits into two disconnected *partial local graphs* α_1 and α_2: $\gamma = \alpha_1 \# \alpha_2$. The partial local graphs have special univalent vertices at the cuts: we call them *boundary vertices*, and their adjacent edges are called *legs*. We think of the boundary vertices as the boundary of a partial local graph.

The partial local graphs inherit the decorations of γ, except that their boundary vertices are not assigned matrix factorizations. To a decorated partial local graph α, we associate a matrix factorization $(M_\alpha, D_\alpha, W_\alpha)$ which is the tensor product of all the matrix factorizations of its non-boundary vertices. However this time instead of Equation (5.10), we have

$$W_\alpha = \sum_{\epsilon\in\mathcal{L}_\alpha} W_\epsilon, \tag{5.18}$$

where $\mathcal{L}_\alpha$ is the set of legs of α, and we assume that the legs are oriented away from the boundary vertices.

Let $\boldsymbol{\phi}_\alpha = (\boldsymbol{\phi}_\epsilon \,|\, \epsilon \in \mathcal{L}_\alpha)$ be the list of all variables of the legs of a partial local graph α, and let $R_\alpha = \mathbb{C}[\boldsymbol{\phi}_\alpha]$ be their polynomial ring. Obviously,

$$R_\alpha = \bigotimes_{\epsilon \in \mathcal{L}_\alpha} \mathbb{C}[\boldsymbol{\phi}_\epsilon]. \tag{5.19}$$

Since the super-potential W_α of Equation (5.18) depends only on the "external" variables $\boldsymbol{\phi}_\alpha$, from now on we will consider $(M_\alpha, D_\alpha, W_\alpha)$ to be a matrix factorization over the ring R_α. However this poses a problem: if we ignore the "internal" variables of α, then the module M_α is infinite-dimensional as a module over R_α. Indeed, the multiplication by the powers of an internal variable now produces an infinite sequence of linearly independent elements of M_α. In order to resolve this problem, we can contract M_α homotopically to a finite-dimensional R_α-module. Here is the relevant definition: two matrix factorizations (M_i, D_i, W), $i = 1, 2$ are considered *homotopically equivalent* over the polynomial ring $R \ni W$, if there exist two R-linear maps f_{12}, f_{21}

$$M_1 \xrightarrow{f_{12}} M_2 \xrightarrow{f_{21}} M_1 \tag{5.20}$$

commuting with the twisted differential D, such that the compositions $f_{21} f_{12} \in \mathrm{End}_R(M_1)$ and $f_{12} f_{21} \in \mathrm{End}_R(M_2)$ are BRST-equivalent to the identity maps. We showed in [5] that under some mild assumptions an infinite rank matrix factorization is homotopically equivalent to a finite rank one.

If a decorated local graph is split: $\gamma = \alpha_1 \# \alpha_2$, then according to Equation (5.9)

$$M_\gamma = M_{\alpha_1} \otimes_R M_{\alpha_2}, \quad R = R_{\alpha_1} = R_{\alpha_2}, \tag{5.21}$$

and we used the notation $\otimes_R$ in order to emphasize that the tensor product is taken over the polynomial ring of all leg variables of α_1 (or, equivalently, α_2). If we replace the R-modules M_{α_1} and M_{α_2} in Equation (5.21) by their finite-dimensional homotopic equivalents, then the module M_γ will change, but its D cohomology H_γ will stay the same. Therefore we will use the same notation M_α for the whole homotopy equivalence class of R_α-modules related to a decorated partial local graph α.

Let (Σ, Γ) be a decorated world-sheet foam with a puncture set $\mathcal{P}$. Let us pick a vertex $v \in \mathcal{V}$ with a local graph γ and denote $\mathcal{V}' = \mathcal{V} \setminus \{v\}$. A choice of the operators at the punctures of $\mathcal{P}$ and at the vertices of $\mathcal{V}'$ determines an element $A[\gamma] \in H_{\gamma^*}$ by the formula

$$I_{\gamma,\gamma^*}(O, A[\gamma]) = \left\langle O \prod_{P \in \mathcal{P}} O_P \prod_{v' \in \mathcal{V}'} O_{v'} \right\rangle_{(\Sigma,\Gamma)} \qquad \text{for any} \quad O \in H_\gamma. \tag{5.22}$$

Suppose that γ splits: $\gamma = \alpha_1 \# \alpha_2$. Then, in view of Equation (5.21),

$$M_{\gamma^*} = M_{\alpha_1^*} \otimes_R M_{\alpha_2^*} = \mathrm{Hom}_R(M_{\alpha_1}, M_{\alpha_2^*}), \tag{5.23}$$

and hence

$$H_{\gamma^*} = \mathrm{Ext}(M_{\alpha_1}, M_{\alpha_2^*}) \tag{5.24}$$

(cf. definition (2.16)). The latter isomorphism allows us to translate the element $A[\gamma]$ of Equation (5.22) into a transition amplitude $A[\alpha_1, \alpha_2^*] \in \mathrm{Ext}(M_{\alpha_1}, M_{\alpha_2^*})$. This transition amplitude is the analog of the amplitude (5.15): $A[\gamma_1, \gamma_2^*]$ describes the transition between two closed space sections, while $A[\alpha_1, \alpha_2^*]$ describes the transition between two space sections with boundary.

The distinction between the open and closed-space transitions has a topological and an algebraic manifestation. Topologically, if we cut small neighborhoods of the punctures P_1 and P_2, then the remainder $(\Sigma, \Gamma)_{12}$ has two disconnected boundary components: the "in" space γ_1 and the "out" space γ_2. The foam topology is more complicated. If we cut out a small neighborhood of the seam vertex v, then the boundary of the remainder $(\Sigma, \Gamma)_v$ is obviously the local graph γ. If we cut γ just into α_1 and α_2, then these partial local graphs would have common points. However, since we think of α_1 and α_2 as space sections corresponding to different values of "time", we would like them to be completely separated. Therefore, rather than slicing the edges that connect α_1 and α_2, we cut out finite length segments from them. These segments form the *time-like section*. Thus the boundary of $(\Sigma, \Gamma)_v$ consists of three, rather than two, pieces: two "space-like" ones (the "in" space α_1 and the "out" space α_2^*) as well as the time-like section. The algebraic consequence of the presence of a time-like section in the boundary of $(\Sigma, \Gamma)_v$ is that the "in" and "out" spaces of states are not just linear spaces over $\mathbb{C}$, but rather R-modules, and the transition amplitude $A[\alpha_1, \alpha_2^*]$ is R-linear.

Since the world-sheet foam of the open space transition $(\Sigma, \Gamma)_v$ has two types of boundary, the corresponding transition amplitude $A[\alpha_1, \alpha_2^*]$ satisfies two gluing relations. First of all, there is the gluing associated to the composition of transitions, which is similar to Equation (5.17). Consider two world-sheet foams (Σ_j, Γ_j) $(j = 1, 2)$ with marked seam vertices v_j. Suppose that their local graphs γ_j can be split $\gamma_j = \alpha_j \# \alpha'_j$ in such a way that $\alpha_2^* = \alpha'_1$. Then we can glue $(\Sigma_1, \Gamma_1)_{v_1}$ and $(\Sigma_2, \Gamma_2)_{v_2}$ along these matching partial local graphs. Its resulting transition amplitude $A[\alpha_1, \alpha_2^*]$ should be the composition of the elementary ones:

$$A[\alpha_1, \alpha_2^*] = A[\alpha_2, (\alpha'_2)^*]\, A[\alpha_1, (\alpha'_1)^*]. \tag{5.25}$$

One can also glue the world-sheet foams along the time-like sections. Let us describe the corresponding cutting of a world-sheet foam(Σ,Γ). Suppose that it has two marked vertices v_j $(j=1,2)$, whose local graphs γ_j are split: $\gamma_j=\alpha'_j\#\alpha''_j$, and both γ_1 and γ_2 have m slice points. Let $p_{j,k}$ $(1\le k\le m)$ denote the points at which the legs of α'_j and α''_j are joined. Suppose that for all k, the points $p_{1,k}$ and $p_{2,k}$ belong to the same connected component $\Sigma_{i(k)}$ of Σ, and we can choose non-intersecting curves c_k which lie on $\Sigma_{i(k)}$ and join the points $p_{1,k}$ and $p_{2,k}$. Finally, suppose that if we make the cuts along all curves c_k, then $(\Sigma,\Gamma)_{v_1,v_2}$ splits into two disconnected pieces $(\Sigma,\Gamma)'_{v_1,v_2}$, $(\Sigma,\Gamma)''_{v_1,v_2}$ in such a way that $(\Sigma,\Gamma)'_{v_1,v_2}$ is bound by α'_1, α'_2 and the curves c_k, while $(\Sigma,\Gamma)''_{v_1,v_2}$ is bound by α''_1, α''_2 and the curves c_k. Then $(\Sigma,\Gamma)'_{v_1,v_2}$ and $(\Sigma,\Gamma)''_{v_1,v_2}$ produce their own transition amplitudes $A[\alpha'_1,(\alpha'_2)^*]$ and $A[\alpha''_1,(\alpha''_2)^*]$. Their tensor product

$$M_{\alpha'_1}\otimes_R M_{\alpha''_1}\xrightarrow{A[\alpha'_1,(\alpha'_2)^*]\otimes_R A[\alpha''_1,(\alpha''_2)^*]} M_{(\alpha'_2)^*}\otimes_R M_{(\alpha''_2)^*} \tag{5.26}$$

commutes with the twisted differential D and therefore, in view of (5.21), it defines a map from H_{γ_1} to $H_{\gamma_2^*}$. Thus the gluing of two world-sheet foams $(\Sigma,\Gamma)'_{v_1,v_2}$ and $(\Sigma,\Gamma)''_{v_1,v_2}$ along their time-like sections produces the formula

$$A[\gamma_1,\gamma_2^*]=A[\alpha'_1,(\alpha'_2)^*]\otimes_R A[\alpha''_1,(\alpha''_2)^*], \tag{5.27}$$

relating two open-space transition amplitudes to one closed-space transition amplitude. In this gluing formula, the usual compositions (5.17) and (5.25) are replaced by the tensor product over an appropriate ring.

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