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On the nonlinear stability of higher dimensional triaxial Bianchi-IX black holes

Mihalis Dafermos[1] and Gustav Holzegel[2]

[1]Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, U.K.
m.dafermos@dpmms.cam.ac.uk

[2]Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, U.K.
g.holzegel@damtp.cam.ac.uk

Abstract

In this paper, we prove that the five-dimensional Schwarzschild–Tangherlini solution of the Einstein vacuum equations is orbitally stable (in the fully non-linear theory) with respect to vacuum perturbations of initial data preserving triaxial Bianchi-IX symmetry. More generally, we prove that five-dimensional vacuum spacetimes developing from suitable asymptotically flat triaxial Bianchi-IX symmetric initial data and containing a trapped or marginally trapped homogeneous 3-surface necessarily possess a complete null infinity $\mathcal{I}^+$, whose past $J^-(\mathcal{I}^+)$ is bounded to the future by a regular event horizon $\mathcal{H}^+$, whose cross-sectional volume in turn satisfies a Penrose inequality, relating it to the final Bondi mass. In particular, the results of this paper give the first examples of vacuum black holes which are not stationary exact solutions.

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1 Introduction

The study of higher dimensional gravity has attracted much attention in recent years, motivated mainly by speculations from high-energy physics. The variety of possible end-states for vacuum gravitational collapse in higher dimensions appears richer [10] than in four dimensions and gives rise to many interesting questions. All analytical work, thus far, however, has centred on the question of the existence and uniqueness of static [11] or stationary [18, 14] solutions or has been based on study of the linearized equations [13, 15]. While such results are suggestive as to what may occur dynamically, they do not directly address the problem of evolution and leave open the possibility that the non-linear theory admits phenomena of a completely different and unexpected nature.

The purpose of this paper is to initiate the rigorous study of dynamical vacuum black holes in higher dimensions in the *fully non-linear* theory. Specifically, we will study the problem of evolution for the Einstein vacuum equations

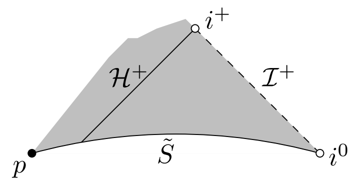
$$R_{\mu\nu} = 0, \tag{1.1}$$

for asymptotically flat initial data possessing triaxial Bianchi-IX symmetry. This model has been recently introduced by Bizon *et al.* [2]. They show that vacuum solutions with this symmetry have two dynamic degrees of freedom, and the Einstein equations can be written (see [2]) as a system of non-linear pde's on a two-dimensional Lorentzian quotient of five-dimensional spacetime by an $SU(2)$ action with three-dimensional orbits.

The system of equations thus obtained is studied numerically in [2], where analogues of critical behaviour have been discovered. Proving rigorously the kind of behaviour suggested by these numerics appears a formidable problem, beyond the scope of current techniques. Implicit in the discussion of [2], however, is the notion that there is an open set of initial data that leads to black hole formation. It is this aspect of [2] that we will formulate and rigorously prove in this paper.

The main result is

Theorem 1.1. *Consider asymptotically flat smooth initial data* $(\mathcal{S}, \bar{g}, K)$ *for the vacuum Einstein equations, possessing triaxial Bianchi-IX symmetry. Let* $(\mathcal{M}, g)$ *denote the maximal Cauchy development, and let* $\pi : \mathcal{M} \to \mathcal{Q}$ *denote the projection map to the two-dimensional Lorentzian quotient* $\mathcal{Q}$. *Suppose there exists an asymptotically flat spacelike Cauchy surface* $\tilde{S} \subset \mathcal{Q}$, *and a point* $p \in \tilde{S}$ *such that* $\pi^{-1}(p)$ *is trapped or marginally trapped, and (at least) one of the connected components* $\tilde{S} \setminus \{p\}$ *contains an asymptotically* *flat end such that $\pi^{-1}(q)$ is not outer antitrapped or marginally antitrapped for any q in the component. Then $\mathcal{Q}$ contains a subset with Penrose diagram:*



Moreover, the null infinity $\mathcal{I}^+$ corresponding to the above end is complete, and the Penrose inequality

$$r \leq \sqrt{2M_{\mathrm{f}}}$$

holds on $\mathcal{H}^+$, where r denotes the volume-radius function and where M_{f} denotes the final Bondi mass.

Note that one can construct a large family of initial data such that the assumptions of the theorem are satisfied with $\tilde{S} = \pi(\mathcal{S})$, for instance.

The region $J^-(\mathcal{I}^+)$ depicted above is what is typically called a black hole *exterior*, the region $\mathcal{Q} \setminus J^-(\mathcal{I}^+)$ is called the *black hole*, and $\mathcal{H}^+$ is the *event horizon*. Thus, the statement of the theorem can be paraphrased by

Asymptotically flat triaxial Bianchi-IX symmetric spacetimes evolving from suitable data, with an $SU(2)$-invariant trapped or marginally trapped 3-surface, possess a black hole with a regular event horizon (satisfying a Penrose inequality) and a complete null infinity.

Theorem 1.1 can in fact be specialized to yield

Corollary 1.1. *Let $(\mathcal{S}, \bar{g}, K)$ denote initial data evolving to the Schwarzschild–Tangherlini metric. Then for smooth triaxial Bianchi-IX symmetric initial data $(\mathcal{S}', \bar{g}', K')$, sufficiently close to $(\mathcal{S}, \bar{g}, K)$ in a suitable norm, the result of the previous theorem holds for the maximal development $(\mathcal{M}', g')$, and moreover, the black hole exterior of $(\mathcal{M}', g')$ remains close in a suitable sense to Schwarzschild–Tangherlini.*

Corollary 1.1 can be paraphrased by the statement:

Schwarzschild–Tangherlini is orbitally stable within the class of triaxial Bianchi-IX symmetric spacetimes.

The results of this paper can be thought to complement previous results of Gibbons and Hartnoll [13] suggesting linear stability[1] and also to the

[1]See also Ishibashi and Kodama [15].

uniqueness of Schwarzschild–Tangherlini as a static black hole vacuum spacetime [11]. Finally, we note that Theorem 1.1 gives, in particular, the first examples of vacuum black holes which are not static or stationary exact solutions.[2]

An outline of the paper is given as follows: In Section 2, the vacuum five-dimensional Einstein equations under triaxial Bianchi-IX symmetry are written as a system of pde's on a quotient two-dimensional Lorentzian manifold, where the latter is endowed with a null coordinate system. Triaxial Bianchi-IX symmetric initial data give rise to a triaxial Bianchi-IX symmetric maximal development, and this is discussed in Sections 3 and 4. Fundamental for understanding the global behaviour of this maximal development is the existence of a quantity m with good monotonicity properties in null directions, analogous to the classical Hawking mass. This quantity was first identified in [2]. This is discussed in Section 5. Borrowing from ideas of [3, 7], one can exploit the structure provided by m to estimate all quantities in the regular region (defined in Section 6) and thus derive an extension criterion which prohibits "boundary" points of the maximal development in the "closure" of the regular region, unless these lie on a future directed null ray emanating from the centre. This is the content of Section 7. The words boundary and closure can be interpreted in the topology of the Penrose diagrams. The completeness of null infinity is then shown in Sections 8 and 9, adapting ideas of [6] to higher dimensions. This completes the proof of Theorem 1.1. Corollary 1.1 is shown in Section 10, and final comments are given in Section 11.

2 Triaxial Bianchi-IX

We will say that a globally hyperbolic spacetime $(\mathcal{M}, g)$ admits triaxial Bianchi-IX symmetry if $\mathcal{M} = \mathcal{Q} \times SU(2)$ topologically, for $\mathcal{Q}$ a two-dimensional manifold possibly with boundary on which global coordinates u and v can be chosen such that

$$g = -\Omega^2 \, du \, dv + \frac{1}{4} r^2 (\mathrm{e}^{2B} \sigma_1^2 + \mathrm{e}^{2C} \sigma_2^2 + \mathrm{e}^{-2(B+C)} \sigma_3^2) \tag{2.1}$$

where B, C, Ω, and r are functions $\mathcal{Q} \to \mathbb{R}$ and the σ_i are a standard basis of left invariant one-forms on $SU(2)$, i.e., the σ_i are such that coordinates

[2]Such solutions are yet to be constructed in $3+1$-dimensions, as, in view of Birkhoff's theorem, it is impossible to reduce the problem to a $1+1$-dimensional system of pde's. Solutions with a future complete, but not past complete, $\mathcal{I}^+$ have been constructed, however, by Chruściel [5], by solving a certain parabolic problem.

(θ, ϕ, ψ) can be chosen on $SU(2)$ with

$$\begin{aligned}\sigma_1 &= \sin\theta \sin\psi \, d\phi + \cos\psi \, d\theta,\\ \sigma_2 &= \sin\theta \cos\psi \, d\phi - \sin\psi \, d\theta,\\ \sigma_3 &= \cos\theta \, d\phi + d\psi.\end{aligned} \tag{2.2}$$

If there is a boundary Γ to $\mathcal{Q}$, it is to be a timelike curve, characterized by $r = 0$.

From the above, it is clear that the metric (2.1) admits an $SU(2)$ action by isometry. The boundary Γ corresponds to fixed points of the group action. We call it the *centre*. The angular part of the metric can be understood as a "squashed" 3-sphere. In the case that $B = C$, the so-called *biaxial* case, the system enjoys an additional $U(1)$ symmetry. If $B = C = 0$, we have $SO(4)$ symmetry, and the unique solution to the Einstein vacuum equations is five-dimensional Schwarzschild, which we will here refer to as the Schwarzschild–Tangherlini solution.

From the Einstein equations (1.1), we derive the following equations:

$$\partial_u\left(\Omega^{-2}\partial_u r\right) = -\frac{2r}{3\Omega^2}\left((B_{,u})^2 + B_{,u}C_{,u} + (C_{,u})^2\right), \tag{2.3}$$

$$\partial_v\left(\Omega^{-2}\partial_v r\right) = -\frac{2r}{3\Omega^2}\left((B_{,v})^2 + B_{,v}C_{,v} + (C_{,v})^2\right), \tag{2.4}$$

$$-2\partial_u\partial_v \log\Omega - \frac{3}{r}r_{,uv} = B_{,v}\left(2B_{,u} + C_{,u}\right) + C_{,v}\left(2C_{,u} + B_{,u}\right), \tag{2.5}$$

$$\begin{aligned}\partial_u\partial_v \log\Omega + \frac{3}{r}r_{,uv} + 3\frac{r_{,u}r_{,v}}{r^2} = &-\frac{\Omega^2\rho}{2r^2} - \frac{1}{2}(B_{,v}(2B_{,u} + C_{,u})\\ &+ C_{,v}(2C_{,u} + B_{,u})),\end{aligned} \tag{2.6}$$

where ρ denotes the scalar curvature of the group orbit:

$$\rho = \mathrm{e}^{2B+2C} + \mathrm{e}^{-2B} + \mathrm{e}^{-2C} - \frac{1}{2}\mathrm{e}^{-(4B+4C)} - \frac{1}{2}\mathrm{e}^{4B} - \frac{1}{2}\mathrm{e}^{4C}. \tag{2.7}$$

From these equations, we can derive a system of nonlinear wave equations for the four quantities r, Ω, B, and C:

$$r_{,uv} = -\frac{1}{3}\frac{\Omega^2\rho}{r} - \frac{2r_{,u}r_{,v}}{r}, \tag{2.8}$$

$$\partial_u\partial_v \log\Omega = \frac{\Omega^2\rho}{2r^2} + \frac{3}{r^2}r_{,u}r_{,v} - \frac{1}{2}\left(B_{,v}\left(2B_{,u} + C_{,u}\right) + C_{,v}\left(2C_{,u} + B_{,u}\right)\right), \tag{2.9}$$

$$B_{,uv} = -\frac{3}{2}\frac{r_{,u}}{r}B_{,v} - \frac{3}{2}\frac{r_{,v}}{r}B_{,u} + \frac{\Omega^2}{3r^2}(\mathrm{e}^{2B+2C} + \mathrm{e}^{-4B-4C} - 2\mathrm{e}^{-2B} - 2\mathrm{e}^{4B} + \mathrm{e}^{-2C} + \mathrm{e}^{4C}), \tag{2.10}$$

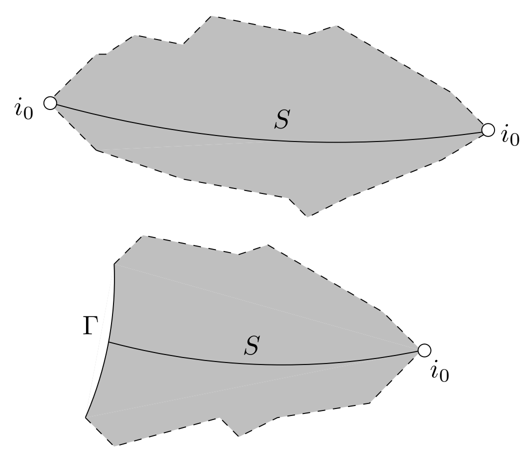
$$C_{,uv} = -\frac{3}{2}\frac{r_{,u}}{r}C_{,v} - \frac{3}{2}\frac{r_{,v}}{r}C_{,u} + \frac{\Omega^2}{3r^2}(\mathrm{e}^{2B+2C} + \mathrm{e}^{-4B-4C} - 2\mathrm{e}^{-2C} - 2\mathrm{e}^{4C} + \mathrm{e}^{-2B} + \mathrm{e}^{4B}). \tag{2.11}$$

Note that the last two equations become identical in the biaxial case. Equations (2.3) and (2.4) are to be thought of as constraints which are preserved by the evolution of (2.8)–(2.11).

The system of equations (2.3)–(2.11) should be compared to the equations originally derived in [2] in r, t coordinates.

3 The initial value problem

Consider an asymptotically flat triaxial Bianchi-IX vacuum initial data set[3] $(\mathcal{S}, \bar{g}, K)$. Let $(\mathcal{M}, g)$ denote the maximal development of $(\mathcal{S}, \bar{g}, K)$. By standard arguments, it follows that $(\mathcal{M}, g)$ is triaxial Bianchi-IX symmetric in the sense of the previous section. Moreover, the range of the null coordinates can be chosen to be bounded, defining *i.e.*, a conformal embedding of $\mathcal{Q}$ into a bounded subset of $\mathbb{R}^{1+1}$. The two possibilities for the global structure of the image of such an embedding are depicted,



depending on the number of asymptotically flat ends. S denotes $\pi(\mathcal{S})$. In what follows, the notations J^+, closure, etc., will refer to the topology and causal structure of $\mathbb{R}^{1+1}$. By the definition of asymptotic flatness, it follows that r tends monotonically to infinity along S, sufficiently close to the points

[3]We leave to the reader the correct formulation of this notion.

labeled i_0. Moreover, $\mathcal{Q} \cap J^+(S)$ is foliated by constant-v curves emanating from S and constant-u curves emanating from $S \cup \Gamma$.

4 Local existence and extension

We wish to understand future boundary points of $\mathcal{Q}$ (in the topology of the Penrose diagram) which do not "emanate" from the centre Γ.[4] For this, the following local existence theorem in null coordinates shall suffice for our purposes.

Proposition 4.1. *Let Ω, r, B, and C be functions defined on $X = [0,d] \times \{0\} \cup \{0\} \times [0,d]$. Let $k \geq 0$, and assume $r > 0$ is $C^{k+2}(u)$ on $[0,d] \times \{0\}$ and $C^{k+2}(v)$ on $\{0\} \times [0,d]$, and assume that Ω, B, and C are $C^{k+1}(u)$ on $[0,d] \times \{0\}$ and $C^{k+1}(v)$ on $\{0\} \times [0,d]$. Suppose that equations* (2.3) *and* (2.4) *hold initially on $[0,d] \times \{0\}$ and $\{0\} \times [0,d]$, respectively. Let $|f|_{n,u}$ denote the $C^n(u)$ norm of f on $[0,d] \times \{0\}$, $|f|_{n,v}$ the $C^n(v)$ norm of f on $\{0\} \times [0,d]$, etc. Define*

$$N = \sup\{|\Omega|_{1,u}, |\Omega|_{1,v}, |\Omega^{-1}|_0, |r|_{2,u}, |r|_{2,v}, |r^{-1}|_0, |B|_{1,u}, |B|_{1,v}, |C|_{1,u}, |C|_{1,v}\}.$$

Then there exists a δ, depending only on N, and a C^{k+2} function (*unique among C^2 functions*) *r, and C^{k+1} functions* (*unique among C^1 functions*) *Ω, B, and C, satisfying equations* (2.3)–(2.11) *in $[0,\delta^*] \times [0,\delta^*]$, where $\delta^* = \min\{d,\delta\}$, such that the restriction of these functions to $[0,d] \times \{0\} \cup \{0\} \times [0,d]$ is as prescribed.*

Proof. The proof is by standard methods and is omitted. For a similar proof of a local existence theorem in this framework, see Appendix B of [8]. □

From Proposition 4.1 and the maximality of the Cauchy development, the following extension principle follows. Given a subset $Y \subset \mathcal{Q} \setminus \Gamma$, define

$$N(Y) = \sup\{|\Omega|_1, |\Omega^{-1}|_0, |r|_2, |r^{-1}|_0, |B|_1, |C|_1\},$$

where, for f defined on $\mathcal{Q}^+$, $|f|_k$ denotes the restriction of the C^k norm to Y.

Proposition 4.2. *Let $p \in \overline{\mathcal{Q}} \setminus \overline{\Gamma}$ and $q \in \mathcal{Q} \cap I^-(p)$ such that $J^-(p) \cap J^+(q) \setminus \{p\} \subset \mathcal{Q}$, and $N(J^-(p) \cap J^+(q) \setminus \{p\}) < \infty$. Then $p \in \mathcal{Q}$.*

[4] Such "boundary" points can also be described without reference to this topology, in the language of TIPs. See [16].

5 The Hawking mass

A remarkable feature of the system of equations (2.8)–(2.11) is the existence of energy estimates for B and C. For this, we first define the so-called *Hawking mass*

$$m = \frac{r^2}{2}\left(1 + \frac{4r_{,u}r_{,v}}{\Omega^2}\right). \tag{5.1}$$

We compute the identities:

$$\partial_u m = -\frac{4}{3}\frac{r^3}{\Omega^2} r_{,v}[(B_{,u})^2 + B_{,u}C_{,u} + (C_{,u})^2] + r \cdot r_{,u}\left[1 - \frac{2}{3}\rho\right], \tag{5.2}$$

$$\partial_v m = -\frac{4}{3}\frac{r^3}{\Omega^2} r_{,u}[(B_{,v})^2 + B_{,v}C_{,v} + (C_{,v})^2] + r \cdot r_{,v}\left[1 - \frac{2}{3}\rho\right]. \tag{5.3}$$

Note that ρ is bounded above:

$$\rho \leq \frac{3}{2}. \tag{5.4}$$

(A straightforward way to show this is to set $x = \mathrm{e}^{2B}$, $y = \mathrm{e}^{2C}$, and to study the function $\rho(x,y)$. First one shows that $\rho(x,y) < 3/2$ in the region

$$R = \left\{x \leq \frac{1}{10}, y \leq \frac{1}{10}\right\} \cup \{x \geq 10, y \geq 10\}. \tag{5.5}$$

Next, one determines the critical points of $\rho(x,y)$. It turns out that that there is only one extremum at $x = 1$, $y = 1$, which is shown to be a maximum. This proves $\rho(x,y) \leq (3/2)$ with equality only for the round sphere, $B = C = 0$.)

By (5.4), we now see that all terms in square brackets are manifestly non-negative. Thus, if, say $\partial_u r < 0$ and $\partial_v r \geq 0$, we have

$$\partial_u m \leq 0, \qquad \partial_v m \geq 0. \tag{5.6}$$

The monotonicity (5.6) can be compared with the monotonicity in the r-direction for the (r,t) coordinates given by formula (2.7) of [2].

6 The regions $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{A}$

Let us define the *regular region*

$$\mathcal{R} = \{p \in \mathcal{Q} \text{ such that } \partial_v r > 0, \partial_u r < 0\},$$

the *trapped region*

$$\mathcal{T} = \{p \in \mathcal{Q} \text{ such that } \partial_v r < 0, \partial_u r < 0\},$$

and the *marginally trapped region*

$$\mathcal{A} = \{p \in \mathcal{Q} \text{ such that } \partial_v r = 0, \partial_u r < 0\}.$$

The reader is warned that the term *regular* is meant with reference to the asymptotically flat end in the direction of which the vector ∂_v points. By the results of the previous section, the inequalities (5.6) hold in $\mathcal{R} \cup \mathcal{A}$. In the next section, we will show how this leads to a stronger extension theorem than Proposition 4.2.

7 Extension in the non-trapped region

The monotonicity (5.6) indicates that our systems (2.3)–(2.11) share a formal similarity with spherically symmetric $3+1$-dimensional Einstein-matter systems for suitable matter fields satisfying the dominant energy condition [6, 17]. In particular, one might conjecture that an extension principle analogous to the one formulated in [6] holds in the non-trapped region. This is what we show in this section.

We have

Proposition 7.1. *Let $p \in \overline{\mathcal{R}} \setminus \overline{\Gamma}$ and $q \in \mathcal{R} \cup \mathcal{A} \cap I^-(p)$ such that $J^-(p) \cap J^+(q) \setminus \{p\} \subset \mathcal{R} \cup \mathcal{A}$. Then $p \in \mathcal{R} \cup \mathcal{A}$.*

Proof. The proof adapts techniques introduced in [7]. The strategy is as follows: one first obtains bounds for the mass via the monotonicity (5.6). Then one revisits equations (5.2) and (5.3) to obtain a priori L^2-type bounds on weighted null derivatives of B and C. Finally, one controls from below these weights with the help of the Raychaudhuri equation (7.6), and this allows one to estimate all quantities needed to apply Proposition 4.2.

Let us introduce the following notation:

$$
\begin{aligned}
\nu &= \partial_u r, \\
\lambda &= \partial_v r, \\
\kappa &= -\frac{1}{4}\frac{\Omega^2}{\nu}, \\
\mu &= \frac{2m}{r^2}, \\
\zeta_B &= r^{3/2}\partial_u B, \\
\zeta_C &= r^{3/2}\partial_u C, \\
\theta_B &= r^{3/2}\partial_v B, \\
\theta_C &= r^{3/2}\partial_v C.
\end{aligned}
$$

Note that $\kappa(1-\mu) = \lambda$.

Let X denote $(J^+(q) \setminus I^+(q)) \cap \mathcal{Q}$. Setting $p = (u_1, v_1)$ and $q = (u_\epsilon, v_\epsilon)$, we have $X = \{u_\epsilon\} \times [v_\epsilon, v_1] \cup [u_\epsilon, u_1] \times \{v_\epsilon\}$. Since X is compact, the quantities

$$
r, \kappa, \lambda, \nu, m, B, C, \zeta_B, \zeta_C, \theta_B, \theta_C, \partial_u\Omega, \partial_v\Omega, \partial_v\lambda, \partial_u\nu \tag{7.1}
$$

are uniformly bounded above and below on X:

$$
\begin{aligned}
0 < r_0 \le r &\le R, \\
0 \le \lambda &\le \Lambda, \\
0 > \nu_0 \ge \nu &\ge N, \\
|B| &\le P_B, \\
|C| &\le P_C, \\
|\theta_B| &\le T_B, \\
|\theta_C| &\le T_C, \\
|\zeta_B| &\le Z_B, \\
|\zeta_C| &\le Z_C, \\
|m| &\le M, \\
0 < \kappa_0 \le \kappa &\le K, \\
|\partial_u\Omega| &\le H, \\
|\partial_v\Omega| &\le H, \\
|\partial_u\nu| &\le H, \\
|\partial_v\lambda| &\le H.
\end{aligned} \tag{7.2}
$$

By Proposition 4.2, in view also of the fact that $\Omega^2 = -4\kappa\nu$, to prove Proposition 7.1, it suffices to show that the quantities (7.1) are uniformly bounded everywhere in $[u_\epsilon, u_1] \times [v_\epsilon, v_1] \setminus \{(u_1, v_1)\}$, with bounds similar to (7.2).

We first derive a bound for r. Integrating ν along u and λ along v, we obtain from (7.2), in view of the signs of ν, λ in $\mathcal{R} \cup \mathcal{A}$, that

$$0 < r_0 \leq r \leq R \tag{7.3}$$

in $[u_\epsilon, u_1] \times [v_\epsilon, v_1] \setminus \{(u_1, v_1)\}$. A similar argument can be given for the mass: Integrating (5.2) along u yields

$$m(u^\star, v^\star) - m(u_\epsilon, v^\star) \leq 0$$

and integrating (5.3) along v yields

$$m(u^\star, v^\star) - m(u^\star, v_\epsilon) \geq 0. \tag{7.4}$$

We conclude the bound

$$-M \leq m \leq M \tag{7.5}$$

on $[u_\epsilon, u_1] \times [v_\epsilon, v_1] \setminus \{(u_1, v_1)\}$.

A bound on κ can be derived as follows: Note that $\kappa > 0$ by definition, in view of the $\nu < 0$. On the other hand, we compute from (2.3)

$$\kappa_{,u} = -\frac{1}{6}\frac{r}{\nu^2}\Omega^2 \left((B_{,u})^2 + B_{,u}C_{,u} + (C_{,u})^2\right) \leq 0. \tag{7.6}$$

Thus, integrating in u from X, in view of (7.2), we obtain

$$0 < \kappa \leq K \tag{7.7}$$

in $[u_\epsilon, u_1] \times [v_\epsilon, v_1] \setminus \{(u_1, v_1)\}$.

Next, we bound the quantity ν using the evolution equation (2.8), written:

$$\partial_v \nu = r_{,uv} = -\frac{1}{3}\frac{\Omega^2 \rho}{r} - \frac{2\nu\lambda}{r} = \nu\left(\frac{4\kappa\rho}{3r} - \frac{2\lambda}{r}\right).$$

Integrating this equation in v, we get

$$\nu(u^\star, v^\star) = \nu(u^\star, v_\epsilon) \exp\left(\int_{v_\epsilon}^{v^\star} \left(\frac{4\kappa\rho}{3r} - \frac{2\lambda}{r}\right)(u^\star, v)dv\right). \tag{7.8}$$

Since $\rho \leq (3/2)$, $\lambda \geq 0$, we obtain the upper bound

$$-\nu \leq |N| \cdot \exp\left(\frac{2\epsilon K}{r_0}\right) \equiv N'.$$

From the above and (7.7), it follows that the quantity $\Omega^2 = -4\kappa\nu$ is also bounded from above.

To estimate B and C, we revisit equations (5.2) and (5.3), in view of (7.5), to infer a priori integral estimates for derivatives of these quantities. Equation (5.3) gives

$$\int_{v_\epsilon}^{v^\star}\left(-\frac{4}{3}\nu\frac{r^3}{\Omega^2}\left((B_{,v})^2+B_{,v}C_{,v}+(C_{,v})^2\right)+\lambda r\left(1-\frac{2}{3}\rho\right)\right)(u^\star,v)\,dv\le 2M, \tag{7.9}$$

and therefore, since

$$(B_{,v})^2+B_{,v}C_{,v}+(C_{,v})^2\ge\frac{1}{2}(B_{,v})^2+\frac{1}{2}(C_{,v})^2\ge 0,$$

we have

$$\int_{v_\epsilon}^{v^\star}\frac{1}{3}\frac{r^3}{\kappa}(B_{,v})^2\ (u^\star,v)\,dv=\int_{v_\epsilon}^{v^\star}\frac{1}{3\kappa}\left(\theta_B\right)^2\ (u^\star,v)\,dv\le 4M. \tag{7.10}$$

Obviously, the same inequality holds with B replaced by C. In the same way, integrating equation (5.2) along u using the mass-bound (7.5) leads to the estimate

$$\int_{u_\epsilon}^{u^\star}\frac{1}{3}\left(1-\mu\right)\left(\frac{\zeta_B}{\nu}\right)^2(-\nu)\ (u,v^\star)\ du\le 4M. \tag{7.11}$$

Again, the same inequality holds with B replaced by C.

We may now integrate the equation $B_{,v}=r^{-3/2}\theta_B$ in v to obtain

$$\begin{aligned}|B(u^\star,v^\star)|&\le|B(u^\star,v_\epsilon)|+\left|\int_{v_\epsilon}^{v^\star}\frac{\theta_B}{r^{3/2}}(u^\star,v)\,dv\right|\\&\le P_B+\sqrt{\int\frac{\theta_B^2}{\kappa}\,dv}\sqrt{\int\frac{\kappa}{r^3}\,dv}\le P_B+\sqrt{12M}\sqrt{\frac{K\epsilon}{r_0^3}}\equiv P_b,\end{aligned}$$

where we used the Schwarz inequality in the step from the first to the second line and (7.10) for the last step. In a completely analogous fashion — integrating $C_{,v}=r^{-3/2}\theta_C$ in v — we obtain the same bound for C. Having bounded B and C, it follows from (2.7) that ρ is also bounded in $[u_\epsilon,u_1]\times[v_\epsilon,v_1]\setminus\{(u_1,v_1)\}$. This enables us to bound λ. Rewriting the evolution equation (2.8) for $r_{,uv}$ in terms of quantities we already control, we obtain

$$\partial_u\lambda=r_{,uv}=\nu\left(\frac{4\kappa}{3r}\rho-\frac{2}{r}\kappa\left(1-\frac{2m}{r^2}\right)\right)$$

which we can integrate along u. Because we already control all the quantities appearing in the integrand, we immediately obtain a bound for λ in $[u_\epsilon,u_1]\times[v_\epsilon,v_1]\setminus\{(u_1,v_1)\}$:

$$\lambda\le L.$$

The determination of a suitable constant L is left to the reader.

We turn to bound $|\nu|$ and κ from below, away from zero. In view of the bound on $|\rho|$, we may derive immediately from (7.8) a bound

$$\nu \le \tilde{\nu}_0 < 0.$$

For κ, we integrate (7.6), rewritten as

$$\partial_u \kappa = \kappa \left(\frac{2}{3} r\nu^{-1}((B_{,u})^2 + B_{,u}C_{,u} + (C_{,u})^2) \right) \tag{7.12}$$

to obtain

$$\begin{aligned}\kappa(u,v) &= \kappa(u_\epsilon, v) \exp \int_{u_\epsilon}^{u} \frac{2}{3} r\nu^{-1}((B_{,u})^2 + B_{,u}C_{,u} + (C_{,u})^2)\, du \\ &\ge \tilde{\kappa}_0,\end{aligned}$$

where, for the last inequality, we use (7.2) and the bounds proved above, in particular, the u-analogue of (7.9).

Finally, we note at this stage that from (2.8), it follows immediately $r_{,uv}$ is bounded in $[u_\epsilon, u_1] \times [v_\epsilon, v_1] \setminus \{(u_1, v_1)\}$.

We turn now to bound the derivatives of B and C. First let us consider $\partial_v B$, $\partial_v C$. Differentiating $\theta_B = r^{3/2}\partial_v B$ in u and using the evolution equation (2.10), we get

$$\begin{aligned}\partial_u \theta_B = &-\frac{3}{2}\frac{\lambda\zeta_B}{r} + \frac{\Omega^2}{3\sqrt{r}}\big(\mathrm{e}^{2B+2C} + \mathrm{e}^{-4B-4C} - 2\mathrm{e}^{-2B} - 2\mathrm{e}^{4B} \\ &+ \mathrm{e}^{-2C} + \mathrm{e}^{4C}\big),\end{aligned}$$

which can be integrated in u to give

$$\begin{aligned}|\theta_B(u^\star, v^\star)| \le |\theta_B(u_\epsilon, v^\star)| &+ \frac{3}{2}\left| \int_{u_\epsilon}^{u^\star} \frac{\lambda\zeta_B}{r}\,(u, v^\star)\, du \right| + \left| \int_{u_\epsilon}^{u^\star} \frac{\Omega^2}{3\sqrt{r}}(\mathrm{e}^{2B+2C} \right. \\ &\left. + \mathrm{e}^{-4B-4C} - 2\mathrm{e}^{-2B} - 2\mathrm{e}^{4B} + \mathrm{e}^{-2C} + \mathrm{e}^{4C})\ (u, v^\star)\, du \right|.\end{aligned}$$

The third term on the right hand side is bounded because we control all quantities in the integrand. We estimate it say by the constant F. For the second term, we use the Schwarz inequality and the a priori bound (7.11):

$$\begin{aligned}|\theta_B(u^\star, v^\star)| &\le T_B + F + \frac{3}{2}\left|\int_{u_\epsilon}^{u^\star} \frac{\nu\kappa(1-\mu)\zeta_B}{r\nu}\,(u, v^\star)\,du\right| \\ &\le T_B + F + \frac{3}{2}\sqrt{\int_{u_\epsilon}^{u^\star} (-\nu)(1-\mu)\left(\frac{\zeta_B}{\nu}\right)^2 (u, v^\star)\,du} \\ &\quad\cdot\sqrt{\int_{u_\epsilon}^{u^\star} \frac{(-\nu)\kappa^2(1-\mu)}{r^2}\,(u, v^\star)\,du} \\ &\le T_B + F + \frac{3}{2}\sqrt{12M}\cdot K\cdot\sqrt{r_0^{-1} + Mr_0^{-3}} \equiv V.\end{aligned}$$

Hence, we bounded θ_B and therefore $\partial_v B$. The bound for $\partial_v C$ is obtained completely analogously.

Next we turn to $\partial_u B$, $\partial_u C$. Differentiating $\zeta_B = r^{3/2}\partial_u B$ with respect to v using the evolution equation (2.10), we obtain

$$\partial_v\zeta_B = -\frac{3}{2}\frac{\nu\theta_B}{r} + \frac{\Omega^2}{3\sqrt{r}}(\mathrm{e}^{2B+2C} + \mathrm{e}^{-4B-4C} - 2\mathrm{e}^{-2B} - 2\mathrm{e}^{4B} + \mathrm{e}^{-2C} + \mathrm{e}^{4C}).$$

Integration in v now yields a bound for ζ_B since all the quantities on the right have already been shown to be bounded. (Alternatively, we could use the Schwarz inequality and the a priori bound (7.10).) The bound for ζ_C and therefore $C_{,u}$ is obtained in a completely analogous manner. Having bounded B, C and their first derivatives, equation (2.10) yields that $B_{,uv}$ is also bounded.

Bounds for $\Omega_{,u}$ and $\Omega_{,v}$ follow by integrating (2.9) in v and u, respectively. Finally, bounds for $r_{,uu}$ and $r_{,vv}$ follow from (2.3), respectively, (2.4) and the previous bounds.

As remarked at the beginning, the proof now follows by applying Proposition 4.2. □

8 Null infinity

Let $\tilde{S}$ be as in the statement of Theorem 1.1. Without loss of generality, let the asymptotically flat end in question be such that ∂_v points "outwards". We define a set $\mathcal{I}^+ \subset (\overline{\mathcal{Q}} \setminus \mathcal{Q}) \cap J^+(\tilde{S})$, as follows: Let

$$\mathcal{U} = \left\{ u : \sup_{v\,:\,(u,v)\in\mathcal{Q}^+} r(u, v) = \infty \right\}. \tag{8.1}$$

For each $u \in \mathcal{U}$, there is a unique $v^\star(u)$ such that

$$(u, v^\star(u)) \in (\overline{\mathcal{Q}} \setminus \mathcal{Q}) \cap J^+(\tilde{S}). \tag{8.2}$$

Let the end in question have limit point on S given by $i_0 = (\hat{u}, V)$. Then the *null infinity* corresponding to i_0 is defined as the set

$$\mathcal{I}^+ = \bigcup_{u \in \mathcal{U} : v^\star(u) = V} (u, v^\star(u)). \tag{8.3}$$

Arguments similar to [3] show that $\mathcal{I}^+$ is non-empty for the data considered here.[5] It is straightforward to show, adapting [6], that $\mathcal{I}^+$ is then a connected ingoing null ray with past-limit point i_0. Denote the future limit point of $\mathcal{I}^+$ by i^+. A priori, it could be that $i^+ \in \mathcal{I}^+$.

Adapting [6], one shows from the monotonicity properties (5.6) of m that the *Bondi mass*[6]

$$M(u) = \limsup_{v \to V} m(u, v)$$

is a finite (not necessarily continuous) function on $\mathcal{I}^+$, non-increasing in u. We define $M_{\mathrm{f}} = \inf\, M(u)$ to be the *final Bondi mass*.

9 Proof of Theorem 1.1

This proof is an adaptation of methods introduced in [6].

As above, let $\tilde{S}$ be as in the statement of Theorem 1.1, let ∂_v be the outward direction, and consider the set

$$\mathcal{D} = J^+(\tilde{S}) \cap J^-(\mathcal{I}^+) \cap \mathcal{Q}.$$

This set is non-empty. On the other hand, by the Raychaudhuri equations (2.3) and (2.4) and the assumption that $\partial_u r < 0$ along $\tilde{S}$, it follows that $\partial_u r < 0$ along future-directed constant-v curves in $\mathcal{Q}$ emanating from $\tilde{S} \cap \{v \geq v(p)\}$, and

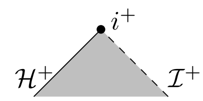
$$\mathcal{D} \subset \mathcal{R}.$$

Since, by assumption, $p \in \mathcal{T} \cup \mathcal{A}$, it follows that $p \notin \mathcal{D}$, and thus $\mathcal{D}$ has a non-empty future boundary in $\mathcal{Q}$. Denote this boundary $\mathcal{H}^+$. Note also that $m \geq r^2(p) > 0$ in $\mathcal{D}$, and thus, in particular, $M_{\mathrm{f}} > 0$.

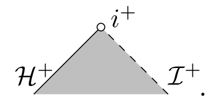
[5] If one further specializes to data for which B, C, ∇B, ∇C have compact support initially, then this follows immediately from the domain of dependence theorem.

[6] Note that with our normalization of m, this differs from its standard definition by a constant factor.

Proposition 7.1 shows immediately that $\mathcal{H}^+$ cannot terminate before reaching i^+, i.e., the Penrose diagram is as:



or



We will first show that the latter is the case, i.e., $i^+ \notin \mathcal{I}^+$, in fact, that the Penrose inequality

$$r^2 \leq 2M_{\mathrm{f}} \tag{9.1}$$

holds on the event horizon $\mathcal{H}^+$.

To show (9.1) on $\mathcal{H}^+$, one assumes the contrary, i.e., the existence of a point $(\widetilde{U}, \widetilde{V})$ with $r^2(\widetilde{U}, \widetilde{V}) = R^2 > 2M_{\mathrm{f}}$ on the horizon, and as in [6], one infers (using monotonicity properties of r and m, together with Proposition 7.1) the existence of a neighbourhood of the horizon which is part of the regular region:

$$\mathcal{N} := \left[u_0, u''\right] \times [\widetilde{V}, V) \subset \mathcal{R}$$

with $u_0 < \widetilde{U} < u''$. In particular, this neighbourhood can be chosen such that there exists an $R' < R$ with the property that in $\left[\widetilde{U}, u''\right] \times [\widetilde{V}, V) \subset \mathcal{R}$

$$r \geq R' \quad \text{and} \quad 1 - \frac{2m}{r^2} \geq 1 - \frac{2M}{(R')^2} \tag{9.2}$$

holds. The last step is to show that for any $u^\star \in [u_0, u'']$, $\lim_{v^\star \to \infty} r(u^\star, v^\star) = \infty$, *i.e.*, $\mathcal{H}^+$ cannot be the event horizon, as defined, a contradiction.

To show this last step, having shown (9.2), we proceed as follows: Integrating (5.2) along u from u_0 to a point $u^\star < u''$, we obtain the estimate

$$\sup_{\overline{v} \geq \widetilde{V}} \int_{u_0}^{u^\star} \frac{4r^3}{3\Omega^2} \lambda((B_{,u})^2 + B_{,u}C_{,u} + (C_{,u})^2)(\overline{u}, \overline{v})\, d\overline{u} \leq M,$$

which can be written as

$$\sup_{\overline{v} \geq \widetilde{V}} \int_{u_0}^{u^\star} \frac{r^3}{3(-\nu)} (1-\mu)((B_{,u})^2 + B_{,u}C_{,u} + (C_{,u})^2)\,(\overline{u}, \overline{v})\, d\overline{u} \leq M.$$

Taking (9.2) into account, we can derive the estimate

$$\sup_{\overline{v} \geq \widetilde{V}} \int_{u_0}^{u^\star} \frac{1}{3} \frac{r((B_{,u})^2 + B_{,u}C_{,u} + (C_{,u})^2)}{\nu} (\overline{u}, \overline{v})\, d\overline{u} \geq \frac{-M}{(R')^2 - 2M} \tag{9.3}$$

valid for any $u^\star \in [u_0, u'')$. Integrating (7.12), we obtain

$$\kappa(u^\star, v^\star) \geq \kappa(u_0, v^\star) \cdot \exp\left(\frac{-2M}{(R')^2 - 2M}\right),$$

and therefore

$$\lambda(u^\star, v^\star) \geq \left(1 - \frac{2M}{R^2}\right) \exp\left(\frac{-2M}{(R')^2 - 2M}\right) \lambda(u_0, v^\star). \tag{9.4}$$

Integrating (9.4) in v, we see that

$$\lim_{v^\star \to V} r(u^\star, v^\star) \longrightarrow \infty,$$

since

$$\lim_{v^\star \to V} r(u_0, v^\star) \longrightarrow \infty$$

by the definition of $\mathcal{I}^+$. We conclude $(u^\star, V) \in \mathcal{I}^+$. Therefore, $\mathcal{H}^+$ is not the event horizon and we have arrived at the desired contradiction.

The only thing left in the proof of Theorem 1.1 is to show the completeness of $\mathcal{I}^+$. (Completeness here refers to an adaptation in [6] of the concept defined in [4].) When restricted to the present symmetry class, this notion of completeness states that the suitably normalized affine length, as measured from a fixed outgoing null curve $u = u_0$, of the null curves $v = \text{const}$ in $J^-(\mathcal{I}^+) \cap J^+(\tilde{\mathcal{S}})$ should tend to infinity in both past and future, as $v \to V$. The past completeness can be deduced easily from the methods of [3],[7] so we discuss here only future completeness. Define the vector field

$$X(u,v) = \frac{\Omega^2(u_0, v)}{\Omega^2(u,v)} \frac{\partial}{\partial u}$$

on $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+$. Note that this vector field is parallel along all ingoing null rays and along the curve $u = u_0$. The desired statement of future completeness to be proven here is precisely

$$\lim_{v \to V} \int_{u_0}^{\widetilde{U}} (X(u,v) \cdot u)^{-1} \, du = \infty. \tag{9.5}$$

From equation (2.3), we can derive

$$\begin{aligned} \Omega^2(u,v)\Omega^{-2}(u_0,v) &= \nu(u,v)\nu^{-1}(u_0,v) \\ &\quad \cdot \exp\left(\int_{u_0}^{u} \frac{2r}{3\nu} \left((B_{,u})^2 + B_{,u}C_{,u} + (C_{,u})^2\right)(\overline{u}, v)\, d\overline{u}\right). \end{aligned} \tag{9.6}$$

[7] Again, if one restricts to data where B, C, ∇B, ∇C have compact support, then past completeness is immediate.

Let M be the Bondi mass at u_0. We choose an R such that $R^2 > 2M \geq 2M_{\mathrm{f}}$ and consider the curve $\{r = R\} \cap J^-(\mathcal{I}^+)$. For sufficiently large $v_0 < V$, all ingoing null curves with $v > v_0$ intersect $\{r = R\} \cap J^-(\mathcal{I}^+)$ at a unique point $(u^\star(v), v)$, depending on v.

Analogously to (9.3), we derive the bound

$$\int_{u_0}^{u} \frac{2r}{3\nu}\left((B_{,u})^2 + B_{,u}C_{,u} + (C_{,u})^2\right)(\overline{u}, v)\, d\overline{u} \geq \frac{-2M}{R^2 - 2M}, \tag{9.7}$$

which we use to estimate

$$\begin{aligned}\int_{u_0}^{\widetilde{U}} (X(u,v)\cdot u)^{-1}\, du &\geq \int_{u_0}^{u^\star(v)} (X(u,v)\cdot u)^{-1}\, du \\ &= \nu^{-1}(u_0, v)\cdot \int_{u_0}^{u^\star(v)} \exp\left(\int_{u_0}^{u} \frac{2r}{3(\partial_u r)}((B_{,u})^2 \right.\\ &\quad \left. + B_{,u}C_{,u} + (C_{,u})^2)(\overline{u}, v)\, d\overline{u}\right) \nu\, du \\ &\geq \frac{r(u_0,v) - R}{(-\nu)(u_0,v)} \exp\left(\frac{-2M}{R^2 - 2M}\right).\end{aligned} \tag{9.8}$$

Since $r(u_0, v) \to \infty$ as $v \to \infty$, to show (9.5), we only need to show that $(-\nu)(u_0, v)$ is uniformly bounded in v. The quantity

$$\frac{\nu}{1-\mu}$$

satisfies

$$\partial_v \frac{\nu}{1-\mu} = \frac{\nu}{1-\mu}\frac{2r}{3\lambda}\left((B_{,v})^2 + B_{,v}C_{,v} + (C_{,v})^2\right)$$

which integrates to

$$\begin{aligned}\frac{-\nu}{1-\mu}(u_0, v) = \exp&\left(\int_{v_0}^{v} \frac{2r}{3}\frac{1}{\lambda}((B_{,v})^2 \right.\\ &\left. + B_{,v}C_{,v} + (C_{,v})^2)(u_0, \overline{v})\, d\overline{v}\right) \frac{-\nu}{1-\mu}(u_0, v_0).\end{aligned} \tag{9.9}$$

We can choose v_0 (so large) such that

$$1 - \frac{2M}{(r(u_0, v_0))^2} > 0.$$

Set $R' = r(u_0, v_0)$. Analogously to (9.3) and (9.7), we derive the bound

$$\int_{v_0}^{v} \frac{2r}{3}\frac{1}{\lambda}\left((B_{,v})^2 + B_{,v}C_{,v} + (C_{,v})^2\right)(u_0, \overline{v})\, d\overline{v} \leq \frac{2M}{(R')^2 - 2M},$$

which enables us to obtain from (9.9) the estimate

$$-\nu(u_0, v) \leq \left(1 - \frac{2M}{(R')^2}\right)^{-1} \exp\left(\frac{2M}{(R')^2 - 2M}\right)$$

for $v \geq v_0$, which in turn shows uniform boundedness of $(-\nu)(u_0, v)$ in v.

We have shown (9.5) and thus, the desired completeness of $\mathcal{I}^+$.

10 Proof of Corollary 1.1

Let S denote the projection of an arbitrary spherically symmetric Cauchy surface in a Schwarzschild spacetime, and let $\tilde{S}$ denote the projection of a second asymptotically flat spherically symmetric Cauchy surface, with the property that $\tilde{S}$ contains a p satisfying the conditions of Theorem 1.1. (Such Cauchy surfaces clearly exist.) By Cauchy stability, sufficiently small triaxial Bianchi-IX perturbations of Schwarzschild data on $\pi^{-1}(S)$ yield solutions $(\mathcal{M}', g')$ possessing a triaxial Bianchi-IX symmetric Cauchy surface $\tilde{S}'$ with geometry arbitrarily close to that of $\tilde{S}$, in particular, also satisfying the assumptions of Theorem 1.1. We apply thus this theorem.

Finally, we note that the Hawking mass on $\tilde{S}'$ is arbitrarily close to the constant value M it takes on Schwarzschild, i.e., we have $M - \epsilon \leq m \leq M + \epsilon$ on $\tilde{S}$. By the monotonicity (5.6), it follows that this bound is preserved in $J^+(\tilde{S}') \cap J^-(\mathcal{I}^+)$. It is this statement — together with the stability of the Penrose diagram and the completeness of $\mathcal{I}^+$ — that we term "orbital stability".

11 Final comments

Besides orbital stability, one is interested in what could be called *asymptotic stability* of the Schwarzschild family, i.e., the statement that perturbations of a Schwarzschild initial data set asymptotically approach another Schwarzschild solution. An even more ambitious problem would be to understand the rates of approach, as in [9]. These problems remain open.

Another interesting and partly related question is to understand the structure of the outermost apparent horizon.[8] In analogy to [6], we may define

[8]Here, outermost is with respect to the double null foliation.

this as the set

$$\mathcal{A}' = \{(u,v) \in \mathcal{A} \ : \ (u^\star, v) \in \mathcal{R} \quad \text{for all} \ \ u^\star < u$$
$$\text{and} \ \exists u' \ : \ (u', v) \in J^-(\mathcal{I}^+) \cap \mathcal{Q} \cap J^+(\tilde{S})\}.$$

As in [6], $\mathcal{A}'$ is now easily shown to be an achronal curve intersecting all ingoing null curves for $v \geq v_0$ for sufficiently large v_0. In addition, one shows easily that on $\mathcal{A}'$, the Penrose inequality (9.1) holds. There are many other issues, however, which are not settled: is it a connected set in a neighbourhood of i^+? Is it "generically" a strictly spacelike curve in a neighbourhood of i^+? Does it terminate at i^+ in the topology of the Penrose diagram? For more on these questions, the reader can consult the literature on so-called *dynamical horizons*, in particular [1].

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References

[1] A. Ashtekar and G. Galloway, *Some uniqueness results for dynamical horizons*, Adv. Theor. Math. Phys. **9** (2005).

[2] P. Bizon, T. Chmaj, and B. Schmidt, *Critical behavior in vacuum gravitational collapse in 4+1 dimensions*, Phys. Rev. Lett. **95** (2005), 071102.

[3] D. Christodoulou, *The problem of a self-gravitating scalar field*, Commun. Math. Phys. **105** (1986), 337–361.

[4] D. Christodoulou, *The global initial value problem and the issue of singularities*, Class. Quantum Grav. **19** (1999), A23–A35.

[5] P. Chruściel, *On the global structure of Robinson-Trautman space-times*, Proc. Roy. Soc. London Ser. A **436** (1992), 299–316.

[6] M. Dafermos, *Spherically symmetric spacetimes with a trapped surface*, Class. Quantum Grav. **22** (2005), 2221–2232.

[7] M. Dafermos, *On naked singularities and the collapse of self-gravitating Higgs fields*, Adv. Theor. Math. Phys. **9** (2005), 575–591.

[8] M. Dafermos and A. Rendall, *An extension principle for the Einstein-Vlasov equatinos in spherical symmetry*, Ann. Henri Poincaré **6**(6) (2005), 1137–1155.

[9] M. Dafermos, and I. Rodnianski, *A proof of Price's law for the collapse of a self-gravitating scalar field*, Invent. Math. 162 (2005), 381–457.

[10] R. Emparan and H. Reall, *A rotating black ring solution in five dimensions*, Phys. Rev. Lett. **88**(10) (2002), 101101, 4 pp.

[11] G. Gibbons, D. Ida, and T. Shiromizu, *Uniqueness and nonuniqueness of static black holes in higher dimensions*, Phys. Rev. Lett. **89**(4) (2002), 041101, 4 pp.

[12] G. Gibbons, *Black holes in higher dimensions*, Talk given at the Newton Institute, September 5th 2005, http://www.newton.cam.ac.uk/webseminars/pg+ws/2005/gmr/0905/gibbons/.

[13] G. Gibbons, and S.A. Hartnoll, *Gravitational instability in higher dimensions*, Phys. Rev. D (3) **66**(6) (2002), 064024, 17 pp.

[14] G. Gibbons, H. Lü, D.N. Page, and C.N. Pope, *The general Kerr-de Sitter metrics in all dimensions*, J. Geom. Phys. **53**(1) (2005), 49–73.

[15] A. Ishibashi and H. Kodama, *Stability of higher-dimensional Schwarzschild black holes*, Progr. Theoret. Phys. **110**(5) (2003), 901–919.

[16] S.W. Hakwing and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, London-New York, 1973.

[17] P. Langfelder and R. Mann, *A note on spherically symmetric naked singularities in general dimension*, Classical Quantum Gravity **22**(11) (2005), 1917–1932.

[18] R.C. Myers and M.J. Perry, *Black holes in higher dimensional space-times*, Ann. Physics **172**(2) (1986), 304-347.