

# T-duality for principal torus bundles and dimensionally reduced Gysin sequences

Peter Bouwknegt<sup>1</sup>, Keith Hannabuss<sup>2</sup>, and Varghese Mathai<sup>3</sup>

<sup>1</sup>Department of Theoretical Physics, Research School of Physical Sciences  
and Engineering and Department of Mathematics,  
Mathematical Sciences Institute,  
Australian National University, Canberra, ACT 0200, Australia  
peter.bouwknegt@anu.edu.au

<sup>2</sup>Mathematical Institute, 24–29 St. Giles', Oxford, OX1 3LB, England  
kch@balliol.oxford.ac.uk

<sup>3</sup>Department of Pure Mathematics,  
University of Adelaide, Adelaide, SA 5005, Australia  
mathai.varghese@adelaide.edu.au

## Abstract

We reexamine the effects of T-duality on the global properties for principal circle bundles in the context of a dimensionally reduced Gysin sequence. We then construct a Gysin sequence for principal torus bundles and examine the consequences. In particular, we argue that the T-dual of a principal torus bundle with nontrivial H-flux is, in general, a continuous field of noncommutative, nonassociative tori.

## 1 Introduction

T-duality in String Theory, certainly from a local perspective, is an important and well-studied subject (see, e.g., [1–3] and references therein for a comprehensive review), but only recently have people begun to study the global properties of T-duality, in particular, in the presence of background fluxes.

In this paper, we study global properties of T-duality for principal torus bundles in the background of NS H-flux. We will do this by constructing a Gysin sequence for principal torus bundles, which encodes the T-dual in terms of its invariants, e.g., a generalized Chern class. We will argue that in the most general case, the T-dual is a bundle (or more precisely a continuous field) of noncommutative, nonassociative tori, generalizing all earlier partial results, [4–9]. In a companion paper [7], we study the algebraic structures of the T-dual as arising from the results of this paper. A similar conclusion, in the context of deformation quantization, was reached in [10].

The main observation in this paper is that fluxes  $[H] \in H^3(E)$  can be characterized by vector-valued forms  $(H_3, H_2, H_1, H_0)$  on the base manifold. This is referred to as “dimensional reduction” or, mathematically, as the Chern–Weil homomorphism.

T-duality for principal circle bundles was treated geometrically in [4, 5, 11], and dimensional considerations force the  $H_0$  and  $H_1$  components of the H-flux to vanish in this case. The T-dual turns out to be another principal circle bundle with T-dual H-flux. The arguments were extended in [6] to principal  $\mathbb{T}^n$ -bundles with H-flux satisfying the condition that the  $H_0$  and  $H_1$  components vanish. Then, the T-dual turns out to be another principal  $\mathbb{T}^n$ -bundle with T-dual H-flux having vanishing  $H_0$  and  $H_1$  components. The analysis in [8, 9] shows that if one considers principal  $\mathbb{T}^n$ -bundles with H-flux satisfying condition that just the  $H_0$  component vanishes, then one arrives at the surprising conclusion that the T-dual bundle has to have noncommutative tori as fibres, provided the  $H_1$  component is nonzero. The weaker condition in [8, 9] permits nonvanishing  $H_1$ , but still excludes nonzero  $H_0$ . The emergence of noncommutative structures in T-dual descriptions has also arisen in various other contexts, such as mirror symmetry [12], and matrix theory compactifications [13]. In this paper, we will remove the last of these constraints and will allow for a nonvanishing  $H_0$  component. In this case, we arrive at the astonishing conclusion that the T-dual bundle has to have nonassociative tori as fibres, taking it even beyond the normal range of noncommutative geometry. The algebraic structure, generalizing that of [8, 9], is discussed in [7].

An alternative approach to make global sense of the T-duality transformation in the presence of a ‘nongeometric’ flux  $H_1$  has been advocated in, e.g. [14, 15], in terms of ‘doubled torus bundles’ with T-duality transformations in the overlaps and a local choice of polarization. We believe these so-called nongeometric string compactifications are equivalent, in an appropriate sense, to our global description by means of a Seiberg-Witten map. We leave this for further investigation.

The paper is organized as follows. In order to explain our ideas as carefully as possible, we first recall in Section 2 how the Buscher rules encode the global aspects of T-duality. Then, we review how those rules are related to the Gysin sequence for principal circle bundles. In the latter part of Section 2, we make closer contact with the usual form of the Buscher rules by dimensionally reducing the Gysin sequence. We also discuss how T-duality gives rise to an isomorphism on twisted cohomology in this dimensionally reduced setting. Motivated by the discussion in Section 2, we generalize the dimensional reduction to (higher rank) principal torus bundles in Section 3. We derive a Gysin sequence for principal torus bundles (which, to the best of our knowledge, has not appeared elsewhere). In Section 4, we deduce from the Gysin sequence of Section 3 that the T-dual of a principal torus bundle with H-flux is, in general, a continuous field of noncommutative, nonassociative tori over the base. We discuss both the T-duality group and a simple example in this setting. We end, in Section 5, with some conclusions and open problems. In an appendix, we briefly discuss duality from the operator algebraic perspective, which might be useful for the reader to relate the results of this paper to the results of, in particular, [8, 9], as well as the companion paper [7] where we explore the nonassociative structures arising in this paper in more detail.

## 2 Principal circle bundles

To motivate our construction, in the case of principal torus bundles, we first look at the simplest case, name that of a principal circle bundle.

### 2.1 A bit of notation

Throughout this paper, we will denote by  $\Omega^k(E)$ ,  $Z^k(E)$ , and  $B^k(E)$ , the spaces of  $k$ -forms, closed  $k$ -forms, and exact  $k$ -forms on a smooth manifold  $E$ , respectively. The de-Rham cohomology of  $E$  is defined as  $H_{\text{dR}}^k(E) = Z^k(E)/B^k(E)$ . The integer, or Čech, cohomology of  $E$  will be denoted by  $H^k(E, \mathbb{Z})$ . For clarity, we will restrict the presentation in this paper to

de-Rham cohomology only, or rather to the image  $H^k(E) = i(H^k(E, \mathbb{Z})) \subset H_{\text{dR}}^k(E)$  of integer cohomology in de-Rham cohomology (i.e., forms with integral periods), but most of the results of this paper will generalize to integer cohomology without too much effort.

## 2.2 Buscher rules revisited

We start with a principal  $S^1$ -bundle  $\pi : E \rightarrow M$ , supported by an H-flux  $[H] \in H^3(E)$ . If  $A$  denotes a connection 1-form on  $E$  and  $\bar{g}$  a metric on the base  $M$ , then  $E$  carries a canonical metric  $g = \bar{g} + A \otimes A$ . If  $\kappa$  denotes the Killing vector corresponding to the  $S^1$ -isometry, we may choose a (de-Rham) representative  $H$  of  $[H]$ , satisfying the invariance condition  $\mathcal{L}_\kappa H = 0$ . Again, locally  $H = dB$  for a 2-form  $B$ , and we will assume that  $B$  can be chosen such that  $\mathcal{L}_\kappa B = 0$  (this is not a necessary requirement, but it will slightly simplify the discussion below).

Locally, we can choose coordinates  $x^M = (x^\mu, x^0) \equiv (x^\mu, \theta)$  on  $E$  such that the Killing vector of the  $S^1$ -isometry is given by  $\kappa = \partial/\partial\theta$ . The invariance conditions  $\mathcal{L}_\kappa H = \mathcal{L}_\kappa B = 0$  then simply imply that the components  $H_{LMN}$  and  $B_{MN}$  do not depend on  $\theta$ .

Furthermore, locally we can choose the connection  $A = A_M dx^M = d\theta + A_\mu dx^\mu$ , where  $A_\mu dx^\mu \in \Omega^1(M)$ . That is,

$$\begin{aligned} g &= \bar{g} + A \otimes A = \bar{g}_{\mu\nu} dx^\mu \otimes dx^\nu + (d\theta + A_\mu dx^\mu)^2, \\ B &= \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu + B_\mu dx^\mu \wedge (d\theta + A_\nu dx^\nu), \end{aligned} \quad (2.1)$$

where the components  $A_\mu$ ,  $B_{\mu\nu}$ , and  $B_\mu$  do not depend on  $\theta$ . Physically, the decomposition of  $g$  and  $B$  above is referred to as dimensional reduction. In terms of matrices, the metric and B-field components are

$$g_{MN} = \begin{pmatrix} \bar{g}_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}, \quad B_{MN} = \begin{pmatrix} B_{\mu\nu} + (B_\mu A_\nu - A_\mu B_\nu) & B_\mu \\ -B_\nu & 0 \end{pmatrix}. \quad (2.2)$$

The Buscher rules [16]

$$\begin{aligned} \hat{g}_{\mu\nu} &= g_{\mu\nu} - \frac{1}{g_{00}} (g_{\mu 0} g_{\nu 0} - B_{\mu 0} B_{\nu 0}), & \hat{g}_{00} &= \frac{1}{g_{00}}, & \hat{g}_{\mu 0} &= \frac{B_{\mu 0}}{g_{00}}, \\ \hat{B}_{\mu\nu} &= B_{\mu\nu} - \frac{1}{g_{00}} (g_{\mu 0} B_{\nu 0} - g_{\nu 0} B_{\mu 0}), & \hat{B}_{\mu 0} &= \frac{g_{\mu 0}}{g_{00}} \end{aligned} \quad (2.3)$$

give

$$\widehat{g}_{MN} = \begin{pmatrix} \bar{g}_{\mu\nu} + B_\mu B_\nu & B_\mu \\ B_\nu & 1 \end{pmatrix}, \quad \widehat{B}_{MN} = \begin{pmatrix} B_{\mu\nu} & A_\mu \\ -A_\nu & 0 \end{pmatrix}. \quad (2.4)$$

That is, with the choices above, T-duality locally simply corresponds to the interchange  $A_\mu \leftrightarrow B_\mu$ .

Denoting the coordinate of the dual circle by  $\hat{\theta}$ , we can interpret  $\widehat{A} = d\hat{\theta} + B_\mu dx^\mu$ , locally, as a connection on a dual circle bundle  $\widehat{\pi} : \widehat{E} \rightarrow M$ . We deduce from equation (2.4) that on the correspondence space  $E \times_M \widehat{E} = \{(x, \hat{x}) \in E \times \widehat{E} \mid \pi(x) = \widehat{\pi}(\hat{x})\}$ , with local coordinates  $(x^\mu, \theta, \hat{\theta})$ ,

$$\widehat{B} = B + A \wedge \widehat{A} - d\theta \wedge d\hat{\theta}, \quad (2.5)$$

so that

$$\widehat{H} - H = d(A \wedge \widehat{A}) = F \wedge \widehat{A} - A \wedge \widehat{F}, \quad (2.6)$$

where  $F = dA$  and  $\widehat{F} = d\widehat{A}$  are the curvatures of  $A$  and  $\widehat{A}$ , respectively, and are (globally) defined forms on  $M$ . Equation (2.6) actually makes sense globally on  $E \times_M \widehat{E}$ . Rewriting this equation as

$$H - \widehat{F} \wedge A = \widehat{H} - F \wedge \widehat{A},$$

we see that the left-hand side is a form on  $E$  while the right-hand side is a form on  $\widehat{E}$ . Thus, in order to have equality, we conclude that both have to equal a form  $H_3$  defined on  $M$ . That is

$$\begin{aligned} H &= H_3 + \widehat{F} \wedge A, \\ \widehat{H} &= H_3 + F \wedge \widehat{A}. \end{aligned} \quad (2.7)$$

We note that these equations imply that

$$F = \widehat{\pi}_* \widehat{H}, \quad \widehat{F} = \pi_* H, \quad (2.8)$$

which is the statement that H-flux and first Chern class of the circle bundle are exchanged under T-duality [4, 5].

### 2.3 Dimensionally reduced Gysin sequence

Let us first review how the global content of the Buscher rules, i.e., equation (2.8), is encoded in the Gysin sequence for the principal circle bundle  $\pi: E \rightarrow M$  (cf. [4, 5]). Principal circle bundle are classified, up to isomorphism, by the Euler class  $\chi(E) \in H^2(M, \mathbb{Z})$  or, equivalently, by the first Chern class  $c_1(L_E) \in H^2(M, \mathbb{Z})$  of the associated line bundle  $L_E = E \times_{U(1)} \mathbb{C}$ . Given a principal circle bundle  $\pi: E \rightarrow M$ , we have the pull-back map  $\pi^*: H^k(M) \rightarrow H^k(E)$ , as well as the push-forward map (“integration over the  $S^1$ -fibre”)

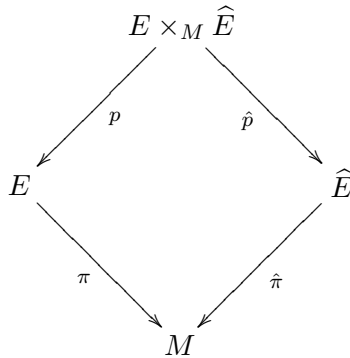
$\pi_*: H^k(E) \rightarrow H^{k-1}(M)$ . These maps nicely fit into a long exact sequence in cohomology, the so-called Gysin sequence [17].<sup>1</sup>

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^k(M) & \xrightarrow{\pi^*} & H^k(E) & \xrightarrow{\pi_*} & H^{k-1}(M) \\ & & \searrow^{\delta} & & \longrightarrow & & \dots \end{array} \tag{2.9}$$

where the map  $\delta: H^{k-1}(M) \rightarrow H^{k+1}(M)$  is given, on a representative  $\omega$  of a class in  $H^{k-1}(M)$ , by  $\delta\omega = F \wedge \omega$ . Here,  $F$  is (a representative for) the Euler class of  $E$ , i.e., the curvature of a connection on  $E$ .

Considering the  $k = 3$  segment of the Gysin sequence (2.9), we see that any class  $[H] \in H^3(E)$ , i.e., any H-flux on  $E$ , gives rise to a class  $\pi_*[H] \in H^2(M)$ , which can be interpreted as the equivalence class  $[\widehat{F}]$  of the curvature  $\widehat{F}$  of a T-dual circle bundle  $\widehat{\pi}: \widehat{E} \rightarrow M$ .<sup>2</sup> Furthermore, we have  $[F] \wedge [\widehat{F}] \equiv 0$  in  $H^4(M)$ . Conversely, by considering the Gysin sequence corresponding to the T-dual circle bundle  $\widehat{\pi}: \widehat{E} \rightarrow M$ , we conclude from  $[F] \wedge [\widehat{F}] = [\widehat{F}] \wedge [F] \equiv 0$  that  $[F] = \widehat{\pi}_*[\widehat{H}]$  for some class  $[\widehat{H}] \in H^3(\widehat{E})$ . This is precisely the content of Equation (2.8).

From the Gysin sequence, we can of course only determine the element  $[\widehat{H}] \in H^3(\widehat{E})$  up to an element in  $\pi^*(H^3(M))$ . To fix this ambiguity, we need some extra input. The extra input, of course, is that T-duality should not affect that part of the H-flux that “lives” on the base manifold  $M$ . This is equivalent to demanding that  $p^*[H] - \widehat{p}^*[\widehat{H}] \equiv 0$  in  $H^3(E \times_M \widehat{E})$ , where  $E \times_M \widehat{E} = \{(x, \widehat{x}) \in E \times \widehat{E} \mid \pi(x) = \widehat{\pi}(\widehat{x})\}$  is the correspondence space, and the projections  $p$  and  $\widehat{p}$  are defined in the following commutative diagram



<sup>1</sup>This Gysin sequence also holds in integer cohomology, but for simplicity of presentation, we restrict ourselves to de-Rham cohomology throughout the paper.

<sup>2</sup>The isomorphism class of this T-dual bundle is only unique up to torsion, but would be unique if we would have presented the analysis in integer cohomology.

With a bit more work, using again the Gysin sequence, one can actually argue that at the level of forms, we reproduce Equations (2.6)–(2.8) (see [4] for details).

Obviously, there is a great deal of similarity between the analysis in Section 2.2 and the discussion from the point of view of the Gysin sequence above. This similarity can be further illuminated by “dimensionally reducing” the Gysin sequence (2.9). The resulting dimensionally reduced Gysin sequence will immediately present to us how the above analysis should be generalized to higher rank principal torus bundles.

To explain the ideas, let us start by considering the trivial principal circle bundle  $\pi: M \times S^1 \rightarrow M$ . By the Künneth theorem, we have

$$H^k(E) \cong \bigoplus_{p+q=k} (H^p(M) \oplus H^q(S^1)) \cong H^k(M) \oplus H^{k-1}(M). \tag{2.10}$$

Explicitly, if we denote the generator of  $H^1(S^1)$  by  $d\theta$ , normalized such that  $\pi_*(d\theta) = \int d\theta = 1$ , then the isomorphism (2.10) is given by

$$(\omega_k, \omega_{k-1}) \mapsto \omega_k + d\theta \wedge \omega_{k-1}, \tag{2.11}$$

with inverse, for  $\Omega \in \Omega^k(E)_{\text{inv}}$

$$\Omega \mapsto (\omega_k, \omega_{k-1}) \equiv (\Omega - d\theta \wedge \pi_*\Omega, \pi_*\Omega). \tag{2.12}$$

Note that, in particular,  $\pi_*(\Omega - d\theta \wedge \pi_*\Omega) = 0$ , so that  $\omega_k$  can indeed be identified with an element of  $\Omega^k(M)$ .

Now consider a nontrivial circle bundle  $\pi: E \rightarrow M$ . Choose a representative curvature  $F \in \Omega^2(M)$  such that  $[F] = c_1(L_E)$ , with connection  $A \in \Omega^1(E)$ , i.e.,  $\pi^*F = dA$ , normalized such that  $\pi_*A = 1$ . The question arises whether we can still characterize classes in  $H^k(E)$  by forms living on the base manifold.

Let  $\Omega$  be a representative of an element in  $H^k(E)$ . We would like to mimic (2.12), but of course, in the more general case, the element  $d\theta$  is not a globally defined 1-form on  $E$ . Instead, we can make use of the connection  $A$ . Again, let  $\kappa$  denote the (globally defined) Killing vector field corresponding to the  $U(1)$ -isometry, and let  $\Omega^k(E)_{\text{inv}}$  denote the space of  $k$ -forms invariant under the isometry, i.e.,  $\mathcal{L}_\kappa\Omega = 0$ . Then, we have a map

$$f_A: \Omega^k(M) \oplus \Omega^{k-1}(M) \rightarrow \Omega^k(E)_{\text{inv}}, \quad (\omega_k, \omega_{k-1}) \mapsto \omega_k + A \wedge \omega_{k-1}, \tag{2.13}$$

with inverse

$$f_A^{-1}: \Omega^k(E)_{\text{inv}} \rightarrow \Omega^k(M) \oplus \Omega^{k-1}(M), \quad \Omega \mapsto (\Omega - A \wedge \pi_*\Omega, \pi_*\Omega). \tag{2.14}$$

A simple computation shows

$$(d \circ f_A)(\omega_k, \omega_{k-1}) = (d\omega_k + F \wedge \omega_{k-1}) - A \wedge d\omega_{k-1},$$

and thus, upon defining a modified differential  $D: \Omega^k(M) \oplus \Omega^{k-1}(M) \rightarrow \Omega^{k+1}(M) \oplus \Omega^k(M)$  by

$$D(\omega_k, \omega_{k-1}) = (d\omega_k + F \wedge \omega_{k-1}, -d\omega_{k-1}), \tag{2.15}$$

we have  $d \circ f_A = f_A \circ D$ . It is straightforward to check that  $D^2 = 0$  and hence that  $D$  defines a cohomology  $H_D^k(M) \equiv H^k(\Omega^\bullet(M) \oplus \Omega^{\bullet-1}(M), D)$ . Furthermore, because of the commutativity of the diagram

$$\begin{array}{ccc} \Omega^k(M) \oplus \Omega^{k-1}(M) & \xrightarrow[\cong]{f_A} & \Omega^k(E)_{\text{inv}} \\ D \downarrow & & \downarrow d \\ \Omega^{k+1}(M) \oplus \Omega^k(M) & \xrightarrow[\cong]{f_A} & \Omega^{k+1}(E)_{\text{inv}} \end{array}$$

we have the result

$$H^k(E) \cong H_D^k(M). \tag{2.16}$$

While the explicit isomorphism (2.13) depends on the choice of connection  $A$ , it is easily verified that isomorphism (2.16) is independent of the choice of  $A$ .

Now that we have a globally defined dimensional reduction of forms, equation (2.14), and an identification of cohomology, equation (2.16), it is straightforward to dimensionally reduce the Gysin sequence (2.9). The result is the following exact sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^k(M) & \xrightarrow{\pi^*} & H_D^k(M) & \xrightarrow{\pi_*} & H^{k-1}(M) \\ & & \xrightarrow{\delta} & H^{k+1}(M) & \longrightarrow & \dots & \end{array} \tag{2.17}$$

where the various maps, on representatives of the cohomology, are given by

$$\begin{aligned} \pi^*: H^k(M) &\longrightarrow H_D^k(M), & \pi^*(\omega_k) &= (\omega_k, 0), \\ \pi_*: H_D^k(M) &\longrightarrow H^{k-1}(M), & \pi_*(\omega_k, \omega_{k-1}) &= \omega_{k-1}, \\ \delta: H^{k-1}(M) &\longrightarrow H^{k+1}(M), & \delta(\omega_{k-1}) &= F \wedge \omega_{k-1}. \end{aligned}$$

It is interesting to observe that we can consider the sequence (2.17) in itself, without making reference to any principal circle bundle, but just defined by a certain representative of  $[F] \in H^2(M)$ .

Usually, the proof that (2.9) defines a long exact sequence in cohomology is quite involved. The standard proof is given by examining the Leray spectral sequence corresponding to the principal circle bundle (see e.g., [17]). After dimensional reduction, however, the proof is remarkably simple and does



not require sophisticated techniques. To illustrate this point, we present the proof below.

**Theorem 2.1.** *The sequence (2.17) defines an exact complex.*

*Proof.* First we show that (2.17) defines a complex.

- a. Let  $\omega_k \in Z^k(M)$  be a representative of a class in  $H^k(M)$ . It is obvious, from the definitions, that  $\pi_*(\pi^*\omega_k) = \pi_*(\omega_k, 0) = 0$ .
- b. Let  $(\omega_k, \omega_{k-1}) \in Z_D^k(M)$  represent a class in  $H_D^k(M)$ . We have  $\delta(\pi_*(\omega_k, \omega_{k-1})) = F \wedge \omega_{k-1} = -d\omega_k$ , hence  $\delta(\pi_*(\omega_k, \omega_{k-1})) \equiv 0$  in  $H^{k+1}(M)$ .
- c. Finally, let  $\omega_{k-1} \in Z^{k-1}(M)$  represent a class in  $H^{k-1}(M)$ . Then,  $\pi^*(\delta(\omega_{k-1})) = \pi^*(F \wedge \omega_{k-1}) = (F \wedge \omega_{k-1}, 0) = D(0, \omega_{k-1})$ , hence  $\pi^*(\delta(\omega_{k-1})) \equiv 0$  in  $H_D^{k+1}(M)$ .

To show that the complex is exact, consider

1. Let  $\omega_k \in Z^k(M)$  be such that  $\pi^*(\omega_k) \equiv 0$  in  $H_D^k(M)$ , i.e.,  $\pi^*(\omega_k) = (\omega_k, 0) = D(\nu_{k-1}, \nu_{k-2}) = (d\nu_{k-1} + F \wedge \nu_{k-2}, -d\nu_{k-2})$  for some  $(\nu_{k-1}, \nu_{k-2})$ . Then,  $\omega_k \equiv F \wedge \nu_{k-2}$  in  $H^k(M)$  for some  $\nu_{k-2} \in Z^{k-2}(M)$ .
2. Let  $(\omega_k, \omega_{k-1}) \in Z_D^k(M)$  be such that  $\pi_*(\omega_k, \omega_{k-1}) = \omega_{k-1} \equiv 0$  in  $H^{k-1}(M)$ , i.e.,  $\omega_{k-1} = d\nu_{k-2}$  for some  $\nu_{k-2}$ . Thus,  $(\omega_k, \omega_{k-1}) = (\omega_k + F \wedge \nu_{k-2}, 0) - D(0, \nu_{k-2})$ , and  $(\omega_k, \omega_{k-1}) \equiv (\omega_k + F \wedge \nu_{k-2}, 0)$  in  $H_D^k(M)$ . But,  $(\omega_k + F \wedge \nu_{k-2}, 0) = \pi^*(\omega_k + F \wedge \nu_{k-2})$ .
3. Finally, let  $\omega_{k-1} \in Z^{k-1}(M)$  be such that  $\delta(\omega_{k-1}) = F \wedge \omega_{k-1} \equiv 0$  in  $H^{k+1}(M)$ , i.e.,  $F \wedge \omega_{k-1} = -d\nu_k$  for some  $\nu_k$ . Then,  $(\nu_k, \omega_{k-1}) \in Z_D^k(M)$  while  $\pi_*(\nu_k, \omega_{k-1}) = \omega_{k-1}$ . □

The dimensionally reduced Gysin sequence in the case of principal circle bundles, as discussed in this section, is of course not new. It is apparently well known in the context of foliations (see e.g. [18]), and also occurs in the context of M-theory reduction on a principal circle bundle to type IIA string theory in [19] (section 3.1).

## 2.4 Twisted cohomology

Let us denote the space of even and odd forms on  $E$  by  $\Omega^{\bar{0}}(E)$  and  $\Omega^{\bar{1}}(E)$ , respectively. That is,

$$\Omega^{\bar{i}}(E) = \bigoplus_{i=\bar{i} \bmod 2} \Omega^i(E). \tag{2.18}$$

Then, given a representative  $H$  for a class  $[H] \in H^3(E)$ , we can construct a “twisted differential”  $d_H : \Omega^i \rightarrow \Omega^{i+1}$  by

$$d_H \Omega = d\Omega + H \wedge \Omega. \tag{2.19}$$

Clearly,  $(d_H)^2 = 0$  (since  $dH = 0$ ). The cohomology of the  $\mathbb{Z}_2$ -graded complex  $(\Omega^\bullet(E), d_H)$  is known as the twisted cohomology  $H^i(E, [H])$  of  $E$ , with respect to the 3-form  $H$ . It is easy to see that while explicit representatives for twisted cohomology classes depend on the choice of  $H$ , the twisted cohomology itself only depends on the class  $[H]$ .

Let us now examine what a twisted cohomology class looks like under the dimensional reduction of Section 2.2.

Decomposing  $H = H_3 + A \wedge H_2$ , and  $\Omega = \Omega' + A \wedge \Omega''$ , as in equation (2.13), we have

$$d_H \Omega = (d\Omega' + H_3 \wedge \Omega' + F \wedge \Omega'') + A \wedge (-d\Omega'' - H_3 \wedge \Omega'' + H_2 \wedge \Omega').$$

Thus, the condition for  $\Omega$  to be a twisted cohomology class, i.e.,  $d_H \Omega = 0$ , decomposes into two equations

$$\begin{aligned} d\Omega' + H_3 \wedge \Omega' + F \wedge \Omega'' &= 0, \\ d\Omega'' + H_3 \wedge \Omega'' - H_2 \wedge \Omega' &= 0. \end{aligned} \tag{2.20}$$

Note that both equations do not depend on the choice of  $A$  and are described completely in terms of forms on  $M$ .

Now, consider the pair  $((H_3, H_2), F) \in H_D^3(M) \oplus H^2(M)$ . As shown in the previous section, T-duality is the transformation

$$((\widehat{H}_3, \widehat{H}_2), \widehat{F}) = ((H_3, F), H_2). \tag{2.21}$$

Therefore, T-duality provides an isomorphism on twisted cohomology, which is explicitly given by

$$(\widehat{\Omega}', \widehat{\Omega}'') = (\Omega'', -\Omega'). \tag{2.22}$$

That is,  $d_H \Omega = 0$  iff  $d_{\widehat{H}} \widehat{\Omega} = 0$ . Of course, equation (2.22) agrees with the “Hori formula” [4, 20]

$$\widehat{\Omega} = \int_{S^1} e^{-A \wedge \widehat{A}} \Omega. \tag{2.23}$$

### 3 Principal torus bundles

In this section, we will generalize the construction of the “dimensionally reduced” Gysin sequence of the previous section to principal torus bundles. In order to do this, we will first have to establish how forms on the bundle space of a principal torus bundle are related to forms on the base space. This

is a special case of the so-called Chern–Weil homomorphism which holds for general principal G-bundles.

### 3.1 Dimensional reduction—the Chern–Weil homomorphism

Let  $\mathbb{T} = \mathbb{T}^n$  denote a rank- $n$  torus. Let  $\mathfrak{t}$  denote the Lie algebra of  $\mathbb{T}$  and denote by  $\mathfrak{t}^*$  the dual Lie algebra. Suppose we are given a principal  $\mathbb{T}$ -bundle  $\pi: E \rightarrow M$ . The action of  $\mathbb{T}$  on  $E$  associates to each element  $X \in \mathfrak{t}$  a vector field on  $E$  which, by abuse of notation, we will also denote as  $X$ . The Lie derivative and contraction with respect to the vector field  $X$  will be denoted as  $\mathcal{L}_X$  and  $\iota_X$ , respectively.

For each cohomology class in  $H^k(E)$ , we may choose a closed representative  $\Omega \in \Omega^k(E)$ , such that  $\mathcal{L}_X \Omega = 0$  for all  $X \in \mathfrak{t}$ . Locally, we can choose coordinates  $x^M = (x^\mu, x^a)$  ( $\mu = 1, \dots, N - n, a = 1, \dots, n$ ) such that a basis of Killing vectors for the  $\mathbb{T}^n$ -isometry is given by  $X_a = \partial/\partial x^a$ . Then,  $\mathcal{L}_{X_a} \Omega = 0$ , for all  $a = 1, \dots, n$ , translates into  $\Omega(x^\mu, x^a) = \Omega(x^\mu)$ . We can decompose  $\Omega = \Omega_{M_1 M_2 \dots M_k} dx^{M_1} \wedge \dots \wedge dx^{M_k}$  with respect to the number of directions in  $\mathbb{T}$  (‘dimensional reduction’), as  $\Omega = (\Omega_{\mu_1 \mu_2 \dots \mu_k}, \Omega_{\mu_1 \dots \mu_{k-1} a_1}, \dots, \Omega_{a_1 \dots a_k})$ . We can think of the component  $\Omega_{\mu_1 \dots \mu_q a_1 \dots a_p}, p + q = k$ , as defining an element  $\omega_{p,q} \in \Omega^p(M) \otimes \wedge^q \mathfrak{t}^*$  by

$$\omega_{p,q} = \frac{1}{p!q!} \Omega_{\mu_1 \dots \mu_q a_1 \dots a_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q} \otimes (X^{*a_1} \wedge \dots \wedge X^{*a_p}), \quad (3.1)$$

where  $X^{*a}, a = 1, \dots, n$ , denotes a basis of  $\mathfrak{t}^*$ . Obviously, the original form  $\Omega \in \Omega^k(E)$  can be reconstructed locally from its components  $\omega_{p,q} \in \Omega^p(M) \otimes \wedge^q \mathfrak{t}^*, p + q = k$ .

The local construction above is of course reminiscent of the Künneth theorem for trivial torus bundles  $E = M \times \mathbb{T}^n$ , in which case

$$H^k(E) \cong \bigoplus_{p+q=k} (H^p(M) \otimes H^q(\mathbb{T}^n)) \cong \bigoplus_{p+q=k} (H^p(M) \otimes \wedge^q \mathfrak{t}^*). \quad (3.2)$$

As in the case of circle bundles, the local construction above can be made global. To this end, we need a choice of (principal) connection  $A \in \Omega^1(E, \mathfrak{t})$  on  $E$ , i.e., a connection  $A$  satisfying

$$\iota_X A = X, \quad (3.3)$$

for all  $X \in \mathfrak{t}$  (such a connection always exists, see e.g., [21]). Or, equivalently, if we think of a connection  $A$  as defining a map  $A: \mathfrak{t}^* \rightarrow \Omega^1(E)$ , the

principality condition can be expressed as

$$\iota_X A(Y^*) = \langle Y^*, X \rangle, \tag{3.4}$$

for all  $X \in \mathfrak{t}$ ,  $Y^* \in \mathfrak{t}^*$ .

In addition, let us introduce the following notation for invariant, horizontal, and basic forms on  $E$ , respectively,

$$\begin{aligned} \Omega(E)_{\text{inv}} &= \{\omega \in \Omega(E) \mid \mathcal{L}_X \omega = 0, \forall X \in \mathfrak{t}\}, \\ \Omega(E)_{\text{hor}} &= \{\omega \in \Omega(E) \mid \iota_X \omega = 0, \forall X \in \mathfrak{t}\}, \\ \Omega(E)_{\text{bas}} &= \{\omega \in \Omega(E) \mid \mathcal{L}_X \omega = \iota_X \omega = 0, \forall X \in \mathfrak{t}\} = \Omega(E)_{\text{hor}} \cap \Omega(E)_{\text{inv}}. \end{aligned} \tag{3.5}$$

As remarked before, we have an isomorphism  $H(\Omega(E)_{\text{inv}}, d) \cong H(E)$ , i.e., every class in  $H(E)$  can be represented by a closed, invariant, form [21]. In addition, basic forms are in 1–1 correspondence with forms on  $M$  through the pull-back map, i.e.,  $\pi^*: \Omega(M) \rightarrow \Omega(E)_{\text{bas}}$  is an isomorphism.

We have an isomorphism  $f_A: \Omega(E)_{\text{hor}} \otimes \wedge \mathfrak{t}^* \rightarrow \Omega(E)$ , given by

$$f_A(\omega \otimes (X_1^* \wedge \cdots \wedge X_q^*)) = \omega \wedge A(X_1^*) \wedge \cdots \wedge A(X_q^*), \tag{3.6}$$

which, since the group is Abelian, restricts to an isomorphism  $f_A: \Omega(E)_{\text{bas}} \otimes \wedge \mathfrak{t}^* \rightarrow \Omega(E)_{\text{inv}}$ . Or, since  $\Omega(E)_{\text{bas}} \cong \Omega(M)$ , we have

$$\bigoplus_{p+q=k} (\Omega^p(M) \otimes \wedge^q \mathfrak{t}^*) \cong \Omega^k(E)_{\text{inv}}. \tag{3.7}$$

The inverse of  $f_A$  is more cumbersome to write down. First of all, given  $\omega \in \Omega^k(E)_{\text{inv}}$ , we can define  $\tilde{\omega}_{p,q} \in \Omega^p(E) \otimes \wedge^q \mathfrak{t}^*$ , with  $p + q = k$ , by

$$\tilde{\omega}_{p,q}(X_1, \dots, X_q) = \iota_{X_1} \cdots \iota_{X_q} \omega,$$

where  $X_1, \dots, X_q \in \mathfrak{t}$ . Obviously,  $\mathcal{L}_X \tilde{\omega}_p = 0$ , thus  $\tilde{\omega}_{p,q} \in \Omega^p(E)_{\text{inv}} \otimes \wedge^q \mathfrak{t}^*$ . However,

$$\iota_X \tilde{\omega}_{p,q}(X_1, \dots, X_q) = \tilde{\omega}_{p-1,q+1}(X, X_1, \dots, X_q).$$

Thus, upon defining,  $\omega_{p,q} \in \Omega^p(E) \otimes \wedge^q \mathfrak{t}^*$  by

$$\omega_{p,q}(X_1, \dots, X_q) = \sum_{r=0}^p \frac{1}{r!} (-1)^{rp - (1/2)r(r+1)} \tilde{\omega}_{p-r,q+r}(\underbrace{A, \dots, A}_r, X_1, \dots, X_q), \tag{3.8}$$

we have  $\omega_{p,q} \in \Omega^p(E)_{\text{bas}} \otimes \wedge^q \mathfrak{t}^*$ , where we have used

$$\begin{aligned} \iota_X \widetilde{\omega}_{p-r,q+r}(\underbrace{A, \dots, A}_r, X_1, \dots, X_q) &= \widetilde{\omega}_{p-r-1,q+r+1}(X, \underbrace{A, \dots, A}_r, X_1, \dots, X_q) \\ &+ (-1)^{p-r} r \widetilde{\omega}_{p-r,q+r}(X, \underbrace{A, \dots, A}_{r-1}, X_1, \dots, X_q). \end{aligned}$$

Hence,  $f_A^{-1}: \Omega^k(E)_{\text{inv}} \rightarrow \bigoplus_{p+q=k} (\Omega^p(M) \otimes \wedge^q \mathfrak{t}^*)$  is given by

$$\omega \longmapsto (\omega_{k,0}, \omega_{k-1,1}, \dots, \omega_{0,k}), \quad (3.9)$$

with  $\omega_{p,q} \in \Omega^p(M) \otimes \wedge^q \mathfrak{t}^*$  and  $p+q=k$ . We will often simplify the notation and simply write

$$\omega \longmapsto (\omega_k, \omega_{k-1}, \dots, \omega_0), \quad (3.10)$$

with  $\omega_p \in \Omega^p(M) \otimes \wedge^{k-p} \mathfrak{t}^*$ .

We think of the curvature  $F \in Z^2(M, \mathfrak{t}) \cong Z^2(M) \otimes \mathfrak{t}$ , and thus as defining a map  $F: \mathfrak{t}^* \rightarrow \Omega^2(M)$ , satisfying  $dF(X^*) = 0$  for all  $X^* \in \mathfrak{t}^*$ . The differential on  $\Omega^p(M) \otimes \wedge^q \mathfrak{t}^*$  is then defined by

$$\begin{aligned} D(\omega \otimes (X_1^* \wedge \dots \wedge X_q^*)) &= d\omega \otimes (X_1^* \wedge \dots \wedge X_q^*) \\ &+ (-1)^p \sum_{i=1}^q (-1)^{i-1} (\omega \wedge F(X_i^*)) \otimes (X_1^* \wedge \dots \wedge \widehat{X_i^*} \wedge \dots \wedge X_q^*). \end{aligned} \quad (3.11)$$

It is easily verified that  $D^2 = 0$  using the fact that  $F$  is closed and the symmetry of  $F(X_i^*) \wedge F(X_j^*)$  in  $i$  and  $j$ .

Next, we show that  $f_A \circ D = d \circ f_A$ . On the one hand, we have, for  $\omega \otimes (X_1^* \wedge \dots \wedge X_q^*) \in \Omega^p(M) \otimes \wedge^q \mathfrak{t}^*$ ,

$$\begin{aligned} (d \circ f_A)(\omega \otimes (X_1^* \wedge \dots \wedge X_q^*)) &= d(\omega \wedge A(X_1^*) \wedge \dots \wedge A(X_q^*)), = d\omega \wedge A(X_1^*) \\ &\wedge \dots \wedge A(X_q^*) + (-1)^p \sum_{i=1}^q (-1)^{i-1} \omega \wedge A(X_1^*) \wedge \dots \wedge F(X_i^*) \wedge \dots \wedge A(X_q^*), \end{aligned}$$

while

$$\begin{aligned} (f_A \circ D)(\omega \otimes (X_1^* \wedge \dots \wedge X_q^*)) &= f_A(d\omega \otimes (X_1^* \wedge \dots \wedge X_q^*) + (-1)^p \sum_{i=1}^q (-1)^{i-1} \\ &(\omega \wedge F(X_i^*)) \otimes (X_1^* \wedge \dots \wedge \widehat{X_i^*} \wedge \dots \wedge X_q^*)) = d\omega \wedge A(X_1^*) \wedge \dots \wedge A(X_q^*) \\ &+ (-1)^p \sum_{i=1}^q (-1)^{i-1} (\omega \wedge F(X_i^*)) \wedge A(X_1^*) \wedge \dots \wedge \widehat{A(X_i^*)} \wedge \dots \wedge A(X_q^*). \end{aligned}$$

Obviously, the two expressions are equal, and thus we have a commutative diagram

$$\begin{array}{ccc}
 \bigoplus_{p+q=k} (\Omega^p(M) \otimes \wedge^q \mathfrak{t}^*) & \xrightarrow[f_A]{\cong} & \Omega^k(E)_{\text{inv}} \\
 D \downarrow & & \downarrow d \\
 \bigoplus_{p+q=k+1} (\Omega^p(M) \otimes \wedge^q \mathfrak{t}^*) & \xrightarrow[f_A]{\cong} & \Omega^{k+1}(E)_{\text{inv}}
 \end{array}$$

To summarize, if we denote

$$\Omega^k(M, \mathfrak{t}^*) \cong \bigoplus_{p+q=k} (\Omega^p(M) \otimes \wedge^q \mathfrak{t}^*), \tag{3.12}$$

we have a complex  $(\Omega(M, \mathfrak{t}^*), D)$  whose cohomology,  $H_D^k(M, \mathfrak{t}^*)$ , is isomorphic to the cohomology  $H^k(E)$ .<sup>3</sup>

**Remark.** It can be shown that while the Chern–Weil homomorphism depends on the choice of connection, the isomorphism on cohomology does not (see, e.g., [21]).

**Remark.** As mentioned before, the construction we have described here is merely a special case of a theory constructed by Chevalley, Chern, Weil, Cartan, and others (see, in particular, [21–24]). Given a Lie group  $G$ , its cohomology can be calculated as  $H(G) \cong \wedge P(\mathfrak{g})$ , where  $P(\mathfrak{g})$  is a set of primitive elements in the symmetric algebra  $S\mathfrak{g}^*$  and  $\mathfrak{g}$  is the Lie algebra of  $G$ . Now, given a principal  $G$ -bundle  $E \rightarrow M$ , there exists an isomorphism between the cohomology  $H(E)$  and the cohomology of a complex  $\Omega(M) \otimes \wedge P(\mathfrak{g})$  with Koszul differential  $D$ . The case of Abelian  $G$ , in this paper, is special since in that case the Chern–Weil homomorphism  $\Omega(M) \otimes \wedge P(\mathfrak{g}) \rightarrow \Omega(E)_{\text{inv}}$  is actually an isomorphism, unlike in the more general case.

### 3.2 Dimensionally reduced Gysin sequences

In order to write down a Gysin sequence for principal  $T$ -bundles  $\pi: E \rightarrow M$ , part of which is the pull-back map  $\pi^*: H^k(M) \rightarrow H^k(E) \cong H_D^k(M, \mathfrak{t}^*)$  given simply by  $\pi^*\omega = \omega \in \Omega^k(M) \otimes \wedge^0 \mathfrak{t}^*$ . We need to define a map  $\pi_*$  on  $H_D^k(M, \mathfrak{t}^*)$  by “stripping off” the component in  $\Omega^k(M) \otimes \wedge^0 \mathfrak{t}^*$ . Thus, in

---

<sup>3</sup>We denote this cohomology by  $H_D^k(M, \mathfrak{t}^*)$  to distinguish it from the cohomology  $H^k(M, \mathfrak{t}^*)$  of  $k$ -forms with coefficients in  $\mathfrak{t}^*$ .

addition to

$$\Omega^k(M, \mathfrak{t}^*) \cong \bigoplus_{p+q=k} (\Omega^p(M) \otimes \wedge^q \mathfrak{t}^*), \tag{3.13}$$

let us define the following truncated versions of this space (for  $0 \leq r \leq s \leq n$ )

$$\Omega^{k,(r,s)}(M, \mathfrak{t}^*) \cong \bigoplus_{i=r}^s (\Omega^{k-i}(M) \otimes \wedge^i \mathfrak{t}^*), \tag{3.14}$$

such that  $\Omega^k(M, \mathfrak{t}^*) \cong \Omega^{k,(0,n)}(M, \mathfrak{t}^*)$  and  $\Omega^k(M) \cong \Omega^{k,(0,0)}(M, \mathfrak{t}^*)$ . Also denote the basic truncation as

$$\overline{\Omega}^k(M, \mathfrak{t}^*) \equiv \Omega^{k,(1,n)}(M, \mathfrak{t}^*). \tag{3.15}$$

The differential  $D$  of equation (3.11) restricts to a differential on  $\Omega^{\bullet,(r,s)}(M, \mathfrak{t}^*)$  and defines a cohomology  $H_D^{k,(r,s)}(M, \mathfrak{t}^*)$ . In particular, we have  $H^k(M) \cong H_D^{k,(0,0)}(M, \mathfrak{t}^*)$  and  $H_D^k(M, \mathfrak{t}^*) \cong H_D^{k,(0,n)}(M, \mathfrak{t}^*)$ . We also define<sup>4</sup>

$$\overline{H}_D^{k-1}(M, \mathfrak{t}^*) \equiv H_D^{k,(1,n)}(M, \mathfrak{t}^*). \tag{3.16}$$

We then have

**Theorem 3.1.** *We have the following long exact sequence for cohomologies related to a principal torus bundle  $\pi: E \rightarrow M$ .*

$$\begin{array}{ccccccc} \longrightarrow & H^k(M) & \xrightarrow{\pi^*} & H_D^k(M, \mathfrak{t}^*) & \xrightarrow{\pi_*} & \overline{H}_D^{k-1}(M, \mathfrak{t}^*) & \\ & \xrightarrow{\delta} & H^{k+1}(M) & \longrightarrow & & & \end{array} \tag{3.17}$$

where the maps are given, on representatives, by

$$\begin{aligned} \pi^*: H^k(M) &\longrightarrow H_D^k(M, \mathfrak{t}^*), & \pi^*(\omega_k) &= (\omega_k, 0, \dots, 0), \\ \pi_*: H_D^k(M, \mathfrak{t}^*) &\longrightarrow \overline{H}_D^{k-1}(M, \mathfrak{t}^*), & \pi_*(\omega_k, \dots, \omega_0) &= (\omega_{k-1}, \dots, \omega_0), \\ \delta: \overline{H}_D^{k-1}(M, \mathfrak{t}^*) &\longrightarrow H^{k+1}(M), & \delta(\omega_{k-1}, \dots, \omega_0) &= F(X^*) \wedge \tilde{\omega}_{k-1}, \\ & & & \text{(if } \omega_{k-1} \equiv \tilde{\omega}_{k-1} \otimes X^* \text{)}. \end{aligned}$$

*Proof.* The proof is exactly analogous to the proof in Section 2.3 for the circle bundle case. The proof that equation (3.17) defines a complex is straightforward. The hardest part in the proof of exactness is at  $\overline{H}_D^{k-1}(M, \mathfrak{t}^*)$ . Thereto, suppose  $\omega = (\omega_{k-1}, \dots, \omega_0)$ , with  $\omega_{k-1} = \tilde{\omega}_{k-1} \otimes X^*$ , is a representative of a class in  $\overline{H}_D^{k-1}(M, \mathfrak{t}^*)$  such that  $\delta\omega = 0$  in  $H^{k+1}(M)$ , i.e., we have  $F(X^*) \wedge \tilde{\omega}_{k-1} = -d\nu$  for some  $\nu \in \Omega^k(M)$ . Then,  $\nu = (\nu, \omega_{k-1}, \dots, \omega_0) \in \Omega^k(M, \mathfrak{t}^*)$  satisfies  $D\nu = 0$  while  $\pi_*\nu = \omega$ .  $\square$

---

<sup>4</sup>Note the shift in degree, chosen such that  $\overline{H}_D^{k-1}(M, \mathfrak{t}^*) \cong H^{k-1}(M)$  for principal circle bundles.

In fact, Theorem 3.2 is easily generalized by considering different truncations as in equation (3.14). Namely

**Theorem 3.2.** *For  $0 \leq r < s \leq n$ , we have exact sequences*

$$\begin{CD} H_D^{k,(r,r)}(M, \mathfrak{t}^*) @>\pi^*>> H_D^{k,(r,s)}(M, \mathfrak{t}^*) @>\pi_*>> H_D^{k,(r+1,s)}(M, \mathfrak{t}^*) \\ @. @VV\delta V @. @. \\ @. @>> H_D^{k+1,(r,r)}(M, \mathfrak{t}^*) @>>> @. \end{CD} \tag{3.18}$$

where

$$\begin{aligned} \pi^*: H_D^{k,(r,r)}(M, \mathfrak{t}^*) &\longrightarrow H_D^{k,(r,s)}(M, \mathfrak{t}^*), & \pi^*(\omega_{k-r}) &= (\omega_{k-r}, 0, \dots, 0), \\ \pi_*: H_D^{k,(r,s)}(M, \mathfrak{t}^*) &\longrightarrow H_D^{k,(r+1,s)}(M, \mathfrak{t}^*), & \pi_*(\omega_{k-r}, \dots, \omega_{k-s}) &= \\ & & & (\omega_{k-r-1}, \dots, \omega_{k-s}), \\ \delta: H_D^{k,(r+1,s)}(M, \mathfrak{t}^*) &\longrightarrow H_D^{k+1,(r,r)}(M, \mathfrak{t}^*), \end{aligned}$$

$$\begin{aligned} \delta(\omega_{k-r-1}, \dots, \omega_{k-s}) &= \sum_i (-1)^{i-1} F(X_i^*) \wedge \tilde{\omega}_{k-r-1} \\ &\quad \otimes (X_1^* \wedge \dots \wedge \widehat{X_i^*} \wedge \dots \wedge X_{r+1}^*), \\ \text{if } \omega_{k-r-1} &= \tilde{\omega}_{k-r-1} \otimes (X_1^* \wedge \dots \wedge X_{r+1}^*). \end{aligned}$$

The basic Gysin sequence for  $\pi: E \rightarrow M$  is the one given by  $r = 0, s = n$  in Theorem 3.2. The other sequences can be viewed as truncated Gysin sequences (from the left if  $r > 0$  and from the right if  $s < n$ ). In particular, we note that for  $r = 0, s = 1$  we get the Gysin sequence of [25], which was discussed in the de-Rham framework in equation [6]. The image of  $i: H^{3,(0,1)}(M, \mathfrak{t}^*) \hookrightarrow H^3(E)$  by means of the Chern–Weil homomorphism  $f_A$  (equation (3.6)) is what was called a T-dualizable H-flux in [6].

## 4 Application to T-duality

### 4.1 T-duality for principal torus bundles

Suppose we are given a principal torus bundle  $\pi: E \rightarrow M$ , with curvature class  $[F] \in H^2(M)$  and with H-flux  $[H] \in H^3(E)$ . We will think of this as specifying an element  $([H], [F]) \in H^3(E) \oplus H^2(M)$ . We will choose a representative  $(H, F)$  which, upon dimensional reduction, can be viewed as a tuple  $((H_3, H_2, H_1, H_0), (F_2, 0, 0))$ , with  $H_i \in \Omega^i(M) \otimes \wedge^{3-i} \mathfrak{t}^*$ ,  $F_i \in \Omega^i(M) \otimes \wedge^{2-i} \mathfrak{t}$ , both closed under  $D$ .



The image of  $H \equiv (H_3, H_2, H_1, H_0)$ , in the Gysin sequence (3.17), is given by  $\widehat{F} \equiv (\widehat{F}_2, \widehat{F}_1, \widehat{F}_0) = (H_2, H_1, H_0)$ . This 3-tuple is supposed to classify (up to torsion) our T-dual object. Subsequently, one would expect the T-dual H-flux carried by this object to be given by the 4-tuple  $\widehat{H} \equiv (\widehat{H}_3, \widehat{H}_2, \widehat{H}_1, \widehat{H}_0) = (H_3, F_2, F_1, F_0) = (H_3, F_2, 0, 0)$ .

In the case  $H_1 = H_0 = 0$ , our T-dual object is characterized by  $\widehat{F} = (H_2, 0, 0)$ , where  $H_2 \in \Omega^2(M) \otimes \mathfrak{t}^*$  and hence can be identified with the curvature of a principal  $\widehat{T}$ -bundle  $\widehat{\pi}: \widehat{E} \rightarrow M$ . This case was analyzed in detail in [6], and the corresponding H-fluxes were dubbed ‘‘T-dualizable.’’

As soon as  $H_1 \neq 0$  or  $H_0 \neq 0$ , the T-dual object takes us outside the realm of principal torus bundles. The case  $H_1 \neq 0, H_0 = 0$ , was analyzed in detail in [8, 9]. In this case, the T-dual object was argued to be a continuous field of noncommutative tori over the base space  $M$ . Concretely, in this case,  $H_1$  determines an integral class  $[H_1] \in H^1(M, \wedge^2 \mathfrak{t}^*)$  or, equivalently, a homotopy class in  $[M, \wedge^2 \mathfrak{t}^*]$ . If  $f: M \rightarrow \wedge^2 \mathfrak{t}^*$  is a representative for this class, then the fibre over a point  $z$  in the base  $M$  is given by the noncommutative torus  $A_{f(z)}$ .<sup>5</sup> A global description of this field of noncommutative tori is given in terms of a crossed product algebra  $\mathfrak{A} \rtimes \mathbb{R}^n$ , such that  $\text{spec}(\mathfrak{A}) = E$  (see Appendix A for some details). The results of this paper suggest that appropriate equivalence classes of these objects are classified by a triple  $(F_2, F_1, 0)$ ,  $F_i \in \Omega^i(M) \otimes \wedge^{2-i} \mathfrak{t}$ , closed under  $D$ . It would be extremely interesting to establish this directly and to find a more ‘‘geometric’’ description of the T-dual object even in this case.

In the most general case,  $H_0 \neq 0$ , the T-dual object carries information about an integral class  $[H_0] \in H^0(M, \wedge^3 \mathfrak{t}^*)$ , i.e., a locally constant function with values in  $\wedge^3 \mathfrak{T}^*$ . It is well known that such classes often correspond to nonassociative structures (cf. [26, 27]). In [7], we construct a  $C^*$ -algebra  $\mathfrak{A}$ , with Dixmier–Douady invariant  $H = (H_3, H_2, H_1, H_0)$  and  $\text{spec}(\mathfrak{A}) = E$ , as a twisted induced algebra that carries a twisted action of  $\mathbb{R}^n$ . This generalizes a construction in [8, 9], which applies to  $H = (H_3, H_2, H_1, 0)$ , by introducing a twisting  $u$  given by the  $H_0$ -component.<sup>6</sup>

We argue that the T-dual is given by the twisted crossed product  $\mathfrak{A} \rtimes_u \mathbb{R}^n$  and that this twisted crossed product can be interpreted as a continuous

<sup>5</sup>Locally, one can think of the commutativity parameter of the torus as given by the components of the B-field in the torus directions.

<sup>6</sup>Note that while the analysis in this paper is carried out in the de-Rham cohomology, the component  $H_0$  does not carry torsion, it can in fact simply be described as the pull-back of  $H$  from  $H^3(E, \mathbb{Z})$  to  $H^3(\mathbb{T}, \mathbb{Z})$  under fibre inclusion  $i: \mathbb{T} \hookrightarrow E$ . Hence, the construction of [7], by adding an  $H_0$  part, is valid for arbitrary integer H-fluxes in  $H^3(E, \mathbb{Z})$ .

field of noncommutative, nonassociative tori over the base  $M$ . It would be interesting to establish directly and precisely which objects are classified by a 3-tuple  $(F_2, F_1, F_0)$ ,  $F_i \in \Omega^i(M) \otimes \wedge^{2-i}\mathfrak{t}$ , closed under  $D$ , and in which sense the T-dual H-flux can be interpreted as a flux on that object.

### 4.2 The T-duality group

The T-duality group is  $O(n, n; \mathbb{Z})$ . It turns out that the action of the T-duality group on the tuples

$$((H_3, H_2, H_1, H_0), (F_2, F_1, F_0)),$$

where  $H_i \in \Omega^i(M, \wedge^{3-i}\mathfrak{t}^*)$  for  $i = 0, 1, 2, 3$  and  $F_i \in \Omega^i(M, \wedge^{2-i}\mathfrak{t})$  for  $i = 0, 1, 2$  have very simple expression.

Given  $g \in O(n, n; \mathbb{Z})$ , it acts naturally on  $\mathfrak{t}^* \oplus \mathfrak{t}$ , preserving the natural quadratic form. It induces an action on  $\wedge^i \mathfrak{t}^* \oplus \wedge^i \mathfrak{t}$ . Let us denote by  $g_i = \wedge^{3-i} g$  the action of  $O(n, n; \mathbb{Z})$  on  $\wedge^{3-i}\mathfrak{t}^* \oplus \wedge^{3-i}\mathfrak{t}$  (for  $i = 0, 1, 2$ ). We have a corresponding induced action on  $\Omega^i(M, \wedge^{3-i}\mathfrak{t}^*) \oplus \Omega^i(M, \wedge^{3-i}\mathfrak{t})$  for  $i = 0, 1, 2$ , which we also denote by  $g_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ .

The action of  $g \in O(n, n; \mathbb{Z})$  on  $((H_3, H_2, H_1, H_0), (F_2, F_1, F_0))$  is then explicitly by

$$\begin{aligned} g \cdot ((H_3, H_2, H_1, H_0), (F_2, F_1, F_0)) &\cong \left( H_3, g_2 \cdot \begin{pmatrix} H_2 \\ F_2 \end{pmatrix}, g_1 \cdot \begin{pmatrix} H_1 \\ F_1 \end{pmatrix}, g_0 \cdot \begin{pmatrix} H_0 \\ F_0 \end{pmatrix} \right) \\ &\cong ((H_3, A_2 H_2 + B_2 F_2, A_1 H_1 + B_1 F_1, A_0 H_0 + B_0 F_0), (C_2 H_2 + D_2 F_2, C_1 H_1 \\ &\quad + D_1 F_1, C_0 H_0 + D_0 F_0)). \end{aligned} \tag{4.1}$$

Note that the action of the T-duality group resembles fractional linear transformations.

In particular, if we start out with a principal torus bundle with H-flux, then the action of  $g \in O(n, n; \mathbb{Z})$  on  $((H_3, H_2, H_1, H_0), (F_2, 0, 0))$  is given explicitly by

$$((H_3, A_2 H_2 + B_2 F_2, A_1 H_1, A_0 H_0), (C_2 H_2 + D_2 F_2, C_1 H_1, C_0 H_0)). \tag{4.2}$$

The T-duality transformation discussed in Section 4.1 corresponds to the element  $g \in O(n, n, \mathbb{Z})$  given by

$$g = \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}. \tag{4.3}$$

### 4.3 Example

The simplest example of the various cases of T-duality is given by the three torus  $\mathbb{T}^3$ , with H-flux  $k dV = k dx \wedge dy \wedge dz$ , which can be considered as a principal torus bundle over a (strictly) lower dimensional torus in three different ways

- (1)  $(\mathbb{T}^3, k dx \wedge dy \wedge dz)$  considered as a trivial circle bundle over  $\mathbb{T}^2$ . The T-dual of  $(\mathbb{T}^3, k dx \wedge dy \wedge dz)$  is the nilmanifold  $(H_{\mathbb{R}}/H_{\mathbb{Z}}, 0)$ , where  $H_{\mathbb{R}}$  is the three-dimensional Heisenberg group and  $H_{\mathbb{Z}}$  the lattice in it defined by

$$H_{\mathbb{Z}} = \left( \begin{array}{ccc} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) : x, y, z \in \mathbb{Z} \tag{4.4}$$

- (2)  $(\mathbb{T}^3, k dx \wedge dy \wedge dz)$  considered as a trivial  $\mathbb{T}^2$ -bundle over  $\mathbb{T}$ . The T-dual of  $(\mathbb{T}^3, k dx \wedge dy \wedge dz)$  is a continuous field of stabilized noncommutative tori,  $C^*(H_{\mathbb{Z}}) \otimes \mathcal{K}$ , since

$$H_1 \sim \int_{\mathbb{T}^2=\{(y,z)\}} k dx \wedge dy \wedge dz = k dx \neq 0.$$

- (3)  $(\mathbb{T}^3, k dx \wedge dy \wedge dz)$  considered as a trivial  $\mathbb{T}^3$ -bundle over a point. The T-dual of  $(\mathbb{T}^3, k dx \wedge dy \wedge dz)$  is a nonassociative torus,  $A_{\phi}$ , where  $\phi$  is the tricharacter associated to  $k dx \wedge dy \wedge dz$ , since

$$H_0 \sim \int_{\mathbb{T}^3} k dx \wedge dy \wedge dz = k \neq 0.$$

Other examples, treated in previous papers, such as the nilmanifold [4] and the group manifold, viewed as a principal torus bundle over the flag manifold [6], both with H-flux, can be re-interpreted similarly.

## 5 Conclusions and further generalizations

In this paper, we have shown how Gysin sequences encode the global properties of T-duality, building on previous work [4–6]. We have constructed a Gysin sequence for principal torus bundles, using what is known in physics as dimensional reduction and in mathematics as a special case of the Chern–Weil homomorphism, and have used it to determine the T-dual of a principal torus bundle with arbitrary H-flux, generalizing the special cases considered in [4–6, 8, 9]. The algebraic structures arising in the T-dual have been discussed in a separate paper [7]. Gysin sequences are useful in a more general context as well, e.g., in cases where we do not have a principal torus bundle,

but only an infinitesimally free torus action (e.g., Seifert fibred spaces, in which case the base manifold is a two-dimensional orbifold).

Once one realizes that the T-dual of a principal torus bundle, characterized by a curvature  $F = (F_2, 0, 0)$ ,  $F_2 \in \Omega^2(M) \otimes \mathfrak{t}$ , with background H-flux  $H = (H_3, H_2, H_1, H_0)$ ,  $H_i \in \Omega^i(M) \otimes \wedge^{3-i}\mathfrak{t}^*$ , is an object characterized by the 3-tuple  $(H_2, H_1, H_0)$ , it seems natural to somehow study a Gysin sequence related to such an object and to interpret the dual tuple  $\tilde{H} = (H_3, F_2, 0, 0)$  as an H-flux on this dual object by reversing the argument in the Gysin sequence. This we leave as a problem for further investigation. From the physics point of view, the results of this paper show that if one is trying to build a manifestly T-duality invariant description of M-theory, one is forced to include not only noncommutative structures but also nonassociative structures into the game.

Another obvious extension of this work is to incorporate torsion by generalizing the analysis of this paper from de-Rham cohomology to integer, i.e., Čech cohomology. Although there are some subtleties, we believe this is possible and we hope to come back to this in a future publication (cf. [28, 29] for some relevant results in this direction). One could even wonder whether a similar dimensional reduction might be possible at the level of K-theory.

Maybe one of the most striking results of this paper is that we are led to T-dual objects which appear to be quite “pathological”—noncommutative or even nonassociative—but nevertheless are characterized by forms on the base manifold. Thus one realizes that this approach might be useful in getting a hold on even more exotic structures such as torus fibrations (manifolds with nonfree torus actions), as long as the base manifold has a well-understood theory of differential forms and de-Rham cohomology.

## Appendix A Duality from the operator algebraic perspective

In this paper, as well as previous ones [4, 6], we have attempted to sketch the geometric counterpart, as well as applications to T-duality, of results established in the context of, in particular, operator algebras, derived in a series of beautiful papers (see, in particular, [7–9, 25, 30–35]). For the benefit of readers wishing to familiarize themselves with the algebraic perspective, we include this appendix.

We begin with a reformulation of [4, 5]. Given a circle bundle  $S^1 \hookrightarrow E \rightarrow M$  over  $M$ , and a closed, integral 3-form  $H$  on  $E$ , then there is a unique

algebra bundle  $\mathcal{E} \rightarrow E$  with fibre equal to the algebra of compact operators  $\mathcal{K}$  and Dixmier–Douady invariant equal to  $[H] \in H^3(E, \mathbb{Z})$ . Then, the space of continuous sections  $\mathfrak{A} = C(E, \mathcal{E})$  is a stable, continuous trace  $C^*$ -algebra with spectrum  $\text{spec}(\mathfrak{A}) = E$ . The  $\mathbb{R}$  action on  $E$  lifts uniquely to an  $\mathbb{R}$  action on  $\mathfrak{A}$  (cf. [30]), and one has a commutative diagram,

$$\begin{array}{ccc}
 & \text{spec}(\mathfrak{A} \rtimes \mathbb{Z}) & \\
 & \swarrow p \qquad \searrow \hat{p} & \\
 \text{spec}(\mathfrak{A}) & & \text{spec}(\mathfrak{A} \rtimes \mathbb{R}) \\
 & \swarrow \pi \qquad \searrow \hat{\pi} & \\
 & \text{spec}(\mathfrak{A})/\mathbb{R} &
 \end{array} \tag{A.1}$$

That is,  $\mathfrak{A} \rtimes \mathbb{Z}$  and  $\mathfrak{A} \rtimes \mathbb{R}$  are also continuous trace  $C^*$ -algebras with  $\text{spec}(\mathfrak{A} \rtimes \mathbb{R}) = \hat{E}$ , a circle bundle over  $M = \text{spec}(\mathfrak{A})/\mathbb{R}$ , such that  $c_1(\hat{E}) = \pi_*[H]$  and the Dixmier–Douady invariant of  $\mathfrak{A} \rtimes \mathbb{R}$  is  $[\hat{H}] \in H^3(\hat{E}, \mathbb{Z})$ , such that  $c_1(E) = \hat{\pi}_*[\hat{H}]$ , and  $\text{spec}(\mathfrak{A} \rtimes \mathbb{Z}) = E \times_M \hat{E}$  is the correspondence space. Now the T-dual of the continuous trace  $C^*$ -algebra  $\mathfrak{A} \rtimes \mathbb{R}$  is the crossed product  $(\mathfrak{A} \rtimes \mathbb{R}) \rtimes \hat{\mathbb{R}}$ , which by Takai duality is Morita equivalent to  $\mathfrak{A}$ , and in particular,  $\text{spec}((\mathfrak{A} \rtimes \mathbb{R}) \rtimes \hat{\mathbb{R}}) = \text{spec}(\mathfrak{A})$ . That is, applying T-duality twice gets us back to where we started off. As a result, we also get the horizontal isomorphisms (Connes–Thom isomorphisms in K-theory and in cyclic homology) and the commutativity of the diagram,

$$\begin{array}{ccc}
 K_\bullet(\mathfrak{A}) & \xrightarrow[\cong]{T_!} & K_\bullet(\mathfrak{A} \rtimes \mathbb{R}) \\
 Ch_H \downarrow & & \downarrow Ch \\
 HP_\bullet(\mathfrak{A}^\infty) & \xrightarrow[\cong]{T_*} & HP_\bullet(\mathfrak{A}^\infty \rtimes \mathbb{R})
 \end{array} \tag{A.2}$$

where  $\mathfrak{A}^\infty = C^\infty(E, \mathcal{E}^\infty)$  is a smooth subalgebra of  $\mathfrak{A}$ . This motivates the definition of the T-dual of a principal torus bundle with H-flux, when the T-dual is not classical, viz, when the T-dual is not another principal torus bundle with H-flux.

In dealing with higher rank torus bundles, we will use the notation  $G = \mathbb{R}^n$ ,  $N = \mathbb{Z}^n$ , and  $T = G/N = \mathbb{T}^n$ . Now, given a torus bundle  $T \hookrightarrow E \rightarrow M$  over  $M$ , and a closed, integral 3-form  $H$  on  $E$ , then there is a unique algebra bundle  $\mathcal{E} \rightarrow E$  with fibre equal to the algebra of compact operators  $\mathcal{K}$  and Dixmier–Douady invariant equal to  $[H] \in H^3(E, \mathbb{Z})$ . Let  $\mathfrak{A}$  denote the space of all continuous sections of  $\mathcal{E}$ , which is a stable, continuous trace  $C^*$ -algebra with spectrum  $\text{spec}(\mathfrak{A}) = E$ . Now if (and only if) the restriction of  $[H]$  to each fibre is trivial (i.e.,  $H_0 = 0$ ), then the  $G$ -action on  $E$  lifts to an  $G$ -action on the total space  $\mathcal{E}$ , i.e., there is an induced  $G$ -action on  $\mathfrak{A}$ . The lift of the  $G$ -action on  $E$  to  $\mathfrak{A}$  is not unique, cf. [8,9], but this does not affect the  $K$ -theory. The  $T$ -dual of the torus bundle with  $H$ -flux  $(E, H)$  is then defined as  $\mathfrak{A} \rtimes G$ . Note that this is in general not a stable, continuous trace  $C^*$ -algebra. Then we have an analogous diagram for this situation as in (A.2). More importantly, by Takai duality  $(\mathfrak{A} \rtimes G) \rtimes \widehat{G}$  is Morita equivalent to  $\mathfrak{A}$ , and in particular,  $\text{spec}((\mathfrak{A} \rtimes G) \rtimes \widehat{G}) = \text{spec}(\mathfrak{A})$ . That is, applying  $T$ -duality twice gets us again back to where we started off, as was established in [8,9].

Given a torus bundle  $T \hookrightarrow E \rightarrow M$  over  $M$ , and a closed, integral 3-form  $H$  on  $E$ , let  $\mathcal{E} \rightarrow E$  be the unique algebra bundle with fibre equal to the algebra of compact operators  $\mathcal{K}$  and Dixmier–Douady invariant equal to  $[H] \in H^3(E, \mathbb{Z})$ . Let  $\mathfrak{A}$  denote the space of all continuous sections of  $\mathcal{E}$ . Now, if the restriction of  $[H]$  to each fibre is not trivial (i.e.,  $H_0 \neq 0$ ), then the  $G$ -action on  $E$  lifts to a twisted  $G$ -action on  $\mathcal{E}$  and hence on  $\mathfrak{A}$  [7]. Again, this lifted action is not unique. The  $T$ -dual of the torus bundle with  $H$ -flux  $(E, H)$  is defined as the twisted crossed product  $\mathfrak{A} \rtimes_u G$ . The twisted crossed product  $\mathfrak{A} \rtimes_u G$  is in general a nonassociative, noncommutative algebra, which is a field of nonassociative tori on the base. By twisted Takai duality [7],  $(\mathfrak{A} \rtimes_u G) \rtimes_{\widehat{u}} \widehat{G}$  is Morita equivalent to  $\mathfrak{A}$ , and in particular,  $\text{spec}((\mathfrak{A} \rtimes_u G) \rtimes_{\widehat{u}} \widehat{G}) = \text{spec}(\mathfrak{A})$ . That is, applying  $T$ -duality twice gets us again back to where we started off, as was established in [7].

## Acknowledgments

P.B. would like to thank the Erwin Schrödinger Institute in Vienna, the CIRM in Luminy, and the Aspen Center for Physics for hospitality and financial support during various stages of this project. K.C.H would like to thank the University of Adelaide for hospitality. P.B. and V.M. were financially supported by the Australian Research Council. We would also like to thank Aaron Bergman for correspondence on an earlier version of the manuscript.

## References

- [1] J. Polchinski, *String theory, Vol. 1 and 2*, Cambridge, Cambridge University Press, 2000.
- [2] C.V. Johnson, *D-branes*, Cambridge, Cambridge University Press, 2003.
- [3] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in String Theory*, Phys. Reports **244** (1994), 77–202, hep-th/9401139.
- [4] P. Bouwknegt, J. Evslin and V. Mathai, *T-duality: Topology change from H-flux*, Commun. Math. Phys. **249** (2004), 383–415, hep-th/0306062.
- [5] P. Bouwknegt, J. Evslin and V. Mathai, *On the topology and H-flux of T-dual manifolds*, Phys. Rev. Lett. **92** (2004), 181601, hep-th/0312052.
- [6] P. Bouwknegt, K. Hannabuss and V. Mathai, *T-duality for principal torus bundles*, J. High Energy Phys. **03** (2004), 018, hep-th/0312284.
- [7] P. Bouwknegt, K. Hannabuss and V. Mathai, *Nonassociative tori and applications to T-duality*, Commun. Math. Phys., to appear, hep-th/0412092.
- [8] V. Mathai and J. Rosenberg, *T-duality for torus bundles with H-fluxes through noncommutative topology*, Comm. Math. Phys. **253** (2005), 705–721, hep-th/0401168.
- [9] V. Mathai and J. Rosenberg, *On mysteriously missing T-duals, H-flux and the T-duality group*, hep-th/0409073, to appear in Proceedings of the XXXIII International Conference of Differential Geometric Methods in Mathematical Physics, (August 2005), editors Mo-Lin Ge and Weiping Zhang, World Scientific 2006.
- [10] L. Cornalba and R. Schiappa, *Nonassociative star product deformations for D-brane worldvolumes in curved backgrounds*, Commun. Math. Phys. **225** (2002), 33–66, hep-th/0101219.
- [11] U. Bunke and T. Schick, *On the topology of T-duality*, math.GT/0405132.
- [12] R. Donagi and T. Pantev, *Torus fibrations, gerbes and duality*, math.AG/0306213.
- [13] D.A. Lowe, H. Nastase and S. Ramgoolam, *Massive IIA string theory and matrix theory compactification*, Nucl. Phys. **B667** (2003) 55–89, hep-th/0303173.
- [14] S. Kachru, M.B. Schulz, P.K. Tripathy and S.P. Trivedi, *New supersymmetric string compactifications*, J. High Energy Phys. **03** (2003) 061, hep-th/0211182.
- [15] C. Hull, *A geometry for non-geometric string backgrounds*, hep-th/0406102.

- [16] T. Buscher, *A symmetry of the string background field equations*, Phys. Lett. **B194** (1987), 59–62;  
ibid., *Path integral derivation of quantum duality in nonlinear sigma models*, Phys. Lett. **B201** (1988), 466–472.
- [17] R. Bott and L.W. Tu, *Differential forms in algebraic topology*, New York, Springer Verlag, 1982.
- [18] F.W. Kamber and P. Tondeur, *Duality theorems for foliations*, Astérisque **116** (1984) 108–116. Transversal structure of foliations (Toulouse, 1982).
- [19] A. Bergman and U. Varadarajan, *Loop groups, Kaluza-Klein reduction and M-theory*, J. High Energy Phys. **06** (2005) 043, hep-th/0406218.
- [20] K. Hori, *D-branes, T-duality, and index theory*, Adv. Theor. Math. Phys. **3** (1999), 281–342, hep-th/9902102.
- [21] W. Greub, S. Halperin and R. Vanstone, *Connections, curvature, and cohomology*, Vol. 2 and 3, New York, Academic Press, 1976.
- [22] H. Cartan, *Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère in groupe de Lie*, Colloque de Topologie (1950), 15–27 (reprinted in [24]).
- [23] H. Cartan, *La transgression dans un groupe de Lie et dans un espace fibré principal*, Colloque de Topologie (1950), 57–71 (reprinted in [24]).
- [24] V.W. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Berlin, Springer Verlag, 1999.
- [25] J. Packer, I. Raeburn and D.P. Williams *The equivariant Brauer group of principal bundles*, J. Oper. Theory **36** (1996), 73–105.
- [26] R. Jackiw, *Three-cocycles in mathematics and physics*, Phys. Rev. Lett. **54** (1985), 159–162.
- [27] A.L. Carey, *The origin of three-cocycles in quantum field theory*, Phys. Lett. **B194** (1987), 267–270.
- [28] M. Franz, *Koszul duality and equivariant cohomology for tori*, math.AT/0301083.
- [29] M. Franz, *Koszul duality and equivariant cohomology*, math.AT/0307115.
- [30] I. Raeburn and J. Rosenberg, *Crossed products of continuous-trace  $C^*$ -algebras by smooth actions*, Trans. Amer. Math. Soc. **305** (1988), 1–45.
- [31] I. Raeburn and D.P. Williams, *Moore cohomology, principal bundles, and actions of groups on  $C^*$  algebras*, Indiana U. Math. J. **40** (1991), 707–740.



- [32] I. Raeburn and D.P. Williams, *Topological invariants associated with the spectrum of crossed product  $C^*$ -algebras*, J. Funct. Anal. **116** (1993), 245–276.
- [33] I. Raeburn and D.P. Williams, *Dixmier–Douady classes of dynamical systems and crossed products*, Can. J. Math. **45** (1993), 1032–1066.
- [34] J. Phillips and I. Raeburn, *Crossed products by locally unitary automorphism groups and principal bundles*, J. Oper. Theory **11** (1984), 215–241.
- [35] D. Crocker, A. Kumjian, I. Raeburn and D. Williams, *An equivariant Brauer group and actions of groups on  $C^*$ -algebras*, J. Funct. Anal. **146** (1997), 151–184.

