Adv. Theor. Math. Phys. **9** (2005) 129–172

A gluing construction for non-vacuum solutions of the Einstein-constraint equations

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Abstract

We extend the conformal gluing construction of Isenberg *et al.* [19] by establishing an analogous gluing result for field theories obtained by minimally coupling Einstein's gravitational theory with matter fields. We treat classical fields such as perfect fluids and the Yang–Mills equations as well as the Einstein–Vlasov system, which is an important example

e-print archive: http://lanl.arXiv.org/abs/gr-qc/0501083

coming from kinetic theory. In carrying out these extensions, we extend the conformal gluing technique to higher dimensions and codify it in such a way as to make more transparent where it can, and cannot, be applied. In particular, we show exactly what criteria need to be met in order to apply the construction, in its present form, to any other non-vacuum field theory.

1 Introduction

One of the effective tools which have been recently developed and employed for the construction and study of initial data for solutions of Einstein's gravitational field equations is the method of gluing. This is a technique which has had a long and fruitful history in geometric analysis, but which has been only recently successfully applied to general relativity. The idea of the gluing method (in its simplest form) is that given a pair of sets of initial data which satisfy the Einstein constraint equations, we may use it to construct a new set of initial data, which (a) lives on a connected sum of the manifolds of the given sets of data, (b) solves the constraint equations and (c) closely approximates the original sets of data on the parts of the new manifold, which correspond to the original manifolds (i.e., away from the tubular "neck" of the connected sum). The work of Isenberg *et al.* [19, 20] shows that this sort of gluing can be carried out for fairly general sets of *vacuum* initial data. That work also it describes a number of applications of gluing, such as producing multi-black hole initial data, adding wormholes to given sets of data and showing that an arbitrary closed manifold with a point removed always admits both asymptotically Euclidean and asymptotically hyperbolic solutions of the vacuum constraint equations.

In this work, we show that the gluing results and the gluing applications which are discussed for vacuum initial data in [19] can be extended to fairly general sets of non-vacuum data as well. We do this here for a number of special cases, including Einstein–Maxwell, Einstein–Yang–Mills, Einstein-fluids and Einstein-Vlasov, as well as for any of these theories with a cosmological constant added. We also discuss the features which a general field theory should have if, when it is coupled to Einstein's equations, solutions of the corresponding constraint equations should allow gluing.

As for vacuum data, the gluing procedure applied to non-vacuum data relies quite heavily on the conformal method for obtaining solutions of the constraint equations. Thus, after commenting in Section 2 on the general form of the constraint equations for non-vacuum field theories, we proceed in that section to describe the application of the conformal method to such theories. For the non-vacuum theories listed earlier, Einstein–Maxwell, etc., the conformal method leads to determined sets of (non-linear) elliptic equations, which, at least for constant mean curvature (CMC) data, are readily analyzed for solubility. This is not true for all non-vacuum field theories; indeed, for any non-vacuum field theory which involves derivative coupling (e.g., the Einstein-vector–Klein–Gordon theory), the conformal method leads to equations which are intractable using known techniques [18]. Thus, the gluing procedure which we use does not work for such theories.

Note that in our discussion of the conformal method, we focus on the situation in which the initial data have CMC. We do this because, as with the vacuum case, even when the constraint solutions we are gluing together have non-CMC, the analysis we rely on to carry out the gluing is based primarily on the CMC version of the conformal treatment of the constraints [20].

While the details of the analysis differ from one non-vacuum field theory to another, the basic steps of the gluing procedure are largely the same for those non-vacuum field theories with the appropriate form for the constraint equations (in conformal form). Thus, for these field theories, we can present a general discussion of these basic steps: (i) conformal blowup at the gluing points; (ii) connected sum of the manifolds and patching of the conformal metrics; (iii) patching of the non-gravitational fields, solution of the non-gravitational constraints and deformation estimates; (iv) patching of the extrinsic curvatures, solution of the momentum constraint and deformation estimates; (v) patching of the conformal factor, solution of the Hamiltonian constraint (in Lichnerowicz form) and deformation estimates; (vi) conformal recomposition of the initial data, forming the glued solution (Section 3). Also in that section, we outline the general analysis which leads to a proof that the gluing can be carried out for appropriate sets of initial data for these theories. The section culminates with a general statement of our gluing results for general theories.

We discuss some of the details of gluing for various particular non-vacuum field theories in Section 4. Included are discussions of the field theories listed earlier. However, to avoid repetition, we focus primarily on two of them: Einstein perfect fluids and Einstein–Yang–Mills.

Although the physically important versions of most of the field theories we discuss here are defined on $3+1$ dimensional spacetimes, our results hold for arbitrary dimension. Hence, we state most of our formulations and our results for $n+1$ dimensional spacetimes where $n \geq 3$.

A very different and important type of gluing construction has been developed by Corvino and Schoen [10, 12] and adapted and applied by Chruściel and Delay [6, 7]. These results exploit the underdetermined nature of the constraint equations as opposed to using the conformal method to convert them into a determined system. This has led to a number of remarkable results beginning with the existence of a large class of asymptotically Euclidean spacetimes, which are exactly Schwarzchild near infinity. Recently, by combining these techniques with the results of [19] and with the previous work of Bartnik [2], Chruściel *et al.* [8, 9] have obtained a gluing construction of the type described here, which is optimal in two distinct ways. First, it applies to *generic* initial data sets and the required (generically satisfied) hypotheses are geometrically and physically natural. Second, the construction is completely *local* in the sense that the initial data are left unaltered on the complement of arbitrarily small neighborhoods of the points about which the gluing takes place. Using this construction, they have been able to establish the existence of cosmological, maximal globally hyperbolic, vacuum spacetimes with no CMC space-like Cauchy surfaces. Except for the case of generic non-gravitational fields described entirely by an energy density function ρ and a current density vector field J (satisfying a strict energy condition $\rho > |J|$), the Corvino–Schoen techniques have not yet been generalized away from the vacuum case. It is, however, expected that this can be done; it would then follow that the gluing theorems obtained for vacuum data in [9] would extend to non-vacuum data.

We end this introduction by remarking that we have, for simplicity, restricted ourselves here to the consideration of initial data on compact manifolds (i.e., the *cosmological* setting). The gluing results presented here have analogous statements, which are valid for either asymptotically Euclidean or asymptotically hyperbolic initial data sets. The required adaptations, which are similar to those discussed in detail in [19], are left to the interested reader.

2 Constraint equations and conformal method

We restrict our attention in this work to classical field theories, which are obtained by minimally coupling a spacetime covariant field theory to Einstein's gravitational theory and which have a well-posed Cauchy formulation. For such theories, if one is given a set of initial data which satisfy the constraint equations corresponding to that theory, one can always evolve to obtain a spacetime solution of the full PDE system. Our main interest here is primarily in the construction of solutions of the constraints.

For the theories we are interested in here, the initial data consist of a choice of an n-dimensional manifold Σ^n, together with a Riemannian metric

γ, a symmetric tensor K and a collection of non-gravitational fields which we collectively label $\mathcal{F}$, all specified on Σ^n. These non-gravitational fields are usually, but not always, sections of a bundle over Σ^n. The constraint equations which these initial data must satisfy generally take the form

$$\operatorname{div} K - d \operatorname{tr} K = J(\mathcal{F}, \gamma) \tag{2.1}$$

$$R_\gamma - |K|^2_\gamma + (\operatorname{tr} K)^2 = 2\rho(\mathcal{F}, \gamma) \tag{2.2}$$

$$\mathcal{C}(\mathcal{F}, \gamma) = 0, \tag{2.3}$$

where J is the current density of the non-gravitational fields, ρ their energy density and $\mathcal{C}$ denotes the set of additional constraints that come from the non-gravitational part of the theory.[1] Note that the first of these constraint equations is known as the momentum constraint, the second is often referred to as the Hamiltonian constraint while the last are collectively labeled as the non-gravitational constraints.

As an example, for the Einstein–Maxwell theory in $3 + 1$ dimensions, the non-gravitational fields consist of the electric and magnetic vector fields E and B, respectively, we have $\rho = \frac{1}{2}(|E|^2_\gamma + |B|^2_\gamma)$ and $J = (E \times B)_\gamma$ and we have the extra (non-gravitational) constraints $\operatorname{div}_\gamma LE = 0$ and $\operatorname{div}_\gamma FB = 0$.

The system of constraint Eqs. (2.1) to (2.3), vacuum or non-vacuum, is an underdetermined PDE system. The idea of the conformal method is to split the initial data fields into two sets of fields: the "conformal data," which is freely chosen, and the "determined data," which is to be found by solving the constraints. In the familiar vacuum case [5], the conformal data consist of the manifold Σ^n, a Riemannian metric γ, a divergence-free trace-free symmetric tensor σ and a function τ, whereas the determined data consist of a positive definite function ϕ and a vector field W. With the conformal data $(\Sigma^n, \gamma, \sigma, \tau)$ chosen, one determines (ϕ, W) by solving the equations

$$\operatorname{div}_\gamma (\mathcal{D}W) = \frac{n-1}{n}\phi^{q+2}\nabla\tau \tag{2.4}$$

and

$$\Delta_\gamma \phi - \frac{1}{q(n-1)}R_\gamma\phi + \frac{1}{q(n-1)}\left|\sigma + \mathcal{D}W\right|^2_\gamma \phi^{-q-3} - \frac{1}{qn}\tau^2\psi^{q+1} = 0, \tag{2.5}$$

where $\mathcal{D}W$ is the conformal Killing (CK) operator, with coordinate representation

$$\mathcal{D}W_{ab} = \nabla_a W_b + \nabla_b W_a - \frac{2}{n}\gamma_{ab}\nabla_c W^c, \tag{2.6}$$

[1] If we use T to denote the stress-energy tensor for the non-gravitational fields and we use $e_\perp$ to denote the unit normal to the hypersurface Σ embedded in the spacetime solution generated from the initial data $(\Sigma^n, \gamma, K, \mathcal{F})$, then $J = -T(e_\perp, \ )$ and $\rho = T(e_\perp, e_\perp)$. Note that we have chosen units so that $8\pi G = 1 = c$.

and $q = \frac{4}{n-2}$ is a dimensional constant. If for a given choice of the conformal data one does determine W and ϕ satisfying (2.4) and (2.5), then the fields

$$\tilde{\gamma} = \phi^q \gamma \tag{2.7}$$

$$\tilde{K} = \phi^{-2}(\sigma + \mathcal{D}W) + \frac{\tau}{n}\phi^q \gamma \tag{2.8}$$

satisfy the vacuum constraints, consisting of (2.1) and (2.2) with vanishing ρ and J. Note that the explicit form of Eqs. (2.4) and (2.5) is determined by the form of the field decomposition, expressed in (2.7) and (2.8). The explicit choice of the form of the field decomposition is in turn determined to a large extent by the two identities (for $\tilde{\gamma} = \phi^q \gamma$)

$$R_{\tilde{\gamma}} = -\phi^{-q-1}(q(n-1)\Delta_\gamma \phi - R_\gamma \phi) \tag{2.9}$$

(with $q = \frac{4}{n-2}$ being the unique exponent, which avoids $|\nabla\phi|^2$ terms in (2.9)) and

$$\nabla^a_{\tilde{\gamma}}(\phi^{-2} B_{ab}) = \phi^{-q-2}\nabla^a_\gamma B_{ab}, \tag{2.10}$$

which holds for any trace-free tensor B.

The extent to which there exist unique solutions to Eqs. (2.4) and (2.5) for various classes of conformal data has been studied extensively; see [3] for a recent review. Here, we note primarily that while although the issue is fairly well understood for CMC conformal data and near-CMC conformal data, very little is known more generally. Consequently, the earliest gluing results for the vacuum constraints [19] pertain to CMC data sets, and later results rely primarily on CMC analysis. Hence, we focus on CMC data sets here.

A set of initial data has CMC if tr $\tilde{K} = \tau$ is constant on Σ^n. This condition significantly simplifies the analysis of the vacuum constraint equations, because it decouples Eqs. (2.4) and (2.5). As $W \equiv 0$ is a solution to (2.4) with vanishing right-hand side, the analysis reduces to first finding a divergence-free trace-free symmetric tensor and then solving (2.5) (often referred to as the Lichnerowicz equation) with $\mathcal{D}W = 0$.

To extend the conformal method to the non-vacuum constraints, with non-gravitational fields $\mathcal{F}$ present, one needs to extend the field decomposition (2.7) and (2.8) to $\mathcal{F}$. The chief criteria generally used to decide how to do this are the following [18]: (a) In the CMC case, the constraint system should be semi-decoupled, in the sense that one can first solve the non-gravitational constraints independently of W and ϕ, then one can solve the momentum constraint for W independently of ϕ, and then finally, one solves the Lichnerowicz equation for ϕ. (b) The addition of the non-gravitational terms to the Hamiltonian constraint should not result in the Lichnerowicz equation containing either derivatives of ϕ or those powers of ϕ which would lead to insurmountable difficulties in the subsequent analysis.

To make these criteria more precise, let us presume that the field decomposition for the non-gravitational fields is defined by an action Φ of the group of conformal factors ($C^\infty_+(\Sigma)$ under multiplication) on the set of matter fields, so that the physical fields $\tilde{\mathcal{F}}$ which together with $\tilde{\gamma}$ and $\tilde{K}$ must satisfy the constraints (2.1) to (2.3) are given by $\tilde{\mathcal{F}} = \Phi(\mathcal{F}, \phi)$, where $\Phi(\mathcal{F}, 1) = \mathcal{F}$ and $\Phi(\Phi(\mathcal{F}, \phi_1), \phi_2) = \Phi(\mathcal{F}, \phi_1\phi_2)$. Note that the data $\mathcal{F}$ is included in the set of conformal data (along with γ, σ and τ); the explicit form of the action Φ is to be chosen for each theory.

In this language, the first of our criteria is satisfied so long as $\mathcal{C}, \Phi$ and J satisfy the conditions

$$\mathcal{C}(\Phi(\mathcal{F}, \phi), \phi^q\gamma) = \phi^p\mathcal{C}(\mathcal{F}, \gamma) \tag{C1}$$

and

$$J(\Phi(\mathcal{F}, \phi), \phi^q\gamma) = \phi^{-q-2}J(\mathcal{F}, \gamma), \tag{C2}$$

for some number p. As a consequence of (C1), if we choose the conformal data $(\gamma, \sigma, \tau, \mathcal{F})$, so that $\mathcal{C}\left(\mathcal{F}, \gamma\right) = 0$, then whatever ϕ and W are determined to be, the constraint $\mathcal{C}\left(\tilde{\mathcal{F}}, \tilde{\gamma}\right) = 0$ is satisfied by the physical initial data. As a consequence of (C2), the conformal form of the momentum constraint is (cf. (2.4))

$$\operatorname{div}_\gamma\left(\mathcal{D}W\right) = \frac{n-1}{n}\phi^{q+2}\nabla\tau + J(\mathcal{F}, \gamma), \tag{2.11}$$

which, in the CMC case, can be solved for W independent of ϕ.

Satisfaction of the second criteria depends on the form of the term $\rho(\Phi(\mathcal{F}, \phi), \phi^q\gamma)$. It is crucial, first of all, that this quantity involves neither W nor any derivatives of ϕ. We formalize this by assuming that for each choice of the matter field $\mathcal{F}$ and the metric γ, there exists a function $n_{\mathcal{F},\gamma}\colon \mathbb{R}^+ \times \Sigma \to \mathbb{R}$ such that at any point p of Σ,

$$\phi^{q+1}2\rho(\Phi(\mathcal{F}, \phi), \phi^q\gamma) = n_{\mathcal{F},\gamma}(\phi(p), p). \tag{C3}$$

Note that in practice, to ensure that this condition holds for a given field theory, it is generally sufficient that the non-gravitational fields do not involve derivative coupling.

Assuming that condition (C3) holds, the Lichnerowicz equation takes the form

$$\Delta_\gamma \phi - a_1 R_\gamma \phi + a_1 \left|\sigma + \mathcal{D}W\right|^2_\gamma \phi^{-q-3} + -a_2 \tau^2 \phi^{q+1} + a_1 n_{\mathcal{F},\gamma}(\phi) = 0, \tag{2.12}$$

where for convenience, we set $a_1 = \frac{1}{q(n-1)} = \frac{n-2}{4(n-1)}$ and $a_2 = \frac{1}{qn} = \frac{n-2}{4n}$ and also we suppress the dependence of $n_{\mathcal{F},\gamma}$ on its second argument. To ensure that this equation is analytically tractable, it is important that for fixed γ and $\mathcal{F}$, $n_{\mathcal{F},\gamma}(\phi)$ is a monotonically decreasing function of ϕ. In practice, we find that this condition is often satisfied by $n_{\mathcal{F},\gamma}(\phi)$ being expressible as a sum of negative powers of ϕ, with non-negative coefficients.

While the two criteria and their consequent conditions (C1) to (C3) on $\Phi(\mathcal{F}, \phi)$ and its interaction with $\mathcal{C}, J$ and ρ appear quite restrictive, it is shown in [18] that for most familiar physical fields, a choice of $\Phi(\mathcal{F}, \phi)$ which satisfies these criteria can be found. We shall discuss a number of such examples in this article. Before proceeding, we illustrate how this works for a simple example: Einstein–Maxwell in three space dimensions.

The conformal data for the Einstein–Maxwell theory consists of $(\Sigma^3, \gamma, \sigma, \tau, B, E)$, where $(\Sigma^n, \gamma, \sigma, \tau)$ are the usual vacuum conformal data and both B and E are vector fields, which are required to be divergence free with respect to the metric γ. To satisfy the two criteria, we choose $\Phi(B^a, \phi) = B^a \phi^{-6}$ and $\Phi(E^a, \phi) = E^a \phi^{-6}$. It follows then that (i) the extra constraints $\operatorname{div}_\gamma \tilde{B} = 0$ and $\operatorname{div}_\gamma \tilde{E} = 0$ are satisfied automatically so long as B and E are both divergence free with respect to the metric γ; (ii) the momentum constraint takes the form

$$\operatorname{div}_\gamma (\mathcal{D}W) = \frac{2}{3}\phi^6 \nabla\tau + (E \times B), \tag{2.13}$$

which is independent of ϕ in the CMC case; and (iii) the Lichnerowicz equation takes the form

$$\Delta_\lambda \phi - \frac{1}{8} R_\gamma \phi + \frac{1}{8} \left|\sigma + \mathcal{D}W\right|^2_\gamma \phi^{-7} - \frac{1}{12}\tau^2 \phi^5 + \frac{1}{8}(E^2 + B^2)\phi^{-3} = 0. \tag{2.14}$$

We note that the extra term in (2.14) involves ϕ with a negative power and a positive coefficient, much like the $|\sigma + LW|$ term. As discussed in [16], existence and uniqueness of solutions of (2.13) and (2.14) in the CMC case then closely follows the pattern of the vacuum case.

3 Gluing construction for non-vacuum solutions

3.1 Overview

Before generalizing the gluing construction from [19] to apply to non-vacuum fields, we briefly summarize the original technique, modified to arbitrary spatial dimension $n \geq 3$. We start with an n-manifold Σ and a CMC solution (γ, K) of the vacuum Einstein constraint equations (so $K = \sigma + \frac{\tau}{n}\gamma$, with σ divergence free and trace free and with τ constant). We fix two points p_1 and p_2 of Σ and a small radius R. Let $B_j = B_R(p_j)$ be the balls on which we will do surgery, let $\Sigma^* = \Sigma \backslash \{p_1, p_2\}$ and let $\Sigma^*_r = \Sigma \backslash (B_r(p_1) \cup B_r(p_2))$. The construction then proceeds as follows.

- We first construct a conformally related metric on Σ^* agreeing with γ away from the surgery site and having two asymptotically cylindrical ends at the puncture locations. Let ψ_c be a conformal factor equal to 1 on Σ^*_{2R} and equal to $r_j^{2/q}$ on B_j, where r_j is the geodesic distance from p_j and where $q = \frac{4}{n-2}$ as earlier. Then, $\gamma_c = \psi_c^{-q}\gamma$ is the desired metric. Setting $\sigma_c = \psi_c^2\sigma$ and $K_c = \sigma_c + \frac{\tau}{n}\gamma$, we see that (γ_c, K_c) satisfy the momentum constraint and that ψ_c satisfies the Lichnerowicz equation with respect to (γ_c, K_c).
- We next perform surgery on the cylindrical ends by identifying finite segments of length T, to construct a family of topologically identical manifolds Σ_T. To do this, we first construct maps from $B_j \backslash p_j$ to the half cylinder $(0, \infty) \times S^{n-1}$ by sending points at the ball radius r_j to the cylinder length $t_j = -\log r_j + \log R$ and by using Riemann normal coordinates to determine the projections onto S^{n-1}. We then identify the finite segments $\{(t_j, \theta)\colon 0 < t_j < T\}$ via the map $(t_1, \theta) \mapsto (T - t_1, -\theta)$, resulting in the smooth manifold Σ_T. Letting $s = t_1 - T/2 = T/2 - t_2$, we denote by $Q_{l,a}$ the cylindrical segment $\{(s, \theta) : a - l < s < a + l\}$ of length $2l$ centered at a. We use the shorthand notation $Q_l = Q_{l,0}$ for centered segments, $Q = Q_1$ for a short collar in the middle and $C_T = Q_{T/2}$ for the entire cylindrical region.
- We now construct approximate solutions (γ_T, K_T, ψ_T) of the momentum constraint and the Lichnerowicz equation on Σ_T by using the conformally modified solution away from the surgery site and using cutoff functions to piece together an approximation along the identified cylindrical segment.
- In preparation for using the CMC-conformal technique, to map the approximate solutions to full solutions, we perturb the trace-free part of K_T to obtain a constant trace second fundamental form $\hat{K}_T = \hat{\sigma}_T + \frac{\tau}{n}\gamma_T$ which, together with γ_T, satisfies the momentum constraint.

- Finally, we solve the Lichnerowicz equation using a contraction-map argument to arrive at a T parameterized set of solutions of the constraints, which, for large T, is "close" to the original one away from the surgery site.

The last step is quite delicate; the contraction-map argument demands some analytic conditions on the linearization $\mathcal{L}_T$ about ψ_T of the Lichnerowicz operator $\mathcal{N}_T$ on Σ_T. In particular, $\mathcal{L}_T$ must be surjective, and there must exist bounds uniform in T for the norm of its inverse on certain weighted Hölder spaces. The argument also imposes some stringent conditions on the size of the error terms which arise from the earlier approximations and corrections. Substantial work in [19] is devoted to constructing and obtaining precise estimates for the perturbation of σ_T to $\hat{\sigma}_T$ that enter into this analysis.

To extend the technique to include matter fields, we start as earlier with a CMC solution $(\gamma, K, \mathcal{F})$ of the Einstein-matter constraints on an n-manifold Σ and a fiber bundle E over Σ. We also assume that we have chosen a conformal group action Φ for the non-gravitational fields, which satisfies the criteria discussed in Section 2.

The topological step that constructs Σ_T must generally be supplemented with a construction of appropriate fiber bundles E_T over Σ_T. As each of the balls B_i is contractible, there exists a local trivialization over each of them. We can then use these trivializations to identify fibers: if $q_1 \in B_1$ is identified with $q_2 \in B_2$ in the connected sum to create Σ_T, we can identify the fiber over q_1 with the fiber over q_2 via our pair of fixed local trivializations. On the other hand, if $q \in \Sigma_T$ is outside of the neck, then we take the fiber to be the one over q in E. The result is a smooth fiber bundle E_T over Σ_T.

Together with the conformally modified gravitational fields γ_c and K_c, we define on Σ^* the conformally modified non-gravitational fields

$$\mathcal{F}_c = \Phi(\mathcal{F}, \psi_c^{-1}).$$

Then, as a consequence of the choice of Φ, we verify that $(\gamma_c, K_c, \mathcal{F}_c)$ satisfy the non-gravitational and momentum constraints on Σ^*, and in addition, ψ_c satisfies the Lichnerowicz equation corresponding to these data:

$$\Delta_{\gamma_c} \psi_c - a_1 R_{\gamma_c} \psi_c + a_1 |\sigma_c + \mathcal{D}W|^2_{\gamma_c} \psi_c^{-q-3} - a_2 \tau^2 \psi_c^{q+1} + a_1 n_{\mathcal{F}_c, \gamma_c}(\psi_c) = 0. \tag{3.1}$$

To construct from $(\gamma_c, K_c, \mathcal{F}_c)$ on Σ^* a parameterized set of conformal data $(\gamma_T, K_T, \mathcal{F}_T)$ on Σ_T, we use a cutoff function procedure as in the vacuum case. That is, we first set $(\gamma_T, K_T, \mathcal{F}_T) = (\gamma_c, K_c, \mathcal{F}_c)$ on $\Sigma_T \backslash Q$. Then, on Q (recalling the definition of s in terms of T), we let $\chi(s)$ be a cutoff function on $\mathbb{R}$ equal to 0 for $s > 1$ and equal to 1 for $s < -1$ and we define (i) $\gamma_T = \chi(s)\gamma_1 + (1-\chi(s))\gamma_2$; (ii) $\sigma_T = \chi(s)\sigma_1 + (1-\chi(s))\sigma_2$ in Q and $K_T = \sigma_T + \frac{\tau}{n}\gamma_T$ (recall that τ is a constant); and finally (iii) $\mathcal{F}_T = \chi(s)\mathcal{F}_1 + (1-\chi(s))\mathcal{F}_2$. Note that γ_i, σ_i and $\mathcal{F}_i$ (for $i \in \{1,2\}$) are all defined by identifying C_T with a subset first of B_1 and then of B_2. If we do the same with ψ_c, and then set $\psi_T = \chi(t_2-1)\psi_1 + \chi(t_1-1)\psi_2$ in C_T and 1 outside, then we can verify (as discussed in further detail subsequently) that for each value of T, $(\gamma_T, K_T, \mathcal{F}_T)$ together with ψ_T constitute an approximate solution of the constraints, including the Lichnerowicz equation. Note that $\psi_T = \psi_1 + \psi_2$ on most of C_T.

To go from these approximate solutions to a parameterized set of exact solutions, we proceed as follows. First, we perturb $\mathcal{F}_T$ and thereby obtain a set of matter fields $\hat{\mathcal{F}}_T$, which closely approximate $\mathcal{F}_c$ outside the neck and which together with γ_T solve the non-gravitational constraints globally on Σ_T. Then, we perturb K_T in order to obtain a set of symmetric tensors $\hat{K}_T$, which closely approximate K outside the neck and which together with γ_T and $\hat{\mathcal{F}}_T$ satisfy the momentum constraint globally on Σ_T. Note that both of these two perturbations involve solving PDEs for the perturbation terms (in order to satisfy the non-gravitational constraints and the momentum constraint respectively). The solvability of these equations (for the perturbation terms, given the approximate solutions) is an issue that must be addressed at this stage. Finally, provided that various error terms in the Lichnerowicz equation have been kept under control after substituting in γ_T, $\hat{K}_T$, $\hat{\mathcal{F}}_T$ and ψ_T, we use a contraction mapping argument to show that there exists a solution $\hat{\psi}_T$ to this equation with the conformal data $(\gamma_T,\ \hat{K}_T,\ \hat{\mathcal{F}}_T)$ and we show further that away from the gluing region Q, the solution data $(\tilde{\gamma}_T = \hat{\psi}_T^q\gamma_T,\ \tilde{K}_T = \hat{\psi}_T^{-2}\hat{\sigma}_T + \frac{1}{n}\hat{\psi}_T^q\gamma_T\tau,\ \tilde{\mathcal{F}}_T = \Phi(\hat{\mathcal{F}}_T, \hat{\psi}_T))$ approaches arbitrarily closely the original data $(\gamma, K, \mathcal{F})$.

We discuss some of the details of these steps for generic non-gravitational fields in the rest of this section and discuss them for particular examples in Section 4.

3.2 Satisfying the non-gravitational constraints

As we presumably have chosen the action of the conformal map Φ on the matter fields, so that the non-gravitational constraints decouple from the others in the conformal representation, the first step in going from $(\gamma_T, K_T, \mathcal{F}_T)$ to $(\gamma_T,\ \hat{K}_T,\ \hat{\mathcal{F}}_T)$ is to choose $\hat{\mathcal{F}}_T$, so that $\mathcal{C}(\hat{\mathcal{F}}_T, \gamma_T) = 0$. In obtaining $\hat{\mathcal{F}}_T$, we want it to be arbitrarily close (for T sufficiently large) to $\mathcal{F}_T$ on Σ_R^*, in order to minimize the errors that are introduced into the other constraints. In particular, it is crucial for the gluing procedure that the error estimates (M1) and (M2), (E1) and (E2) and (N1)–(N4) described subsequently be satisfied.

The details of the construction of $\hat{\mathcal{F}}_T$ are field specific; we discuss a number of cases in Section 4. Here, we note what happens in the Einstein–Maxwell case: while although the original E and B fields are divergence free with respect to γ, and consequently, the conformally mapped fields E_c and B_c are divergence free with respect to γ_c, the fields E_T and B_T constructed using cutoff functions are not divergence free with respect to γ_T (or any metric). Effectively, the non-gravitational constraints are equivalent to this divergence-free property. We obtain new fields $\hat{E}_T$ and $\hat{B}_T$, which satisfy the conditions $\mathrm{div}_{\gamma_T} \hat{E}_T = 0$ and $\mathrm{div}_{\gamma_T} \hat{B}_T = 0$ by carrying through the standard linear procedure. That is, we solve the linear equation

$$\Delta_{\gamma_T}\ \mu_T = \mathrm{div}_{\gamma_T}\ E_T \tag{3.2}$$

for the scalar μ_T and then set

$$\hat{E}_T = E_T - \nabla\mu_T; \tag{3.3}$$

the divergence-free condition immediately follows. We carry out a similar procedure to obtain $\hat{B}_T$. Noting that the supports of $\mathrm{div}_{\gamma_T} E_T$ and of $\mathrm{div}_{\gamma_T} B_T$ are contained in Q, one can carry through the analysis which verifies the estimates (M1) and (M2), (E1) and (E2), and (N1)–(N4) as detailed subsequently.

3.3 Repairing the momentum constraint

With $\hat{\mathcal{F}}_T$ determined, we next need to find $\hat{K}_T$, for which the momentum constraint is satisfied. We may do this by finding a symmetric trace-free (0,2) tensor $\hat{\nu}_T$, which satisfies

$$\mathrm{div}_{\gamma_T}(\hat{\nu}_T) = J(\hat{\mathcal{F}}_T, \gamma_T) - \mathrm{div}_{\gamma_T}\sigma_T. \tag{3.4}$$

If we can obtain a tensor $\hat{\nu}_T$ which satisfies this condition, and if we then set $\hat{K}_T = \sigma_T + \hat{\nu}_T + \frac{\tau}{n}\gamma_T$, then we do have a solution of the momentum constraint.

To solve this equation for $\hat{\nu}_T$, we let $\mathcal{D}$ be the (γ_T compatible) CK operator on vector fields X, so $\mathcal{D}X = \mathcal{L}_X\gamma_T - \frac{2}{n}\mathrm{div}_{\gamma_T}(X)\gamma_T$, where $\mathcal{L}_X$ is the Lie derivative. Its formal adjoint $\mathcal{D}^*$ is $-\mathrm{div}_{\gamma_T}$ and we set $L = \mathcal{D}^*\mathcal{D}$. If W_T is a vector field solving

$$LW_T = J(\hat{\mathcal{F}}_T, \gamma_T) - \mathrm{div}_{\gamma_T} \sigma_T$$

(where we freely identify, via γ_T, vectors and covectors), then we can set $\hat{\nu}_T = -\mathcal{D}W_T$ to obtain a solution to (3.4), and consequently, a solution $\hat{K}_T$ to the momentum constraint.

Besides obtaining $\hat{\nu}_T$, we need to establish control of its size. Lemma 3.2, proved in [19] for 3-manifolds Σ, provides this estimate in terms of the following Hölder norm.

Definition 3.1. Let $||X||_{k,\alpha,\Omega}$ denote the Hölder norm (computed with respect to the metric γ_T) of the vector field X on an open subset Ω of Σ. Then, we define

$$||X||_{k,\alpha} = ||X||_{k,\alpha,\Sigma^*_{R/2}} + \sup_{-T/2+1 \leq a \leq T/2-1} ||X||_{k,\alpha,Q_{1,a}}.$$

Lemma 3.2. *Suppose there are no CK fields that vanish at the points p_j of Σ. Then, for T sufficiently large and for each $X \in \mathcal{C}^{k,\alpha}(\Sigma_T)$, there is a unique solution $W \in \mathcal{C}^{k+2,\alpha}(\Sigma_T)$ to $LW = X$. Moreover, there exists a constant C independent of W and T such that*

$$||W||_{k+2,\alpha} \leq CT^3 ||X||_{k,\alpha}.$$

The proof of Lemma 3.2 in general dimensions, which we skip now for the sake of exposition, is presented in Section 5.

Since we wish $\hat{\nu}_T$ to be small, we therefore require that $J(\hat{\mathcal{F}}_T, \gamma_T) - \mathrm{div}_{\gamma_T} \sigma_T$ be small. Outside Q, we have $\mathrm{div}_{\gamma_T} \sigma_T = J(\mathcal{F}_c, \gamma_c)$. Inside Q, it is relatively easy to see that σ_T has norm and derivatives comparable to $\psi_T^{2+q} \sim \mathrm{e}^{-nT/2}$. This motivates the following conditions on the matter field $\hat{\mathcal{F}}_T$.

Definition 3.3. We say that $\hat{\mathcal{F}}_T$ satisfies the momentum error estimates if for each k and α, there exist constants $C > 0$ and $\kappa > \frac{n-1}{2}$ independent of T such that

$$||J(\hat{\mathcal{F}}_T, \gamma_T) - J(\mathcal{F}_c, \gamma_c)||_{k,\alpha,\Sigma_T \setminus \overline{Q}} < C\,\mathrm{e}^{-\kappa T} \tag{M1}$$

and such that (recalling that Q_r is the centered collar of length $2r$)

$$||J(\hat{\mathcal{F}}_T, \gamma_T)||_{k,\alpha,Q_2} < C\,\mathrm{e}^{-\kappa T}. \tag{M2}$$

If $\hat{\mathcal{F}}_T$ satisfies the momentum error estimates, it follows that

$$||J(\hat{\mathcal{F}}_T, \gamma_T) - \mathrm{div}_{\gamma_T}\ \sigma_T||_{k,\alpha} < C\ \mathrm{e}^{-\kappa T}.$$

The threshold $\kappa > \frac{n-1}{2}$ is the lower limit, which will allow the subsequent analysis of the Lichnerowicz equation to go forward. From the momentum error estimates and Lemma 3.2 we immediately obtain Proposition 3.4.

Proposition 3.4. *Suppose $\hat{\mathcal{F}}_T$ satisfies the momentum error estimates. Then, there exists a tensor $\hat{\nu}_T$ on Σ_T such that*

$$\mathrm{div}_{\gamma_T}(\hat{\nu}_T) = J(\hat{\mathcal{F}}_T, \gamma_T) - \mathrm{div}_{\gamma_T}(\sigma_T).$$

In particular, $\hat{K}_T = \sigma_T + \hat{\nu}_T + \frac{\tau}{n}\gamma_T$ together with γ_T and $\hat{\mathcal{F}}_T$ is a CMC solution of the momentum and non-gravitational constraints. Moreover, defining $\hat{\sigma}_T = \hat{\nu}_T + \sigma_T$, we find that there exist constants $C > 0$ and $\kappa > \frac{n-1}{2}$ independent of T such that

$$||\hat{\sigma}_T - \sigma_T||_{k,\alpha} < C\ \mathrm{e}^{-\kappa T}. \tag{3.5}$$

The method of proof has been outlined earlier; the details can be found in [19]. We note that the existence of $\hat{\nu}_T$– and therefore the existence of $\hat{\sigma}_T$–follows even without $\hat{\mathcal{F}}_T$ satisfying the momentum error estimates. The key point is that if we impose these estimates, the solution necessarily satisfies (3.5).

3.4 Repairing the energy constraint

Up to this point, we have constructed a CMC solution $(\gamma_T, \hat{K}_T, \hat{\mathcal{F}}_T)$ of the momentum and non-gravitational constraints; we also have an approximate solution ψ_T of the Lichnerowicz equation in terms of the conformal data $(\gamma_T, \hat{K}_T, \hat{\mathcal{F}}_T)$. Our goal is to find a perturbation η_T of ψ_T, so that $\psi_T + \eta_T$ solves the Lichnerowicz equation.

Let $\mathcal{N}_T$ be the Lichnerowicz operator with respect to γ_T, $\hat{\sigma}_T$ and $\hat{\mathcal{F}}_T$ on Σ_T, we write

$$\mathcal{N}_T(\psi) = \Delta_{\gamma_T}\psi - a_1 R_{\gamma_T}\psi + a_1\,|\hat{\sigma}_T|^2_{\gamma_T}\,\psi^{-q-3} - a_2\tau^2\psi^{q+1} + a_1\hat{n}_T(\psi), \tag{3.6}$$

where we abbreviate $n_{\hat{\mathcal{F}}_T,\gamma_T}(\psi)$ as $\hat{n}_T(\psi)$. Similarly, we let $\mathcal{N}$ denote the Lichnerowicz operator with respect to γ, σ and $\mathcal{F}$ on Σ.

Since we will be working with the linearization of $\mathcal{N}_T$, it is convenient to denote by $\hat{n}'_T$ the derivative of $\hat{n}_T$ with respect to its last argument. Then, the linearization $\mathcal{L}_T$ of $\mathcal{N}_T$ about ψ_T is

$$\begin{aligned}\mathcal{L}_T = \Delta_{\gamma_T} &- a_1 R_{\gamma_T} + a_1(-q-3)\,|\hat\sigma_T|^2_{\gamma_T}\,\psi_T^{-q-4} \\ &- a_2(q+1)\tau^2\psi_T^q + a_1\;\hat n'_T(\psi_T).\end{aligned} \tag{3.7}$$

We will similarly denote by $\mathcal{L}$ the linearization of $\mathcal{N}$ about 1.

The contraction mapping argument we use here for the existence of a solution to the Lichnerowicz equation with data $(\gamma_T, \hat K_T, \hat{\mathcal{F}}_T)$ requires that the error $\mathcal{E}_T := \mathcal{N}_T(\psi_T)$ decays faster as a function of T than a prescribed exponential rate; it also requires control of certain mapping properties of $\mathcal{L}_T$. The proofs of these properties follow closely those given in [19], taking into account the new features of $\mathcal{N}_T$, which result from the introduction of the term n_T, as well as the use of higher dimensions. We now outline this argument.

3.4.1 Bound for the error $\mathcal{E}_T$

Corresponding to the momentum error estimates of Definition 3.3, we make the following definition.

Definition 3.5. Recalling that $\hat n_T(\psi_T) := n_{\hat{\mathcal{F}}_T,\gamma_T}(\psi_T)$ depends implicitly on $\hat{\mathcal{F}}_T$ and setting $n_c(\psi_c) := n_{\mathcal{F}_c,\gamma_c}(\psi_c)$, we say that $\hat{\mathcal{F}}_T$ satisfies the energy error estimates if for each k and α, there exist constants $C > 0$ and $\lambda > 1/q$ independent of T such that

$$||\hat n_T(\psi_T) - n_c(\psi_c)||_{k,\alpha,\Sigma_T\setminus\overline{Q}} < C\,\mathrm{e}^{-\lambda T} \tag{E1}$$

and such that

$$||\hat n_T(\psi_T)||_{k,\alpha,Q_2} < C\,\mathrm{e}^{-\lambda T}. \tag{E2}$$

Proposition 3.6. *Suppose $\hat{\mathcal{F}}_T$ satisfies the momentum and energy error estimates. Then, for every k and α, there exists a constant $C > 0$ and a constant $\rho > \frac{1}{q}$ not depending on T such that $||\mathcal{E}_T||_{k,\alpha} < C\,\mathrm{e}^{-\rho T}$.*

Proof. The proof follows the approach of the corresponding result in [19]. However, the computations are more involved, because we wish to prove the result for $n \geq 3$ and because of the effects of $\hat{\mathcal{F}}_T$ in (3.6), both directly via $\hat n_T$ and indirectly via $\hat\sigma_T$; hence, we outline the proof now.

We divide Σ_T into three regions, $\Sigma_T\setminus C_T$, $C_T\setminus Q$ and Q, and estimate the various terms of $\mathcal{E}_T$ in each. For simplicity of presentation, we only discuss estimates of the C^0 norm of $\mathcal{E}_T$; because of the exponential decay rates of the terms involved, the estimates for the higher derivative norms are essentially the same. It is also convenient to pick in advance the exponent

$$\rho := \min\left\{\lambda, \frac{1}{2}+\frac{1}{q}, \frac{2}{q}, 2\kappa - 1 - \frac{3}{q}\right\}.$$

One can readily verify that $\rho > \frac{1}{q}$. In fact, the restriction $2\kappa - 1 - \frac{3}{q} > \frac{1}{q}$ is the source of the choice $\kappa > \frac{n-1}{2}$ in Definition 3.3.

We first verify the bound in $\Sigma_T \backslash C_T$, where we have $\psi_T = \psi_c$, $\gamma_T = \gamma_c$ and $\sigma_T = \sigma_c$. Since ψ_c solves the Lichnerowicz equation with respect to $(\gamma_c, K_c, \mathcal{F}_c)$, it follows that in this region,

$$\mathcal{E}_T = \mathcal{N}_T(\psi_T) = a_1\left(|\hat\sigma_T|_T^2 - |\sigma_T|_T^2\right)\psi_T^{-q-3} + a_1\left(\hat n_T(\psi_T) - n_c(\psi_c)\right),$$

where the definition of n_c is analogous to that of $\hat n_T$. From Proposition 3.1, we know that $\left||\hat\sigma_T|^2 - |\sigma_T|^2\right| \le C\ \mathrm{e}^{-\kappa T}$ in $\Sigma_T \backslash C_T$. From the energy error estimate (E1), it easily follows that in this region,

$$\begin{aligned}|\mathcal{E}_T| &\le C\ \max(\mathrm{e}^{-\kappa T}, \mathrm{e}^{-\lambda T})\\ &\le C\,\mathrm{e}^{-\rho T}.\end{aligned}$$

Turning now to bounds in the region C_T, we first note the following easy estimates which hold on all of C_T:

$$\gamma_T = ds^2 + h + \mathcal{O}\left(\mathrm{e}^{-T/2}\cosh(s)\right) \tag{3.8}$$

$$\psi_T \le C\,\mathrm{e}^{-T/q}\cosh\left(\frac{2s}{q}\right) \tag{3.9}$$

$$|\sigma_T| \le C\,\mathrm{e}^{-nT/2}\left(\cosh\left(\frac{2s}{q}\right)\right)^{q+2} \tag{3.10}$$

$$|\hat\sigma_T - \sigma_T| \le C\,\mathrm{e}^{-\kappa T}, \tag{3.11}$$

where h denotes the round metric on the sphere. In the subregion Q, we have (recall that $\psi_T = \psi_1 + \psi_2$ in most of Q_T)

$$\begin{aligned}\mathcal{E}_T = \mathcal{N}_T(\psi_T) &= (\Delta_T - a_1 R_T)\psi_1 + (\Delta_T - a_1 R_T)\psi_2\\ &+ a_1\,|\hat\sigma_T|^2\,\psi_T^{-q-3} - q_2\tau^2\psi_T^{q+1} + a_1\hat n_T(\psi_T)\end{aligned}$$

From (3.8) and the definition of γ_T, it follows that $\Delta_T - a_1 R_T = \Delta_1 - a_1 R_1 + \mathcal{O}\left(\mathrm{e}^{-T/2}\right)$. Hence,

$$\begin{aligned}(\Delta_T - a_1 R_T)\psi_1 = -a_1\,|\sigma_1|^2\,\psi_1^{-q-3} + a_2\tau^2\psi_1^{q+1} - a_1 n_1(\psi_1)\\ + \mathcal{O}\left(\mathrm{e}^{-(1/2+1/q)T}\right),\end{aligned}$$

where $n_1(\psi_1) := \rho(\Phi(\mathcal{F}_1, \psi_1), \psi_1^q \gamma_1)\psi_1^{q+1}$. Since $n_1(\psi_1) = \psi_1^{q+1} n_0(1)$, for $n_0(1) = \rho(\mathcal{F}, \gamma)$, we have the easy bound $|n_1(\psi_1)| \le C\,\mathrm{e}^{-(1+1/q)T}$. Also, since $|\sigma_1| \le C\,\mathrm{e}^{-n/2T}$ and $\psi_j \le C\,\mathrm{e}^{-1/qT}$, it follows that

$$\begin{aligned}|(\Delta_T - a_1 R_T)\psi_1| &\le C\,\max(\mathrm{e}^{-(1/2+1/q)T}, \mathrm{e}^{-\lambda T})\\ &\le C\,\mathrm{e}^{-\rho T};\end{aligned}$$

an analogous estimate holds for ψ_2.

From (3.10) and (3.11), we determine that

$$|\hat{\sigma}_T|^2 \le C\,\max(\mathrm{e}^{-nT}, \mathrm{e}^{-2\kappa T}),$$

and hence from (3.9), we have

$$|\hat{\sigma}_T|^2\,\psi_T^{-q-3} \le C\,\max(\mathrm{e}^{-(1+1/q)T}, \mathrm{e}^{-(2\kappa-1-3/q)T}).$$

From (3.9), we also know that $\tau^2\psi_T^{q+1} \le C\,\mathrm{e}^{-(1+1/q)T}$, and from (E2), that $|\hat{n}_T(\psi_T)| \le C\,\mathrm{e}^{-\lambda T}$. Hence, we verify that the estimate

$$|\mathcal{E}_T| \le C\,\mathrm{e}^{-\rho T}$$

holds in the region Q.

The remaining region $C_T\backslash Q$ has two components, $C_T^{(1)} = [-T/2, 1] \times S^{n-1}$ and $C_T^{(2)} = [1, T/2] \times S^{n-1}$. By symmetry, it suffices to prove the bound on just one component. In $C_T^{(2)}$, we have (recall that $\psi_T = \psi_2 + \chi_1\psi_1$ in $C_T^{(2)}$)

$$\begin{aligned}\mathcal{E}_T = \mathcal{N}_T(\psi_T) = (\Delta_T - a_1 R_T)(\chi_1\psi_1) + a_1\left(|\sigma_T|^2\,\psi_2^{-q-3} - |\hat{\sigma}_T|^2\,\psi_T^{q-3}\right)\\ - a_2\tau^2\left(\psi_2^{q+1} - \psi_T^{q+1}\right) + a_1\left(\hat{n}_T(\psi_T) - n_c(\psi_c)\right).\end{aligned} \tag{3.12}$$

The last two terms of (3.12) are easy to estimate. From (E1), we know

$$|\hat{n}_T(\psi_T) - n_c(\psi_c)| \le C\,\mathrm{e}^{-\lambda T}. \tag{3.13}$$

As $\psi_T = \psi_2 + \chi_1\psi_1$ and $\chi_1\psi_1 = \mathcal{O}\left(\mathrm{e}^{-T/q-2s/q}\right)$, we have

$$\begin{aligned}\left|\psi_T^{q+1} - \psi_2^{q+1}\right| &= \mathcal{O}\left(\mathrm{e}^{-(1+1/q)T+(2-2/q)s}\right)\\ &\le C\max(\mathrm{e}^{-2/qT}, \mathrm{e}^{-(1+1/q)T}).\end{aligned} \tag{3.14}$$

We now turn to the second term of the right-hand side of (3.12). From (3.9) to (3.11), we have

$$\left||\hat{\sigma}_T|^2 - |\sigma_T|^2\right|\psi_T^{-q-3} \le C\max(\mathrm{e}^{-(\kappa-1/q)T}, \mathrm{e}^{-(2\kappa-1-3/q)T}). \tag{3.15}$$

Also, as

$$\psi_T^{-q-3} - \psi_2^{-q-3} = \mathcal{O}\left(\mathrm{e}^{(1+3/q)T-(2+10/q)s}\right),$$

we have from (3.10) that

$$|\sigma_T|^2(\psi_T^{-q-3} - \psi_2^{-q-3}) \leq C\max(\mathrm{e}^{-2/qT}, \mathrm{e}^{-(1+1/q)T}). \tag{3.16}$$

From (3.15) and (3.16), it then follows that

$$\left||\hat{\sigma}_T|^2\,\psi_T^{-q-3} - |\sigma_T|^2\,\psi_2^{-q-3}\right| \leq C\mathrm{e}^{-\rho T}. \tag{3.17}$$

Hence, it remains to estimate

$$(\Delta_T - a_1 R_T)(\chi_1\psi_1).$$

Since $\chi_1 \equiv 1$ except near $s = T/2$ and as $\psi_1 = \mathrm{e}^{-T/q-2s/q}$ on $C_T^{(2)}$, it follows that

$$\chi_1\psi_1 = \mathrm{e}^{-T/q-2s/q} + \mathcal{O}\left(\mathrm{e}^{-2/qT}\right). \tag{3.18}$$

Letting γ_0 be the round metric on the cylinder, it follows from (3.8) and from calculation of the scalar curvature of a round metric that

$$\Delta_T - c_1 R_T = \Delta_0 - a_1(n-1)(n-2) + \mathcal{O}\left(\mathrm{e}^{s-T/2}\right). \tag{3.19}$$

Since $(\Delta_0 - a_1(n-1)(n-2))\mathrm{e}^{-T/q-2s/q} = 0$, we see from (3.18) and (3.19) that

$$(\Delta_T - a_1 R_T)(\chi_1\psi_1) \leq C\ \max(\mathrm{e}^{-(1/2+1/q)T}, \mathrm{e}^{-2/qT}). \tag{3.20}$$

Hence, from (3.13), (3.14), (3.17) and (3.20), it follows that in $C_T^{(2)}$ (as in the rest of Σ_T),

$$|\mathcal{E}_T| \leq C\ \mathrm{e}^{-\rho T}.$$

□

3.4.2 Mapping properties of the linearization of $\mathcal{N}_T$

The goal of this section is to verify certain properties of the linearization operator $\mathcal{L}_T$: specifically, that it is invertible and that the norm of its inverse on certain weighted spaces is bounded independent of T. These properties, which hold in the vacuum case provided $K \not\equiv 0$, depend on the form of the matter term $n(\gamma, \mathcal{F}, \psi)$ and its derivative n' with respect to ψ. To guarantee these properties, we add the following assumptions to the momentum error estimates and the energy error estimates already made.

- For each $p \in \Sigma$,

$$n_{\mathcal{F},\gamma}(1,p) - n'_{\mathcal{F},\gamma}(1,p) \geq 0. \tag{N1}$$

- As $T \to \infty$,

$$\left|\left|\hat{n}'_T(\psi_T) - n'_c(\psi_c)\right|\right|_{k,\alpha,\Sigma_T \setminus \overline{Q}} \to 0. \tag{N2}$$

- As $T \to \infty$,

$$\left|\left|\hat{n}'_T(\psi_T)\right|\right|_{k,\alpha,Q_2} \to 0. \tag{N3}$$

We use property (N1) to establish that $\mathcal{L}$ (the linearization of $\mathcal{N}$ on Σ about $\psi = 1$) is invertible, and we use properties (N2) and (N3) to ensure the convergence of $\mathcal{L}_T$ to known operators away from the middle of the neck and on the middle of the neck, respectively.

To see that $\mathcal{L}$ is invertible, we note that

$$\mathcal{L} = \Delta_\gamma \; - a_1 \left(R_\gamma + (q+3)\,|\sigma|^2_\gamma + \frac{a_2}{a_1}(q+1)\tau^2 - n'_{\mathcal{F},\gamma}(1) \right).$$

Since $(\gamma, \sigma, \tau, \mathcal{F})$ is a solution of the constraints, we can write R_γ in terms of γ, σ, τ and $\mathcal{F}$ to obtain

$$\mathcal{L} = \Delta_\gamma \; - a_1 \left((q+4)\,|\sigma|^2_\gamma + q\frac{a_2}{a_1}\tau^2 + \left(n_{\mathcal{F},\gamma}(1) - n'_{\mathcal{F},\gamma}(1)\right) \right). \tag{3.21}$$

As a consequence of our assumption (N1), we know that the term in this expression which involves $\mathcal{F}$ is non-negative. Moreover, if the term in parentheses is not identically zero, then it follows from the maximum principle that $\mathcal{L}$ has a trivial kernel on $\mathcal{C}^{k+2,\alpha}(\Omega)$. This is true if either $K \not\equiv 0$ or $n_{\mathcal{F},\gamma}(1) - n'_{\mathcal{F},\gamma}(1) \not\equiv 0$; we will henceforth assume this non-degeneracy condition. Note that it often holds that $-n'_{\mathcal{F},\gamma}(1) \geq 0$. In this case, as $n_{\mathcal{F},\gamma}(1) = 2\rho(\mathcal{F},\gamma)$, the non-degeneracy condition holds if either $K \not\equiv 0$ or $\rho(\mathcal{F},\gamma) \not\equiv 0$.

We now show that for sufficiently large T, $\mathcal{L}_T$ also has trivial kernel. We also want to control the norm of its inverse, which we can do in terms of the following weighted Hölder space norm.

Definition 3.7. Let w_T be an everywhere positive smooth function on Σ_T which equals $\mathrm{e}^{-T/q}\cosh(\frac{2s}{q})$ on C_T, which is uniformly bounded away from zero on Σ^*_R, and equals one on Σ^*_{2R}. For any $\delta \in \mathbb{R}$, and any $\phi \in \mathcal{C}^{k,\alpha}(\Sigma_T)$, we set

$$||\phi||_{k,\alpha,\delta} = ||w_T^{-\delta}\phi||_{k,\alpha},$$

and we let $\mathcal{C}^{k,\alpha}_\delta(\Sigma_T)$ be the corresponding normed space.

Proposition 3.8. *Fix any $\delta \in \mathbb{R}$. For T sufficiently large, the mapping*

$$\mathcal{L}_T \colon \mathcal{C}^{k+2,\alpha}_\delta(\Sigma_T) \longrightarrow \mathcal{C}^{k,\alpha}_\delta(\Sigma_T)$$

is an isomorphism.

Proof. Since the spaces $\mathcal{C}_\delta^{k+2,\alpha}(\Sigma_T)$ and $\mathcal{C}^{k+2,\alpha}(\Sigma_T)$ are identical as sets and the associated norms are equivalent to each other for fixed T and δ, it is enough to show that $\mathcal{L}_T$ is an isomorphism for $\delta = 0$. Since $\mathcal{L}_T$ is Fredholm of index zero on $\mathcal{C}^{k+2,\alpha}(\Sigma_T)$, we need only to show that for T sufficiently large, it has trivial kernel.

Suppose not, there exists a sequence of increasing parameters T_k and functions η_k (not identically zero) such that $\mathcal{L}_{T_k}\eta_k = 0$. We assume without loss of generality that $\sup |\eta_k| = 1$.

Suppose now that $\sup |\eta_k|$ is bounded away from zero on Σ_r^* for some $r \leq R$. As a consequence of elliptic regularity, together with property (N2), we conclude that there exists a non-zero function η on Σ^* such that $\mathcal{L}_c\eta = 0$. Using the conformal covariance property $\Delta_{\gamma_c}\eta - a_1 R_{\gamma_c}\eta = \psi_c^{q+1}(\Delta_\gamma(\eta/\psi_c) - a_1 R_\gamma \eta/\psi_c)$, we conclude that η/ψ_c satisfies

$$[\Delta_\gamma - a_1 R_\gamma - (q+3)a_1 |\sigma|_\gamma^2 + a_2(q+1)\tau^2 - a_1\psi_c^{-q} n_c'(\psi_c)] \left(\frac{\eta}{\psi_c}\right) = 0.$$

Now, it follows from the definition of n that $n_c(\psi) = \psi_c^{q+1} n(\psi\psi_c^{-1})$, and hence, that $\psi_c^{-q} n_c'(\psi_c) = n'(1)$. So η/ψ_c solves $\mathcal{L}(\eta/\psi_c) = 0$ on Σ^*. Since η is bounded on Σ^* and since ψ_c decays like $r_j^{(n-2)/2}$ near p_j, it follows that η/ψ_c is less singular than $r_j^{(n-2)}$ at p_j and hence extends to a non-trivial solution of $\mathcal{L}(\eta/\psi_c) = 0$ on all of Σ, which is a contradiction.

If on the other hand η_T converges uniformly to 0 on Σ_r^* for any $R > r > 0$, we can consider a sequence of functions on increasing finite length sections of the cylinder $\mathbb{R} \times S^{n-1}$ by translating η_T, so that its maximum occurs at $s = 0$. Then, it follows from elliptic regularity and the properties (N2) and (N3) that we can extract a subsequence that converges in $\mathcal{C}^2$ on compact sets of the cylinder to a non-trivial bounded solution of the equation

$$\Delta_{\gamma_0}\eta - \left(\frac{n-2}{2}\right)^2 \eta = 0.$$

Since there are no such solutions, we again have a contradiction. □

Let $\mathcal{G}_T$ denote the inverse of $\mathcal{L}_T$ on $\mathcal{C}_\delta^{k+2,\alpha}$, which exists for T sufficiently large, as a consequence of Proposition 3.8. The proof that $\mathcal{G}_T$ has bounded norm as T goes to ∞ is, using properties (N2) and (N3), identical to the one appearing in [19], and hence, we will not repeat it.

Proposition 3.9. *If $0 < \delta < 1$, then the norm of the operator $\mathcal{G}_T$ is uniformly bounded as $T \to \infty$.*

3.4.3 Deforming the conformal factor ψ_T to a solution

We deform ψ_T to a true solution $\hat{\psi}_T = \psi_T + \eta_T$ of the Lichnerowicz equation using a contraction map. That is, we wish to solve

$$\mathcal{N}_T(\psi_T + \eta_T) = 0.$$

To do this, we introduce the quadratically vanishing non-linearity $\mathcal{Q}_T$ defined by

$$\mathcal{N}_T(\psi_T + \eta) = \mathcal{N}_T(\psi_T) + \mathcal{L}_T(\eta) + \mathcal{Q}_T(\eta).$$

Then, $\psi_T + \eta_T$ is a solution of the Lichnerowicz equation if and only if η_T satisfies

$$\eta_T = -\mathcal{G}_T\left(\mathcal{E}_T + \mathcal{Q}_T(\eta_T)\right),$$

and hence is a fixed point of the map $\mathcal{T}_T$ defined by

$$\mathcal{T}_T(\eta) = -\mathcal{G}_T\left(\mathcal{E}_T + \mathcal{Q}_T(\eta)\right).$$

The key estimate which leads to a verification that $\mathcal{T}_T$ is a contraction map is that for η near zero, the non-linear operator $\mathcal{N}_T$ differs from its linear approximation $\mathcal{L}_T$ by a quadratically small amount. That is, we wish to show that there is a constant C independent of T such that

$$||\mathcal{Q}_T(\eta)||_{k,\alpha,\delta} \leq C||\eta||^2_{k,\alpha,\delta}. \tag{3.22}$$

For the terms in $\mathcal{Q}_T$ involving $\hat{\sigma}$ and τ, this estimate follows from the explicit form of their dependence on η. To argue this, let us, for convenience, define

$$w_T(x) := a_1|\hat{\sigma}_T|^2\left(x^{-q-3} - a_2\tau^2 x^{q+1}\right).$$

We then have

$$\begin{aligned} Q_T(\eta) = {} & w_T(\psi_T + \eta) - w_T(\psi_T) - w_T'(\psi_T)\eta + n_T(\psi_T + \eta) \\ & - n_T(\psi_T) - n_T'(\psi_T)\eta. \end{aligned}$$

The terms involving w_T can be rewritten in integral form as

$$w_T(\psi_T + \eta_T) - w_T(\psi_T) - w_T'(\psi_T)\eta_T = \int_0^{\eta_T} (\eta_T - s)w_T''(\psi_T + s)\, ds;$$

hence, an estimate of the type (3.22) for these terms readily follows from appropriate controls on $w_T''(\psi_T + s)$. The only serious difficulty stems from the negative exponents appearing in the definition of w_T. Following [19], we require that there is a constant $c < 1$ independent of T such that $|\eta| < c\psi_T$. This restriction ensures that $1 + \eta/\psi_T$ remains uniformly bounded away from zero and a straightforward computation using this fact shows that an estimate of the form (3.22) holds for the w_T terms.

Since there is no a priori form for the dependence of the matter terms n_T on ψ_T, to complete the estimate for Q_T, we need to make a further assumption on n_T (thus extending our list of assumptions from Section 3.4.2).

There exist constants $C > 0$ and $c < 1$ independent of T such that

$$\big|\big|n_T(\psi_T + \eta) - n_T(\psi_T) - n'_T(\psi_T)\eta\big|\big|_{k,\alpha,\delta} \leq C\,||\eta||^2_{k,\alpha,\delta} \tag{N4}$$

whenever $|\eta| < c|\psi_T|$.

With the estimate (3.22) conditionally in hand, we now seek an open neighborhood about 0 on which $\mathcal{T}_T$ (as defined earlier) is a contraction map. This neighborhood must be mapped to itself under the action of $\mathcal{T}_T$, and it must be small enough so that (for all T) it only contains functions η such that $|\eta| < c\,\psi_T$ and hence the quadratic estimate holds for it. Fixing $\delta > 0$, we find from Propositions 3.6 and 3.9 that there are constants C and M, respectively, such that $||\mathcal{G}(\mathcal{E}_T)||_{k,\alpha,\delta} \leq CM\,\mathrm{e}^{(-\lambda+\delta/q)T}$. It is therefore convenient to set

$$\mathbb{B}_\nu = \{u \in \mathcal{C}^{k,\alpha}_\delta : ||u||_{k,\alpha,\delta} \leq \nu\,\mathrm{e}^{(-\lambda+\delta/q)T}\},$$

and we note for future reference that if $\nu = 2CM$, then $\mathcal{G}_T(\mathcal{E}_T) \in B_{\nu/2}$.

Now, for $\eta \in \mathbb{B}_\nu$, we see that inside the neck

$$\begin{aligned}|\eta| &\leq \nu\,\mathrm{e}^{(-\lambda+\delta/q)T}\psi_T^{\delta-1}\psi_T\\ &\leq \nu\,\mathrm{e}^{(-\lambda+\delta/q)T}\ \mathrm{e}^{-T/q(\delta-1)}\psi_T\\ &\leq \nu\,\mathrm{e}^{(-\lambda+1/q)T}\psi_T;\end{aligned}$$

whereas outside the neck

$$|\eta| \leq \nu\,\mathrm{e}^{(-\lambda+1/q)T}.$$

Since (we recall from the energy error estimate) $\lambda > \frac{1}{q}$, it follows that for any fixed values of $c < 1$ and ν, there exists a T large enough, so that if $\eta \in \mathbb{B}_\nu$, then $|\eta| < c\,\psi_T$ in all Σ_T. Combining this inequality with estimate (3.22), we find that there exists large enough T, so that

$$||\mathcal{Q}_T(\eta_1) - \mathcal{Q}_T(\eta_2)||_{k,\alpha,\delta} \leq \frac{1}{2M}||\eta_1 - \eta_2||_{k,\alpha,\delta} \tag{23}$$

for η_1 and η_2 in $\mathbb{B}_\nu$ and $\nu = 2CM$.

For $\nu = 2CM$, we have $\mathcal{G}(\mathcal{E}_T) \in \mathbb{B}_{\nu/2}$. Moreover, setting $\eta_1 = \eta$ and $\eta_2 = 0$ in (23), it follows that $\mathcal{G}(\mathcal{Q}_T(\eta)) \in \mathbb{B}_{\nu/2}$ for $\eta \in \mathbb{B}_\nu$ and for sufficiently large T. It follows that $\mathcal{T}_T$ maps $\mathbb{B}_\nu$ to itself for sufficiently large T. Equation (23) also implies that $\mathcal{T}_T$ is a contraction map on $\mathbb{B}_\nu$ for T sufficiently large. (See [19] for further details.)

If $\mathcal{T}_T$ is a contraction map, it follows that it has a unique fixed point η_T. It then follows that $\hat{\psi}_T = \psi_T + \eta_T$ satisfies the Lichnerowicz equation

$$\mathcal{N}_T(\hat{\psi}_T) = 0 \tag{3.24}$$

with $\mathcal{N}_T$ defined (for the glued data $\gamma_T, \hat{\sigma}_T, \hat{\mathcal{F}}_T, \tau$) as in (3.6). Moreover, the difference between the approximate solution ψ_T and the exact one $\hat{\psi}_T$ is contained in the space $\mathbb{B}_\nu$, which we see from its definition gets arbitrarily small as T gets large. We are thus led to our main result.

Theorem 3.10. *Let $(\Sigma^n, \gamma, K, \mathcal{F})$ be a smooth CMC solution of the n-dimensional constraint equations for an Einstein-matter field theory, which satisfies the two criteria of Section 2. Let Σ^n be compact, and let p_1 and p_2 be a pair of points in Σ^n. We assume that these solution data satisfy the following conditions. (i) The metric is non-degenerate with respect to p_1 and p_2 in the sense of* [19],[2] *and either the quantity K or the quantity $n_{\mathcal{F},\gamma}(1) - n'_{\mathcal{F},\gamma}(1)$ appearing in condition (*N1*) is not identically zero.*[3] *(ii) The momentum error estimates (Definition 3.3), the energy error estimates (Definition 3.5) and the matter term estimates (*N1*) to (*N4*) hold. Then, for T sufficiently large, there is a one-parameter family of solutions $(\Sigma^n_T, \Gamma_T, K_T, F_T)$ of the Einstein-matter field constraints with the following properties: the manifolds Σ^n_T (all diffeomeorphic to each other) are constructed by adding a handle or neck to Σ^n, connecting the two points p_1 and p_2. On the region of Σ^n_T outside of the neck, the fields (Γ_T, K_T, F_T) approach arbitrarily closely to $(\Sigma^n, \gamma, K, \mathcal{F})$ as T tends to infinity.*

If we compare the hypotheses of this theorem for gluing solutions of the Einstein-matter field constraints with the hypotheses of the corresponding gluing theorem for the Einstein vacuum data (see Theorem 1 in [19]), it appears as if there are far more conditions which must be satisfied by the Einstein-matter data. We note, however, that for most of the Einstein-matter theories of physical interest (see Section 4), all of the conditions except non-degeneracy are satisfied automatically by solutions of the constraints. That is, the nature of the matter fields and how they couple to the Einstein-matter constraints for most such theories guarantees satisfaction of most of these hypotheses. We shall see this in the discussion of example theories in the next section.

[2]That is, we assume that there are no non-trivial CK fields on Σ^n, which vanish at either p_1 or p_2.

[3]Note that for most Einstein-matter field theories, it follows from the momentum constraint that if $\mathcal{F}$ is not identically zero, then K is not identically zero.

3.5 Gluing in the presence of a cosmological constant

What happens if we seek to glue solutions with a cosmological constant Λ present? We find that so long as the constant $a_2\tau^2 - a_1\Lambda$ is positive (or zero in certain appropriate cases), our results are unchanged; otherwise, gluing via the methods we describe here generally cannot be carried out.

To see this, we recall that since the field equations with Λ present take the form $G = T + \Lambda g$, it follows that the momentum constraint (4.4) is unchanged by the presence of Λ, while the Hamiltonian equation (4.5) is changed to the following:

$$R_\gamma - |K|_\gamma^2 + (\text{tr } K)^2 = 2\rho + 2\Lambda. \tag{3.25}$$

Then, as we require Λ to be invariant under the conformal action Φ (otherwise we could not guarantee that it would remain a constant), we find that the Lichnerowicz equation with Λ present takes the form

$$\Delta_\lambda \phi - a_1 R_\gamma \phi + a_1 |\sigma + \mathcal{D}W|_\gamma^2 \phi^{-q-3} - (a_2\tau^2 - a_1\Lambda)\phi^{q+1} + n_{\mathcal{F},\gamma}(\phi) = 0. \tag{3.26}$$

The pairing of Λ with τ^2 which we see in (3.26) persists throughout the gluing analysis. Since it is crucial to this analysis that $a_2\tau^2 - a_1\Lambda \geq 0$, our stated result follows.

4 Applications to example Einstein-matter field theories

The work done in Section 3, culminating in Theorem 3.1, provides criteria for determining whether gluing can be carried out for given solutions of the constraints of a specified Einstein-matter field theory. While these criteria involve the data of the particular solutions, they depend predominantly on the features of the specified theory. In this section, we discuss some example field theories for which those solutions of the constraints which satisfy the mild non-degeneracy conditions stated in hypothesis (i) of Theorem 3.1 automatically satisfy hypothesis (ii) as well and can therefore be glued.

4.1 Einstein perfect fluids

The Einstein-perfect fluid field theories are especially simple from the point of view of gluing, because they do not require any extra bundle structure, they introduce no additional constraints beyond the Hamiltonian and the momentum constraints of the vacuum theory and the fluid fields can be readily scaled in such a way that the conformal method analysis is without complication.

The additional fields needed for an Einstein-perfect fluid field theory are the fluid energy density ζ, the pressure p and the fluid velocity u (which satisfies the condition $g(u,u) = -1$). In terms of these quantities, the stress-energy tensor for the field theory is given by

$$T = \zeta u \otimes u + p(g + u \otimes u), \tag{4.1}$$

and the field equations consist of

$$G = T \tag{4.2}$$

together with

$$\operatorname{div} T = 0, \tag{4.3}$$

which is a consequence of (4.1). The system is complete, with a well-posed Cauchy problem, once an equation of state specifying the pressure p as a function of the density, $p = F(\zeta)$, has been given.

Regardless of the choice of equation of state, initial data for a solution of the Einstein-perfect fluid equations consist of the vacuum initial data (Σ, γ, K) together with a scalar field ζ and a spatial vector field v (the spatial projection of the fluid velocity u). The constraint equations take the form

$$\operatorname{div} K - d \ \operatorname{tr} K = J = -(1 + |v|^2)^{1/2}(\zeta + F(\zeta))v \tag{4.4}$$

$$R_\gamma - |K|^2_\gamma + (\operatorname{tr} K)^2 = 2\rho = 2\zeta + 2(\zeta + F(\zeta))|v|^2. \tag{4.5}$$

To extend the conformal method to these Einstein-perfect fluid constraints, we need to define the conformal action Φ on the matter fields. While one could do this working directly with the fluid initial data fields ζ and v, the analysis is much simpler if we work instead with the composite functions ρ and J. If we are to do this, we need the following result.

Lemma 4.1. *The map from (ζ, v) to (ρ, J) given by $\rho(\zeta, v) = \zeta + (\zeta + F(\zeta))|v|^2$ and by $J(\zeta, v) = -(1 + |v|^2)^{\frac{1}{2}}(\zeta + F(\zeta))v$ is invertible, so long as we assume the physical conditions $\rho^2 > |J|^2$, $p = F(\zeta) \geq 0$, $F'(\zeta) < 1$ and $\zeta > 0$.*

Remark 4.2. See Theorem 9 of [13], where a similar result was obtained.

Proof. To prove this lemma, let us presume that we are given (ρ, J) with $\rho > 0$ and that we want to solve for (ζ, v). We first notice that it follows from Eq. (4.4) that J and v are parallel. Hence, it is useful to calculate $|J|^2 =$

$|v|^2(1+|v|^2)(\zeta+F(\zeta))^2 = (\rho-\zeta)(\rho+F(\zeta))$ and then focus on solving for $(\zeta, |v|^2)$ in terms of $(\rho, |J|^2)$.

Let us fix ρ and define the function $H_\rho(\zeta) := (\rho-\zeta)(\rho+F(\zeta))$. Note from the form of the equation $\rho(\zeta, v) = \zeta + (\zeta+F(\zeta))|v|^2$ and the hypotheses that $\zeta \geq 0$ and $F(\zeta) \geq 0$ be non-negative that if a solution ζ exists, we must have $0 \leq \zeta \leq \rho$. So we seek to solve the equation $H_\rho(\zeta) = |J|^2$ (fixed J) for ζ with $0 \leq \zeta \leq \rho$. Using the conditions $F(\zeta) \geq 0$ and $\rho^2 \geq |J|^2$, we find $H_\rho(0) = \rho(\rho+F(0)) \geq \rho^2 > |J|^2$ and also $H_\rho(\rho) = 0 \leq |J|^2$. We conclude from the continuity of H_ρ that there exists $\zeta \in (0, \rho]$ such that $H_\rho(\zeta) = |J|^2$. Then, calculating that, as a consequence of the condition $F'(\zeta) < 1$, we have $H'_\rho(\zeta) < 0$, it follows that there is a unique $\zeta(\rho, |J|^2)$, which satisfies $H_\rho(\zeta) = |J|^2$, and moreover, that ζ is as smooth a function of ρ and $|J|^2$ as F allows. We may use this result together with Eq. (4.5) to solve for $|v|^2(\rho, |J|^2)$, and then finally, we may use Eq. (4.4) to solve for $v(\rho, J)$. □

As a consequence of this lemma, we may now treat ρ and J as the initial data variables for the fluid field, in place of ζ and v. Hence, to extend the conformal method to the Einstein-fluid field, we define conformal action Φ on ρ and J: we set $\Phi(\rho, \phi) = \rho\phi^{-3/2q-2}$ and $\Phi(J, \phi) = J\phi^{-q-2}$. This choice has three consequences. (i) The quantity $\frac{\gamma^{ab}J_aJ_b}{\rho^2}$ is conformally invariant, and hence the satisfaction of the dominant energy condition by the fluid field is also conformally invariant. (ii) The quantity J satisfies the condition (C2) and hence the conformal momentum constraint

$$\mathrm{div}_\gamma\,(LW) = \frac{n-1}{n}\phi^{q+2}\nabla\tau + J \tag{4.6}$$

is independent of ϕ so long as τ is constant. (iii) The matter-dependent term in the Lichnerowicz equation

$$\Delta_\lambda\,\phi - a_1 R_\gamma \phi + a_1 \left|\sigma + \mathcal{D}W\right|^2_\gamma \phi^{-q-3} - a_2\tau^2\phi^{q+1} + a_1\rho\phi^{-q/2-1} = 0, \tag{4.7}$$

like the $|\sigma + \mathcal{D}W|$ term, has a positive sign and contains a negative power of ϕ; hence, its role in the solvability analysis of (4.7) is essentially the same as the $|\sigma + \mathcal{D}W|^2_\gamma$ term. Thus, we find, with this choice of the conformal action on ρ and J, that the conformal method works for the Einstein-perfect fluid constraints more or less the same as it works for the Einstein vacuum constraints. This is a prerequisite for carrying out the gluing of solutions of the constraints as described in Section 3.

Let us say that we are given a CMC solution $(\Sigma, \gamma, K, \zeta, v)$ of the Einstein-perfect fluid constraints, together with pair of points $p_1, p_2 \in \Sigma$ at which we would like to carry out a gluing operation. We first rewrite the initial data in the $(\Sigma, \gamma, K, \rho, J)$ form; we work entirely in this form until the end, at which time, we may invert to recover the glued solution in $(\Sigma, \gamma, K, \zeta, v)$ form.

The steps which take us from $(\Sigma, \gamma, K, \rho, J)$ to the preliminary glued data sets $(\gamma_T, K_T, \rho_T, J_T)$ are straightforward, as described in Section 3.1. With no non-gravitational constraints $\mathcal{C}$ present, we set $\hat{\rho}_T = \rho_T$ and $\hat{J}_T = J_T$, and we next proceed to repair the momentum constraint, as in Section 3.3. To continue with the gluing at this stage, we need to check the CK field non-degeneracy condition (described in footnote 2) and we need to verify the momentum error estimates (M1) and (M2). The CK non-degeneracy condition is (as with the vacuum case) a mild restriction on the class of solutions of the constraints which admit gluing. The momentum error estimates, on the other hand, follow immediately from the choice of the conformal action Φ on (ρ, J). They are automatic for any solution: we verify

$$||J_T - J)||_{k,\alpha,\Sigma_T \setminus \overline{Q}} = 0 \tag{4.8}$$

and

$$||J_T||_{k,\alpha,Q_2} < C\,\mathrm{e}^{n/2T}, \tag{4.9}$$

thereby confirming that these estimates hold. Consequently, the estimate (3.5) holds as well. We emphasize that this is true for any chosen solution $(\Sigma,\ \gamma, K, \rho, J)$ of the Einstein-perfect fluid constraints, which satisfies the CK non-degeneracy condition relative to the chosen gluing points p_1 and p_2.

We next need to check that the energy error estimates (E1) and (E2) are satisfied. Noting that for the Einstein-perfect fluid theories, $n_T(\gamma_T, \rho_T, J_T, \psi_T) = \rho_T \psi^{-q/2-1}$, these estimates are readily verified for any choice of data, much like the momentum error estimates just discussed.

To ensure that the linearized Lichnerowicz operator $\mathcal{L}_T$ is invertible for the chosen data $(\Sigma, \gamma, K, \rho, J)$, we need the quantity $n_{\mathcal{F},\gamma}$ to satisfy condition (N1). This follows immediately from the expression $n_T(\gamma_T, \rho_T, J_T, \psi_T) = \rho_T \psi^{-q/2-1-n/n-2}$, regardless of the choice of data. If we are working with data on a closed manifold, we also need one or the other of the conditions $K \neq 0$ or $\rho \neq 0$ to be satisfied. This is the only restriction beyond the CK non-degeneracy condition that we need to make on the data itself. Note that it is analogous to the restriction imposed on the data for gluing in the vacuum case.

The remaining assumptions which need to be verified are (N2), (N3) and (N4). The first two of these essentially follow from the conformal invariance of $\frac{\gamma^{ab} J_a J_b}{\rho^2}$. The last of these is a straightforward calculation. We thus conclude that any set of CMC initial data satisfying the CK non-degeneracy condition relative to p_1 and p_2 and having either K or ρ non-vanishing can be glued at p_1 and p_2.

4.2 Einstein–Maxwell and Einstein–Yang–Mills

We first briefly review the Yang–Mills field theory (on a fixed background spacetime) to establish our notation. Let $\mathcal{M}$ be a vector bundle with compact structure group G over a Lorentz manifold (M, η). The sub-bundle of $\mathrm{End}(\mathcal{M})$ consisting of those transformations associated with the adjoint representation of G is denoted by $\mathrm{ad}(\mathcal{M})$; note that each fiber of $\mathrm{ad}(\mathcal{M})$ is isomorphic to $\mathfrak{g}$, the Lie algebra of G. We fix a bi-invariant metric on G, which induces a metric on the fibers of $\mathrm{ad}(\mathcal{M})$. If $\mathcal{D}$ is a connection on $\mathcal{M}$, then its curvature $F_{\mathcal{D}} = \mathcal{D}^2$ is a 2-form taking values in $\mathrm{ad}(\mathcal{M})$ and therefore is a section of $\Omega^2(\mathrm{ad}(\mathcal{M}))$.

A solution to the Yang–Mills equations is a connection $\mathcal{D}$ such that $\mathcal{D}^* F_{\mathcal{D}} = 0$, where $\mathcal{D}^* = (-1)^{\dim(M)+1} * \mathcal{D} *$ and where "$*$" is the Hodge star operator. (Note that to define the operator "$*$", we need the specified metric η.) One readily verifies that the Yang–Mills system has a well-posed Cauchy problem, for which the initial data consist of a G-vector bundle $\mathcal{V}$ over a Riemannian manifold Σ with a connection D and a section E of $\Omega^1(\mathrm{ad}(\mathcal{V}))$. The variable E plays the part of the time derivative of D and necessarily satisfies the constraint equation $D^* E = 0$, where D^* is defined analogously to $\mathcal{D}^*$. In the discussion subsequently, the curvature of D appears; we denote this curvature by B_D, recalling its familiar role in Maxwell's equations. Note that B_D is a section of $\Omega^2(\mathrm{ad}(\mathcal{V}))$

Maxwell's theory is a special case of Yang–Mills theory, in which the structure group is chosen to be $U(1)$. As $\mathfrak{g}$ in this case is naturally isomorphic to $\mathbb{R}$, E and B_D are real-valued differential forms. In three dimensions, E and $*B_D$ are the covector fields corresponding to the electric and magnetic fields.

Minimally coupling Yang–Mills to Einstein, we obtain the Einstein–Yang–Mills theory (with Einstein–Maxwell as a special case), for which the field equations are $\mathcal{D}^* F_{\mathcal{D}} = 0$ and $G = T$, where $\mathcal{D}$ is now the covariant derivative corresponding to the gravitational as well as the Yang–Mills connection and where the Yang–Mills stress-energy tensor is (in component form) $T_{\alpha\beta} = F^{\mu}_{\alpha} F_{\mu\beta} - \frac{1}{n} g_{\alpha\beta} F^{\mu\nu} F_{\mu\nu}$. Using techniques similar to those discussed in [15], one readily verifies that the Einstein–Yang–Mills field equations have a well-posed Cauchy formulation. A set of initial data for Einstein–Yang–Mills consists of a G-vector bundle $\mathcal{V}$ over an n-manifold Σ, with Einstein data γ

and K, together with Yang–Mills data D and E, all satisfying the coupled constraint equations

$$\begin{aligned}\operatorname{div} K - d \operatorname{tr} K &= 2(-1)^{n+1} * \langle E, *B_D \rangle \\ R_\gamma - |K|_\gamma^2 + (\operatorname{tr} K)^2 &= |E|^2 + |B_D|^2 \\ D^* E &= 0.\end{aligned}$$

Here, the bilinear map $\langle \cdot, \cdot \rangle$ on sections of $\Omega^*(\mathrm{ad}(\mathcal{V}))$ is the one which is induced from the bi-invariant metric on $\mathfrak{g}$, and the norm on sections of $\Omega^k(\mathrm{ad}(\mathcal{V}))$ is defined by $|W|^2 = * < W, *W >$. For Maxwell's equations, the equation $D^* E = 0$ is the familiar constraint $\operatorname{div} E = 0$ (the magnetic field constraint $\operatorname{div} B = 0$ is absent, since it is automatically satisfied when the Maxwell theory is formulated as a special case of the Yang–Mills theory).

The choice of the conformal action Φ on the Yang–Mills fields D and E is essentially determined by the criteria discussed in Section 2. In particular, noting that $\gamma_c = \psi^q \gamma$ implies that the Hodge star operator acting on k-forms transforms via $*_c = \psi^{q(n/2-k)} *$, we find that in order to satisfy (C1) and thereby prevent the appearance of ψ in the conformal version of the non-gravitational constraint $D^* E = 0$, we require that $\Phi(D, \psi) = D$ and $\Phi(E, \psi) = \psi^{-2} E$. It then follows that if the conformal data satisfy $D^* E = 0$, then the reconstituted data satisfy $\tilde{D}^* \tilde{E} = 0$. For Maxwell's equations in three dimensions, the conformal action on the connection $\Phi(D, \psi) = D$ is equivalent to a conformal action on the magnetic covector field B given by $\Phi(B, \psi) = \psi^{-2} B$.

With this choice of the conformal action, we may proceed to check that (C2) and (C3) hold as well. We calculate

$$\begin{aligned}*_c \langle \Phi(E, \psi), *_c B_D \rangle &= \psi^{q(n/2-(n-1))} \psi^{-2} \psi^{q(n/2-2)} * \langle E, *B_D \rangle \\ &= \psi^{-q-2} * \langle E, *B_D \rangle ,\end{aligned}$$

thus satisfying condition (C2); we calculate

$$\psi^{q+1} \left(|\Phi(E, \psi)|_c^2 + |B_D|_c^2 \right) = \psi^{-3} |E|^2 + \psi^{1-q} |B_D|^2,$$

from which (C3) follows, with

$$n_{E,D}(\psi) = \psi^{-3} |E|^2 + \psi^{1-q} |B_D|^2.$$

To apply the gluing construction, we start with a solution (γ, K, E, D) of the Einstein–Yang–Mills constraints on a G-vector bundle $\mathcal{V}$ over Σ. From the induced vector bundle $\mathcal{V}_c$ over Σ^* with connection D_c, we construct the vector bundles $\mathcal{V}_T$ over Σ_T as described in Section 3 by fixing local trivializations on the balls B_j and using them to identify fibers over identified points in the connected sum. We then construct a connection D_T on $\mathcal{V}_T$ given by D_c except over C_T, where $D_T = \chi(s)\, D_1 + (1 - \chi(s))\, D_2$ (here, χ is the cutoff function used to define γ_T). Hence, $D_T = D_c$ except over Q. In terms of the classical notation for Maxwell's equations in three dimensions, the construction of this connection results in setting the magnetic field to $\mathrm{curl}(\chi(s)A_1 + (1 - \chi(s))A_2)$ over C_T, where each A_j is a vector potential for the magnetic field over the ball B_j.

To construct E_T, we first note that we may define local sections E_c, E_1, and E_2 of the bundle $\mathrm{ad}(\mathcal{V}_T)$ restricted to $\Sigma\backslash Q$, C_T and C_T, respectively, by identifying $\mathrm{ad}(\mathcal{E}_T)$ with $\mathrm{ad}(\mathcal{E}_c)$ over the appropriate subdomains. We then set $E_T = E_c$ over $\Sigma\backslash Q$ and we set $E_T = \chi_1 E_1 + \chi_2 E_2$ over C_T, where $\chi_1 = \chi(t_2 - 1)$ and $\chi_2 = \chi(t_1 - 1)$. Thus, E_T is pieced together in the same way that the conformal factor ψ_T is (Section 3.1) and one has $E_T = E_1 + E_2$ over most of C_T. It would be more convenient to stitch together E_T over Q alone. However, doing so results in unacceptably large error terms and would result in a failure to satisfy the momentum error estimates.

The next step in the gluing construction is the repair of the non-gravitational constraint: we need to find a section $\hat{E}_T$ of $\Omega^1(\mathrm{ad}(\mathcal{V}_T))$, which satisfies $D_T^*\, \hat{E}_T = 0$. To do this, we seek a section μ_T of $\mathrm{ad}(\mathcal{V}_T)$ that satisfies

$$D_T^*\, D_T\, \mu_T = -\, D_T^*\, E_T.$$

Then, $\hat{E}_T = E_T + D_T\, \mu_T$ is the section we need. The operator $D_T^*\, D_T$ is self-adjoint and elliptic; therefore, as $-\, D_T^*\, E_T$ is L^2 orthogonal to the kernel of $D_T^*\, D_T$, we can solve this equation.

To proceed further, we need bounds on μ_T as well as existence. While it is not clear that such bounds always hold, we can prove that they do, so long as either the Yang–Mills group G is $U(1)$ (the Maxwell case) or so long as $\mathrm{ad}(\mathcal{V})$ has no globally parallel sections. We note that this last condition holds generically for the groups $SU(2)$ and $SU(3)$ of primary physical interest. The precise statement of the necessary boundedness result is as follows, where the Hölder spaces used here are defined analogously to those in Definition 3.1.

Proposition 4.3. *Suppose that either D acting on $\mathrm{ad}(\mathcal{V})$ has trivial kernel or $G = U(1)$. Then, the unique solution μ_T of $D_T^*\, D_T\, \mu_T = -D_T^*\, E_T$ satisfies $||D_T\, \mu_T||_{k,\alpha} < CT^{5/2}\, ||E_T||_{k,\alpha}$ for some constant C independent of T.*

We prove Proposition 4.3 later in this section. For now, let us assume the result and proceed with the verification of the gluing construction conditions. To determine the size of μ_T and, therefore, the size of the correction $|\hat{E}_T - E_T|$, we need to estimate the quantity $D_T^* E_T$ appearing on the right-hand side of the equation for μ_T. We readily verify that the following bounds hold in the regions Q and $C_T^{(2)}$ (recall from Section 3.4.1 that $C_T \backslash Q$ can be divided into the two components, $C_T^{(1)} = [-T/2, 1] \times S^{n-1}$ and $C_T^{(2)} = [1, T/2] \times S^{n-1}$):

$$D_T = D_1 + \mathcal{O}\left(\mathrm{e}^{-T/2+s}\right) \tag{4.10}$$

$$*_T = *_1 + \mathcal{O}\left(\mathrm{e}^{-T/2+s}\right) \tag{4.11}$$

$$E_1 = \mathcal{O}\left(\mathrm{e}^{-T(n-1/2)-s(n-1)}\right). \tag{4.12}$$

Now, in the complement of C_T, we have $D_T^* E_T = 0$, while in Q, $D_T^* E_T = D_T^*(E_1 + E_2)$. Combining (4.10)–(4.12) together with the identity $D_1 *_1 E_1 = 0$, we calculate

$$\begin{aligned} D_T^* E_1 &= *_T(D_1 + \mathcal{O}\left(\mathrm{e}^{-T/2}\right))(*_1 + \mathcal{O}\left(\mathrm{e}^{-T/2}\right))E_1 \\ &= \mathcal{O}\left(\mathrm{e}^{-nT/2}\right). \end{aligned}$$

A similar estimate holds for $D_T^* E_2$; so in Q, we have $D_T^* E_T = \mathcal{O}\left(\mathrm{e}^{-nT/2}\right)$. On $C_T^{(2)}$, we have $*_T = *_2$, $D_T = D_2$ and $E_T = \chi_1 E_1 + E_2$. Thus, combining (4.10)–(4.12) together with the identity $D_T *_T E_2 = 0$, we calculate

$$\begin{aligned} D_T^* E_T &= *_T(D_1 + \mathcal{O}\left(\mathrm{e}^{-T/2+s}\right))\left(*_1 + \mathcal{O}\left(\mathrm{e}^{-T/2}\right)\right)\chi_1 E_1 \\ &= d\chi_1 \wedge \left(*_1 + \mathcal{O}\left(\mathrm{e}^{-T/2}\right)\right) E_1 \\ &\quad + \chi_1 *_T \left(D_1 + \mathcal{O}\left(\mathrm{e}^{-T/2+s}\right)\right)\left(*_1 + \mathcal{O}\left(\mathrm{e}^{-T/2}\right)\right) E_1 \\ &= \mathcal{O}\left(\mathrm{e}^{-T(n-1}\right) + \mathcal{O}\left(\mathrm{e}^{-nT/2}\right), \end{aligned}$$

with an analogous estimate on $C_T^{(1)}$. So finally, we obtain $D_T^* E_T = \mathcal{O}$ $(\mathrm{e}^{-nT/2})$, from which it follows that

$$||\hat{E}_T - E_T||_{k,\alpha} = \mathcal{O}\left(T^{5/2}\,\mathrm{e}^{-nT/2}\right).$$

Each of the remaining error estimates now needs to be computed. The techniques are not different from those used earlier or in the proof of Proposition 3.6. Hence, we just summarize the results of the computation as follows:

Estimate	$\Sigma \setminus C_T$	$C_T \setminus Q$	Q
$D^*\, E_T$	0	$\mathrm{e}^{-nT/2}$	$\mathrm{e}^{-nT/2}$
M1 and M2	$T^{5/2}\, \mathrm{e}^{-nT/2}$	$T^{5/2}\, \mathrm{e}^{-nT/2}$	$T^{5/2}\, \mathrm{e}^{-nT/2}$
E1 and E2	$T^{5/2}\, \mathrm{e}^{-nT/2}$	$\mathrm{e}^{-T(n-2/2)}$	$\mathrm{e}^{-T(n+2/4)}$
N2 and N3	$T^{5/2}\, \mathrm{e}^{-6T/2}$	$\mathrm{e}^{-T(\min(1,n-2/2))}$	e^{-T}

We note, in particular, that $\hat{E}_T$ and D_T satisfy the momentum and energy error estimates for any constants κ and ρ for which $\frac{n-1}{2} < \kappa < \frac{n}{2}$ and $\frac{n-2}{4} < \rho < \min\left(\frac{n+2}{4}, \frac{n-2}{2}\right)$.

The only remaining gluing conditions which need to be verified are (N1) and (N4). Since

$$n'_{E,D}(1) = -4\,|E|^2 + (1-q)\,|B_D|^2\,,$$

it easily follows that for any choice of initial data,

$$n_{E,D}(1) = |E|^2 + |B_D|^2 \geq n'_{E,D}(1),$$

which establishes (N1). As well, it is clear from the form of n' that (N4) is satisfied for any $0 < c < 1$. Hence, we conclude that a given set of Einstein–Yang–Mills initial data can be glued at a chosen pair of points so long as the CK non-degeneracy condition holds for the metric, so long as, if the manifold is compact, either $K \neq 0$ or $E \neq 0$ or $B_D \neq 0$, and so long as the hypotheses of Proposition 4.3 are satisfied.

We now return to the proof of Proposition 4.3. We do this first (Lemma 4.4) for the case in which $\mathrm{ad}(\Sigma)$ has no parallel sections, and then (Lemma 4.5) for the case in which $G = U(1)$. The key step (at least when no parallel sections are present) is the establishment of lower bounds for the principle eigenvalue of the operator $D_T^*\, D_T$.

Lemma 4.4. *Suppose* $\mathrm{ad}(\Sigma)$ *has no parallel sections. Then, there exists a constant* C *such that for* T *sufficiently large, the lowest non-zero eigenvalue* λ_T *of* $D_T^*\, D_T$ *on sections of* $\mathrm{ad}(\Sigma_T)$ *satisfies*

$$\lambda_T \geq \frac{C}{T^2}\,.$$

Proof. If the statement was false, there would exist a sequence of sections u_k on Σ_{T_k} satisfying $D^*_{T_k} D_{T_k} u_k = \lambda_k u_k$ with $\lambda_k < \frac{1}{kT_k^2}$. Without loss of generality, we can assume $\sup |u_k| = 1$. Reducing to a subsequence and relabeling, we conclude using elliptic regularity that u_k converges uniformly on compact subsets of Σ^* to a solution u of the equation $D^*_c D_c u = 0$.

We claim that u is not identically zero. Suppose that u_k converges uniformly to 0 on $\Sigma_T \setminus C_T$. We will show that u_k also converges uniformly to 0 on C_T, which contradicts the assumption $\sup |u_k| = 1$. First, we note that

$$|u_k(\theta, t)| = \int_0^t \partial_s |u_k(\theta, s)| \, ds + |u_k(\theta, 0)|,$$

where the derivatives are meant in the weak sense. Hence, by Hölder's inequality

$$|u_k(\theta, t)| \leq T^{1/2} \left(\int_0^T |d\,|u_k|\,|^2 \, ds \right)^{1/2} + m_k,$$

where $m_k := \sup_{\partial C_T} |u_k|$. The volume element dV on C_T satisfies the condition $c_1\, dV_0 \leq dV \leq c_2\, dV_0$ for constants c_1 and c_2 independent of T, where dV_0 is the volume element of the round cylinder. Letting S_t be the spherical cross-section of C_T at length parameter value t, we find that

$$\begin{aligned} \int_{S_t} |u_k|^2 &\leq CT \int_{C_T} |d\,|u_k||^2 \, ds + Cm_k^2 \\ &\leq CT \int_{\Sigma_T} |d\,|u_k||^2 \, dV + Cm_k^2. \end{aligned}$$

Now, Kato's inequality implies that

$$\begin{aligned} \int_{\Sigma_T} |d\,|u_k||^2 \, dV &\leq \int_{\Sigma_T} |D_{T_k} u_k|^2 \, dV \\ &= \lambda_k \int_{\Sigma_T} |u_k|^2 \, dV, \end{aligned}$$

and hence

$$\begin{aligned} \int_{S_t} |u_k|^2 &\leq CT\lambda_k \int_{\Sigma_T} |u_k|^2 \, dV + Cm_k^2 \\ &\leq CT\lambda_k \operatorname{Vol}(\Sigma_T) + Cm_k^2. \end{aligned}$$

Since $\operatorname{Vol}(\Sigma_T) \leq CT$ and as (by hypothesis) $\lambda_k \leq \frac{1}{kT^2}$, it follows that

$$\int_{S_t} |u_k|^2 \leq C \left(\frac{1}{k} + m_k^2 \right),$$

and we conclude from interior Schauder estimates applied to $D_c^* D_c$ that in fact,

$$\sup_{C_T} |u_k| \leq C\left(\frac{1}{k} + m_k^2\right).$$

Since m_k converges to 0, this proves that u_k converges uniformly to 0 on C_T, which is the desired contradiction. We conclude that u is not identically 0.

We now show that if this non-zero limit u exists, then it extends to a non-trivial solution of $D\,u = 0$ on Σ, which contradicts the hypothesis that such parallel sections do not exist. If Ω is a compact subset of Σ^*, then

$$\begin{aligned}\int_\Omega |\,D\,u|^2 = \lim_{k\to\infty}\int_\Omega |D_{T_k}u_k|^2 \leq \liminf_{k\to\infty}\int_{\Sigma_T}|D_{T_k}u_k|^2 \\ = \liminf_{k\to\infty}\lambda_k\int_{\Sigma_T}|u_k|^2 \leq \frac{C}{k},\end{aligned}$$

since $|u_k| \leq 1$, $\mathrm{Vol}(\Sigma_{T_k}) < CT$ and $\lambda_k < \frac{1}{kT^2}$. It follows that $\int_\Omega |D\,u|^2 = 0$, and we conclude that $D\,u = 0$ on Σ_*. Since u is bounded, it extends to a weak solution of $D^*\,D\,u = 0$ on Σ and hence u is a non-trivial solution of $D\,u = 0$ on Σ, which is a contradiction. We therefore conclude that for T large enough, $\lambda_k \geq \frac{C}{T^2}$. □

The conversion of the eigenvalue estimate of Lemma 4.4 to the Hölder estimate of Proposition 4.3 proceeds in a similar fashion as was done for the vector Laplacian in [19]. From Lemma 4.4, we have $||\mu_T||_{L^2} \leq CT^2||D_T^*E_T||_{L^2}$. Since the volume of Σ_T grows linearly in T, we obtain $||D_T^*E_T||_{L^2} \leq T^{1/2}||D_T^*E_T||_{\mathcal{C}^{0,\alpha}}$. Finally, local Schauder estimates for $D_c^*D_c$ yield $||\mu_T||_{C^{2,\alpha}} \leq C(||D_T^*E_T||_{C^{0,\alpha}} + ||\mu_T||_{L^2})$. Hence, we conclude

$$||\,D_T\,\mu_T||_{\mathcal{C}^{1,\alpha}} \leq C||\mu_T||_{\mathcal{C}^{2,\alpha}} \leq CT^2||D_T^*E_T||_{\mathcal{C}^{0,\alpha}} \leq CT^{5/2}||E_T||_{\mathcal{C}^{1,\alpha}}.$$

The estimate of Proposition 4.3 for higher order spaces $\mathcal{C}^{k,\alpha}$ follows from another application of local Schauder estimates.

Lemma 4.4 does not apply to $U(1)$ bundles, which do admit parallel sections. In this case, D^*D is the Hodge Laplacian d^*d acting on scalar fields (or equivalently $-\Delta$ using our earlier notation and sign convention). The following lemma implies Proposition 4.3 for $U(1)$ bundles.

Lemma 4.5. *Any solution μ_T of $d^*d\mu_T = -d^*E_T$ satisfies $||d\mu_T||_{k,\alpha} < CT^{1/2}\,||E_T||_{k,\alpha}$ for some constant C independent of T.*

Proof. Multiplying both sides of the equation $d^* d\mu_T = d^* E_T$ by μ_T and integrating by parts, we have

$$\int_{\Sigma_T} |d\mu_T|^2 \; dV_T = \int_{\Sigma_T} \langle E, d\mu_T \rangle \; dV_T.$$

The Cauchy–Schwartz inequality then implies

$$||d\mu_T||_{L^2} \leq ||E_T||_{L^2}.$$

Since the volume of Σ_T grows linearly, we obtain

$$||d\mu_T||_{L^2} \leq CT^{1/2} ||E_T||_{\mathcal{C}^{0,\alpha}}.$$

This L^2 estimate replaces the eigenvalue estimate of Lemma 4.4. But to use effectively, we have to show that $d\mu_T$ satisfies an appropriate elliptic equation.

Applying the Hodge Laplacian $d^*d + dd^*$ to $d\mu_T$, we have

$$(d^*d + dd^*)d\mu_T = dd^* d\mu_T = dd^* E.$$

So from local Schauder estimates applied to the Hodge Laplacian, we obtain

$$\begin{aligned} ||d\mu_T||_{\mathcal{C}^{2,\alpha}} &\leq C(||dd^* E_T||_{\mathcal{C}^{0,\alpha}} + ||d\mu_T||_{L^2}) \\ &\leq C(||E_T||_{\mathcal{C}^{2,\alpha}} + T^{1/2}||E_T||_{\mathcal{C}^{0,\alpha}}) \\ &\leq CT^{1/2}||E_T||_{\mathcal{C}^{2,\alpha}}. \end{aligned}$$

The proof of the lemma in the case $k = 2$ is complete, and the the higher order estimates follow from another application of local Schauder estimates. □

4.3 Einstein–Vlasov

The Einstein–Vlasov system is used in general relativity to model self-gravitating systems of collisionless matter, i.e., matter which interacts only by means of the collective gravitational field. There has been a renewed interest recently in establishing rigorous results for the dynamics of solutions of the Einstein–Vlasov system. Two useful survey papers on the subject are given in [21, 1].

On a Lorentz manifold (M, g), the additional field specified by the Einstein–Vlasov system is the distribution function representing the density of particles with a given spacetime position and a given momentum. Each particle is assumed to travel along a time-like future directed geodesic, so its momentum at each point is mv, where m is the mass of the particle and v is a future pointing, unit time-like vector. For simplicity, we assume that all the masses are taken to be one; however, the theory can be easily adapted to account for continuous, non-constant masses. The collection

$$\mathcal{P} = \{(x, v) \,|\, x \in M, v \text{ a future pointing, unit time-like vector}\}$$

forms a Riemannian hypersurface, called the mass shell, in the tangent bundle of M. The distribution function is then given by a non-negative function

$$f \colon \mathcal{P} \to \mathbb{R}^+.$$

We assume for simplicity that f has compact support (again this can be relaxed). The Einstein–Vlasov system is then

$$G_{\mu\nu} = T_{\mu\nu},$$

where we have again chosen units, so that the speed of light and 8π times the gravitational constant are one. The Einstein–Vlasov stress-energy tensor is given by

$$T_{\mu\nu}(x) = -\int_{\mathcal{P}_x} f(x, v) v_\mu v_\nu \, dv_g$$

for each $x \in M$, where dv_g is the induced Riemannian volume measure on the fiber $\mathcal{P}_x$ of $\mathcal{P}$ over x.

The non-gravitational initial data for the Einstein–Vlasov system now consist of a (compactly supported) function f_0 on the tangent bundle of Σ. The energy density and current density of the non-gravitational field are then given by

$$\begin{aligned} \rho(x) &= \int_{T_x\Sigma} f_0(v)\,(1 + |v|^2)\, dv_\gamma \\ J_a(x) &= \int_{T_x\Sigma} f_0(v)\, \gamma_{ab} v^b \, dv_\gamma, \end{aligned}$$

where γ is the Riemannian metric on Σ and dv_γ is the induced Riemannian volume form on the tangent spaces $T_x\Sigma$. Note that there are no additional non-gravitational constraints which need to be satisfied.

One readily checks that that if, under the conformal change $\tilde{\gamma} = \psi^q \gamma$, we set

$$\tilde{f}_0(v) = \phi^{-(3/2)q-2} f_0(\phi^{q/2} v),$$

then

$$\tilde{J}_a = \phi^{-q-2} J_a \quad \text{and} \quad \tilde{\rho} = \phi^{-(3/2)q-2} \rho.$$

From this, it follows that $n(\phi) = \phi^{q+1}\tilde{\rho} = \phi^{-(q/2)-1}\rho$ and we observe that the construction now proceeds exactly as in the Einstein perfect fluids case.

Remark 4.6. Note that the results of this section are applicable beyond the Einstein–Vlasov system. The only aspect of Einstein–Vlasov which is used is the nature of the unknown in the Vlasov equation and the way it enters into the energy-momentum tensor. Since the dynamic equations of motion play no role here, these results apply equally well to, for example, the Boltzmann equation.

5 Correcting σ_T in higher dimensions

In Section 3.3, the following lemma was left unproved.

Lemma 5.1. *Suppose there are no CK that vanish at points p_j of Σ. Then, for T sufficiently large and for each $X \in \mathcal{C}^{k,\alpha}(\Sigma_T)$, there is a unique solution $W \in \mathcal{C}^{k+2,\alpha}(\Sigma_T)$ to $LW = X$. Moreover, there exists a constant C independent of W and T such that*

$$||X||_{k+2,\alpha} \leq CT^3||W||_{k,\alpha}.$$

The proof of the lemma is identical to the case when $n = 3$ found in [19], so long as one can establish the following lower bound on the size of the smallest eigenvalue of the vector Laplacian.

Theorem 5.2. *For T sufficiently large, the lowest eigenvalue $\lambda_0 = \lambda_0(T)$ for L on Σ_T satisfies $\lambda_0 \geq CT^{-2}$ for some constant C independent of T.*

In the case $n = 3$, Theorem 5.2 follows from a perturbation argument for the lowest eigenvalue of the vector Laplacian on the round cylinder. Here, the proof when $n > 3$ is identical to the case when $n = 3$, once one has obtained specific estimates for the vector Laplacian on the round cylinder, to which we turn now our attention.

For convenience, we take the vector Laplacian to operate on 1-forms rather than on vector fields. Let X be a covector field on the cylinder $\mathbb{R} \times S^{n-1}$, which we write as

$$X = f ds + Y(s),$$

where $Y(s)$ is a covector field on S^{n-1}. The vector Laplacian applied to X can then be written in terms of its action on f and Y. We obtain

$$LX = L\begin{pmatrix} f \\ Y \end{pmatrix} = \begin{pmatrix} \frac{1-n}{n}\partial_s^2 + \frac{1}{2}\Delta_\theta & \frac{n-2}{2n}\partial_s\delta_\theta \\ \frac{2-n}{2n}\partial_s d_\theta & -\frac{1}{2}\partial_s^2 + \frac{1}{2}\delta_\theta d_\theta + \frac{n-1}{n}d_\theta\delta_\theta + (2-n) \end{pmatrix}\begin{pmatrix} f \\ Y \end{pmatrix}.$$

An orthonormal basis of 1-forms on S^{n-1} is given by the eigenfunctions of the Laplacian. Moreover, if η is a 1-form such that

$$\Delta\,\eta = \lambda\eta, \tag{5.1}$$

then from the Hodge decomposition and the topology of the sphere, it follows that

$$\eta = d\phi + \psi, \tag{5.2}$$

where ϕ is a function, ψ is a divergence-free covector field and both are eigenfunctions of the Laplacian with eigenvalue λ. So if $\{\phi_j\}$ is an orthonormal basis of eigenfunctions of the scalar Laplacian with eigenvalues λ_j, we have an orthonormal basis of one-forms given by $\{\frac{1}{\sqrt{\lambda_j}}d\phi_j\} \cup \{\psi_j\}$, where $\delta\psi_j = 0$ and $\Delta\,\psi_j = \mu_j\psi_j$. Finally, we note from [14] that for each j, there exists $k, l \in \mathbb{N}$ such that $\lambda_j = k(k+n-2)$ and $\mu_j = (l+1)(l+n-3)$.

For a covector field X of the form $u(s)\phi_j ds + v(s)\frac{1}{\sqrt{\lambda_j}}d_\theta\phi_j$, the vector Laplacian acts on the column vector $(u, v)^t$ via

$$\begin{aligned} L'_j = &\begin{pmatrix} \frac{1-n}{n} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\partial_s^2 + \begin{pmatrix} 0 & \frac{n-2}{2n}\sqrt{\lambda_j} \\ -\frac{n-2}{2n}\sqrt{\lambda_j} & 0 \end{pmatrix}\partial_s \\ &+ \begin{pmatrix} \frac{\lambda_j}{2} & 0 \\ 0 & \frac{n-1}{n}\lambda_j + (2-n) \end{pmatrix} \end{aligned} \tag{5.3}$$

except when $j = 0$. In this case, ϕ_0 is constant, $X = u(s)ds$, and we simply have

$$L_0 = \frac{1-n}{n}\partial_s^2.$$

Finally, for covector fields of the form $X = w(s)\psi_j$, we have

$$L''_j = -\frac{1}{2}\partial_s^2 + (\frac{\mu_j}{2} + 2 - n).$$

So together

$$L = L_0 \oplus \bigoplus_{j\geq 1}(L'_j \oplus L''_j).$$

An analysis parallel to that of [19] shows that the temperate solutions of $LX = 0$ on the cylinder (i.e., those with slower than exponential growth at both ends) are spanned by ds, $s\,ds$, ω_{ij} and $s\omega_{ij}$, where for $i < j$, $\omega_{ij} = x_i dx_j - x_j dx_i$ is the generator of a rotation on the sphere. So there is a $n(n-1)+2$ dimensional family of temperate solutions, and the bounded ones are also CK fields on the cylinder.

The remainder of the differences in the analysis are contained in the following propositions, which are analogous to Propositions 2 and 3 of [19].

Proposition 5.3. *Let $\lambda_0 = \lambda_0(T)$ denote the first Dirichlet eigenvalue for L on $C_T = [-T/2, T/2] \times S^{n-1}$. Then,*

$$\lambda_0(T) \geq \frac{C}{T^2}$$

for some constant C independent of T.

Proof. The estimate is obvious for L_0 and for L_j'' when $\mu_j = 2n(n-2)$ (i.e., when $L_j'' = -\frac{1}{2}\partial_s^2$). On the other hand when $\mu_j = (l+1)(l+n-3)$ for $l \geq 2$, the lowest Dirichlet eigenvalue of L_j'' on C_T converges to $\frac{\mu_j}{2} - (n-2) > 0$. So it remains to consider the operators L_j'.

Let X denote a lowest eigenfunction for L_j' with components (u, v). Then,

$$\begin{aligned}
\left\langle L_j' X, X \right\rangle &= \int_{-T/2}^{T/2} \frac{n-1}{n}(u')^2 + \frac{1}{2}(v')^2 + \frac{n-2}{2n}\sqrt{\lambda_j}(uv' - u'v) \\
&\quad + \frac{\lambda_j}{2}u^2 + (\frac{n-1}{n}\lambda_j + (2-n))v^2\, ds \\
&\geq \int_{-T/2}^{T/2} \frac{n-1}{n}(u')^2 + \frac{1}{2}(v')^2 + \frac{2-n}{n}\left(\frac{(v')^2}{2} + \lambda_j \frac{u^2}{2}\right) \\
&\quad + \frac{\lambda_j}{2}u^2 + (\frac{n-1}{n}\lambda_j + (2-n))v^2\, ds \\
&= \int_{-T/2}^{T/2} \frac{n-1}{n}(u')^2 + \frac{1}{n}(v')^2 + \frac{\lambda_j}{n}u^2 + (\frac{n-1}{n}\lambda_j + (2-n))v^2\, ds,
\end{aligned} \tag{5.4}$$

where we have integrated by parts and applied Young's inequality. As $\lambda_j \geq n-1$, it follows that $\frac{n-1}{n}\lambda_j + (2-n) \geq \frac{1}{n}$. Hence, $\left\langle L_j' X, X \right\rangle \geq \frac{1}{n}\langle X, X\rangle$. □

Proposition 5.4. *Suppose $LX = \mu X$ on C_T. Then, for T sufficiently large, there exists a constant c independent of T such that if $\mu \leq \frac{c}{T^2}$ and if $\int_{S^{n-2}}(|X(-T/2,\theta)|^2 + |X(T/2,\theta)|^2)d\theta \leq C_1$, then for any $a \in [-T/2+1, T/2-1]$, we have*

$$\int_{a-1}^{a+1}\int_{S^{n-2}} |X(s,\theta)|^2\, ds\, d\theta \leq C_2,$$

where C_2 depends only on C_1 but not on T or a, and $C_2 \to 0$ as $C_1 \to 0$.

Proof. Following the proof of the corresponding result in [19], it is enough to prove the result for each component of the separation of variables decomposition. Moreover, the estimate is clear for L_0 and L''_j; so we restrict our attention to L'_j.

We can write L'_j in the form

$$L'_j = -A\partial_s^2 + B\partial_s + C,$$

where A, B and C can be determined from Eq. (5.3). Write $X = (u, v)^t$ and $f(s) = \langle AX, X\rangle$. It follows from the equation $L'_j X = \mu X$ that

$$\frac{1}{2}\partial_s^2 f(s) = |A^{1/2}X' - \frac{1}{2}A^{-1/2}BX|^2 + \langle DX, X\rangle,$$

where

$$D = \begin{bmatrix} -1/8\,\frac{(n-2)^2\lambda_j}{n^2} + 1/2\,\lambda_j - \mu & 0 \\ 0 & -1/16\,\frac{(n-2)^2\lambda_j}{(n-1)n} + \frac{(n-1)\lambda_j}{n} + 2 - n - \mu \end{bmatrix}.$$

Clearly, $D_{11} \geq 3\lambda_j/8 - \mu$. Also, D_{22} can be rewritten as

$$\frac{(n-1)\lambda_j}{n}\left[1 - \frac{1}{16}\left(\frac{n-2}{n-1}\right)^2\right] + 2 - n - \mu \geq \frac{15}{16}\frac{(n-1)\lambda_j}{n} + 2 - n - \mu.$$

When λ_j is $2n$ (its second non-zero value), we have

$$\frac{15}{16}\frac{(n-1)\lambda_j}{n} + 2 - n - \mu = \frac{7n+1}{8} - \mu,$$

which is positive when T is sufficiently large (forcing μ to be sufficiently small). Since D_{22} is increasing in λ_j, we have therefore proved that $f(s)$ is convex and hence the L^2 norm of X at $s = \pm T/2$ controls the L^2 norm of X over any strip $a - 1 \leq s \leq a + 1$, except possibly when $\lambda_j = n - 1$.

To handle the case $\lambda_j = n - 1$, we proceed as in Proposition 5.3. The steps leading to estimate (5.4) follow as earlier, except now we pick up boundary terms from the integration by parts. Specifically, we have

$$\begin{aligned} &\int_{-T/2}^{T/2}\int_{S^{n-1}} \left\langle (L'_j - \mu)X, X\right\rangle \, d\theta\, ds \\ &\quad = \int_{-T/2}^{T/2} \frac{n-1}{n}(u')^2 + \frac{1}{n}(v')^2 + \left(\frac{n-1}{n} - \mu\right)u^2 + \left(\frac{1}{n} - \mu\right)v^2\, ds \\ &\qquad + b\left(\frac{T}{2}\right) - b\left(\frac{-T}{2}\right), \end{aligned}$$

where $b = \frac{1-n}{n}u'u - \frac{1}{2}v'v - \frac{n-2}{2n}\sqrt{n-1}uv$. Hence,

$$\int_{-T/2}^{T/2}\int_{S^{n-1}} \frac{1}{n}\left|X'\right|^2 + \left(\frac{1}{n} - \mu\right)|X|^2 \; d\theta\, ds \leq b\left(\frac{T}{2}\right) - b\left(\frac{-T}{2}\right). \quad (5.5)$$

Let $\chi(s)$ be a cutoff function equal to 0 for $s < 0$ and equal to 1 for $s > 1$ and let $\chi_T(s) = \chi(s - T/2)$. Then,

$$\begin{aligned}
&\int_{S^{n-1}} \langle AX'(T/2), X'(T/2)\rangle \; d\theta \\
&\quad = \int_{(T/2)-1}^{T/2}\int_{S^{n-1}} \frac{d}{ds}\left\langle AX', \chi_T X'\left(\frac{T}{2}\right)\right\rangle \; d\theta\, ds \\
&\quad = \int_{(T/2)-1}^{T/2}\int_{S^{n-1}} \left\langle AX'', \chi_T X'\left(\frac{T}{2}\right)\right\rangle + \left\langle AX', \chi_T' X'\left(\frac{T}{2}\right)\right\rangle \; d\theta\, ds \\
&\quad = \int_{(T/2)-1}^{T/2}\int_{S^{n-1}} \left\langle BX' + (C-\mu)X, \chi_T X'\left(\frac{T}{2}\right)\right\rangle \\
&\qquad + \left\langle AX', \chi_T' X'\left(\frac{T}{2}\right\rangle \; d\theta\, ds \\
&\quad \leq \frac{c}{\epsilon}\int_{(T/2)-1}^{T/2}\int_{S^{n-1}} \left|X'\right|^2 + |X|^2 \; d\theta\, ds + \epsilon\int_{S^{n-1}} \left|X'\left(\frac{T}{2}\right)\right| \; d\theta,
\end{aligned}$$

where the constant c is independent of T. Taking ϵ sufficiently small, we obtain

$$\int_{S^{n-1}} \left|X'\left(\frac{T}{2}\right)\right|^2 \leq c\int_{(T/2)-1}^{T/2}\int_{S^{n-1}} \left|X'\right|^2 + |X|^2 \; d\theta\, ds, \quad (5.6)$$

where c is independent of T. Finally, we note that

$$\left|b\left(\frac{T}{2}\right)\right| \leq \frac{c}{\epsilon}\int_{S^{n-1}} \left|X\left(\frac{T}{2}\right)\right|^2 \; d\theta + \epsilon\int_{S^{n-1}} \left|X'\left(\frac{T}{2}\right)\right|^2 \; d\theta \quad (5.7)$$

for any $\epsilon > 0$. Similar estimates hold at $s = -T/2$, and combining (5.5)–(5.7), we conclude that there exists a constant c independent of T such that

$$\int_{-T/2}^{T/2}\int_{S^{n-1}} \left(\frac{1}{n} - \mu\right)|X|^2 \leq c\int_{S^{n-1}} \left|X\left(\frac{-T}{2}\right)\right|^2 + \left|X\left(\frac{T}{2}\right)\right|^2 \; d\theta$$

for μ sufficiently small, which completes the proof. □

The remainder of the proof of Theorem 5.2 now follows exactly as in [19], and the reader is referred there for details.

6 Conclusions

Our discussion in Section 4 of the Einstein fluid, Einstein–Yang–Mills and Einstein–Vlasov cases provides a sample collection of Einstein-matter field theories for which solutions of the relevant constraints can be generally glued together, provided the solutions satisfy mild non-degeneracy hypotheses. The same is likely true for a wide collection of other Einstein-matter field theories as well; one need only to check that the conformal method can be applied and that the the various criteria discussed in Sections 2 and 3, and summarized in Theorem 3.1, are met.

It is not always easy to check these criteria. Indeed, for the Einstein–Klein–Gordon theory, with the standard spacetime action principle $S[g, \chi] = \int_M (R + \frac{1}{2}|\nabla\chi|^2 + \frac{1}{2}m^2|\chi|^2)$, where χ is a $\mathbb{C}$-valued scalar field, one runs into serious difficulty (even in the massless case when $m = 0$). We note, however, that for this Einstein-matter model, it is also not straightforward to use the standard conformal method to construct solutions of the constraint equations. (For very recent work related to this, see, [4].) The difficulty arises in verifying solvability for the Einstein–Klein–Gordon version of the Lichnerowicz equation

$$\Delta\psi = \frac{1}{8}(R - 2|D\chi|^2)\psi - \frac{1}{8}[(\sigma + \mathcal{D}W)^2 + 2|P|^2]\psi^{-7} + \left(\frac{1}{12}\tau^2 - \frac{1}{8}m^2|\chi|^2\right)\psi^5, \tag{6.1}$$

where P is a $\mathbb{C}$-valued scalar field, representing the time derivative of χ. For solutions of the constraint equations of other field theories, such as the Einstein–Dirac theory [17], one can readily check whether the conditions needed for the gluing construction presented here are satisfied.

For those Einstein-matter theories which do satisfy our gluing criteria, can we proceed further and obtain stronger gluing results of the sort established in [8, 9] ("CIP-gluing") for the vacuum case? To be able to show this, one needs to prove that some appropriate version of the Corvino–Schoen gluing results [12] holds for the field theory of interest. Corvino appears to have done this for the Einstein–Maxwell theory [11], so we should be able to obtain CIP-gluing results in this case.

Acknowledgment

We thank the referee for some useful suggestions. J. I. is partially supported by the NSF under Grants PHY-0099373, PHY-0354659 and the American

Institute of Mathematics. D.M. and D.P. are partially supported by the NSF under Grant DMS-0305048 and the UW Royalty Research Fund.

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