

$U(n)$ vector bundles on Calabi–Yau three-folds for string theory compactifications

Björn Andreas¹, Daniel Hernández Ruipérez²

¹Institut für Mathematik, Humboldt Universität zu Berlin
D-10115 Berlin, Germany
andreas@mathematik.hu-berlin.de

²Departamento de Matemáticas and Instituto Universitario de
Física Fundamental y Matemáticas (IUFFYM),
Universidad de Salamanca
Plaza de la Merced 1-4, 37008 Salamanca, Spain
ruiperez@usal.es

Abstract

An explicit description of the spectral data of stable $U(n)$ vector bundles on elliptically fibered Calabi–Yau three-folds is given, extending previous work of Friedman, Morgan and Witten. The characteristic classes are computed and it is shown that part of the bundle cohomology vanishes. The stability and the dimension of the moduli space of the $U(n)$ bundles are discussed. As an application, it is shown that the $U(n)$

bundles are capable to solve the basic topological constraints imposed by heterotic string theory. Various explicit solutions of the Donaldson–Uhlenbeck–Yau equation are given. The heterotic anomaly cancellation condition is analyzed; as a result, an integral change in the number of fiber wrapping 5-branes is found. This gives a definite prediction for the number of 3-branes in a dual F -theory model. The net-generation number is evaluated, showing more flexibility compared with the $SU(n)$ case.

1 Introduction

Three approaches to construct holomorphic vector bundles, with structure group the complexification $G_{\mathbb{C}}$ of a compact Lie group G , on elliptically fibered Calabi–Yau three-folds have been introduced in ref. [40]. The parabolic bundle approach applies for any simple G . One considers deformations of certain minimally unstable G -bundles corresponding to special maximal parabolic subgroups of G . The spectral cover approach applies for $SU(n)$ and $Sp(n)$ bundles and can be essentially understood as a relative Fourier–Mukai transformation. The del Pezzo surface approach applies for E_6 , E_7 and E_8 bundles and uses the relation between subgroups of G and singularities of del Pezzo surfaces. Various aspects of these approaches have been further explored in refs. [1, 10, 12, 13, 15, 16, 17, 23, 24, 28, 30, 31, 41, 42, 53].

The present paper continues the discussion of the spectral cover approach. Our aim is to give an explicit description of the spectral cover data and characteristic classes of stable $U(n)$ vector bundles V on elliptically fibered Calabi–Yau three-folds X . As an application, we will study the question whether the $U(n)$ bundles, which have non-zero first Chern class, are capable to solve the topological conditions

$$c_1(V) = 0 \pmod{2}, \quad \text{ch}_2(X) = \text{ch}_2(V), \quad (1.1)$$

that have to be imposed [26, 47, 60] for a consistent heterotic string compactification. The first condition guarantees that the bundle V admits spinors, while the second condition is the known anomaly cancellation condition which has to be generalized in the presence of 5-branes.

Moreover, in order to have unbroken supersymmetry in four dimensions, the vector bundle has to satisfy the Donaldson–Uhlenbeck–Yau equation [22]. A solution to this equation is guaranteed [59] if V is a stable and satisfies the integrability (or zero slope) condition (J denotes the Kähler form on X)

$$\int_X c_1(V) \wedge J^2 = 0. \quad (1.2)$$

Due to the zero-slope condition, the search for explicit solutions of equation (1.1) has been restricted mainly to vector bundles with vanishing first Chern class (see, for instance [26, 27]). If $c_1(V)$ is non-zero, then one typically has to work on manifolds for which the dimension of $H^{1,1}(X)$ is greater than or equal to 2; a condition which is satisfied for the class of elliptically fibered Calabi-Yau three-folds considered in this paper.

Let us also note, while the importance of vector bundles with vanishing first Chern class for the heterotic string theory is clear from a physics point of view, that there is no reason to prefer bundles with vanishing first Chern class as long as mathematics is concerned. Actually, moduli spaces of stable or semistable vector bundles (or sheaves) with arbitrary Chern classes have been constructed. Also most mathematical operations with vector bundles and sheaves, for instance, twisting by a line bundle or elementary transforms, do not preserve the vanishing of the first Chern class.

Clearly solving the topological constraints is only a first step to understand heterotic string compactifications based on $U(n)$ vector bundles. Further, issues such as the embedding properties of the $U(n)$ structure group into E_8 or the cancellation of an $U(1)$ anomaly arising on the string world sheet have to be addressed [6].

1.1 Significance for D -branes at large volume

To conclude this section, let us mention a second application of $U(n)$ vector bundles (or more generally sheaves) on Calabi-Yau manifolds.

It has been shown that D -branes on Calabi-Yau manifolds, in the large volume limit, can be described by vector bundles (or sheaves) [18, 32–34]. For D -branes wrapping all the Calabi-Yau manifold, supersymmetry requires that the holomorphic connection A on V has to satisfy

$$F_A^{1,1} \wedge J^2 = \frac{2\pi}{i} \mu(V) I_V. \tag{1.3}$$

Since the $U(n)$ vector bundles (and sheaves) we consider in this paper are stable, they provide solutions to equation (1.3), in contrast to heterotic string theory where one has to impose the stronger (zero-slope) condition $\mu(V) = 0$.

In particular, vector bundles (or sheaves) with non-zero first Chern class have been predicted, using the correspondence between the spectrum of D -branes at large volume and the Gepner point [18], at which a particular conformal field theory is exactly solvable. The basic strategy which had

been applied to establish this correspondence relied on a comparison of two central charges: the central charge corresponding to an integral vector in the middle cohomology lattice $H^3(Y, \mathbb{Z})$ of the mirror manifold Y of a given Calabi–Yau three-fold and the central charge associated to a D -brane state represented by a K-theory class and measured by the Mukai vector in the even cohomology. The comparison of these central charges gives a relation between the low-energy charges and the topological invariants of the K-theory class and leads in many cases to a prediction of vector bundles (and sheaves) that have non-zero first Chern class. For elliptically and $K3$ -fibered Calabi–Yau manifolds, this correspondence has been worked out in many examples [7, 25, 36, 48].

Thus finding stable vector bundles (or sheaves) with non-zero first Chern class gives a direct proof that the corresponding moduli spaces, predicted in this correspondence, are not empty.

Another advantage to treat vector bundles (or sheaves) as D -branes on elliptically fibered Calabi–Yau three-folds is that the relative Fourier–Mukai transformation can be viewed as a fiberwise T-duality transformation [7, 8, 31]. This gives a concrete physical interpretation of the spectral cover approach for vector bundles.

This paper is organized as follows. In Section 2, we first review the spectral cover approach as originally introduced in ref. [40], then we review the more general description of vector bundles and sheaves in terms of a relative Fourier–Mukai transformation. In Section 3.1, we analyze the relation of the Fourier–Mukai transformation and its inverse, and this leads to a first simple example of an $U(n)$ vector bundle. It also shows that every stable $SU(n)$ vector bundle constructed in ref. [40] automatically determines a stable $U(n)$ vector bundle via the inverse Fourier–Mukai transformation. In Section 3.2, we compute the characteristic classes of $U(n)$ vector bundles and comment on the finiteness of vector bundles (and sheaves) in this construction. In Section 3.3, we derive the precise conditions which the spectral data of $U(n)$ vector bundle have to satisfy. In Section 3.4, we study the reduction to $SU(n)$ bundles and recover the known expressions for the spectral data and characteristic classes derived in ref. [40]. In Section 3.5, we review the known arguments and conditions for the stability of vector bundles which have been constructed in the spectral cover approach, or equivalently, by a relative Fourier–Mukai transformation. This section also prepares the analysis of the Donaldson–Uhlenbeck–Yau equation presented in Section 4.1. In Section 3.6, we compute the bundle cohomology and show that the $U(n)$ bundles have no sections which is relevant for the evaluation of the net-number of generations in a heterotic string compactification. In Section 3.7, we discuss

the moduli of $U(n)$ vector bundles. In Section 4.1, we show that $U(n)$ bundles are capable to solve the Donaldson–Uhlenbeck–Yau equation and give several explicit examples illustrating that. In Section 4.2, we discuss the anomaly constraint and find a change in the number of 5-branes wrapping the elliptic fiber. This leads to a definite prediction for the number of 3-branes required for a consistent F -theory compactification. In Section 4.3, we give the expression for the net-number of generations which receives a new term compared to the $SU(n)$ case.

There are three appendices. In Appendix A, we provide some background on the Hermite–Einstein equations. In Appendix B, we review the Grothendieck–Riemann–Roch computation which led to the characteristic classes of $U(n)$ vector bundles in Section 3.2. In Appendix C, we give a simplified derivation of the net-generation number for $SU(n)$ vector bundles which are constructed via the inverse Fourier–Mukai transformation. The same result has been obtained in ref. [24] using Leray spectral sequence technics.

2 Review

In this section, we review the basic strategy of Friedman *et al.* [40] which led to the construction of vector bundles on elliptically fibered Calabi–Yau three-folds X with a section σ . In the following, we will denote by E a single elliptic curve, i.e., a 2-torus with a complex structure and a distinguished point p , the identity element in the group law on E . We are interested in G -bundles on E and X . We also recall the coherent description of vector bundles (and sheaves) in terms of a relative Fourier–Mukai transformation which will be our essential tool in this paper.

2.1 $SU(n)$ bundles on E

An $SU(n)$ bundle on E is a rank n vector bundle V of trivial determinant. Any $SU(n)$ vector bundle V on E can be expressed, as a smooth bundle, in the form $V = \oplus_{i=1}^n \mathcal{L}_i$, where \mathcal{L}_i are holomorphic line bundles.¹ Note that, as a holomorphic vector bundle, V has only a filtration with quotients given by the \mathcal{L}_i 's. The $SU(n)$ condition is imposed if $\otimes_{i=1}^n \mathcal{L}_i$ is the trivial

¹In fact, Atiyah [11] showed for a smooth vector bundle V on E and $G = GL_r(\mathbb{C})$ that $V = \oplus_{i=1}^n \mathcal{L}_i$ as smooth bundles, where each \mathcal{L}_i is indecomposable and of degree zero.

line bundle. Further, V is semistable if all \mathcal{L}_i are of degree zero. The \mathcal{L}_i are uniquely determined up to permutations and further determine a unique point q_i in E ; conversely, every q_i in E determines a degree zero line bundle $\mathcal{L}_i = \mathcal{O}(q_i - p)$ on E . Further, a semistable smooth $SU(n)$ bundle $V = \bigoplus_{i=1}^n \mathcal{L}_i$ is determined due to the additional condition $\sum_{i=1}^n (q_i - p) = 0$ (as divisors). The q_i can be realized as roots of

$$s = a_0 + a_2x + a_3y + a_4x^2 + a_5x^2y + \cdots + a_nx^{n/2} = 0, \quad (2.1)$$

and give a moduli space of bundles \mathbb{P}^{n-1} , as stated by a theorem of Looijenga [50, 51]. Note that if n is odd, the last term in $s = 0$ is given by $a_nx^{(n-3)/2}y$. The roots are determined by the coefficients a_i only up to an overall scale factor so that the a_i become the homogeneous coordinates on \mathbb{P}^{n-1} .

2.2 $SU(n)$ bundles on X

The basic strategy of Friedman *et al.* [40] to construct $SU(n)$ vector bundles on X is a two-step process. First, they use the bundle description of $SU(n)$ bundles on E and then “glue” the bundle data together over the base manifold B of X . More precisely, the variation of the n points in the fiber over B leads to a hypersurface C embedded in X , i.e., C is a ramified n -fold cover — the spectral cover — of the base. The line bundle on X determined by C is given by $\mathcal{O}_X(C) = \mathcal{O}_X(n\sigma) \otimes \mathcal{M}$, where σ denotes the section and \mathcal{M} is a line bundle on X whose restriction to the fiber is of degree zero. It follows that the cohomology class of C in $H^2(X, \mathbb{Z})$ is given by $[C] = n\sigma + \eta$, where $\eta = c_1(\mathcal{M})$. Let $\pi_C: C \rightarrow B$ and denote by E_b the general elliptic fiber over a point b in B , then we have $C \cap E_b = \pi_C^{-1}(b) = q_1 + \cdots + q_n$ and $\sigma \cap E_b = p$. Now, each q_i determines a line bundle \mathcal{L}_i of degree zero on E_b whose sections are the meromorphic functions on E_b with first-order poles at q_i and vanishing at p . The restriction of V to E_b is then $V|_{E_b} = \bigoplus_{i=1}^n \mathcal{L}_i$ as smooth bundles. As b moves in the base, the \mathcal{L}_i move in one-to-one correspondence with the n points q_i above b . This specifies a unique line bundle L on C such that $\pi_{C*}L = V|_B$. Thus in addition to C one has to specify a line bundle L on C to completely specify a rank n vector bundle on X .

This procedure leads typically to $U(n)$ vector bundles and will be our starting point. In ref. [40], the authors are interested in $SU(n)$ vector bundles. Therefore, they impose two further conditions: \mathcal{M} if restricted to the fiber is the trivial bundle and L has to be specified such that $c_1(V) = 0$. For the study of $U(n)$ vector bundles, we shall keep the first condition.

2.3 Coherent description via Fourier–Mukai transform

Fourier–Mukai transformations have been introduced originally in ref. [56] as a tool for studying coherent sheaves on abelian varieties (or complex tori) and the corresponding derived categories. The transform has been extended later to other classes of manifolds, the first example being a Fourier–Mukai transformation for certain K3 surfaces [12]. Also the relative situation for families of abelian varieties has been analyzed in ref. [57]. After that the Fourier–Mukai transformation has been studied in many references, both in the absolute and the relative cases [13, 14, 16, 17, 46, 53, 62] and in special connection with mirror symmetry and D-branes [5, 7, 8, 10] or heterotic/M-theory [30, 31].

Now a coherent description of V can be given using a relative Fourier–Mukai transform [7, 8, 10, 13, 16]. Note that any sheaf constructed as a relative Fourier–Mukai transformation in terms of spectral data has trivial restriction to the fibers, so that it may be considered as a parametrization of $SU(n)$ bundles. Moreover, sheaves on an elliptic fibrations having degree zero on the fibers admit a spectral cover description and can be reconstructed by the inverse Fourier–Mukai transformation from their spectral data. This is the reason why the condition of vanishing degree on the fibers is usually imposed, an assumption we will adopt also in this paper.

For the description of the Fourier–Mukai transform, it is appropriate to work on $X \times_B \tilde{X}$, where \tilde{X} is the compactified relative Jacobian of X . \tilde{X} parameterizes torsion-free rank 1 and degree zero sheaves on the fibers of $X \rightarrow B$ and it is actually isomorphic with X so that we can identify \tilde{X} with X . We have a diagram:

$$\begin{array}{ccc}
 X \times_B X & \xrightarrow{\pi_2} & X \\
 \downarrow \pi_1 & & \downarrow \pi \\
 X & \xrightarrow{\pi} & B
 \end{array} \tag{2.2}$$

and the *Poincaré* sheaf

$$\mathcal{P} = \mathcal{O}(\Delta) \otimes \mathcal{O}(-\pi_1^* \sigma) \otimes \mathcal{O}(-\pi_2^* \sigma) \otimes q^* K_B^{-1} \tag{2.3}$$

normalized to make \mathcal{P} trivial along $\sigma \times \tilde{X}$ and $X \times \sigma$. Here $q = \pi_1 \circ \pi_1 = \pi_2 \circ \pi_2$ and $\mathcal{O}(\Delta)$ is the dual of the ideal sheaf of the diagonal, which is torsion-free of rank 1.

We define two Fourier–Mukai transforms on the derived category $D(X)$ of bounded complexes of coherent sheaves on X in the usual way

$$\begin{aligned}\Phi(\mathcal{G}) &= R\pi_{1*}(\pi_2^*(\mathcal{G}) \otimes \mathcal{P}), \\ \hat{\Phi}(\mathcal{G}) &= R\pi_{1*}(\pi_2^*(\mathcal{G}) \otimes \hat{\mathcal{P}}).\end{aligned}\tag{2.4}$$

Here

$$\hat{\mathcal{P}} = \mathcal{P}^* \otimes q^* K_B^{-1}.\tag{2.5}$$

We can also define the Fourier–Mukai functors Φ^i and $\hat{\Phi}^i$, $i = 0, 1$, in terms of single sheaves V by taking $\Phi^i(V)$ and $\hat{\Phi}^i(V)$ as the i th cohomology sheaves of the complexes $\Phi(V)$ and $\hat{\Phi}(V)$, we have

$$\begin{aligned}\Phi^i(V) &= R^i\pi_{1*}(\pi_2^*(V) \otimes \mathcal{P}), \\ \hat{\Phi}^i(V) &= R^i\pi_{1*}(\pi_2^*(V) \otimes \hat{\mathcal{P}}).\end{aligned}\tag{2.6}$$

As it is usual, we say that a sheaf is WIT_j with respect to Φ (or $\hat{\Phi}$) if $\Phi^j(F) = 0$ (or $\hat{\Phi}^j(F) = 0$) for $j \neq i$.

Further, we note that one obtains for any complex \mathcal{G} (or any element in the category) an invertibility result

$$\Phi(\hat{\Phi}(\mathcal{G})) = \mathcal{G}[-1], \quad \hat{\Phi}(\Phi(\mathcal{G})) = \mathcal{G}[-1].\tag{2.7}$$

Concerning the meaning of the -1 shift, we first recall that if we denote by \mathcal{G}^i the cohomology sheaves of a complex \mathcal{G} , the cohomology sheaves of the shifted complex $\mathcal{G}[n]$ are $\mathcal{G}[n]^i = \mathcal{G}^{i+n}$; then if we consider a complex given by a single sheaf V located at the “degree zero” position. $\hat{\Phi}(\Phi(V)) = V[-1]$ means that the complex $\hat{\Phi}(\Phi(V))$ has only one cohomology sheaf, i.e., V , but located at “degree 1”, $[\hat{\Phi}(\Phi(V))]^1 = V$, $[\hat{\Phi}(\Phi(V))]^i = 0$, $i \neq 1$. When $\Phi^0(V) = 0$ the complex $\Phi(V)$ reduces to a single sheaf, i.e., the unique Fourier–Mukai transform $\Phi^1(V)$, but located at “degree 1”, that is, $S\Phi(V) = \Phi^1(V)[-1]$ and the complex $\hat{\Phi}(\Phi(V)) = \hat{\Phi}(\Phi^1(V))[-1]$ has two cohomology sheaves, one at degree 1, given by $\hat{\Phi}^0(\Phi^1(V))$, and one at degree 2, given by $\hat{\Phi}^1(\Phi^1(V))$. So one has $\hat{\Phi}^0(\Phi^1(V)) = V$ and $\hat{\Phi}^1(\Phi^1(V)) = 0$.

3 $U(n)$ vector bundles on elliptic Calabi–Yau three-folds

In what follows, we require that our elliptically fibered Calabi–Yau three-fold $\pi: X \rightarrow B$ has a section σ (in addition to the smoothness of B and X). This (and the Calabi–Yau condition) restricts the base B to be a Hirzebruch surface $(F_m, m \geq 0)$, a del Pezzo surface $(dP_k, k = 0, \dots, 8)$, a

rational elliptic surface (dP_9), blown-up Hirzebruch surfaces or an Enriques surface [30, 55].

From equation (2.6), it follows that we can construct vector bundles on elliptically fibered Calabi–Yau three-folds X using either a relative Fourier–Mukai transform or its inverse. In both cases, one starts from the spectral data (C, L) , where C is a surface in X and L is a torsion-free rank 1 sheaf on C .² We take $C = n\sigma + \pi^*\eta$, more precisely $C \in |n\sigma + \pi^*\eta|$ with η some effective curve in B . So we have two possibilities to define a sheaf on X

$$V = \Phi^0(i_*L), \quad \pi_{C*}L = \sigma^*V, \tag{3.1}$$

$$W = \hat{\Phi}^0(i_*L), \quad \pi_{C*}L = \sigma^*W \otimes K_B, \tag{3.2}$$

where $i: C \hookrightarrow X$ is the closed immersion of C into X . When C is irreducible and L is a line bundle, the first definition agrees with the description of vector bundles given by Friedman *et al.* which we reviewed above.

3.1 Relation of V to W

We will now show that there is a relation between V and W . For this, let us write $\tau: X \rightarrow X$ for the elliptic involution on $X \rightarrow B$. We can consider the induced involution on $X \times_B X$ given by $\tau(x, y) = (\tau(x), y) = (-x, y)$, where x and y are points of X such that $\pi(x) = \pi(y)$. One easily sees that $\pi_1 \circ \tau = \tau \circ \pi_1$, $\tau \circ \pi_2 = \pi_2$ and $\hat{\tau}^*\mathcal{P} = \mathcal{P}^*$. If we denote by $q = \pi_1 \circ \pi = \pi_1 \circ \pi$ the projection of $X \times_B X$ onto B , then for any object \mathcal{G} in the derived category we have

$$\begin{aligned} \hat{\Phi}(\mathcal{G}) &= R\pi_{1*}(\pi_2^*\mathcal{G} \otimes \mathcal{P}^* \otimes q^*K_B^{-1}) = R\pi_{1*}(\tau^*(\pi_2^*\mathcal{G} \otimes \mathcal{P})) \otimes \pi^*K_B^{-1} \\ &= \tau^*(R\pi_{1*}(\pi_2^*\mathcal{G} \otimes \mathcal{P})) \otimes \pi^*K_B^{-1} = \tau^*\Phi(\mathcal{G}) \otimes \pi^*K_B^{-1} \end{aligned} \tag{3.3}$$

by base change in the derived category [44]. Then in our current situation we get

$$W = \tau^*V \otimes \pi^*K_B^{-1}. \tag{3.4}$$

So if we start with a vector bundle V (as given in ref. [40]) with vanishing first Chern class, the new bundle we obtain has non-zero first Chern class and gives our first simple example of an $U(n)$ vector bundle.

²For reducible C , a sheaf is torsion-free if it is pure of dimension 1. A pure sheaf of dimension i is a sheaf whose support has dimension i and it has no subsheaves concentrated on smaller dimension.

3.2 Characteristic classes of V and W

The goal of this section is to compute the characteristic classes of V and W .

3.2.1 Computation of $\text{ch}(V)$

In order to do so, we first recall from Appendix B the formulae giving the topological invariants of both the Fourier–Mukai transform and the inverse Fourier–Mukai transform of a general complex \mathcal{G} in the derived category. Using those formulae we find that, if we start with the sheaf $E = i_*L$ with Chern characters given by

$$\begin{aligned} \text{ch}_0(i_*L) &= 0, \\ \text{ch}_1(i_*L) &= n\sigma + \pi^*\eta, \\ \text{ch}_2(i_*L) &= \sigma\pi^*\eta_E + a_EF, \\ \text{ch}_3(i_*L) &= s_E, \end{aligned} \tag{3.5}$$

with $\eta_E, \eta \in H^2(B)$, then the Chern characters of the Fourier–Mukai transform $V = \Phi^0(i_*L)$ of i_*L are given by

$$\begin{aligned} \text{ch}_0(V) &= n, \\ \text{ch}_1(V) &= \pi^* \left(\eta_E - \frac{1}{2}nc_1 \right), \\ \text{ch}_2(V) &= (-\pi^*\eta)\sigma + \left(s_E - \frac{1}{2}\pi^*\eta_Ec_1\sigma + \frac{1}{12}nc_1^2\sigma \right) F, \\ \text{ch}_3(V) &= -a_E + \frac{1}{2}\sigma c_1\pi^*\eta. \end{aligned} \tag{3.6}$$

3.2.2 Computation of $\text{ch}(W)$

The characteristic classes of W can be determined directly from equation (3.4) using two facts. First, the Chern character is multiplicative, then applying to our situation we get $\text{ch}(W) = \text{ch}(\tau^*V) \text{ch}(\pi^*K_B^{-1})$; second, we note that all the sheaves E we are considering (including V , i_*L and W) have Chern characters of the type (B.5). All those classes are invariant under τ^* because both the fiber and the section are so; it follows that $\text{ch}_i(\tau^*E) = \text{ch}_i(E)$ for every $i \geq 0$. Thus we get $\text{ch}(W) = \text{ch}(V) \text{ch}(\pi^*K_B^{-1})$. Now because B is a surface we have the relations $(\pi^*a)^3 = \pi^*(a^3) = 0$ and $(\pi^*\omega)\pi^*a = \pi^*(\omega a) = 0$, where $a \in H^2(B)$ and $\omega \in H^4(B)$; using these

we find

$$\begin{aligned}
 \text{ch}_0(W) &= n, \\
 \text{ch}_1(W) &= \pi^* \left(\eta_E + \frac{1}{12} n c_1 \right), \\
 \text{ch}_2(W) &= (-\pi^* \eta) \sigma + \left(s_E + \frac{1}{2} \pi^* \eta_E c_1 \sigma + \frac{1}{12} n c_1^2 \sigma \right) F, \\
 \text{ch}_3(W) &= -a_E - \frac{1}{2} \sigma c_1 \pi^* \eta + n \sigma c_1^2.
 \end{aligned} \tag{3.7}$$

The above formulae follow also directly from Appendix B taking into account that $W = \hat{\Phi}^0(i_* L)$ equation (3.2). Note that the number of moduli of W agrees with the number of moduli of the vector bundle V since in general twisting a vector bundle by a line bundle N does not change the dimension of the moduli space of V as $h^1(\text{End}(V)) = h^1(\text{End}(V \otimes N))$.

3.2.3 Remark on finiteness of sheaves

Having computed $\text{ch}(V)$ and $\text{ch}(W)$, one could ask whether there are bounds on possible values of ch_i . This type of question fits into a recently initiated program which studies the vacuum selection problem of string/M-theory from a statistical point of view [9, 29, 35, 37, 38]. A partial answer can be given as a consequence of theorems by Maruyama [54] which characterize the boundedness of families of torsion-free sheaves. Maruyama proved (see [[54], Corollary 4.9]) that if we have a smooth projective complex manifold X of dimension r with a polarization H and fix $n \in \mathbb{Z}$, c_1 the numerical class of a divisor on X and $a \in \mathbb{Z}$, then the set $C_i(n, c_1, a)$ of the algebraic equivalence classes of i th Chern classes $c_i(V)$ of all semistable vector bundles E on X with $c_1(V) = c_1$, $\text{rk}(V) = n$ and $c_2(V)H^{r-2} \leq a$ is finite for all $2 \leq i \leq r$. In particular, if X is a three-dimensional Calabi-Yau manifold, and we fix n , c_1 and c_2 , then the number of possible values of $c_3(V)$ for a semistable vector bundle V of rank n on X with $c_1(V) = c_1$ and $c_2(V) = c_2$ is finite.

3.3 Fixing c_1 of the spectral line bundle

One should note that η_E , a_E and s_E are not completely arbitrary. One needs to ensure that there exists a line bundle L on C whose Chern characters are given by equation (3.5) which we will determine now. We also given an explicit expression for $c_1(L)$.

For this, we analyze the Grothendieck–Riemann–Roch theorem applied to the n -sheeted cover $\pi_C: C \rightarrow B$ which gives

$$\text{ch}(\pi_{C*}L) \text{Td}(B) = \pi_{C*}(\text{ch}(L) \text{Td}(C)) \tag{3.8}$$

from which we derive

$$c_1(\sigma^*V) + \frac{n}{2}c_1(B) = \pi_{C*} \left(c_1(L) + \frac{c_1(C)}{2} \right). \tag{3.9}$$

For $(1, 1)$ classes α on B we have $\pi_{C*}\pi_C^*\alpha = n\alpha$ and σ^* applied to V gives $c_1(\sigma^*V) = \eta_E - (1/2)nc_1(B)$, so we get

$$\pi_{C*}(c_1(L)) = \pi_{C*} \left(-\frac{c_1(C)}{2} + \frac{\pi_C^*\eta_E}{n} \right), \tag{3.10}$$

which gives

$$c_1(L) = -\frac{c_1(C)}{2} + \frac{\pi_C^*\eta_E}{n} + \gamma, \tag{3.11}$$

where $\gamma \in H^{1,1}(C, \mathbb{Z})$ is some cohomology class satisfying $\pi_{C*}\gamma = 0 \in H^{1,1}(B, \mathbb{Z})$. The general solution for γ has been derived in ref. [40] and is given by $\gamma = \lambda(n\sigma|_C - \pi_C^*\eta + n\pi_C^*c_1(B))$ with λ some rational number which we will specify below. Let us also note that γ restricted to $S = C \cap \sigma$ is given by $\gamma|_S = -\lambda\pi^*\eta(\pi^*\eta - n\pi^*c_1(B))\sigma$.

3.3.1 Specification of η_E

So far we assumed that η_E is an element of $H^2(B)$. We need to find now an η_E such that both $c_1(V)$ and $c_1(L)$ are integer classes; otherwise, we had to impose conditions directly on n or $c_1(B)$. It turns out that there are many solutions for η_E satisfying the integer class condition. For our purposes, it will be sufficient to restrict to the case $\eta_E = (n/2)\beta$ with $\beta \in H^2(B, \mathbb{Z})$. This is in agreement with $\text{ch}_2(i_*L)$ being an element of $H^4(X, \mathbb{Q})$ and if we reduce to the $\text{SU}(n)$ case by setting $\eta_E = (1/2)nc_1(B)$, we find that equation (3.11) reduces to the expression for $c_1(L)$ of $\text{SU}(n)$ bundles derived in ref. [40].

For $\eta_E = (1/2)n\beta$ we find that $c_1(V)$ is always an integer class if n is even. If n is odd, we have to impose the additional constraint $\beta = c_1(B) \pmod{2}$.

The conditions for the right-hand side of equation (3.11) to be an integer class are given for $U(n)$ bundles with n odd by $\lambda = m + 1/2$ and $\beta = nc_1(B) \pmod{2}$ (with $m \in \mathbb{Z}$). Thus if n is odd, we have to find simultaneously solutions to $\beta = c_1(B) \pmod{2}$ and $\beta = nc_1(B) \pmod{2}$. This can be solved since $(n - 1)c_1(B) = 0 \pmod{2}$ and $(n - 1)$ is even.

For $U(n)$ bundles with n even we find two solutions; we can either work with $\lambda = m$ and $\eta = \beta \pmod{2}$ or with $\lambda = m + 1/2$ and $\beta = nc_1(B) \pmod{2}$.

Let us briefly discuss how the conditions change if we modify η_E slightly. For $\eta_E = n\beta$ we find again $c_1(V)$ is always an integer class if n is even. However, if n is odd we find $c_1(B) = 0 \pmod{2}$ which restricts the choice of B . There are two solutions for n even such that $c_1(L)$ is an integer class: $\lambda = m$ and $\eta = 0 \pmod{2}$ or $\lambda = m + 1/2$. If n is odd, we find $\lambda = m + 1/2$ and $c_1(B) = 0 \pmod{2}$.

3.3.2 Specification of a_E and s_E

Having fixed $c_1(L)$ we can now go on and determine a_E and s_E in terms of η_E . For this, apply the Grothendieck–Riemann–Roch theorem to $i: C \rightarrow X$ which gives $\text{ch}(i_*L) \text{Td}(X) = i_*(\text{ch}(L) \text{Td}(C))$. We note that $i_*(1) = C$ and using the fact that $i_*(c_1(B)\gamma) = 0$ a simple computation gives

$$\begin{aligned} a_E &= \gamma|_S + \frac{1}{n}\eta_E\eta, \\ s_E &= \frac{1}{24}nc_1^2 + \frac{1}{2n}\eta_E^2 - \omega, \end{aligned} \tag{3.12}$$

where ω is given by

$$\omega = -\frac{1}{24}c_1(B)^2(n^3 - n) + \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\eta(\eta - nc_1(B)). \tag{3.13}$$

3.4 Reduction to $SU(n)$

To make contact with the work of Friedman *et al.* [40], let us describe the reduction from $U(n)$ to $SU(n)$ and thus recover the second Chern class of an $SU(n)$ vector bundles originally computed in ref. [40] and the third Chern class evaluated in refs. [1, 23, 24].

In order to describe the reduction to $SU(n)$, we specify the class $\eta_E = \frac{1}{2}nc_1$ giving $c_1(V) = 0$. If we insert this into equation (3.12), we find the new expressions for a_E and s_E which are given by

$$\begin{aligned} a_E &= \gamma|_S + \frac{1}{2}c_1\eta, \\ s_E &= \frac{1}{6}nc_1^2\sigma - \omega, \end{aligned} \tag{3.14}$$

and with equation (3.6) we find the Chern classes of an $SU(n)$ vector bundle V on the elliptic fibered Calabi–Yau three-fold

$$r(V) = n, \quad c_1(V) = 0, \quad c_2(V) = \pi^*(\eta)\sigma + \pi^*(\omega), \quad c_3(V) = -2\gamma|_S \tag{3.15}$$

in agreement with [1, 23, 24, 40].

As we were enforcing $c_1(V) = 0$ by setting $\eta_E = (1/2)nc_1$, one could ask are there other possibilities to reduce $U(n)$ to $SU(n)$ vector bundles on X such that the Chern classes differ from equation (3.15). For this, recall that it is often possible to twist a given vector bundle with non-zero first Chern class such that the resulting bundle has zero first Chern class. Assume we have V with $c_1(V) \neq 0$. If we twist as $W = V \otimes N$ where N is a line bundle on X , we have $c_1(W) = c_1(V) + nc_1(N)$ (with n the rank of V). Then $c_1(W) = 0$ implies that $c_1(V) = -nc_1(N)$. A natural choice for N would be to take a root of the inverse of the determinant line bundle, i.e., $c_1(N) = -1/nc_1(V) = c_1(\det V^{-1/n})$, assuming that it exists.

Since all our bundles have vanishing first Chern class on the fibers, this implies that the restrictions of N to the fibers have also vanishing first Chern class, so they are trivial (the general fiber being an elliptic curve). Thus $N = \pi^*L$ for some line bundle L on B .

Using $c_1(V) = \pi^*(\eta_E - 1/2nc_1(B))$, the equation $c_1(V) + nc_1(N) = 0$ reduces to $\pi^*(nc_1(L) + \eta_E - 1/2nc_1(B)) = 0$ and using $\eta_E = 1/2n\beta$ we get $c_1(L) + 1/2\beta - 1/2c_1(B) = 0$. This equation can be solved for L if $1/2(\beta - c_1(B))$ is an integer class, and if so, the space of solutions is isomorphic to the Jacobian (or Picard scheme) of B . Thus if $c_1(L)$ is represented by an integer class, all twists lead to $SU(n)$ vector bundles with characteristic classes given by equation (3.15).

3.5 Stability of V and W

The discussion of stability of V and W depends on the properties of the defining data C and L . If C is irreducible and L a line bundle over C , then V and W will be vector bundles stable with respect to

$$J = \epsilon J_0 + \pi^*H_B, \quad \epsilon > 0, \tag{3.16}$$

if ϵ is sufficiently small (cf. [42, Theorem 7.1] where the statement is proven under the additional assumption that the restriction of V to the generic fiber is regular and semistable). Here J_0 refers to some arbitrary Kähler class on X and H_B a Kähler class on the base B . It implies that the bundle V can

be taken to be stable with respect to J while keeping the volume of the fiber F of X arbitrarily small compared to the volumes of effective curves associated with the base. That J is actually a good polarization can be seen by assuming $\epsilon = 0$. Now one observes that π^*H_B is not a Kähler class on X since its integral is non-negative on each effective curve C in X , however, there is one curve, the fiber F , where the integral vanishes. This means that π^*H_B is on the boundary of the Kähler cone and to make V stable, one has to move slightly into the interior of the Kähler cone, i.e., into the chamber which is closest to the boundary point π^*H_B .

Also we note that although π^*H_B is in the boundary of the Kähler cone, we can still define the slope $\mu_{\pi^*H_B}(V)$ with respect to it. Since $(\pi^*H_B)^2$ is some positive multiple of the class of the fiber F , semistability with respect to π^*H_B is implied by semistability of the restrictions $V|_F$ to the fibers. Assume that V is not stable with respect to J , then there is a destabilizing subbundle $V' \subset V$ with $\mu_J(V') \geq \mu_J(V)$. But semistability along the fibers says that $\mu_{\pi^*H_B}(V') \leq \mu_{\pi^*H_B}(V)$. If we had equality, it would follow that V' arises by the spectral construction from a proper sub-variety of the spectral cover of V , contradicting the assumption that this cover is irreducible. So we must have a strict inequality $\mu_{\pi^*H_B}(V') < \mu_{\pi^*H_B}(V)$. Now taking ϵ small enough, we can also ensure that $\mu_J(V') < \mu_J(V)$ thus V' cannot destabilize V .

3.5.1 Generalization to reducible spectral covers

Let us now consider the case that C is flat over B . If C is not irreducible, then there may exist line bundles such that $V = \Phi^0(i_*L)$ is not stable with respect to the polarization given by equation (3.16); however, the condition one has to impose to the spectral data in order that V is a stable sheaf on X with respect to equation (3.16) has been derived in ref. [5]. Actually, if C is flat over B and L is a pure dimension sheaf on C , then V is stable with respect to $\bar{J} = \bar{\epsilon}\sigma + \pi^*H_B$ for sufficiently small $\bar{\epsilon}$ if and only if i_*L is stable with respect to this polarization. Let us note here that stability with respect to equation (3.16) for ϵ sufficiently small is equivalent to stability with respect to \bar{J} for sufficiently small $\bar{\epsilon}$ if we take $J_0 = a\sigma + b\pi^*H_B$ for some positive a and b . Furthermore, note that if C is irreducible and L is a line bundle, the latter condition is automatically satisfied.

Moreover, V and W are simultaneously stable with respect to \tilde{J} . This is not a surprise because from $\text{ch}_i(\tau^*V) = \text{ch}(V)$ we know that V and τ^*V are simultaneously stable and from $W = \tau^*V \otimes \pi^*K_B^{-1}$ we know that stability is the same for W and τ^*V .

Note that assuming C is flat and L a pure dimension 1 sheaf, one finds a larger class of stable sheaves V (and W) than originally constructed in ref. [40], because we do not need an irreducible spectral cover and that the restriction of the bundle (sheaf) to the generic fiber is regular.

3.6 Bundle cohomology and index computation

In this section, we will show that the $U(n)$ bundles constructed via a relative Fourier–Mukai transform have no sections. This simplifies the index of the Dirac operator with values in the respective vector bundle which is related to the net-number of generations of chiral fermions in heterotic string compactifications.

To begin with let us recall, for stable $SU(n)$ vector bundles the Riemann–Roch theorem reduces to $h^2(X, V) - h^1(X, V) = 1/2c_3(V)$ since $h^0(X, V)$ and by Serre duality $h^3(X, V)$ vanish. Otherwise, a non-zero element of $H^0(X, V)$ would define a mapping from the trivial line bundle into V but since the trivial line bundle has rank 1, this would violate the slope inequality. Since V^* is stable as well, Serre duality gives $h^3(X, V) = h^0(X, V^*) = 0$.

In case of vector bundles E with arbitrary first Chern class, the index is given by

$$\sum_{i=0}^3 (-1)^i \dim H^i(X, E) = \int \text{ch}_3(E) + \frac{1}{12} c_2(X) c_1(E). \quad (3.17)$$

Let us now examine the left-hand side of equation (3.17) and show that $h^0(X, E) = h^3(X, E) = 0$ for E given by a relative Fourier–Mukai transformation. That is, we assume that the restriction of E to the fibers are semistable of degree zero. We also assume that the rank of E is bigger than 1. Then the spectral cover of E is different from the section σ because if it were σ , then $\Phi^1(E) = \sigma_*(N)$ for N a line bundle on B and one would find that $E = \pi^*(N \otimes K_B^{-1})$.

Since both E and the structure \mathcal{O}_X sheaf are WIT_1 with respect to Φ with transforms $\Phi^1(E) = i_*L$, $\Phi^1(\mathcal{O}_X) = \sigma_*K_B$, where $\sigma: B \rightarrow X$ is the section, we can apply the Parseval theorem for the relative Fourier–Mukai transform.

Assume that we have sheaves F and \bar{G} that are, respectively, WIT_h and WIT_j for certain h and j . The Parseval theorem says that one has

$$\text{Ext}_X^i(F, G) = \text{Ext}_X^{h-j+i}(\Phi^h(F), \Phi^j(G)), \quad (3.18)$$

thus giving a correspondence between the extensions of F and G and the extensions of their Fourier–Mukai transforms. The proof is very simple, and relies on two facts. The first one is that the extension groups can be computed in terms of the derived category, namely

$$\text{Ext}_X^i(F, G) = \text{Hom}_{D(X)}(F, G[i]). \tag{3.19}$$

The second one is that the Fourier–Mukai transforms of F and G in the derived category $D(X)$ are $\Phi(F) = \Phi^h(F)[-h]$ and $\Phi(G) = \Phi^j(G)[-j]$. Now, since the Fourier–Mukai transform is an equivalence of categories, one has

$$\begin{aligned} \text{Hom}_{D(X)}(F, G[i]) &= \text{Hom}_{D(X)}(\Phi(F), \Phi(G[i])) \\ &= \text{Hom}_{D(X)}(\Phi^h(F)[-h], \Phi^j(G)[-j+i]) \\ &= \text{Hom}_{D(X)}(\Phi^h(F), \Phi^j(G)[h-j+i]) \end{aligned} \tag{3.20}$$

so that equation (3.19) gives the Parseval theorem equation (3.18). Thus we obtain (cf. [7] or [5])

$$\begin{aligned} \text{Ext}_X^i(E, \mathcal{O}_X) &= \text{Ext}_X^i(i_*L, \sigma_*(K_B)), \\ \text{Ext}_X^i(\mathcal{O}_X, E) &= \text{Ext}_X^i(\sigma_*(K_B), i_*L), \quad i \geq 0. \end{aligned} \tag{3.21}$$

Then we have

$$\begin{aligned} H^0(X, E) &= \text{Hom}_X(\mathcal{O}_X, E) = \text{Hom}_X(\sigma_*(K_B), i_*L) \\ &= \text{Hom}_C(i^*(\sigma_*K_B), L) = 0, \end{aligned} \tag{3.22}$$

because $i^*(\sigma_*K_B)$ is concentrated on $S = C \cap \sigma$ and L is a line bundle on C . Analogously we get

$$\begin{aligned} H^3(X, E) &= \text{Hom}_X(E, \mathcal{O}_X) = \text{Hom}_X(i_*L, \sigma_*(K_B)) \\ &= \text{Hom}_B(\sigma^*(i_*L), K_B) = 0, \end{aligned} \tag{3.23}$$

because $\sigma^*(i_*L)$ is concentrated on S and K_B is a line bundle on B .

The same reasoning applies to $\hat{\Phi}$, so that both V and W constructed in equations (3.1) and (3.2) as well as τ^*V have vanishing groups H^0 and H^3 .

3.7 Moduli of $U(n)$ vector bundles

In this section, let us briefly discuss whether the expression for the number of moduli of a rank n vector bundle constructed in the spectral cover approach

depends on the first Chern class or not. In case of an X being a $K3$ surface it is known that the number of bundle moduli is given by

$$h^1(X, \text{End}(V)) = 2nc_2(V) - (n - 1)c_1(V)^2 - 2n^2 + 2, \tag{3.24}$$

which follows from a Riemann–Roch index computation [49] and shows the dependence on $c_1(V)$. This formula can also be derived using the spectral cover approach. For this, let us recall that if V is given by the spectral data (C, L) we get [7, 8]

$$h^1(X, \text{End}(V)) = h^1(\mathcal{O}_C) + h^0(N_C) = C^2 + 2, \tag{3.25}$$

where N_C denotes the normal bundle of the spectral cover in X and the last equality follows from $h^0(N_C) = h^0(K_C)$ (because $X = K3$), which using Serre duality is equal to $h^1(\mathcal{O}_C)$. A Grothendieck–Riemann–Roch index computation gives for $\text{ch}_1(i_*L) = C$ the following expression [46]

$$C = -c_1(V) + r\sigma - (\text{ch}_2(V) - c_1(V)\sigma)F, \tag{3.26}$$

and using equation (3.25) we recover equation (3.24).

Now in case of X being a Calabi–Yau three-fold the spectral cover C does not depend on $c_1(V)$ and so we expect the number of bundle moduli of an $U(n)$ vector bundle to agree with the corresponding number of moduli of an $SU(n)$ vector bundle. This number is related to the number of parameters specifying the spectral cover C and the dimension of the space of holomorphic line bundles L on C determines the moduli. The first number is given by the dimension of the linear system $|C| = |n\sigma + \eta|$. The second number is given by the dimension of the Picard group $\text{Pic}(C) = H^1(C, \mathcal{O}_C^*)$ of C . One thus expects the moduli of V to be given by [19]

$$\dim H^1(X, \text{End}(V)) = \dim |C| + \dim \text{Pic}(C). \tag{3.27}$$

This formula was proven in refs. [7, 8] under the equivalent form

$$h^1(X, \text{End}(V)) = h^1(\mathcal{O}_C) + h^0(N_C) = h^{(0,1)}(C) + h^{(2,0)}(C), \tag{3.28}$$

when C is a divisor in X using the Parseval isomorphism

$$\text{Ext}_X^1(V, V) = \text{Ext}_X^1(i_*L, i_*L), \tag{3.29}$$

and an argument of spectral sequences associated to Grothendieck duality for the closed immersion $i: C \hookrightarrow X$. Here equation (3.29) follows from equation (3.18) since i_*L is WIT_0 with unique transform V .

Still assuming that C is a divisor, we have $N_C = K_C$ since X is Calabi–Yau, and then $h^0(N_C) = h^2(\mathcal{O}_C)$, so that we get

$$h^1(X, \text{End}(V)) = h^1(\mathcal{O}_C) + h^2(\mathcal{O}_C) \tag{3.30}$$

(for C connected). If we assume moreover that C is smooth, then Lefschetz theorem on hyperplane sections [43, p. 156] implies that $h^1(\mathcal{O}_C) = 0$ and then,

$$h^1(X, \text{End}(V)) = h^2(\mathcal{O}_C). \tag{3.31}$$

4 Application to heterotic string theory

In this section, we present a systematic analysis of the Donaldson–Uhlenbeck–Yau equation and of the anomaly cancellation condition for $U(n)$ vector bundles on elliptic Calabi–Yau three-folds. We also give an explicit expression for the net-generation number N_{gen} using the above index computation of Section 3.6.

4.1 Solving the Donaldson–Uhlenbeck–Yau equation

As mentioned in Section 1 a geometric compactification of the 10-dimensional heterotic string is specified by a holomorphic stable G -bundle V over a Calabi–Yau manifold X . The Calabi–Yau condition, the holomorphicity and stability of V are a direct consequence of the required supersymmetry in the uncompactified space–time. More precisely, supersymmetry requires that a connection A on V has to satisfy

$$F_A^{2,0} = F_A^{0,2} = 0, \quad F_A^{1,1} \wedge J^2 = 0, \tag{4.1}$$

where the first condition implies the holomorphicity of V . A theorem by Uhlenbeck and Yau guarantees the existence and uniqueness of a solution to the second equation which can be written as (cf. Appendix A)

$$\int_X c_1(V) \wedge J^2 = 0. \tag{4.2}$$

This equation can be solved if the first Chern class of a given vector bundle vanishes or if the underlying manifold X has $h^{1,1}(X)$ greater than or equal to 2; for $h^{1,1}(X) = 1$ the first Chern class is a multiple of the Kähler form J of X and J^3 is never zero as its integral is the volume of X .

Let us discuss whether the stable $U(n)$ vector bundles with characteristic classes given by equations (3.6) and (3.12) can solve the Donaldson–Uhlenbeck–Yau equation. The $U(n)$ bundles are stable with respect to

$\bar{J} = \bar{e}\sigma + \pi^*H_B$. To simplify the analysis, let us make the following assumptions: we take as before $\eta_E = 1/2n\beta$ with $\beta \in H^2(B, \mathbb{Z})$ and we set $H_B = c_1(B)$ which restricts us to work on base manifolds whose anti-canonical line bundle is ample, i.e., we consider del Pezzo surfaces which are isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$ or \mathbb{P}^2 with k points ($0 \leq k \leq 8$) blown up, usually denoted by dP_k . Some of those surfaces can also be viewed as Hirzebruch surfaces, $\mathbb{P}^1 \times \mathbb{P}^1 = F_0$ and $dP_1 = F_1$. The Donaldson–Uhlenbeck–Yau equation can then be written as

$$\pi^*(c_1(B)(\beta - c_1(B)))\sigma = 0. \quad (4.3)$$

Furthermore, we have to make sure that $c_1(V)$ and $c_1(L)$ are integer classes, thus we have to solve in case n is even equation (4.3) simultaneously with $\beta = \eta \pmod{2}$ or $\beta = nc_1(B) \pmod{2}$ depending on whether we take $\lambda = m$ or $\lambda = m + 1/2$. If n is odd, we have to solve equation (4.3) simultaneously with $\beta = c_1(B) \pmod{2}$ and $\beta = nc_1(B) \pmod{2}$.

To give a concrete example, let us analyze these conditions in more detail for the Hirzebruch surfaces F_m with $m = 0, 1$ and an $U(n)$ bundle with n being odd. Recall that $c_1(F_m) = 2b + (2 + m)f$ where $b^2 = -m$, $bf = 1$ and $f^2 = 0$ and since F_m is a rational surface we have $c_1(F_m)^2 = 8$. Thus we have to look for solutions of (we set $c_1 = c_1(B)$)

$$\beta c_1 = 8, \quad \beta = c_1 \pmod{2}, \quad \beta = nc_1 \pmod{2}. \quad (4.4)$$

Since β is a generic element of $H^2(F_m, \mathbb{Z})$ we take $\beta = xb + yf$ with x and y being not restricted. Thus the first part in equation (4.4) reduces to $x + y = 4$. Now for $m = 0$ we have $c_1 = 0 \pmod{2}$ and thus the last two terms in equation (4.4) reduce to $\beta = 0 \pmod{2}$ and so we have to take $x = 2x'$ and $y = 4 - 2x'$ with $x' \in \mathbb{Z}$.

If $m = 1$ the first term in equation (4.4) reduces to $x + 2y = 8$. The last two terms in equation (4.4) can be written as $\beta = f \pmod{2}$ and $\beta = nf \pmod{2}$ which can be solved since $(n - 1)f = 0 \pmod{2}$ and $n - 1$ is even. Thus we have to require only $\beta = f \pmod{2}$, i.e., $xb + (y - 1)f = 0 \pmod{2}$ which restricts x to be an even and y to be odd number, i.e., we take $x = 2x'$ and $y = 2y' + 1$ with $x', y' \in \mathbb{Z}$, such that $x' + 2y' = 3$.

A similar analysis can be done for $U(n)$ bundles with n even. For this, we only have to note that if we want the spectral surface C to be irreducible we have to make sure that the linear system $|\eta|$ is base point free in B and $\eta - nc_1(B)$ is effective. For $B = F_m$ and $\eta = vb + wf$, the corresponding conditions have been derived in ref. [58] and are given by $v \geq 2m$, $w \geq n(m + 2)$ and $w \geq mv$. Taking these into account, one easily obtains solutions to $\beta c_1 = 8$ and $\eta = \beta \pmod{2}$.

4.1.1 Solutions for arbitrary ample H_B

So far we assumed in our analysis that $H_B = c_1(B)$. Let us now study the situation for arbitrary ample H_B and $\eta_E = 1/2n\beta$ as before. In order to solve the Donaldson-Uhlenbeck-Yau equation, we must solve in addition to equation (4.3) the constraint

$$\pi^*(H_B(\beta - c_1(B)))\sigma = 0. \tag{4.5}$$

Now let us consider again $U(n)$ bundles with n odd and take as base manifolds the Hirzebruch surfaces F_m whose c_1 is even, the odd case will be discussed below. It is instructive to solve first equation (4.4) and then find solutions to equation (4.5). Similar to our analysis above, we start with a generic $\beta \in H^2(F_m, \mathbb{Z})$ and find solutions to equation (4.4) in terms of x, y and m . The first term in equation (4.4) reduces to $2x + 2y - xm = 8$. For m even we have $c_1 = 0 \pmod{2}$ and thus we find again the constraint $\beta = 0 \pmod{2}$ which determines the coefficients of β to be given by $x = 2x'$ and $y = 4 + (m - 2)x'$ and $x' \in \mathbb{Z}$.

We can now solve equation (4.5). For this, let us recall that the Kähler cone (the very ample classes) of F_m is equal to the positive (ample) classes and is given [45] by the numerically effective classes $wb + zf = (w, z)$ with $w > 0, z > mw$. So if we insert $H_B = ub + vf$ and $\beta = (2x')b + (4 + (m - 2)x')f$ into equation (4.5), we find the condition

$$v = \left(1 + \frac{m}{2}\right) u, \tag{4.6}$$

where $u > 0$, so we have to restrict to the Hirzebruch surface F_0 , otherwise $v > mu$ is violated.

To complete the analysis, let us consider $B = F_m$ with m odd. From equation (4.4) we find $\beta = mf \pmod{2}$ and $\beta = nmf \pmod{2}$ which can be solved since $m(n - 1)f = 0 \pmod{2}$ is always satisfied. Thus to solve $\beta = mf \pmod{2}$ we have to take $\beta = xb + yf$ with $x = 2x'$ and $y = 2y' + 1$ with $x', y' \in \mathbb{Z}$. Moreover, from $\beta c_1 = 8$ we find that $y' = 1/2(3 - x'(2 - m))$. If we insert these expressions in equation (4.5), we find the condition (with $u > 0$)

$$v = \left(1 + \frac{x'm}{2x' - 2}\right) u, \tag{4.7}$$

with $x' \neq 1$ and $(x'/(2x' - 2)) > m - 1/m$ which guarantees that H_B is ample. Thus if $x' > 0$ (and $m \neq 1$) we find x' has to satisfy the bound $x' < (2m - 2)/(m - 2)$ and if $x' < 0$ (and $m \neq 1$) we find $x' > (2m - 2)/(m - 2)$. For $m = 1$, equation (4.7) always satisfies $v > u$, so there are no bounds for x' .

4.2 The anomaly constraint

In a heterotic string compactifications (without 5-branes) on a Calabi–Yau three-fold X , one has to specify, in addition to a stable holomorphic G vector bundle V , a B -field on X of field strength H . In order to get an anomaly free theory, the structure group G of V is fixed to be either $E_8 \times E_8$ or $\text{Spin}(32)/\mathbb{Z}_2$ or one of their subgroups and H has to satisfy the generalized Bianchi identity which, if integrated over a 4-cycle in X , gives the topological constraint $c_2(X) = c_2(V)$. This constraint has to be modified in the presence of 5-branes and if V has a non-zero first Chern class.

The inclusion of 5-branes³ changes this topological constraint [26, 39, 40]. The magnetic 5-brane contributes a source term to the Bianchi identity for the 3-form H with $dH = \text{tr}R \wedge R - \text{Tr}F \wedge F - n_5 \sum \delta_5^{(4)}$, where the sum is taken over all 5-branes and $\delta_5^{(4)}$ is a current that integrates to 1 in the direction transverse to a single 5-brane whose class is denoted by $[W]$, an element of $H_2(X, \mathbb{Z})$. Integration over a 4 cycle in X gives $[W] = c_2(TX) - c_2(V)$. Now if V has a non-vanishing first Chern class, we expect this relation to be modified such that

$$[W] = c_2(TX) + \text{ch}_2(V) \quad (4.8)$$

which is an integer class if $c_1(V) = 0 \pmod{2}$ or equivalently if $\beta = c_1(B) \pmod{2}$. Note that the latter condition is automatically given for $U(n)$ bundles with n odd as a consistency condition for $c_1(V)$ and $c_1(L)$. For n even, we have to impose $\beta = c_1(B) \pmod{2}$ as an additional constraint.

Supersymmetry and the condition to get positive magnetic charges which are carried by the 5-brane constrain $[W]$ to be represented by an effective holomorphic curve in X . Following the conventions of Donagi *et al.* [30], we write the 5-brane class as $W = W_B + a_F F$ and using equation (4.8) we find

$$\begin{aligned} W_B &= \sigma\pi^*(12c_1(B) - \eta), \\ a_f &= c_2(B) + 11c_1(B)^2 - \omega + \frac{1}{8}nc_1(B)^2 + \frac{1}{2n}\eta_E(\eta_E - nc_1(B)). \end{aligned} \quad (4.9)$$

As in the $\text{SU}(n)$ case, W is an effective class in X if and only if W_B is effective and $a_F \geq 0$ (this was originally shown in ref. [30] for $\text{SU}(n)$ vector bundles).

³In refs. [20, 21] magnetic 5-brane solutions to heterotic string theory were discussed; the gauge 5-brane, the symmetric 5-brane and the neutral 5-brane. The gauge and symmetric 5-brane solutions involve finite size instantons of an unbroken non-abelian gauge group. Neutral 5-branes can be interpreted as zero size instantons of the $\text{SO}(32)$ heterotic string [61].

4.2.1 An outlook to F-theory

In ref. [40] it has been shown that the number of 5-branes a_f which are required for the anomaly cancellation in case $V = E_8 \times E_8$ agrees with the number of 3-branes N required for anomaly cancellation in F -theory on a Calabi-Yau four-fold Y given by $(\chi(Y)/24)$. If the structure group G of V being contained in E_8 , then the observed physical gauge group in four-dimensions corresponds to the commutant \tilde{H} of G in E_8 , typically being of ADE type. This leads to a generalized physical set-up. The heterotic string compactified on X and a pair of G -bundles is dual to F -theory compactified on a Calabi-Yau fourfold Y with section and a section θ of ADE singularities. The generalized set-up has been analyzed in refs. [2-4, 15, 40]. Now if we consider, for instance, an $U(n) \times E_8$ vector bundle on X we expect a section of singularities in Y corresponding to the commutant of $U(n)$ in the E_8 gauge group and thus a modified relation $(\chi(Y)/24) = a_{E_8} + a_f$, where a_{E_8} corresponds to the number of 5-branes associated to the E_8 vector bundle and a_f as given in equation (4.9).

4.3 The net-generation number

The net-number of generations N_{gen} in a given heterotic string compactification on a Calabi-Yau three-fold is determined by the index (3.17) if the vector bundle V has non-zero first Chern class. If we insert the expressions for $c_2(X)$ and $\text{ch}_i(V)$, we find the net-generation number

$$N_{\text{gen}} = -\gamma|_S + \frac{1}{2}\pi^*((\eta - nc_1)(c_1 - \beta))\sigma. \tag{4.10}$$

For $U(n)$ vector bundles with n even, we found the consistency condition $\lambda = m$ and $\eta = \beta \pmod{2}$ or $\lambda = m + 1/2$ and $\beta = nc_1(B) \pmod{2}$ which guarantees that $c_1(V)$ and $c_1(L)$ are integer classes. In the first case, we can take $\lambda = 0$ which implied for $SU(n)$ vector bundles the vanishing of N_{gen} . However, although the “gamma”-class vanishes for $U(n)$ bundles, we still have N_{gen} non-zero.

4.3.1 The enriques surface revisited

In case of $SU(n)$ vector bundles on elliptically fibered Calabi-Yau three-folds, it has been argued [30] that the Enriques surfaces will never satisfy the effectivity condition for W_B and at the same time lead to a realistic net-number of generations. The argument is as follows: W_B has to be effective but $K_B^{\otimes 12} = \mathcal{O}_B$ because 12 is even and thus $W_B = -\pi^*\eta\sigma$; since η is an effective class it follows W_B can be effective only if η is trivial. Now the

net-generation number N_{gen} for $\text{SU}(n)$ vector bundles is given by equation (3.15) which vanishes for η being trivial.

For $U(n)$ vector bundles, the situation does not change which can be seen as follows. We still have to require that η is trivial to guarantee the effectivity of W_B , which reduces the net-generation number to $N_{\text{gen}} = 1/2n\pi^*c_1(B)\pi^*\beta\sigma$ which is zero since N_{gen} is a number which can be multiplied by any even number.

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Appendix A Notes on the Hermite–Einstein equations

In this section, we recall for the reader’s convenience some well-known facts about the Donaldson–Uhlenbeck–Yau and Hermite–Einstein equations.

Let X be a complex n -dimensional Kähler manifold with Kähler form J . We denote by Ω^p the bundle of complex differential p -forms on X .

Holomorphic vector bundles can be characterized in terms of connections (see, for instance, [49]). We now that if E is a holomorphic vector bundle, then there exist a connection $A: E \rightarrow E \otimes \Omega^1$ whose $(0,1)$ component $A^{0,1}: E \rightarrow E \otimes \Omega^{0,1}$ equals the natural $\bar{\delta}_E$ operator whose kernel is formed by the holomorphic sections of E . The $(0,2)$ component of the curvature $F_A = A \circ A: E \rightarrow E \otimes \Omega^2$ vanishes. Conversely, whenever we have a connection A on a smooth vector bundle E with $F_A^{0,2} = 0$, then there exists a unique holomorphic structure on E such that $A^{0,1} = \bar{\delta}_E$.

The Chern classes of E can be computed in terms of any complex connection. In particular, we have

$$c_1(E) = \frac{i}{2\pi} \text{Tr}(F_A). \quad (\text{A.1})$$

We now consider Hermitian bundles (E, h) , i.e., smooth complex vector bundles E on X endowed with a Hermitian metric h . In this case a connection A on E is Hermitian if it derives the Hermitian metric, i.e.,

$$d(h(e, e')) = h(A(e), e') + h(e, A(e')),$$

where e and e' are local sections of E .

For Hermitian bundles, we can refine the above discussion. A holomorphic Hermitian bundle has a unique Hermitian connection A such that $A^{0,1} = \bar{\partial}_E$. Moreover, the curvature F_A is of type $1, 1$. We call S the associated Hermitian connection to the Hermitian holomorphic bundle (E, h) .

Conversely, we have the following result: if (E, h) is a Hermitian bundle and A is a Hermitian connection on E whose curvature is of type $(1, 1)$, then there exists a unique holomorphic structure in E such that A is the associated Hermitian connection to the Hermitian holomorphic bundle (E, h) .

Let (E, h) be a Hermitian holomorphic vector bundle and A the associated Hermitian connection. Then (E, h) is said to be a *Hermite–Einstein* bundle if the curvature F_A satisfies the *Hermite–Einstein* equations

$$F_A \wedge J^{n-1} = c \cdot I_E \cdot J^n, \tag{A.2}$$

where c is a complex constant and I_E the identity operator on E . When the constant c is zero, we get the *Donaldson–Uhlenbeck–Yau* equations:

$$F_A \wedge J^{n-1} = 0. \tag{A.3}$$

Both the Hermite–Einstein and the Donaldson–Uhlenbeck–Yau equations can be taught as equations on the Hermitian metric h (because the connection A is completely determined by h and the holomorphic structure).

If we take the trace on both sides of equation (A.2), we find

$$\text{Tr}(F_A) \wedge J^{n-1} = c \cdot \text{Tr}(I_E) \cdot J^n = c \cdot r \cdot J^n. \tag{A.4}$$

Now using equation (A.1), we get

$$\int_X c_1(E) \wedge J^{n-1} = \frac{i \cdot c \cdot r}{2\pi} \int_X J^n, \tag{A.5}$$

which determines the constant c (compare [49, eq. (2.7)]). With equation (A.5), we can now write equation (A.2) as

$$F_A \wedge J^{n-1} = \frac{2\pi \cdot c_1(E) \wedge J^{n-1}}{i \cdot \text{rk}(E)} I_E = \frac{2\pi}{i} \mu(E) J^n I_E \tag{A.6}$$

making apparent that for zero slope $\mu(E) = 0$, the Hermite–Einstein equation reduces to the Donaldson–Uhlenbeck–Yau equation from above.

It follows from a deep result due to Donaldson, Uhlenbeck and Yau that the Hermite–Einstein conditions for holomorphic vector bundles are equivalent to slope (poly)stability. The precise statement is as follows (cf. for instance, [52]):

Let (E, h) be a holomorphic Hermite–Einstein vector bundle. Then E is μ -polystable (i.e., is a direct sum of semistable vector bundles with the same slope). Moreover, if it is irreducible (in the sense that it cannot be represented as a sum of Hermite–Einstein bundles), then E is μ -stable.

The converse is more subtle: if E is a holomorphic μ -stable vector bundle on X , then there exists a Hermitian metric h on E such that (E, h) is Hermite–Einstein, i.e., h satisfies the Hermite–Einstein equations (A.2).

Appendix B Characteristic classes of the Fourier–Mukai transform

In this section we recall from ref. [7] the formulae giving the topological invariants of both the FM transform and the inverse FM transform of a general complex G in the derived category. They are obtained of course by simply applying GRR for the projection π_2 to get

$$\text{ch}(\Phi(G)) = \pi_{2*}[\pi_1^*(G) \cdot \text{ch}(\mathcal{P}) \cdot \text{Td}(T_{X/B})], \tag{B.1}$$

where $\text{Td}(T_{X/B}) = 1 - 1/2c_1 + 1/12(13c_1^2 + 12\sigma c_1) - 1/2\sigma c_1^2$ (with $c_1 = \pi^*c_1(B)$) is the Todd class of the relative tangent bundle $T_{X/B} = T_X/\pi^*T_B$.

To compute equation (B.1), we use that the Poincaré sheaf is

$$\mathcal{P} = \mathcal{O}(\Delta) \otimes \mathcal{O}(-\pi_1^*\sigma) \otimes \mathcal{O}(-\pi_2^*\sigma) \otimes q^*K_B^{-1},$$

where $\mathcal{O}(\Delta)$ is the dual of the ideal sheaf \mathcal{I} of the diagonal. Note first that $\text{ch}(\mathcal{I}) = 1 - \text{ch}(\delta_*\mathcal{O}_X)$ with the diagonal immersion δ . Riemann–Roch gives

$$\text{ch}(\delta_*\mathcal{O}_X) \text{Td}(X \times_B X) = \delta_*(\text{ch}(\mathcal{O}_X) \text{Td}(X)), \tag{B.2}$$

where one has the expressions for $\text{Td}(X)$ and $\text{Td}(X \times_B X)$ given by

$$\begin{aligned} \text{Td}(X) &= 1 + \frac{1}{12}(c_2 + 11c_1^2 + 12\sigma c_1), \\ \text{Td}(X \times_B X) &= \pi_2^* \text{Td}(X) \pi_1^* \text{Td}(T_{X/B}). \end{aligned} \tag{B.3}$$

The Chern character of the ideal sheaf is then given by (with the diagonal class $\Delta = \delta_*(1)$)

$$\text{ch}(\mathcal{I}) = 1 - \Delta - \frac{1}{2}\Delta \cdot \pi_2^*c_1 + \Delta \cdot \pi_2^*(\sigma \cdot c_1) + \frac{5}{6}\Delta \cdot \pi_2^*(c_1^2) + \frac{1}{2}\Delta \cdot \pi_2^*(\sigma c_1^2), \tag{B.4}$$

and one can compute $\text{ch}(\mathcal{P})$ from that expression.

Assume now that the Chern characters of the object E of the derived category are of the type

$$\begin{aligned} \text{ch}_0(E) &= n_E, & \text{ch}_1(E) &= x_E\sigma + \pi^*S_E, \\ \text{ch}_2(E) &= \sigma\pi^*\eta_E + a_EF, & \text{ch}_3(E) &= s_E \end{aligned} \tag{B.5}$$

($\eta_E, S_E \in H^2(B)$). We note that all the sheaves we are considering (including V, i_*L and W) have Chern characters of that type.

It follows from ref. [7] that the Chern characters of the FM of E and of the inverse FM of E are

$$\begin{aligned} \text{ch}_0(\Phi(E)) &= x_E, \\ \text{ch}_1(\Phi(E)) &= -n_E\sigma + \pi^*\eta_E - \frac{1}{2}x_Ec_1, \\ \text{ch}_2(\Phi(E)) &= \left(\frac{1}{2}n_Ec_1 - \pi^*S_E\right)\sigma + \left(s_E - \frac{1}{2}\pi^*\eta_Ec_1\sigma + \frac{1}{12}x_Ec_1^2\sigma\right)F, \\ \text{ch}_3(\Phi(E)) &= -\frac{1}{6}n_E\sigma c_1^2 - a_E + \frac{1}{2}\sigma c_1\pi^*S_E, \end{aligned} \tag{B.6}$$

and

$$\begin{aligned} \text{ch}_0(\hat{\Phi}(E)) &= x_E, \\ \text{ch}_1(\hat{\Phi}(E)) &= -n_E\sigma + \pi^*\eta_E + \frac{1}{2}x_Ec_1, \\ \text{ch}_2(\hat{\Phi}(E)) &= \left(-\frac{1}{2}n_Ec_1 - \pi^*S_E\right)\sigma + \left(s_E + \frac{1}{2}\pi^*\eta_Ec_1\sigma + \frac{1}{12}x_Ec_1^2\sigma\right)F, \\ \text{ch}_3(\hat{\Phi}(E)) &= -\frac{1}{6}n_E\sigma c_1^2 - a_E - \frac{1}{2}\sigma c_1\pi^*S_E + x_E\sigma c_1^2. \end{aligned} \tag{B.7}$$

Appendix C Alternative derivation of N_{gen} for $SU(n)$ bundles

The aim of this section is to give a concrete application of the inverse Fourier–Mukai transformation. We give an alternative derivation of the net-generation number N_{gen} to the derivation given in refs. [23, 24] which adopted the bundle construction of Friedman et al. [40], i.e., consider bundles as given by equation (3.1) with $c_1(V) = 0$. It has been shown in refs. [23, 24] that $c_3(V)$ and therefore N_{gen} can be computed from an Riemann–Roch index computation on the matter curve $S = C \cup \sigma$. The index reduction was derived using Leray spectral sequence technics [24]. We will now give a shortened derivation of their result. For this, we start from equation (3.2)

instead of starting from equation (3.1). We start with the vector bundle W on X given by (3.2) and require $c_1(W) = 0$. A similar computation as in Section 3.3 gives

$$c_1(L|_S) = \frac{1}{2}(C^2\sigma + C\sigma^2) + \gamma|_S. \quad (\text{C.1})$$

Now we can apply the Grothendieck–Riemann–Roch theorem and use the expressions for the characteristic classes given in Appendix B and compute $1/2c_3(V)$ as

$$\frac{1}{2}c_3(V) = \frac{1}{2}(C^2\sigma + \sigma^2C) - c_1(L|_S). \quad (\text{C.2})$$

Further, we have

$$2g(S) - 2 = C^2\sigma + C\sigma^2, \quad (\text{C.3})$$

so that

$$\frac{1}{2}c_3(V) = -(c_1(L|_S) - g(S) + 1). \quad (\text{C.4})$$

This shows that the right-hand side of equation (C.2) can be understood as a Riemann–Roch index computation on S which is

$$\begin{aligned} \chi(S, L|_S) &= h^0(S, L|_S) - h^1(S, L|_S) \\ &= \int_S \text{ch}(L|_S) \text{Td}(S) \\ &= c_1(L|_S) - g(S) + 1, \end{aligned} \quad (\text{C.5})$$

thus the standard index computation evaluating $c_3(V)$ reduces to an index computation on the matter curve S

$$\begin{aligned} \frac{1}{2}c_3(V) &= h^2(X, V) - h^1(X, V) \\ &= h^1(S, L|_S) - h^0(S, L|_S), \end{aligned} \quad (\text{C.6})$$

which is in agreement with Diaconescu and Ionesei [24].

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