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The Trautman-Bondi mass of hyperboloidal initial data sets

Piotr T. Chruściel[1,2], Jacek Jezierski[1,3] and Szymon Łęski[1,4]

[1]Albert Einstein Institut für Gravitationsphysik
Golm, Germany

[2]Département de Mathématiques, Parc de Grandmont
F-37200 Tours, France

[3]Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski
ul. Hoża 74, 00-682 Warsaw, Poland

[4]Centrum Fizyki Teoretycznej, Polska Akademia Nauk
Al. Lotników 32/46, 02-668 Warsaw, Poland

chrusciel@univ-tours.fr, Jacek.Jezierski@fuw.edu.pl, szleski@cft.edu.pl

Abstract

We give a definition of mass for conformally compactifiable initial data sets. The asymptotic conditions are compatible with existence of gravitational radiation, and the compactifications are allowed to be polyhomogeneous. We show that the resulting mass is a geometric invariant, and we prove positivity thereof in the case of a spherical conformal infinity. When $R(g)$ – or, equivalently, $\mathrm{tr}_g K$ – tends to a negative constant to order one at infinity, the definition is expressed purely in terms of three-dimensional or two-dimensional objects.

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Contents

1 Introduction

In 1958 Trautman [43] (see also [42]) has introduced a notion of energy suitable for asymptotically Minkowskian radiating gravitational fields, and proved its decay properties; this mass has been further studied by Bondi *et al.* [9] and Sachs [37]. Several other definitions of mass have been given in this setting, and to put our results in proper perspective it is convenient to start with a general overview of the subject. First, there are at least seven methods for defining energy-momentum ("mass" for short) in the current context:

(i) A definition of Trautman [43], based on the Freud integral [22], that involves asymptotically Minkowskian coordinates in space-time. The definition stems from a Hamiltonian analysis in a fixed global coordinate system.

(ii) A definition of Bondi *et al.* [9], which uses space-time Bondi coordinates.

(iii) A definition of Abbott-Deser [1], originally introduced in the context of space-times with negative cosmological constant, which (as we will see) is closely related to the problem at hand. The Abbott-Deser integrand turns out to coincide with the linearisation of the Freud integrand, up to a total divergence [20].

(iv) The space-time "charge integrals", derived in a geometric Hamiltonian framework [13, 15, 19, 27]. A conceptually distinct, but closely related, variational approach, has been presented in [10].

(v) The "initial data charge integrals", presented below, expressed in terms of data (g, K) on an initial data manifold.

(vi) The Hawking mass and its variations such as the Brown-York mass, using two-dimensional spheres.

(vii) A purely Riemannian definition, that provides a notion of mass for asymptotically hyperbolic Riemannian metrics [14, 44].

Each of the above typically comes with several distinct variations.

Those definitions have the following properties:

(i) The Bondi mass m_B requires in principle a space-time on which Bondi coordinates can be introduced. However, a null hypersurface extending to future null infinity suffices. Neither of this is directly adapted to an analysis in terms of usual spacelike initial data sets. m_B is an invariant under Bondi-van der Burg-Metzner-Sachs coordinate transformations. Uniqueness is not clear, because there could exist Bondi coordinates which are not related to each other by a Bondi-van der Burg-Metzner-Sachs coordinate transformation.

(ii) Trautman's definition m_T requires existence of a certain class of asymptotically Minkowskian coordinates, with m_T being invariant under a class of coordinate transformations that arise naturally in this context [43]. The definition is obtained by evaluating the Freud integral in Trautman's coordinates. Asymptotically Minkowskian coordinates associated with the Bondi coordinates belong to the Trautman class, and the definition is invariant under a natural class of coordinate transformations. Trautman's conditions for existence of mass are less stringent, at least in principle[1], than the Bondi ones. m_B equals m_T of the associated quasi-Minkowskian coordinate system, whenever m_B exists as well. Uniqueness is not clear, because there could exist Trautman coordinates which are not related to each other by the coordinate transformations considered by Trautman.

(iii) The Hawking mass, and its variations, are *a priori* highly sensitive to the way that a family of spheres approaches a cut of $\mathscr{I}$. (This is a major problem one faces when trying to generalise the proof of the Penrose inequality to a hyperboloidal setting.) It is known that those masses converge to the Trautman-Bondi mass when evaluated on Bondi spheres, but this result is useless in a Cauchy data context, as general initial data sets will not be collections of Bondi spheres.

(iv) We will show below that the linearisation of the Freud integral coincides with the linearisation of the initial data charge integrals.

(v) We will show below that the linearisation of the Freud integral does *not* coincide in general with the Freud integral. However, we will also show that the resulting numbers coincide when decay conditions, referred to as

[1] It is difficult to make a clear cut statement here because existence theorems that lead to space-times with Trautman coordinates seem to provide Bondi coordinates as well (though perhaps in a form that is weaker than required in the original definition of mass, but still compatible with an extension of Bondi's definition).

strong decay conditions, are imposed. The strong decay conditions turn out to be incompatible with existence of gravitational radiation.

(vi) We will show below that a version of the Brown-York mass, as well as the Hawking mass, evaluated on a specific foliation within the initial data set, converges to the Trautman-Bondi mass.

(vii) We will show below that the Freud integrals coincide with the initial data charge integrals, for asymptotically CMC initial data sets on which a space-equivalent of Bondi coordinates can be constructed. The proof is indirect and uses the result, just mentioned, concerning the Brown-York mass.

(viii) It is important to keep in mind that hyperboloidal initial data sets in general relativity arise in two different contexts: as hyperboloidal hypersurfaces in asymptotically Minkowskian space-times on which K approaches a multiple of g as one recedes to infinity, or for spacelike hypersurfaces in space-times with a negative cosmological constant on which $|K|_g$ approaches zero as one recedes to infinity (compare [30]). This indicates that the Abbott-Deser integrals, which arose in the context of space-times with non-zero cosmological constant, could be related to the Trautman-Bondi mass. It follows from our analysis below that they do, in fact, coincide with the initial data charge integrals under the strong decay conditions. In space-times with a cosmological constant, the strong decay conditions are satisfied on hypersurfaces which are, roughly speaking, orthogonal to high order to the conformal boundary, but will not be satisfied on more general hypersurfaces.

(ix) It has been shown in [14] that the strong decay conditions on g are *necessary* for a well defined Riemannian definition of mass and of momentum. This appears to be paradoxical at first sight, since the strong decay conditions are incompatible with gravitational radiation; on the other hand one expects the Trautman-Bondi mass to be well defined even if there is gravitational radiation. We will show below that the initial data charge integrals involve delicate cancelations between g and K, leading to a well defined notion of mass of initial data sets without the stringent restrictions of the Riemannian definition (which does not involve K).

(x) Recall that for initial data sets which are asymptotically flat in spacelike directions, one can define the mass purely in terms of the induced three-dimensional metric. An unexpected consequence of our analysis below is therefore that any definition of mass of hyperboloidal initial data sets, compatible with initial data sets containing gravitational radiations, must involve the extrinsic curvature K in a non-trivial way.

Let us discuss in some more detail the definition of the Trautman-Bondi mass. The standard current understanding of this object is the following: one introduces a conformal completion at null infinity of (M, g) and assigns a mass m_{TB} to sections of the conformal boundary $\mathscr{I}$ using Bondi coordinates or Newman-Penrose coefficients. The Trautman-Bondi mass m_{TB} has been shown to be the unique functional, within an appropriate class, which is non-increasing with respect to deformations of the section to the future [16]. A formulation of that mass in terms of "quasi-spherical" foliations of null cones has been recently given in [5] under rather weak differentiability conditions.

The above approach raises some significant questions. First, there is a potential ambiguity arising from the possibility of existence of non-equivalent conformal completions of a Lorentzian space-time (see [12] for an explicit example). From a physical point of view, sections of Scri represent the asymptotic properties of a radiating system at a given moment of retarded time. Thus, one faces the curious possibility that two different masses could be assigned to the same state of the system, at the same retarded time, depending upon which of the conformally inequivalent completions one chooses. While there exist some partial results in the literature concerning the equivalence question [25, 38], no uniqueness proof suitable for the problem at hand can be found. The first main purpose of this paper is to show that the Trautman-Bondi mass of a section S of $\mathscr{I}^+$ is a geometric invariant in a sense which is made precise in Section 5.2 below.

Next, recall that several arguments aiming to prove positivity of m_{TB} have been given (*cf.*, amongst others, [29, 33, 36, 39]). However, the published proofs are either incomplete or wrong, or prove positivity of something which might be different from the TB mass, or are not detailed enough to be able to form an opinion. The second main purpose of this work is to give a complete proof of positivity of m_{TB}, see Theorems 5.4 and 5.7.

An interesting property of the Trautman-Bondi mass is that it can be given a Hamiltonian interpretation [15]. Now, from a Hamiltonian point of view, it is natural to assign a Hamiltonian to an initial data set $(\mathscr{S}, g, K)$, where $\mathscr{S}$ is a three-dimensional manifold, without the need of invoking a four-dimensional space-time. If there is an associated conformally completed space-time in which the completion $\overline{\mathscr{S}}$ of $\mathscr{S}$ meets $\mathscr{I}^+$ in a sufficiently regular, say differentiable, section S, then $m_{\mathrm{TB}}(\mathscr{S})$ can be defined as the Trautman-Bondi mass of the section S. (Hypersurfaces satisfying the above are called hyperboloidal.) It is, however, desirable to have a definition which does not involve any space-time constructions. For example, for non-vacuum initial data sets an existence theorem for an associated space-time might be lacking. Further, the initial data might not be sufficiently differentiable to obtain an associated space-time. Next, there might be loss of differentiability during evolution which will not allow one to perform the space-time constructions needed for the space-time definition of mass. Finally, a proof of uniqueness of the definition of mass of an initial data set could perhaps be easier to achieve than the space-time one. Last but not least, most proofs of positivity use three-dimensional hypersurfaces anyway. For all those reasons it seems of interest to obtain a definition of mass, momentum, *etc.*, in an initial data setting. The third main purpose of this work is to present such a definition, see (3.11) below.

The final $(3+1)$-dimensional formulae for the Hamiltonian charges turn out to be rather complicated. We close this paper by deriving a considerably simpler expression for the charges in terms of the geometry of "approximate Bondi spheres" near $\mathscr{I}^+$, Equations (6.1)-(6.2) below. The expression is similar in spirit to that of Hawking and of Brown, Lau and York [10]. It applies to mass as well as momentum, angular momentum and centre of mass.

It should be said that our three-dimensional and two-dimensional versions of the definitions do not cover all possible hyperboloidal initial hypersurfaces, because of a restrictive assumption on the asymptotic behavior of $g^{ij}K_{ij}$, see (C.1). This condition arises from the need to reduce the calculational complexity of our problem; without (C.1) we would probably not have been able to do all the calculations involved. We do not expect this condition to be essential, and we are planning to attempt to remove it using computer algebra in the future; the current calculations seem pretty much to be the limit of what one can calculate by hand with a reasonable degree of confidence in the final formulae. However, the results obtained are sufficient to prove positivity of $m_{\mathrm{TB}}(S)$ for all smooth sections of $\mathscr{I}^+$ which bound some smooth complete hypersurface $\mathscr{S}$, because then $\mathscr{S}$ can be deformed in space-time to a hypersurface which satisfies (C.1), while retaining the same conformal boundary S; compare Theorem 5.7 below.

Let us expand on our comments above concerning the Riemannian definition of mass: consider a CMC initial data set with $\mathrm{tr}_g K = -3$ and $\Lambda = 0$, corresponding to a hyperboloidal hypersurface in an asymptotically Minkowskian space-time as constructed in [2], so that the g-norm of $K_{ij} + g_{ij}$ tends to zero as one approaches $\mathscr{I}^+$. It then follows from the vacuum constraint equations that $R(g)$ approaches -6 as one recedes to infinity, and one can enquire whether the metric satisfies the conditions needed for the Riemannian definition of mass for such metrics [14]. Now, one of the requirements in [14] is that the background derivatives $\mathring{\nabla} g$ of g be in $L^2(M)$. A simple calculation (see Section 4 below) shows that for smoothly compactifiable $(\mathscr{S}, g)$ this will only be the case if the extrinsic curvature χ of the conformally rescaled metric vanishes at the conformal boundary $\partial\mathscr{S}$. It has been shown in [15, Appendix C.3] that the u-derivative of χ coincides with the Bondi "news function", and therefore the Riemannian definition of mass can *not* be used for families of hypersurfaces in space-times with non-zero flux of Trautman-Bondi energy, yielding an unacceptable restriction. Clearly one needs a definition which would allow less stringent conditions than the ones in [14, 19], but this seems incompatible with the examples in [19] which show sharpness of the conditions assumed. The answer to this apparent paradox turns out to be the following: in contradistinction with the asymptotically flat case, in the hyperboloidal one the definition of mass does involve the extrinsic curvature tensor K in a non-trivial way. The leading behavior of the latter combines with the leading behavior of the metric to give a well defined, convergent, geometric invariant. It is only when K (in the $\Lambda < 0$

case) or $K+g$ (in the $\Lambda=0$ case) vanishes to sufficiently high order that one recovers the purely Riemannian definition; however, because the leading order of K, or $K+g$, is coupled to that of g via the constraint equations, one obtains – in the purely Riemannian case – more stringent conditions on $g-b$ than those arising in the general initial data context.

We will analyse invariance and finiteness properties of the charge integrals in any dimension under asymptotic conditions analogous to those in [14, 19]. However, the analysis under boundary conditions appropriate for existence of gravitational radiation will only be done in space-time dimension four.

2 Global charges of initial data sets

2.1 The charge integrals

Let g and b be two Riemannian metrics on an n-dimensional manifold M, $n\geq 2$, and let V be any function there. We set[2]

$$e_{ij} := g_{ij} - b_{ij} \ . \tag{2.1}$$

We denote by $\mathring{D}$ the Levi-Civita connection of b and by R_f the scalar curvature of a metric f. In [14] the following identity has been proved:

$$\sqrt{\det g}\; V(R_g - R_b) = \partial_i \left(\mathbb{U}^i(V)\right) + \sqrt{\det g}\; (s+Q) \ , \tag{2.2}$$

where

$$\mathbb{U}^i(V) := 2\sqrt{\det g}\; \left(V g^{i[k} g^{j]l} \mathring{D}_j g_{kl} + D^{[i} V g^{j]k} e_{jk} \right) \ , \tag{2.3}$$

$$s := (-V\mathrm{Ric}(b)_{ij} + \mathring{D}_i \mathring{D}_j V - \Delta_b V b_{ij}) g^{ik} g^{j\ell} e_{k\ell} \ , \tag{2.4}$$

$$Q := V(g^{ij} - b^{ij} + g^{ik} g^{j\ell} e_{k\ell}) \mathrm{Ric}(b)_{ij} + Q' \ . \tag{2.5}$$

Brackets over a symbol denote anti-symmetrisation, with an appropriate numerical factor (1/2 in the case of two indices).[3] The symbol Δ_f denotes the Laplace operator of a metric f. The result is valid in any dimension $n\geq 2$. Here Q' denotes an expression which is bilinear in e_{ij} and $\mathring{D}_k e_{ij}$, linear in V, dV and Hess$V$, with coefficients which are uniformly bounded in a b-ON frame, as long as g is uniformly equivalent to b. The idea behind the calculation leading to (2.2) is to collect all terms in R_g that contain second derivatives of the metric in $\partial_i \mathbb{U}^i$; in what remains one collects in s the terms which are linear in

[2] The reader is warned that the tensor field e here is *not* a direct Riemannian counterpart of the one in [20]; the latter makes appeal to the *contravariant* and not the *covariant* representation of the metric tensor. On the other hand e_{ij} here coincides with that of [14].

[3] In general relativity a normalising factor $1/16\pi$, arising from physical considerations, is usually thrown in into the definition of $\mathbb{U}^i$. From a geometric point view this seems purposeful when the boundary at infinity is a round two dimensional sphere; however, for other topologies and dimensions, this choice of factor does not seem very useful, and for this reason we do not include it in $\mathbb{U}^i$.

e_{ij}, while the remaining terms are collected in Q; one should note that the first term at the right-hand-side of (2.5) does indeed not contain any terms linear in e_{ij} when Taylor expanded at $g_{ij} = b_{ij}$.

We wish to present a generalisation of this formula — Equation (2.11) below — which takes into account the physical extrinsic curvature tensor K and its background equivalent $\mathring{K}$; this requires introducing some notation. For any scalar field V and vector field Y we define

$$\begin{aligned} \mathring{A}_{kl} \equiv A_{kl}(b,\mathring{K}) &:= \mathscr{L}_Y b_{kl} - 2V\mathring{K}_{kl} \ , && (2.6)\\ A \equiv A(g,K) &:= V\underbrace{\Big(R_g - K^{kl}K_{kl} + (\mathrm{tr}_g K)^2\Big)}_{=\rho(g,K)} - 2Y^k \underbrace{D_l\Big(K^l{}_k - \delta^l{}_k \mathrm{tr}_g K\Big)}_{=-J_k(g,K)/2} \\ &= -\frac{2\mathcal{G}^0{}_\mu X^\mu}{\sqrt{\det g}} = 2G_{\mu\nu}n^\mu X^\nu \ . && (2.7)\end{aligned}$$

The symbol $\mathscr{L}$ denotes a Lie derivative. If we were in a space-time context, then $\mathcal{G}^\lambda{}_\mu$ would be the Einstein tensor density, $G_{\mu\nu}$ would be the Einstein tensor, while n^μ would be the future directed normal to the initial data hypersurface. Finally, an associated space-time vector field X would then be defined as

$$X = Vn^\mu\partial_\mu + Y^k\partial_k = \frac{V}{N}\partial_0 + (Y^k - \frac{V}{N}N^k)\partial_k \ , \qquad (2.8)$$

where N and N^k are the lapse and shift functions. However, as far as possible we will forget about any space-time structures. It should be pointed out that our ρ here can be interpreted as the energy-density of the matter fields when the cosmological constant Λ vanishes; it is, however, shifted by a constant in the general case.

We set

$$P^{kl} := g^{kl}\mathrm{tr}_g K - K^{kl} \ , \qquad \mathrm{tr}_g K := g^{kl}K_{kl} \ ,$$

with a similar definition relating the background quantities $\mathring{K}$ and $\mathring{P}$; indices on K and P are always moved with g while those on $\mathring{K}$ and $\mathring{P}$ are always moved with b.

We shall say that (V,Y) satisfy the (background) *vacuum Killing Initial Data* (KID) equations if

$$A_{ij}(b,\mathring{K}) = 0 = S^{kl}(b,\mathring{K}) \ , \qquad (2.9)$$

where

$$\begin{aligned} \mathring{S}^{kl} \equiv S^{kl}(b,\mathring{K}) &:= V\left(2\mathring{P}^{ml}\mathring{P}_m{}^k - \frac{3}{n-1}\mathrm{tr}_b\mathring{P}\mathring{P}^{kl} + \mathrm{Ric}(b)^{kl} - R_b b^{kl}\right) \\ &\quad - \mathscr{L}_Y\mathring{P}^{kl} + \Delta_b V b^{kl} - \mathring{D}^k\mathring{D}^l V \ . && (2.10)\end{aligned}$$

Vacuum initial data with this property lead to space-times with Killing vectors, see [34] (compare [7]). We will, however, not assume at this stage that we are dealing with vacuum initial data sets, and we will do the calculations in a general case.

In Appendix A we derive the following counterpart of Equation (2.2):

$$\partial_i \left(\mathbb{U}^i(V) + \mathbb{V}^i(Y)\right) = \sqrt{\det g}\,\left[V\left(\rho(g,K) - \rho(b,\mathring{K})\right) + s' + Q''\right] + Y^k\sqrt{\det g}\,\left(J_k(g,K) - J_k(b,\mathring{K})\right)\,, \tag{2.11}$$

where

$$\mathbb{V}^l(Y) := 2\sqrt{\det g}\left[(P^l{}_k - \mathring{P}^l{}_k)Y^k - \frac{1}{2}Y^l\mathring{P}^{mn}e_{mn} + \frac{1}{2}Y^k\mathring{P}^l{}_k b^{mn}e_{mn}\right]\,. \tag{2.12}$$

Further, Q'' contains terms which are quadratic in the deviation of g from b and its derivatives, and in the deviations of K from $\mathring{K}$, while s', obtained by collecting all terms linear or linearised in e_{ij}, except for those involving ρ and J, reads

$$s' = (\mathring{S}^{kl} + \mathring{B}^{kl})e_{kl} + (P^{kl} - \mathring{P}^{kl})\mathring{A}_{kl}\,,$$
$$\mathring{B}^{kl} := \frac{1}{2}\left[b^{kl}\mathring{P}^{mn}\mathring{A}_{mn} - b^{mn}\mathring{A}_{mn}\mathring{P}^{kl}\right]\,. \tag{2.13}$$

3 Initial data sets with rapid decay

3.1 The reference metrics

Consider a manifold M which contains a region $M_{\rm ext} \subset M$ together with a diffeomorphism

$$\Phi^{-1} : M_{\rm ext} \to [R,\infty)\times N\,, \tag{3.1}$$

where N is a compact boundaryless manifold. Suppose that on $[R,\infty)\times N$ we are given a Riemannian metric b_0 of a product form

$$b_0 := \frac{dr^2}{r^2+k} + r^2\breve{h}\,, \tag{3.2}$$

as well as a symmetric tensor field K_0; conditions on K_0 will be imposed later on. We assume that $\breve{h}$ is a Riemannian metric on N with constant scalar curvature $R_{\breve{h}}$ equal to

$$R_{\breve{h}} = \begin{cases} (n-1)(n-2)k\,, \quad k\in\{0,\pm1\}\,, & \text{if } n>2,\\ 0\,, \quad k=1\,, & \text{if } n=2; \end{cases} \tag{3.3}$$

here r is a coordinate running along the $[R,\infty)$ factor of $[R,\infty)\times N$. Here the dimension of N is $(n-1)$; we will later on specialise to the case $n+1=4$ but we allow a general n in this section. There is some freedom in the choice of k

when $n = 2$, associated with the range of the angular variable φ on $N = S^1$, and we make the choice $k = 1$ which corresponds to the usual form of the two-dimensional hyperbolic space. When $(N, \breve{h})$ is the unit round $(n-1)$–dimensional sphere $(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$, then b_0 is the hyperbolic metric.

Pulling-back b_0 using Φ^{-1} we define on $M_{\rm ext}$ a reference metric b,

$$\Phi^* b = b_0 \ . \tag{3.4}$$

Equations (3.2)–(3.3) imply that the scalar curvature R_b of the metric b is constant:

$$R_b = R_{b_0} = n(n-1)k \ .$$

Moreover, the metric b will be Einstein if and only if $\breve{h}$ is. We emphasise that for all our purposes we only need b on $M_{\rm ext}$, and we continue b in an arbitrary way to $M \setminus M_{\rm ext}$ whenever required.

Anticipating, the "charge integrals" will be defined as the integrals of $\mathbb{U} + \mathbb{V}$ over "the boundary at infinity", *cf.* Proposition 3.2 below. The convergence of the integrals there requires appropriate boundary conditions, which are defined using the following b-orthonormal frame $\{f_i\}_{i=1,n}$ on $M_{\rm ext}$:

$$\Phi^{-1}_* f_i = r\epsilon_i \ , \quad i = 1, \ldots, n-1 \ , \quad \Phi^{-1}_* f_n = \sqrt{r^2 + k}\ \partial_r \ , \tag{3.5}$$

where the ϵ_i's form an orthonormal frame for the metric $\breve{h}$. We moreover set

$$g_{ij} := g(f_i, f_j) \ , \quad K_{ij} := K(f_i, f_j) \ , \tag{3.6}$$

etc., and throughout this section only tetrad components will be used.

3.2 The charges

We start by introducing a class of boundary conditions for which convergence and invariance proofs are particularly simple. We emphasise that the asymptotic conditions of Definition 3.1 are too restrictive for general hypersurfaces meeting $\mathscr{I}$ in anti-de Sitter space-time, or – perhaps more annoyingly – for general radiating asymptotically flat metrics. We will return to that last case in Section 5; this requires considerably more work.

Definition 3.1 (Strong asymptotic decay conditions). *We shall say that the initial data* (g, K) *are* strongly asymptotically hyperboloidal *if:*

$$\begin{aligned}\int_{M_{\rm ext}} \Bigg(& \sum_{i,j} \left(|g_{ij} - \delta_{ij}|^2 + |K_{ij} - \mathring{K}_{ij}|^2 \right) + \sum_{i,j,k} |f_k(g_{ij})|^2 \\ & + \sum_{i,j} \left(|\mathring{S}^{ij} + \mathring{B}^{ij}|^2 + |\mathring{A}_{ij}|^2 \right) + \sum_k |J_k(g,K) - J_k(b, \mathring{K})| \\ & + |\rho(g,K) - \rho(b, \mathring{K})| \Bigg) r \circ \Phi \ d\mu_g < \infty \ ,\end{aligned} \tag{3.7}$$

$$\exists\, C > 0 \quad \textit{such that} \quad C^{-1} b(X,X) \le g(X,X) \le C b(X,X)\ . \tag{3.8}$$

Of course, for vacuum metrics g and b (with or without cosmological constant) and for background KIDs (V,Y) (which will be mostly of interest to us) all the quantities appearing in the second and third lines of (3.7) vanish.

For hyperboloids in Minkowski space-time, or for static hypersurfaces in anti de Sitter space-time, the V's and Y's associated to the translational Killing vectors satisfy

$$V = O(r)\ , \quad \sqrt{b^{\#}(dV,dV)} = O(r)\ , \qquad |Y|_b = O(r)\ , \tag{3.9}$$

where $b^{\#}$ is the metric on T^*M associated to b, and this behavior will be assumed in what follows.

Let $\mathcal{N}_{b_0,\mathring{K}_0}$ denote[4] the *space of background KIDs*:

$$\mathcal{N}_{b_0,\mathring{K}_0} := \{(V_0,Y_0) \mid A_{ij}(b_0,\mathring{K}_0) = 0 = S^{kl}(b_0,\mathring{K}_0)\}\ , \tag{3.10}$$

compare Equations (2.6) and (2.10), where it is understood that V_0 and Y_0 have to be used instead of V and Y there. The geometric character of (2.6) and (2.10) shows that if (V_0,Y_0) is a background KID for $(b_0,\mathring{K}_0)$, then

$$(V := V_0 \circ \Phi^{-1}, Y := \Phi_* Y_0)$$

will be a background KID for $(b = (\Phi^{-1})^* b_0, \mathring{K} = (\Phi^{-1})^* \mathring{K}_0)$. The introduction of the (V_0,Y_0)'s provides a natural identification of KIDs for different backgrounds $((\Phi_1^{-1})^* b_0, (\Phi_1^{-1})^* \mathring{K}_0)$ and $((\Phi_2^{-1})^* b_0, (\Phi_2^{-1})^* \mathring{K}_0)$. We have

PROPOSITION 3.2. *Let the reference metric b on $M_{\rm ext}$ be of the form (3.4), suppose that V and Y satisfy (3.9), and assume that Φ is such that Equations (3.7)-(3.8) hold. Then for all $(V_0,Y_0) \in \mathcal{N}_{b_0,\mathring{K}_0}$ the limits*

$$H_\Phi(V_0,Y_0) := \lim_{R\to\infty} \int_{r\circ\Phi^{-1}=R} \big(\mathbb{U}^i(V_0\circ\Phi^{-1}) + \mathbb{V}^i(\Phi_* Y_0)\big)\, dS_i \tag{3.11}$$

exist, and are finite.

The integrals (3.11) will be referred to as *the Riemannian charge integrals*, or simply *charge integrals*.

Proof. We work in coordinates on $M_{\rm ext}$ such that Φ is the identity. For any R_1, R_2 we have

$$\int_{r=R_2} (\mathbb{U}^i + \mathbb{V}^i) dS_i = \int_{r=R_1} (\mathbb{U}^i + \mathbb{V}^i) dS_i + \int_{[R_1,R_2]\times N} \partial_i (\mathbb{U}^i + \mathbb{V}^i)\, d^n x\ , \tag{3.12}$$

[4] We denote by $\mathring{K}$ the background extrinsic curvature on the physical initial data manifold M, and by $\mathring{K}_0$ its equivalent in the model manifold $[R,\infty)\times N$, $\mathring{K}_0 := \Phi^* \mathring{K}$.

and the result follows from (2.11)-(2.13), together with (3.7)-(3.8) and the Cauchy-Schwarz inequality, by passing to the limit $R_2 \to \infty$. □

In order to continue we need some more restrictions on the extrinsic curvature tensor $\mathring{K}$. In the physical applications we have in mind in this section the tensor field $\mathring{K}$ will be pure trace, which is certainly compatible with the following hypothesis:

$$|\mathring{K}^i{}_j - \frac{\mathrm{tr}_{b_0} \mathring{K}}{n}\delta^i_j|_{b_0} = o(r^{-n/2}) \ . \tag{3.13}$$

Under (3.13) and (3.15) one easily finds from (2.12) that

$$\lim_{R\to\infty}\int_{r\circ\Phi^{-1}=R} \mathbb{V}^i(Y)dS_i = 2\lim_{R\to\infty}\int_{r\circ\Phi^{-1}=R} \sqrt{\det b}\left[(P^l{}_k - \mathring{P}^l{}_k)Y^k\right] dS_i \ , \tag{3.14}$$

which gives a slightly simpler expression for the contribution of P to H_Φ.

Under the conditions of Proposition 3.2, the integrals (3.11) define a linear map from $\mathcal{N}_{b_0,\mathring{K}_0}$ to $\mathbb{R}$. Now, each map Φ used in (3.4) defines in general a different background metric b on M_{ext}, so that the maps H_Φ are potentially dependent[5] upon Φ. (It should be clear that, given a fixed $\breve{h}$, (3.11) does not depend upon the choice of the frame ϵ_i in (3.5).) It turns out that this dependence can be controlled:

THEOREM 3.3. *Under (3.13), consider two maps Φ_a, $a = 1, 2$, satisfying (3.7) together with*

$$\sum_{i,j}\Big(|g_{ij} - \delta_{ij}| + |P^i{}_j - \mathring{P}^i{}_j|\Big) + \sum_{i,j,k}|f_k(g_{ij})| =$$
$$= \begin{cases} o(r^{-n/2}) \ , & \text{if } \ n > 2, \\ O(r^{-1-\epsilon}) \ , & \text{if } \ n = 2, \ \text{for some } \epsilon > 0. \end{cases} \tag{3.15}$$

Then there exists an isometry A of b_0, defined perhaps only for r large enough, such that

$$H_{\Phi_2}(V_0, Y_0) = H_{\Phi_1}\left(V_0 \circ A^{-1}, A_* Y_0\right) \ . \tag{3.16}$$

REMARK 3.4. *The examples in [20] show that the decay rate (3.15) is sharp when $\mathring{P}_{ij} = 0$, or when $Y^i = 0$, compare [14].*

Proof. When $\mathring{K} = 0$ the result is proved, using a space-time formalism, at the beginning of Section 4 in [20]. When $Y = 0$ this is Theorem 2.3 of [14]. It turns out that under (3.13) the calculation reduces to the one in that last theorem, and that under the current conditions the integrals of $\mathbb{U}$ and $\mathbb{V}$ are separately covariant, which can be seen as follows: On M_{ext} we have three pairs of fields:

$$(g, K) \ , \quad \Big((\Phi_1^{-1})^* b_0, (\Phi_1^{-1})^*\mathring{K}_0\Big) \ , \quad \text{and} \quad \Big((\Phi_2^{-1})^* b_0, (\Phi_2^{-1})^*\mathring{K}_0\Big) \ .$$

[5]Note that the space of KIDs is fixed, as $\mathcal{N}_{b_0,\mathring{K}_0}$ is tied to $(b_0, \mathring{K}_0)$ which are fixed once and for all.

Pulling back everything by Φ_2 to $[R,\infty)\times N$ we obtain there

$$\Big((\Phi_2)^*g,(\Phi_2)^*K\Big)\ ,\quad \Big((\Phi_1^{-1}\circ\Phi_2)^*b_0,(\Phi_1^{-1}\circ\Phi_2)^*\mathring{K}_0\Big)\ ,\quad \text{and}\quad (b_0,\mathring{K}_0)\ .$$

Now, $(\Phi_2)^*g$ is simply "the metric g as expressed in the coordinate system Φ_2", similarly for $(\Phi_2)^*K$, and following the usual physicist's convention we will instead write

$$(g,K)\ ,\quad (b_1,\mathring{K}_1):=\Big((\Phi_1^{-1}\circ\Phi_2)^*b_0,(\Phi_1^{-1}\circ\Phi_2)^*\mathring{K}_0\Big)\ ,\quad \text{and}\quad (b_2,\mathring{K}_2)=(b_0,\mathring{K}_0)\ ,$$

which should be understood in the sense just explained.

As discussed in more detail in [14, Theorem 2.3], there exists an isometry A of the background metric b_0, defined perhaps only for r large enough, such that $\Phi_1^{-1}\circ\Phi_2$ is a composition of A with a map which approaches the identity as one approaches the conformal boundary, see (3.21)-(3.22) below. It can be checked that the calculation of the proof of [14, Theorem 2.3] remains valid, and yields

$$H_{\Phi_2}(V_0,0)=H_{\Phi_1}(V_0\circ A^{-1},0)\ . \tag{3.17}$$

(In [14, Theorem 2.3] Equation (2.10) with $Y=0$ has been used. However, under the hypothesis (3.13) the supplementary terms involving Y in (3.13) cancel out in that calculation.) It follows directly from the definition of H_Φ that

$$H_{\Phi_1\circ A}(0,Y_0)=H_{\Phi_1}\left(0,A_*Y_0\right)\ . \tag{3.18}$$

Since

$$H_\Phi(V_0,Y_0)=H_\Phi(V_0,0)+H_\Phi(0,Y_0)\ ,$$

we need to show that

$$H_{\Phi_2}(0,Y_0)=H_{\Phi_1}\left(0,A_*Y_0\right)\ . \tag{3.19}$$

In order to establish (3.19) it remains to show that for all Y_0 we have

$$H_{\Phi_1\circ A}(0,Y_0)=H_{\Phi_2}(0,Y_0)\ . \tag{3.20}$$

Now, Corollary 3.5 of [20] shows that the pull-back of the metrics by $\Phi_1\circ A$ has the same decay properties as that by Φ_1, so that — replacing Φ_1 by $\Phi_1\circ A$ — to prove (3.20) it remains to consider two maps $\Phi_1^{-1}=(r_1,v_1^A)$ and $\Phi_2^{-1}=(r_2,v_2^A)$ (where v^A denote abstract local coordinates on N) satisfying

$$r_2 \quad = r_1+o(r_1^{1-\frac{n}{2}}), \tag{3.21}$$

$$v_2^A \quad = v_1^A+o(r_1^{-(1+\frac{n}{2})}), \tag{3.22}$$

together with similar derivative bounds. In that case one has, in tetrad components, by elementary calculations,

$$P^i{}_j-\mathring{P}_1{}^i{}_j=P^i{}_j-\mathring{P}_2{}^i{}_j+o(r^{-n})\ , \tag{3.23}$$

leading immediately to (3.20). We point out that it is essential that $P^i{}_j$ appears in (3.23) with one index up and one index down. For example, the difference

$$P_{ij} - \mathring{P}_{ij} = o(r^{-n/2}) ,$$

transforms as

$$P_{ij} - \mathring{P}_{1ij} = P_{ij} - \mathring{P}_{2ij} - \operatorname{tr}_{b_1} \mathring{P}_1 \mathcal{L}_\zeta b_1 + o(r^{-n}) ,$$

where

$$\zeta = (r_2 - r_1)\frac{\partial}{\partial r_1} + \sum_A (v_2^A - v_1^A)\frac{\partial}{\partial v_1^A} .$$

□

4 Problems with the extrinsic curvature of the conformal boundary

Consider a vacuum space-time $(\mathscr{M}, {}^4g)$, with cosmological constant $\Lambda = 0$, which possesses a smooth conformal completion $(\overline{\mathscr{M}}, {}^4\overline{g})$ with conformal boundary $\mathscr{I}^+$. Consider a hypersurface $\mathscr{S}$ such that its completion $\overline{\mathscr{S}}$ in $\overline{\mathscr{M}}$ is a smooth spacelike hypersurface intersecting $\mathscr{I}^+$ transversally, with $\overline{\mathscr{S}} \cap \mathscr{I}^+$ being smooth two-dimensional sphere; no other completeness conditions upon $\mathscr{M}$, $\overline{\mathscr{M}}$, or upon $\mathscr{I}^+$ are imposed. In a neighborhood of $\overline{\mathscr{S}} \cap \mathscr{I}^+$ one can introduce Bondi coordinates [41], in terms of which ${}^4g$ takes the form

$$ {}^4g = -xV\mathrm{e}^{2\beta}\,du^2 + 2\mathrm{e}^{2\beta}x^{-2}\,du\,dx + x^{-2}h_{AB}\left(dx^A - U^A\,du\right)\left(dx^B - U^B\,du\right) . \tag{4.1}$$

(The usual radial Bondi coordinate r equals $1/x$.) One has

$$h_{AB} = \breve{h}_{AB} + x\chi_{AB} + O(x^2) , \tag{4.2}$$

where $\breve{h}$ is the round unit metric on S^2, and the whole information about gravitational radiation is encoded in the tensor field χ_{AB}. It has been shown in [15, Appendix C.3] that the trace-free part of the extrinsic curvature of $\overline{\mathscr{S}} \cap \mathscr{I}$ within $\mathscr{S}$ is proportional to χ. In coordinate systems on $\mathscr{S}$ of the kind used in (3.2) this leads to a $1/r$ decay of the tensor field e of (2.1), so that the decay condition (3.7) is not satisfied. In fact, (2.1) "doubly fails" as the $K - \mathring{K}$ contribution also falls off too slowly for convergence of the integral. Thus the decay conditions of Definition 3.1 are not suitable for the problem at hand.

Similarly, let $\mathscr{S}$ be a space-like hypersurface in a vacuum space-time $(\mathscr{M}, {}^4g)$ with strictly negative cosmological constant Λ, with a smooth conformal completion $(\overline{M}, {}^4\overline{g})$ and conformal boundary $\mathscr{I}$, as considered *e.g.* in [21, Section 5]. Then a generic smooth deformation of $\mathscr{S}$ at fixed conformal boundary $\overline{\mathscr{S}} \cap \mathscr{I}$ will lead to induced initial data which will not satisfy (3.7).

As already mentioned, in some of the calculations we will not consider the most general hypersurfaces compatible with the set-ups just described, because the calculations required seem to be too formidable to be performed by hand. We will instead impose the following restriction:

$$d(\mathrm{tr}\,_g K) \ \text{ vanishes on the conformal boundary.} \tag{4.3}$$

In other words, $\mathrm{tr}\,_g K$ is constant on $\overline{\mathscr{S}} \cap \mathscr{I}$, with the transverse derivatives of $\mathrm{tr}\,_g K$ vanishing there as well.

Equation (4.3) is certainly a restrictive assumption. We note, however, the following:

(i) It holds for all initial data constructed by the conformal method, both if $\Lambda = 0$ [2] and if $\Lambda < 0$ [30], for then $d(\mathrm{tr}_g K)$ is zero throughout $\mathscr{S}$.

(ii) One immediately sees from the equations in [15, Appendix C.3] that in the case $\Lambda = 0$ Equation (4.3) can be achieved by deforming $\mathscr{S}$ in $\mathscr{M}$, while keeping $\overline{\mathscr{S}} \cap \mathscr{I}^+$ fixed, whenever an associated space-time exists. It follows that for the proof of positivity of m in space-times with a conformal completion it suffices to consider hypersurfaces satisfying (4.3).

5 Convergence, uniqueness, and positivity of the Trautman-Bondi mass

From now on we assume that the space dimension is three, and that

$$\Lambda = 0 \ .$$

An extension to higher dimension would require studying Bondi expansions in $n+1 > 4$, which appears to be quite a tedious undertaking. On the other hand the adaptation of our results here to the case $\Lambda < 0$ should be straightforward, but we have not attempted such a calculation.

The metric g of a Riemannian manifold (M, g) will be said to be C^k *compactifiable* if there exists a compact Riemannian manifold with boundary $(\overline{M} \approx M \cup \partial_\infty M \cup \partial M, \overline{g})$, where $\partial\overline{M} = \partial M \cup \partial_\infty M$ is the metric boundary of $(\overline{M}, \overline{g})$, with ∂M — the metric boundary of (M, g), together with a diffeomorphism

$$\psi : \mathrm{int}\overline{M} \to M$$

such that

$$\psi^* g = x^{-2} \overline{g} \ , \tag{5.1}$$

where x is a defining function for $\partial_\infty M$ (*i.e.*, $x \geq 0$, $\{x = 0\} = \partial_\infty M$, and dx is nowhere vanishing on $\partial_\infty M$), with $\overline{g}$ — a metric which is C^k up–to–boundary on $\overline{M}$. The triple $(\overline{M}, \overline{g}, x)$ will then be called a C^k *conformal completion* of (M, g).

Clearly the definition allows M to have a usual compact boundary. (M, g) will be said to have a *conformally compactifiable end* M_{ext} if M contains an open submanifold M_{ext} (of the same dimension that M) such that $(M_{\mathrm{ext}}, g|_{M_{\mathrm{ext}}})$ is conformally compactifiable, with a connected conformal boundary $\partial_\infty M_{\mathrm{ext}}$.

In the remainder of this work we shall assume for simplicity that the conformally rescaled metric $\overline{g}$ is *polyhomogeneous* and C^1 near the conformal boundary; this means that $\overline{g}$ is C^1 up-to-boundary and has an asymptotic expansion with smooth expansion coefficients to any desired order in terms of powers of x and of $\ln x$. (In particular, smoothly compactifiable metrics belong to the polyhomogeneous class; the reader unfamiliar with polyhomogeneous expansions might wish to assume smoothness throughout.) It should be clear that the conditions here can be adapted to metrics which are polyhomogeneous plus a weighted Hölder or Sobolev lower order term decaying sufficiently fast. In fact, a very conservative estimate, obtained by inspection of the calculations below, shows that relative $O_{\ln^* x}(x^4)$ error terms introduced in the metric because of matter fields or because of sub-leading non-polyhomogeneous behavior do not affect the validity of the calculations below, provided the derivatives of those error term behave under differentiation in the obvious way (an x derivative lowers the powers of x by one, other derivatives preserve the powers). (We use the symbol $f = O_{\ln^* x}(x^p)$ to denote the fact that there exists $N \in \mathbb{N}$ and a constant C such that $|f| \leq C x^p (1 + |\ln x|^N)$.)

An initial data set (M, g, K) will be said to be $C^k(\overline{M}) \times C^\ell(\overline{M})$ *conformally compactifiable* if (M, g) is $C^k(\overline{M})$ conformally compactifiable and if K is of the form

$$K^{ij} = x^3 L^{ij} + \frac{\mathrm{tr}_g K}{3} g^{ij} \ , \tag{5.2}$$

with the trace-free tensor L^{ij} in $C^\ell(\overline{M})$, and with $\mathrm{tr}_g K$ in $C^\ell(\overline{M})$, strictly bounded away from zero on $\overline{M}$. We note that (3.15) would have required $|L|_{\overline{g}} = o(x^{1/2})$, while we allow $|L|_{\overline{g}} = O(1)$. The slower decay rate is necessary in general for compatibility with the constraint equations if the trace-free part of the extrinsic curvature tensor of the conformal boundary does not vanish (equivalently, if the tensor field χ_{AB} in (4.2) does not vanish); this follows from the calculations in Appendix C.

5.1 The Trautman-Bondi four-momentum of asymptotically hyperboloidal initial data sets – the four-dimensional definition

The definition (3.11) of global charges requires a background metric b, a background extrinsic curvature tensor $\mathring{K}$, and a map Φ. For initial data which are vacuum near $\mathscr{I}^+$ all these objects will now be defined using Bondi coordinates, as follows: Let $(\mathscr{S}, g, K)$ be a hyperboloidal initial data set, by [17, 31] the associated vacuum space-time $(\mathscr{M}, {}^4g)$ has a conformal completion $\mathscr{I}^+$, with perhaps a rather low degree of differentiability. One expects that $(\mathscr{M}, {}^4g)$ will indeed be polyhomogeneous, but such a result has not been established so far. However, the analysis of [18] shows that one can formally determine all the expansion coefficients of a polyhomogeneous space-time metric *on* $\mathscr{S}$, as well as all their time-derivatives on $\mathscr{S}$. This is sufficient to carry out all the calculations here *as if the resulting completion were polyhomogeneous*. In all our calculations from now on we shall therefore assume that $(\mathscr{M}, {}^4g)$ has a polyhomogeneous conformal completion, this assumption being understood in the sense just explained.

In $(\mathscr{M}, {}^4g)$ we can always [18] introduce a Bondi coordinate system (u, x, x^A) such that $\mathscr{S}$ is given by an equation

$$u = \alpha(x, x^A) \;, \quad \text{with} \;\; \alpha(0, x^A) = 0 \;, \; \alpha_{,x}(0, x^A) > 0 \;, \tag{5.3}$$

where α is polyhomogeneous. There is *exactly* a six-parameter family of such coordinate systems, parameterised by the Lorentz group (the supertranslation freedom is gotten rid of by requiring that α vanishes on $\overline{\mathscr{S}} \cap \mathscr{I}$). We use the Bondi coordinates to define the background ${}^4b$:

$${}^4b := -\mathrm{d}u^2 + 2x^{-2}\mathrm{d}u\mathrm{d}x + x^{-2}\breve{h}_{AB}\mathrm{d}x^A\mathrm{d}x^B \;. \tag{5.4}$$

This Lorentzian background metric ${}^4b$ is independent of the choice of Bondi coordinates as above. One then defines the Trautman-Bondi four-momentum p_μ of the asymptotically hyperboloidal initial data set we started with as the Trautman-Bondi four-momentum of the cut $u = 0$ of the resulting $\mathscr{I}^+$; recall that the latter is defined as follows: Let X be a translational Killing vector of ${}^4b$, it is shown *e.g.* in [15, Section 6.1, 6.2 and 6.10] that the integrals

$$H(\mathscr{S}, X, {}^4g, {}^4b) := \lim_{\epsilon \to 0} \int_{\{x=\epsilon\}\cap\mathscr{S}} \mathbb{W}^{\nu\lambda}(X, {}^4g, {}^4b) dS_{\nu\lambda}$$

converge. Here $\mathbb{W}^{\nu\lambda}(X, {}^4g, {}^4b)$ is given by (B.2). Choosing an ON basis X_μ for the X's one then sets

$$p_\mu(\mathscr{S}) := H(\mathscr{S}, X_\mu, {}^4g, {}^4b) \;.$$

(The resulting numbers coincide with the Trautman-Bondi four-momentum; we emphasise that the whole construction depends upon the use of Bondi coordinates.)

5.2 Geometric invariance

The definition just given involves two arbitrary elements: the first is the choice of a conformal completion, the second is that of a Bondi coordinate system. While the latter is easily taken care of, the first requires attention. Suppose, for example, that a prescribed region of a space-time $(\mathscr{M}, g)$ admits two completely unrelated conformal completions, as is the case for the Taub-NUT space-time. In such a case the resulting p_μ's might have nothing to do with each other.

Alternatively, suppose that there exist two conformal completions which are homeomorphic but not diffeomorphic. Because the objects occurring in the definition above require derivatives of various tensor fields, one could a priori again obtain different answers. In fact, the construction of the approximate Bondi coordinates above requires expansions to rather high order of the metric at $\mathscr{I}^+$, which is closely related to high differentiability of the metric at $\mathscr{I}^+$, so even if we have two diffeomorphic completions such that the diffeomorphism is not smooth enough, we might still end up with unrelated values of p_μ.

It turns out that none of the above can happen. The key element of the proof is the following result, which is essentially Theorem 6.1 of [14]; the proof there was given for C^∞ completions, but an identical argument applies under the hypotheses here:

THEOREM 5.1. *Let (M,g) be a Riemannian manifold endowed with two C^k, $k \geq 1$ and polyhomogeneous conformal compactifications $(\overline{M}_1, \overline{g}_1, x_1)$ and $(\overline{M}_2, \overline{g}_2, x_2)$ with compactifying maps ψ_1 and ψ_2. Then*

$$\psi_1^{-1} \circ \psi_2 : \mathrm{int} M_2 \to \mathrm{int} M_1$$

extends by continuity to a C^k and polyhomogeneous conformal up-to-boundary diffeomorphism from $(\overline{M}_2, \overline{g}_2)$ to $(\overline{M}_1, \overline{g}_1)$, in particular $\overline{M}_1$ and $\overline{M}_2$ are diffeomorphic as manifolds with boundary.

We are ready now to prove definitional uniqueness of four-momentum. Some remarks are in order:

(i) It should be clear that the proof below generalises to matter fields near $\mathscr{I}^+$ which admit a well posed conformal Cauchy problem *à la* Friedrich, *e.g.* to Einstein-Yang-Mills fields [23].

(ii) The differentiability conditions below have been chosen to ensure that the conformal Cauchy problem of Friedrich [24] is well posed; we have taken a very conservative estimate for the differentiability thresholds, and for simplicity we have chosen to present the results in terms of classical rather than Sobolev differentiability. One expects that $C^1(\overline{M}) \times C^0(\overline{M})$ and polyhomogeneous CMC initial data $(\overline{g}, L)$ will lead to existence of a polyhomogeneous $\mathscr{I}^+$; such a theorem would immediately imply a corresponding equivalent of Theorem 5.2.

(iii) It is clear that there exists a purely three-dimensional version of the proof below, but we have not attempted to find one; the argument given seems to minimise the amount of new calculations needed. Such a three-dimensional proof would certainly provide a result under much weaker asymptotic conditions concerning both the matter fields and the requirements of differentiability at $\mathscr{I}^+$.

THEOREM 5.2. *Let (M,g,K) be a $C^7(\overline{M}) \times C^6(\overline{M})$ and polyhomogeneous conformally compactifiable initial data set which is vacuum near the conformal boundary, and consider two $C^7(\overline{M})$ and polyhomogeneous compactifications thereof as* *in Theorem 5.1, with associated four-momenta p^a_μ, $a = 1, 2$. Then there exists a Lorentz matrix $\Lambda_\mu{}^\nu$ such that*

$$p^1_\mu = \Lambda_\mu{}^\nu p^2_\nu \ .$$

Proof. By the results of Friedrich (see [24] and references therein) the maximal globally hyperbolic development $(\mathscr{M}, {}^4g)$ of (M, g, K) admits C^4 conformal completions $(\overline{\mathscr{M}}_a, {}^4\overline{g}_a)$, $a = 1, 2$ with conformal factors Ω_a and diffeomorphisms $\Psi_a : \mathrm{int}\ \overline{\mathscr{M}}_a \to \mathscr{M}$ such that

$$\Psi^*_a({}^4g) = \Omega_a^{-2}\ {}^4\overline{g}_a \ , \quad \Psi_a|_M = \psi_a \ .$$

The uniqueness-up-to-conformal-diffeomorphism property of the conformal equations of Friedrich together with Theorem 5.1 show that $\Psi_2^{-1} \circ \Psi_1$ extends by continuity to a C^4-up-to-boundary map from a neighborhood of $\Psi_a^{-1}(\overline{M}) \subset \overline{\mathscr{M}}_1$ to $\overline{\mathscr{M}}_2$. Let b_a be the Minkowski background metrics constructed near the respective conformal boundaries $\mathscr{I}^+_a$ as in Section 5.1, we have

$$\left(\Psi_1 \circ \Psi_2^{-1}\right)^* b_2 = b_1 \ ,$$

so that $\left(\Psi_1 \circ \Psi_2^{-1}\right)^*$ defines a Lorentz transformation between the translational Killing vector fields of b_1 and b_2, and the result follows *e.g.* from [15, Section 6.9]. □

5.3 The Trautman-Bondi four-momentum of asymptotically CMC hyperboloidal initial data sets – a three-dimensional definition

Consider a conformally compactifiable initial data set (M, g, K) as defined in Section 5, see (5.2). We shall say that (M, g, K) is *asymptotically CMC* if $\mathrm{tr}_g K$ is in $C^1(\overline{M})$ and if

$$\text{the differential of } \mathrm{tr}_g K \text{ vanishes on } \partial_\infty M_{\mathrm{ext}} \ . \tag{5.5}$$

The vacuum scalar constraint equation ($\rho = 0$ in (2.7)) shows that, for $C^1(\overline{M}) \times C^1(\overline{M})$ (or for $C^1(\overline{M}) \times C^0(\overline{M})$ and polyhomogeneous) conformally compactifiable initial data sets, Equation (5.5) is equivalent to

$$\text{the differential of the Ricci scalar } R(g) \text{ vanishes on } \partial_\infty M_{\mathrm{ext}} \ . \tag{5.6}$$

We wish, now, to show that for asymptotically CMC initial data sets one can define a mass in terms of limits (3.11). The construction is closely related to that presented in Section 5.1, except that everything will be directly read off from the initial data: *If* a space-time as in Section 5.1 exists, then we define the Riemannian background metric b on $\mathscr{S}$ as the metric induced by the metric ${}^4b$ of Section 5.1 on the hypersurface $u = \alpha$, and $\mathring{K}$ is defined as the extrinsic curvature tensor, with respect to ${}^4b$, of that hypersurface. The map Φ needed in (3.11) is defined to be the identity in the Bondi coordinate system above, and the metric b_0 is defined to coincide with b in the coordinate system above. The four translational Killing vectors X_μ of ${}^4b$ induce on $\mathscr{S}$ four KIDs $(V,Y)_\mu$, and one can plug those into (3.11) to obtain a definition of four-momentum. *However*, the question of existence and/or of construction of the space-time there is completely circumvented by the fact that *the asymptotic development of the function α, and that of b, can be read off directly from g and K*, using the equations of [15, Appendix C.3]. The method is then to read-off the restrictions $x|_{\mathscr{S}}$ and $x^A|_{\mathscr{S}}$ of the space-time Bondi functions x and x^A to $\mathscr{S}$ from the initial data, and henceforth the asymptotic expansions of all relevant Bondi quantities in the metric, up to error terms $O_{\ln^* x}(x^4)$ (order $O(x^4)$ in the smoothly compactifiable case); equivalently, one needs an approximation of the Bondi coordinate x on $\mathscr{S}$ up to error terms $O_{\ln^* x}(x^5)$. The relevant coefficients can thus be recursively read from the initial data by solving a finite number of recursive equations. The resulting approximate Bondi function x induces a foliation of a neighborhood of the conformal boundary, which will be called the *approximate Bondi foliation*. The asymptotic expansion of α provides an identification of $\mathscr{S}$ with a hypersurface $u=\alpha$ in Minkowski space-time with the flat metric (5.4). The Riemannian background metric b is defined to be the metric induced by ${}^4b$ on this surface, and Φ is defined to be the identity in the approximate Bondi coordinates. As already indicated, the KIDs are obtained on $\mathscr{S}$ from the translational Killing vector fields of ${}^4b$. The charge integrals (3.11) have to be calculated *on the approximate Bondi spheres $x=\epsilon$* before passing to the limit $\epsilon\to 0$.

The simplest question one can ask is whether the linearisation of the integrals (3.11) reproduces the linearisation of the Freud integrals under the procedure above. We show in Appendix B that this is indeed the case. We also show in that appendix that the linearisation of the Freud integrals *does not* reproduce the Trautman-Bondi four-momentum in general, see Equation (B.21). On the other hand that linearisation provides the right expression for p_μ when the extrinsic curvature χ of the conformal boundary vanishes. Note that under the boundary conditions of Section 3 the extrinsic curvature χ vanishes, and the values of the charge integrals coincide with the values of their linearised counterparts for translations, so that the calculations in Appendix B prove the equality of the $3+1$ charge integrals and the Freud ones for translational background KIDs under the conditions of Section 3.

The main result of this section is the following:

THEOREM 5.3. *Consider an asymptotically CMC initial data set which is C^1 and polyhomogeneously (or smoothly) conformally compactifiable. Let Φ be defined as above and let $(V,Y)_\mu$ be the background KIDs associated to space-time translations ∂_μ. Then the limits (3.11) $H_\Phi((V,Y)_\mu)$ taken along* approximate Bondi spheres *$\{x=\epsilon\}\subset\mathscr{S}$ exist and are finite. Further, the numbers*

$$p_\mu := H_\Phi((V,Y)_\mu) \tag{5.7}$$

coincide with the Trautman-Bondi four-momentum *of the associated cut in the*

Lorentzian space-time, whenever such a space-time exists.

Proof. We will show that $\mathbb{U}^x + \mathbb{V}^x$ coincides with (6.2) below up to a complete divergence and up to lower order terms not contributing in the limit, the result follows then from (6.3)-(6.4). It is convenient to rewrite the last two terms in (2.12) as

$$-\frac{1}{2}Y^x \mathring{P}^m{}_n e^m{}_n + \frac{1}{2}Y^k \mathring{P}^x{}_k b^{mn} e_{mn} \;, \tag{5.8}$$

so that we can use (E.3)-(E.4) with $M = \chi_{AB} = \beta = N^A = 0$ there. We further need the following expansions (all indices are coordinate ones)

$$\begin{aligned}
e^k{}_l &:= b^{km}(g_{ml} - b_{ml}) \;,\\
e^x{}_x &= 2\beta + x^3 \alpha_{,x} M + O_{\ln^* x}(x^4) \;,\\
e^x{}_A &= \frac{1}{4}x^2 \chi_{AC}{}^{||C} + \frac{\alpha_{,A}}{2\alpha_{,x}} - x^3 N_A - \frac{1}{32}x^3 (\chi^{CD}\chi_{CD})_{||A} + O_{\ln^* x}(x^4) \;,\\
e^A{}_x &= \frac{1}{2}x^2 \alpha_{,x}\chi^{AC}{}_{||C} + \alpha^{,A} - 2x^3\alpha_{,x}\left(N^A + \frac{1}{32}(\chi^{CD}\chi_{CD})^{||A}\right) + O_{\ln^* x}(x^4) \;,\\
e^A{}_B &= x\chi^A{}_B + \frac{1}{4}x^2\chi^{CD}\chi_{CD}\delta^A{}_B + x^3\xi^A{}_B + O_{\ln^* x}(x^4) \;,
\end{aligned}$$

with $||$ denoting a covariant derivative with respect to $\breve{h}$. One then finds

$$\begin{aligned}
-\frac{1}{2}Y^x \mathring{P}^{mn} e_{mn} + \frac{1}{2}Y^k \mathring{P}^x{}_k b^{mn} e_{mn} &= Y^x \cdot O_{\ln^* x}(x^4) + Y^B \cdot O_{\ln^* x}(x^5) \;,\\
-\frac{1}{2}Y^A \mathring{P}^{mn} e_{mn} + \frac{1}{2}Y^k \mathring{P}^A{}_k b^{mn} e_{mn} &= Y^x \cdot O_{\ln^* x}(x^5) + Y^B \cdot O_{\ln^* x}(x^4) \;.
\end{aligned}$$

This shows that for Y^i which are $O(1)$ in the (x, x^A) coordinates, as is the case here (see Appendix F), the terms above multiplied by $\sqrt{\det g} = O(x^{-3})$ will give zero contribution in the limit, so that in (2.12) only the first two terms will survive. Those are clearly equal to the first two terms in (6.2) when a minus sign coming from the change of the orientation of the boundary is taken into account.

On the other hand $\mathbb{U}^x$ does not coincide with the remaining terms in (6.2), instead with some work one finds

$$\begin{aligned}
2(\lambda k - \mathring{\lambda}\mathring{k})V - \mathbb{U}^x &= V\Big[\sqrt{\det g}\cdot {}^2g^{AB} g^{xm} \mathring{D}_A g_{mB}\\
&\quad + 2\Big(\sqrt{\frac{b^{xx}}{g^{xx}}} - 1\Big)\mathring{\lambda}\mathring{k} + 2\sqrt{\det g}({}^2g^{AB} - {}^2b^{AB})\mathring{\Gamma}^x_{AB}\Big]\\
&\quad - 2\sqrt{\det g} D^{[x} V g^{j]k} e_{jk}\\
&= V\sqrt{\breve{h}}\left(\frac{x}{4}\chi^{AB}{}_{||AB} + O(x^2)\right) \;,
\end{aligned} \tag{5.9}$$

where ${}^2g^{AB}g_{BC} = \delta^A{}_C$ and similarly for ${}^2b^{AB}$. However, integration over S^2 yields equality in the limit; here one has to use the fact that V is a linear combination of $\ell = 0$ and 1 spherical harmonics in the relevant order in x (see Appendix F), so that the trace-free part of $V_{||AB}$ is $O(x)$. □

5.4 Positivity

Our next main result is the proof of positivity of the Trautman-Bondi mass:

THEOREM 5.4. *Suppose that (M,g) is geodesically complete without boundary. Assume that (M,g,K) contains an end which is $C^4\times C^3$, or C^1 and polyhomogeneously, compactifiable and asymptotically CMC. If*

$$\sqrt{g_{ij}J^iJ^j} \le \rho \in L^1(M) \ , \tag{5.10}$$

then p_μ is timelike future directed or vanishes, *in the following sense:*

$$p_0 \ge \sqrt{\sum p_i^2} \ . \tag{5.11}$$

Further, equality holds if and only if there exists a ∇-covariantly constant spinor field on M.

REMARK 5.5. We emphasise that no assumptions about the geometry or the behavior of the matter fields except geodesic completeness of (M,g) are made on $M\setminus M_{\rm ext}$.

REMARK 5.6. In vacuum one expects that equality in (5.11) is possible only if the future maximal globally hyperbolic development of (M,g,K) is isometrically diffeomorphic to a subset of the Minkowski space-time; compare [8] for the corresponding statement for initial data which are asymptotically flat in space-like directions. When K is pure trace one can use a result of Baum [6] to conclude that (M,g) is the three-dimensional hyperbolic space, which implies the rigidity result. A corresponding result with a general K is still lacking.

Before passing to the proof of Theorem 5.4, we note the following variation thereof, where no restrictions on $\mathrm{tr}_g K$ are made:

THEOREM 5.7. *Suppose that (M,g) is geodesically complete without boundary. If (M,g,K) contains an end which is $C^4\times C^3$, or C^1 and polyhomogeneously, compactifiable and which is vacuum near the conformal boundary, then the conclusions of Theorem 5.4 hold.*

Proof. It follows from [15, Eq. (C.84)] and from the results of Friedrich that one can deform M near $\mathscr{I}^+$ in the maximal globally hyperbolic vacuum development of the initial data there so that the hypotheses of Theorem 5.4 hold. The Trautman-Bondi four-momentum of the original hypersurface coincides with that of the deformed one by [15, Section 6.1]. □

PROOF OF THEOREM 5.4: The proof follows the usual argument as proposed by Witten. While the method of proof is well known, there are tedious technicalities which need to be taken care of to make sure that the argument applies.

Let D be the covariant-derivative operator of the metric g, let ∇ be the covariant-derivative operator of the space-time metric ${}^4g$, and let $\mathfrak{D}$ be the Dirac operator associated with ∇ along M,

$$\mathfrak{D}\psi = \gamma^i \nabla_i \psi$$

(summation over space indices only). Recall the Sen-Witten [40, 45] identity:

$$\int_{M\setminus\{r\geq R\}} \|\nabla\psi\|_g^2 + \langle\psi, (\rho + J^i\gamma_i\gamma_0)\psi\rangle - \|\mathfrak{D}\psi\|_g^2 = \int_{S_R} B^i(\psi) dS_i \ , \tag{5.12}$$

where the boundary integrand is

$$B^i(\psi) = \langle \nabla^i\psi + \gamma^i\mathfrak{D}\psi, \psi\rangle_g \ . \tag{5.13}$$

We have the following:

LEMMA 5.8. *Let ψ be a Killing spinor for the space-time background metric b, and let $A^\mu\partial_\mu$ be the associated translational Killing vector field $A^\mu = \psi^\dagger\gamma^\mu\psi$. We have*

$$\lim_{R\to\infty} \int_{S(R)} < \psi, \nabla^i\psi + \gamma^i\gamma^j\nabla_j\psi > dS_i = \frac{1}{4\pi} p_\mu A^\mu \ . \tag{5.14}$$

Proof. For $A^\mu\partial_\mu = \partial_0$ the calculations are carried out in detail in Appendix C. For general A^μ the result follows then by the well known Lorentz-covariance of p_μ under changes of Bondi frames (see *e.g.* [15, Section 6.8] for a proof under the current asymptotic conditions). □

LEMMA 5.9. *Let ψ be a spinor field on M which vanishes outside of $M_{\rm ext}$, and coincides with a Killing spinor for the background metric b for R large enough. Then*

$$\nabla\psi \in L^2(M) \ .$$

Proof. In Appendix D we show that

$$\mathfrak{D}\psi \in L^2 \ . \tag{5.15}$$

We then rewrite (5.12) as

$$\int_{M\setminus\{r\geq R\}} \|\nabla\psi\|_g^2 + \langle\psi, (\rho + J^i\gamma_i\gamma_0)\psi\rangle = \int_{M\setminus\{r\geq R\}} \|\mathfrak{D}\psi\|_g^2 + \int_{S_R} B^i(\psi) dS_i \ . \tag{5.16}$$

By the dominant energy condition (5.10) the function $\langle\psi, (\rho + J^i\gamma_i)\psi\rangle$ is non-negative. Passing with R to infinity the right-hand-side is bounded by (5.15) and by Lemma 5.8. The result follows from the monotone convergence theorem. □

Lemmata 5.8 and 5.9 are the two elements which are needed to establish positivity of the right-hand-side of (5.14) whatever A^μ, see *e.g.* [3, 14] for a detailed exposition in a related setting, or [4, 11, 35] in the asymptotically flat context. (For instance, Lemma 5.9 justifies the right-hand-side of the implication [14, Equation (4.17)], while Lemma 5.8 replaces all the calculations that follow [14, Equation (4.18)]. The remaining arguments in [14] require only trivial modifications.) □

One also has a version of Theorem 5.4 with trapped boundary, using solutions of the Dirac equation with the boundary conditions of [26] (compare [28]):

THEOREM 5.10. *Let (M, g) be a geodesically complete manifold with compact boundary ∂M, and assume that the remaining hypotheses of Theorem 5.4 or of Theorem 5.7 hold. If ∂M is either outwards-past trapped, or outwards-future trapped, then p^μ is* timelike future directed*:*

$$p^0 > \sqrt{\sum_i p_i^2}\ .$$

□

6 The mass of approximate Bondi foliations

The main tool in our analysis so far was the foliation of the asymptotic region by spheres arising from space-times Bondi coordinates adapted to the initial data surface. Such foliations will be called *Bondi foliations*. The aim of this section is to reformulate our definition of mass as an object directly associated to this foliation. The definition below is somewhat similar to that of Brown, Lau and York [10], but the normalisation (before passing to the limit) used here seems to be different from the one used by those authors.

Let us start by introducing some notation (for ease of reference we collect here all notations, including some which have already been introduced elsewhere in the paper): Let ${}^4g_{\mu\nu}$ be a metric of a spacetime which is asymptotically flat in null directions. Let g_{ij} be the metric induced on a three-dimensional surface $\mathscr{S}$ with extrinsic curvature K_{ij}. We denote by ${}^3\Gamma^i{}_{jk}$ the Christoffel symbols of g_{ij}.

We suppose that we have a function $x > 0$ the level sets $x =$ const. of which provide a foliation of $\mathscr{S}$ by two-dimensional submanifolds, denoted by $\mathscr{S}_x$, each of them homeomorphic to a sphere. The level set "$\{x = 0\}$" (which does not exist in $\mathscr{S}$) should be thought of as corresponding to $\mathscr{I}^+$. The metric g_{ij} induces a metric ${}^2g_{AB}$ on each of these spheres, with area element $\lambda = \sqrt{\det {}^2g_{AB}}$. By k_{AB} we denote the extrinsic curvature of the leaves of the x-foliation, $k_{AB} = {}^3\Gamma^x{}_{AB}/\sqrt{g^{xx}}$, with mean curvature $k = {}^2g^{AB}k_{AB}$. (The reader is warned that in this convention the outwards extrinsic curvature of a sphere in a flat Riemannian metric is negative.) There are also the corresponding objects for the background (Minkowski) metric ${}^4b_{\mu\nu}$, denoted by letters with a circle.

Let X be a field of space-time vectors defined along $\mathscr{S}$. Such a field can be decomposed as $X = Y + Vn$, where Y is a field tangent to $\mathscr{S}$, n — the unit ($n^2 = -1$), future-directed vector normal to $\mathscr{S}$. Motivated by (5.9), we define the following functional depending upon various objects defined on the hypersurface $\mathscr{S}$ and on a vector field X:

$$\frac{1}{16\pi} \lim_{x\to 0} \int_{\mathscr{S}_x} \mathcal{F}(X) \mathrm{d}^2 x \,, \tag{6.1}$$

where

$$\mathcal{F}(X) = 2\sqrt{\det g}(\mathring{P}^x{}_i - P^x{}_i)Y^i + 2(\mathring{\lambda}\mathring{k} - \lambda k)V. \tag{6.2}$$

So far the considerations were rather general, from now on we assume that the initial data set contains an *asymptotically CMC* conformally compactifiable end and that x provides an approximate Bondi foliation, as defined in Section 5.3. (To avoid ambiguities, we emphasise that we do impose the restrictive condition (5.5), which in terms of the function α of (C.2) translates into Equation (C.3).) We then have $\lambda = \mathring{\lambda}$ up to terms which do not affect the limit $x \to 0$ (see Appendix E.1) so that the terms containing λ above can also be written as $\mathring{\lambda}(\mathring{k} - k)$ or $\lambda(\mathring{k} - k)$. We will show that in the limit $x \to 0$ the functional (6.1) will give the Trautman-Bondi mass and momentum, as well as angular momentum and centre of mass, for appropriately chosen fields X corresponding to the relevant generators of Poincaré group.

To study the convergence of the functional when $x \to 0$ we need to calculate several objects on $\mathscr{S}$. We write the spacetime metric in Bondi-Sachs coordinates, as in (C.8) and use the standard expansions for the coefficients of the metric (see *eg.* [15, Equations (5.98)-(5.101)]). The covariant derivative operator associated with the metric $\breve{h}_{AB}$ is denoted by ${}_{||A}$. Some intermediate results needed in those calculations are presented in Appendix E.1, full details of the calculations can be found[6] in [32]. Using formulae (E.2)-(E.5) and the decomposition of Minkowski spacetime Killing vectors given in Appendix F we get (both here and in Appendix F all indices are coordinate ones):

$$\mathcal{F}(X_{time}) = \sqrt{\breve{h}}[4M - \chi^{CD}{}_{||CD} + O(x)] \,,$$

$$\mathcal{F}(X_{trans}) = -v\sqrt{\breve{h}}[4M - \chi^{CD}{}_{||CD} + O(x)] \,,$$

$$\begin{aligned}\mathcal{F}(X_{rot}) &= -\sqrt{\breve{h}}\Bigg[\varepsilon^{AB}\Bigg(\underline{-\frac{\chi^C{}_{A||C}}{x}} + \frac{3(\chi^{CD}\chi_{CD})_{||A}}{16} \\ &\quad + 6N_A + \frac{1}{2}\chi_{AC}\chi^{CD}{}_{||D}\Bigg) v_{,B} + O(x)\Bigg] \,,\end{aligned}$$

$$\begin{aligned}\mathcal{F}(X_{boost}) &= -\sqrt{\breve{h}}\Bigg[\underline{-\frac{\chi^C{}_{A||C}v^{,A}}{x}} + \frac{1}{8}\chi^{CD}\chi_{CD}v \\ &\quad + \left(\frac{3}{16}(\chi^{CD}\chi_{CD})_{||A} + 6N_A + \frac{1}{2}\chi_{AC}\chi^{CD}{}_{||D}\right)v^{,A} + O(x)\Bigg] \,.\end{aligned}$$

[6]The reader is warned that $\mathcal{F}$ there is $-\mathcal{F}$ here.

The underlined terms in integrands corresponding to boosts and rotations diverge when x tends to zero, but they yield zero when integrated over a sphere. In the formulae above we set $u-\alpha$ equal to any constant, except for the last one where $u-\alpha=0$. Moreover, v is a function on the sphere which is a linear combination of $\ell=1$ spherical harmonics and ε^{AB} is an antisymmetric tensor (more precise definitions are given in Appendix F).

In particular we obtain:

$$E_{TB} = \frac{1}{16\pi}\lim_{x\to 0}\int_{\mathscr{S}_x} \mathcal{F}(X_{time})\mathrm{d}^2x \;, \tag{6.3}$$

$$P_{TB} = \frac{1}{16\pi}\lim_{x\to 0}\int_{\partial\mathscr{S}_R} \mathcal{F}(X_{trans})\mathrm{d}^2x \;, \tag{6.4}$$

where the momentum is computed for a space-translation generator corresponding to the function v (see Appendix F). It follows that the integrals of $\mathcal{F}(X)$ are convergent for all four families of fields X.

6.1 Polyhomogeneous metrics

In this section we consider polyhomogeneous metrics. More precisely, we will consider metrics of the form (4.1) for which the V, β, U^A and h_{AB} have asymptotic expansions of the form

$$f \simeq \sum_i \sum_{j=1}^{N_i} f_{ij}(x^A)x^i \log^j x \;,$$

where the coefficients f_{ij} are smooth functions. This means that f can be approximated up to terms $O(x^N)$ (for any N) by a finite sum of terms of the form $f_{ij}(x^A)x^i\log^j x$, and that this property is preserved under differentiation in the obvious way.

As mentioned in [15, Section 6.10], allowing a polyhomogeneous expansion of h_{AB} of the form

$$h_{AB} = \breve{h}_{AB} + x\chi_{AB} + x\log x D_{AB} + o(x)$$

is not compatible with the Hamiltonian approach presented there because the integral defining the symplectic structure diverges. (For such metrics it is still possible to define the Trautman-Bondi mass and the momentum [16].) It is also noticed in [15, Section 6.10], that when logarithmic terms in h_{AB} are allowed only at the x^3 level and higher, then all integrals of interest converge. It is therefore natural to study metrics for which h_{AB} has the intermediate behavior

$$h_{AB} = \breve{h}_{AB} + x\chi_{AB} + x^2\zeta_{AB}(\log x) + O(x^2) \;, \tag{6.5}$$

where $\zeta_{AB}(\log x)$ is a polynomial of order N in $\log x$ with coefficients being smooth, symmetric tensor fields on the sphere. Under (6.5), for the $\mathcal{F}$ functional we find:

$$\mathcal{F}(X_{time}) = \sqrt{\breve{h}}[4M - \chi^{CD}{}_{||CD} + O(x\log^{N+1}x)] \;,$$

$$\mathcal{F}(X_{trans}) = -v\sqrt{\breve{h}}[4M - \chi^{CD}{}_{||CD} + O(x\log^{N+1}x)] \ .$$

The only difference with the power-series case is that terms with $O(x\log^{N+1}x)$ asymptotic behavior appear in the error term, while previously we had $O(x)$. Thus $\mathcal{F}$ reproduces the Trautman-Bondi four-momentum for such metrics as well.

In [15] no detailed analysis of the corresponding expressions for the remaining generators (*i.e.*, X_{rot} and X_{boost}) was carried out, except for a remark in Section 6.10, that the asymptotic behavior (6.5) leads to potentially divergent terms in the Freud potential, which might lead to a logarithmic divergence for boosts and rotations in the relevant Hamiltonians, which then could cease to be well defined. Here we check that the values $\mathcal{F}(X_{rot})$ and $\mathcal{F}(X_{boost})$ remain well defined for certain metrics of the form (6.5). Supposing that h_{AB} is of such form one finds (see [15, Appendix C.1.2])

$$\begin{aligned}
\partial_u\zeta_{AB} &= 0 \ , \\
U^A &= -\frac{1}{2}x^2\chi^{AC}{}_{||C} + x^3W^A(\log x) + O(x) \ , \\
\beta &= -\frac{1}{32}x^2\chi^{CD}\chi_{CD} + O(x^3\log^{2N}x) \ , \\
V &= \frac{1}{x} - 2M + O(x\log^{N+1}x) \ ,
\end{aligned}$$

where $W^A(\log x)$ is a polynomial of order $N+1$ in $\log x$ with coefficients being smooth vector fields on the sphere. The W^A's can be calculated by solving Einstein equations (which are presented in convenient form in [18]).

The calculation in case of polyhomogeneous metrics is very similar to the one in case of metrics allowing power-series expansion (see Appendix E.2), leading to

$$\mathcal{F}(X_{rot}) = -\sqrt{\breve{h}}\left[\varepsilon^{AB}\left(\underline{-\frac{\chi^C{}_{A||C}}{x}} - 3W_A - xW_{A\wr x}\right)v_{,B} + O(1)\right] \ ,$$

$$\mathcal{F}(X_{boost}) = -\sqrt{\breve{h}}\left[\left(\underline{-\frac{\chi^C{}_{A||C}}{x}} - 3W_A - xW_{A\wr x}\right)v^{,A} + O(1)\right] \ ,$$

where ${}_{\wr x}$ denotes derivation at constant u. As in the previous section, the underlined terms tend to infinity for $x \to 0$, but give zero when integrated over the sphere. However some potentially divergent terms remain, which can be rewritten as

$$\frac{1}{x^2}\varepsilon^{AB}(x^3W_A)_{\wr x}v_{,B} \ , \qquad \frac{1}{x^2}(x^3W_A)_{\wr x}v^{,A} \ .$$

To obtain convergence one must therefore have:

$$\lim_{x\to 0}\int_{\mathscr{S}_x}\frac{1}{x^2}\varepsilon^{AB}(x^3W_A)_{\wr x}v_{,B}\sqrt{\breve{h}}\mathrm{d}^2x = 0$$

and

$$\lim_{x\to 0}\int_{\mathscr{S}_x}\frac{1}{x^2}(x^3W_A)_{\wr x}v^{,A}\sqrt{\breve{h}}\mathrm{d}^2x=0$$

for any function v which is a linear combination of the $\ell = 1$ spherical harmonics.[7] It turns out that in the simplest case $N = 0$ those integrals are identically zero when vacuum Einstein equations are imposed [18].

6.2 The Hawking mass of approximate Bondi spheres

One of the objects of interest associated to the two dimensional surfaces $\mathscr{S}_x$ is their Hawking mass,

$$\begin{aligned} m_H(\mathscr{S}_x) &= \sqrt{\frac{A}{16\pi}}\left(1-\frac{1}{16\pi}\int_{\mathscr{S}_x}\theta^-\theta^+d^2\mu\right) \\ &= \sqrt{\frac{A}{16\pi}}\left(1+\frac{1}{16\pi}\int_{\mathscr{S}_x}\lambda\left(\frac{P^{xx}}{g^{xx}}-k\right)\left(\frac{P^{xx}}{g^{xx}}+k\right)d^2x\right). \end{aligned} \tag{6.6}$$

We wish to show that for asymptotically CMC initial data the Hawking mass of approximate Bondi spheres converges to the Trautman Bondi mass. In fact, for $a = (a_\mu)$ let us set

$$v(a)(\theta,\varphi)=a_0+a_i\frac{x^i}{r}\ ,$$

where x^i has to be expressed in terms of the spherical coordinates in the usual way, and

$$\begin{aligned} p_H(a,\mathscr{S}_x) &= p^\mu_H a_\mu \\ &= -\frac{1}{16\pi}\sqrt{\frac{A}{16\pi}}\int_{\mathscr{S}_x}v(a)\left(\frac{16\pi}{A}-\theta^-\theta^+\right)d^2\mu \\ &= -\frac{1}{16\pi}\sqrt{\frac{A}{16\pi}}\int_{\mathscr{S}_x}v(a)\left(\frac{16\pi}{A}+\left(\frac{P^{xx}}{g^{xx}}\right)^2-k^2\right)d^2\mu\ , \end{aligned}$$

with $d^2\mu = \lambda d^2x$. Up to the order needed to calculate the limit of the integral, on approximate Bondi spheres satisfying (C.3) we have (see Appendices E.1 and E.2)

$$\frac{P^{xx}}{g^{xx}}=P^x{}_x+O(x^4)\ ,\quad \lambda=\frac{\sqrt{\breve{h}}}{x^2}+O(x^3)\ ,$$

and

$$\begin{aligned} \left(\frac{P^{xx}}{g^{xx}}\right)^2 &= \frac{2}{\alpha_{,x}}-\frac{4\beta}{\alpha_{,x}}-3x^2(1-2Mx)-x^3\chi^{CD}{}_{||CD}+O_{\ln^* x}(x^4)\ , \\ k^2 &= \frac{2}{\alpha_{,x}}-\frac{4\beta}{\alpha_{,x}}+x^2(1-2Mx)+x^3\chi^{CD}{}_{||CD}+O_{\ln^* x}(x^4)\ . \end{aligned}$$

[7] See Appendix F.

Therefore

$$\lambda\left(\left(\frac{P^{xx}}{g^{xx}}\right)^2 - k^2\right) = -4\sqrt{\breve{h}} + \sqrt{\breve{h}}[2x(4M - \chi^{CD}{}_{||CD}) + O_{\ln^* x}(x^2)] ,$$

which, together with

$$A = \frac{4\pi}{x^2} + O(x^3) , \quad \sqrt{\frac{A}{16\pi}} = \frac{1}{2x} + O(x^4) ,$$

yields

$$p_H(a, \mathscr{S}_x) = -\frac{1}{16\pi}\int_{\mathscr{S}_x} v(a)[\sqrt{\breve{h}}(4M - \chi^{CD}{}_{||CD}) + O_{\ln^* x}(x)]\mathrm{d}^2x .$$

Passing to the limit $x \to 0$ the $\chi^{CD}{}_{||CD}$ term integrates out to zero, leading to equality of the Trautman-Bondi four-momentum $p_{\mathrm{TB}}(a, \mathscr{S})$ with the limit of $p_H(a, \mathscr{S}_x)$ defined above.

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A Proof of (2.11)

From

$$2D_l(Y^k P^l{}_k) = P^{kl}\mathring{A}_{kl} + Y^k J_k(g, K) + P^{kl}\left(2V\mathring{K}_{kl} + \mathscr{L}_Y e_{kl}\right) \tag{A.1}$$

we get

$$\begin{aligned} VR_g + 2D_l(Y^k P^l{}_k) \quad = \quad & V\rho(g, K) + Y^k J_k(g, K) + P^{kl}\mathring{A}_{kl} \\ & -V\left(\mathring{P}^{mn}\mathring{P}_{mn} - \frac{1}{n-1}(\mathrm{tr}_b\mathring{P})^2\right) + \mathring{P}^{kl}\mathscr{L}_Y e_{kl} \\ & +Ve_{kl}\left(2\mathring{P}^{ml}\mathring{P}_{mk} - \frac{2}{n-1}\mathring{P}^{kl}\mathrm{tr}_b\mathring{P}\right) + Q_2 , \end{aligned} \tag{A.2}$$

where, here and below, we use the symbol Q_i to denote terms which are quadratic or higher order in e and $P - \mathring{P}$. Moreover, we have

$$\begin{aligned} \mathring{P}^{kl}\mathscr{L}_Y e_{kl} &= -e_{kl}\mathscr{L}_Y\mathring{P}^{kl} + D_m(Y^m\mathring{P}^{kl}e_{kl}) - \mathring{P}^{kl}e_{kl}D_mY^m \ , & \text{(A.3)} \\ D_mY^m &= \frac{1}{2}g^{kl}(\mathring{A}_{kl} + \mathscr{L}_Y e_{kl}) + V(g^{kl} - b^{kl})\mathring{K}_{kl} + V\mathrm{tr}_g\mathring{K} \\ &=: \frac{1}{2}g^{kl}\mathring{A}_{kl} + \frac{V}{n-1}\mathrm{tr}_b\mathring{P} + L_1 \ , & \text{(A.4)} \\ 2D_l(Y^k\mathring{P}^l{}_k) &= \mathring{P}^{kl}\mathring{A}_{kl} + Y^kJ_k(b,\mathring{K}) - 2V\left(\mathring{P}^{mn}\mathring{P}_{mn} - \frac{1}{n-1}(\mathrm{tr}_b\mathring{P})^2\right) \\ &\quad +D_l\left(Y^k\mathring{P}^l{}_kb^{mn}e_{mn}\right) - b^{mn}e_{mn}\mathring{D}_l(Y^k\mathring{P}^l{}_k) + Q_1 \ . & \text{(A.5)} \end{aligned}$$

(The term L_1 in (A.4) is a linear remainder term which, however, will give a quadratic contribution in equations such as (A.7) below.) Using

$$\begin{aligned} -\mathring{D}_l(Y^k\mathring{P}^l{}_k) &= V\left(\mathring{P}^{mn}\mathring{P}_{mn} - \frac{1}{n-1}(\mathrm{tr}_b\mathring{P})^2\right) - \frac{1}{2}\mathring{P}^{kl}\mathring{A}_{kl} - \frac{1}{2}Y^kJ_k(b,\mathring{K}) \\ &= VR_b - V\rho(b,\mathring{K}) - \frac{1}{2}\mathring{P}^{kl}\mathring{A}_{kl} - \frac{1}{2}Y^kJ_k(b,\mathring{K}) \qquad \text{(A.6)} \end{aligned}$$

and (A.2)-(A.5) we are led to

$$\begin{aligned} V(R_g - R_b) &= V\left(\rho(g,K) - \rho(b,\mathring{K})\right) + Y^k\left(J_k(g,K) - J_k(b,\mathring{K})\right) - \frac{\partial_l\mathbb{V}^l}{\sqrt{\det g}} \\ &\quad +V\left[2\mathring{P}^{ml}\mathring{P}_m{}^k - \frac{3}{n-1}\mathrm{tr}_b\mathring{P}\mathring{P}^{kl} - \left(\mathring{P}^{mn}\mathring{P}_{mn} - \frac{1}{n-1}(\mathrm{tr}_b\mathring{P})^2\right)b^{kl}\right]e_{kl} \\ &\quad -e_{kl}\mathscr{L}_Y\mathring{P}^{kl} + \frac{1}{2}e_{kl}\left[b^{kl}\left(\mathring{P}^{mn}\mathring{A}_{mn} + Y^mJ_m(b,\mathring{K})\right) - b^{mn}\mathring{A}_{mn}\mathring{P}^{kl}\right] \\ &\quad +(P^{kl} - \mathring{P}^{kl})\mathring{A}_{kl} + Q_3 \ , \qquad \text{(A.7)} \end{aligned}$$

where

$$\mathbb{V}^l(Y) := 2\sqrt{\det g}\left[(P^l{}_k - \mathring{P}^l{}_k)Y^k - \frac{1}{2}Y^l\mathring{P}^{mn}e_{mn} + \frac{1}{2}Y^k\mathring{P}^l{}_kb^{mn}e_{mn}\right] \ . \qquad \text{(A.8)}$$

Inserting Equation (2.2) into (A.7) one obtains the following counterpart of Equation (2.2)

$$\begin{aligned} \partial_i\left(\mathbb{U}^i(V) + \mathbb{V}^i(Y)\right) &= \sqrt{\det g}\ \left[V\left(\rho(g,K) - \rho(b,\mathring{K})\right)\right. \\ &\quad \left.+Y^k\sqrt{\det g}\left(J_k(g,K) - J_k(b,\mathring{K})\right) + s' + Q''\right] \ , \qquad \text{(A.9)} \end{aligned}$$

where Q'' contains terms which are quadratic in the deviation of g from b and its derivatives, and in the deviations of K from $\mathring{K}$, while s', obtained by collecting all terms linear or linearised in e_{ij}, except for those involving ρ and J, reads

$$\begin{aligned} s' &= (\mathring{S}^{kl} + \mathring{B}^{kl})e_{kl} + (P^{kl} - \mathring{P}^{kl})\mathring{A}_{kl} \ , \\ \mathring{B}^{kl} &:= \frac{1}{2}\left[b^{kl}\mathring{P}^{mn}\mathring{A}_{mn} - b^{mn}\mathring{A}_{mn}\mathring{P}^{kl}\right] \ . \qquad \text{(A.10)} \end{aligned}$$

B 3 + 1 charge integrals *vs* the Freud integrals

In this appendix we wish to show that under the boundary conditions of Section 3 the numerical value of the Riemannian charge integrals coincides with that of the Hamiltonians derived in a space-time setting in [13, 19]:

$$H^\mu \equiv p^\mu_{\alpha\beta}\mathscr{L}_X \mathfrak{g}^{\alpha\beta} - X^\mu L = \partial_\alpha \mathbb{W}^{\mu\alpha} + \frac{1}{8\pi}\mathscr{G}^\mu{}_\beta(\Lambda)X^\beta \ , \tag{B.1}$$

$$\mathbb{W}^{\nu\lambda} = \mathbb{W}^{\nu\lambda}{}_\beta X^\beta - \frac{1}{8\pi}\sqrt{|\det g_{\rho\sigma}|}g^{\alpha[\nu}\delta^{\lambda]}_\beta X^\beta{}_{;\alpha} \ , \tag{B.2}$$

$$\begin{aligned}\mathbb{W}^{\nu\lambda}{}_\beta &= \frac{2|\det {}^4b_{\mu\nu}|}{16\pi\sqrt{|\det g_{\rho\sigma}|}}g_{\beta\gamma}\left(\frac{|\det g_{\rho\sigma}|}{|\det {}^4b_{\mu\nu}|}g^{\gamma[\lambda}g^{\nu]\kappa}\right)_{;\kappa} \\ &= 2\mathfrak{g}_{\beta\gamma}\left(\mathfrak{g}^{\gamma[\lambda}\mathfrak{g}^{\nu]\kappa}\right)_{;\kappa} \\ &= 2\mathfrak{g}^{\mu[\nu}p^{\lambda]}_{\mu\beta} - 2\delta^{[\nu}_\beta p^{\lambda]}_{\mu\sigma}\mathfrak{g}^{\mu\sigma} - \frac{2}{3}\mathfrak{g}^{\mu[\nu}\delta^{\lambda]}_\beta p^\sigma_{\mu\sigma} \ ,\end{aligned} \tag{B.3}$$

where a semi-colon denotes the covariant derivative of the metric ${}^4b$, square brackets denote anti-symmetrisation (with a factor of 1/2 when two indices are involved), as before $\mathfrak{g}_{\beta\gamma} \equiv (\mathfrak{g}^{\alpha\sigma})^{-1} = 16\pi g_{\beta\gamma}/\sqrt{|\det g_{\rho\sigma}|}$. Further, $\mathscr{G}^\alpha{}_\beta(\Lambda)$ is the Einstein tensor density eventually shifted by a cosmological constant,

$$\mathscr{G}^\alpha{}_\beta(\Lambda) := \sqrt{|\det g_{\rho\sigma}|}\left(R^\alpha{}_\beta - \frac{1}{2}g^{\mu\nu}R_{\mu\nu}\delta^\alpha_\beta + \Lambda\delta^\alpha_\beta\right) \tag{B.4}$$

(equal, of course, to the energy-momentum tensor density of the matter fields in models with matter, and vanishing in vacuum), while

$$\begin{aligned}p^\lambda_{\mu\nu} &= \frac{1}{2}\mathfrak{g}_{\mu\alpha}\mathfrak{g}^{\lambda\alpha}{}_{;\nu} + \frac{1}{2}\mathfrak{g}_{\nu\alpha}\mathfrak{g}^{\lambda\alpha}{}_{;\mu} - \frac{1}{2}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\sigma\mu}\mathfrak{g}_{\rho\nu}\mathfrak{g}^{\sigma\rho}{}_{;\alpha} \\ &\quad + \frac{1}{4}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\mu\nu}\mathfrak{g}_{\sigma\rho}\mathfrak{g}^{\sigma\rho}{}_{;\alpha} \ ,\end{aligned} \tag{B.5}$$

where by $\mathfrak{g}_{\mu\nu}$ we denote the matrix inverse to $\mathfrak{g}^{\mu\nu}$, and

$$L := \mathfrak{g}^{\mu\nu}\left[\left(\Gamma^\alpha_{\sigma\mu} - B^\alpha_{\sigma\mu}\right)\left(\Gamma^\sigma_{\alpha\nu} - B^\sigma_{\alpha\nu}\right) - \left(\Gamma^\alpha_{\mu\nu} - B^\alpha_{\mu\nu}\right)\left(\Gamma^\sigma_{\alpha\sigma} - B^\sigma_{\alpha\sigma}\right) + r_{\mu\nu}\right] \ . \tag{B.6}$$

Finally, $B^\alpha_{\mu\nu}$ is the connection of the background metric, and $r_{\mu\nu}$ its Ricci tensor – zero in our case. For typesetting reasons in the remainder of this appendix we write η for ${}^4b$, while the symbol b will be reserved for the space-part of ${}^4b$ and its inverse. While η is flat, the reader should not assume that it takes the usual diagonal form. Let us denote by $\approx$ the linearisation at ${}^4b \equiv \eta$; we find

$$16\pi\mathbb{W}^{0l}{}_0 \approx \sqrt{-\eta}\left(e^l{}_m{}^{;m} - e^m{}_m{}^{;l}\right) \ , \tag{B.7}$$

$$16\pi\mathbb{W}^{0l}{}_k \approx \sqrt{-\eta}\left[\delta^l{}_k\left(e^m{}_m{}^{;0} - e^0{}_m{}^{;m}\right) + e^0{}_k{}^{;l} - e^l{}_k{}^{;0}\right] \ , \tag{B.8}$$

leading to

$$16\pi\mathbb{W}^{0l}{}_\beta n^\beta \approx \sqrt{b}\left(b^{kn}b^{lm} - b^{km}b^{ln}\right)e_{mk;n} \ , \tag{B.9}$$

$$16\pi \mathbb{W}^{0l}{}_k Y^k \approx \sqrt{-\eta}\left(Y^l b^{km} - Y^k b^{lm}\right)\left(e_{mk}{}^{;0} - e^0{}_{k;m}\right) \ . \tag{B.10}$$

Here

$$e_{\mu\nu} := g_{\mu\nu} - \eta_{\mu\nu} \ ,$$

and in the linearised expressions of this appendix all space-time indices are raised and lowered with η. Similarly,

$$\begin{aligned}
&\left[\sqrt{-g}(g^{l\alpha}\eta^{0\mu} - g^{0\alpha}\eta^{l\mu}) - \sqrt{-\eta}(\eta^{l\alpha}\eta^{0\mu} - \eta^{0\alpha}\eta^{l\mu})\right] X_{\mu;\alpha} \approx \\
&\sqrt{-\eta}\left[(\eta^{l\alpha}\eta^{0\mu} - \eta^{0\alpha}\eta^{l\mu})\frac{1}{2}e^\sigma{}_\sigma - e^{l\alpha}\eta^{0\mu} + e^{0\alpha}\eta^{l\mu}\right] X_{\mu;\alpha} = \\
&\sqrt{-\eta}\left(b^{mn}e_{mn}b^{lk}X^0{}_{;k} - b^{ln}b^{mk}e_{mn}X^0{}_{;k} + e^0{}_m b^{mk}b^{ln}X_{n;k}\right) \ .
\end{aligned} \tag{B.11}$$

We also have

$$X^0{}_{;k} = \frac{1}{\mathring{N}}\left(V_{,k} - Y^l \mathring{K}_{lk}\right) \ , \tag{B.12}$$

$$X_{l;k} = b_{lm}\mathring{D}_k Y^m - V\mathring{K}_{lk} \ , \tag{B.13}$$

and we have of course assumed that

$$X = Vn + Y$$

is a background Killing vector field, with Y tangent to the hypersurface of interest, so that

$$b_{lm}\mathring{D}_k Y^m + b_{km}\mathring{D}_l Y^m = 2V\mathring{K}_{lk} \ . \tag{B.14}$$

We recall that

$$n^\mu = -\frac{\eta^{0\mu}}{\sqrt{-\eta^{00}}} \ ,$$

which gives $X^0 = \frac{V}{\mathring{N}}$, $X^k = Y^k - \frac{V}{\mathring{N}}\mathring{N}^k$, with the lapse and shift given by the formulae $\mathring{N} = \frac{1}{\sqrt{-\eta^{00}}}$, $\mathring{N}^k = -\frac{\eta^{0k}}{\eta^{00}}$. We will also need the $3+1$ decomposition of the Christoffel symbols $B^\alpha_{\beta\gamma}$ of the four-dimensional background metric in terms of those, denoted by $\mathring{\Gamma}^m{}_{kl}(b)$, associated with the three-dimensional one:

$$B^m{}_{kl} = \mathring{\Gamma}^m{}_{kl}(b) + \frac{\mathring{N}^m}{\mathring{N}}\mathring{K}_{lk} \ , \quad B^0{}_{0k} = \partial_k \log \mathring{N} - \frac{\mathring{N}^l}{\mathring{N}}\mathring{K}_{lk} \ ,$$

$$B^0{}_{kl} = -\frac{1}{\mathring{N}}\mathring{K}_{lk} \ , \quad B^l{}_{k0} = \mathring{D}_k\mathring{N}^l - \frac{\mathring{N}^l}{\mathring{N}}\mathring{D}_k\mathring{N} - \mathring{N}\mathring{K}^l{}_k + \frac{\mathring{N}^l}{\mathring{N}}\mathring{N}^m\mathring{K}_{mk} \ .$$

Linearising the equation for K_{ij} one finds

$$\delta K_{kl} := -\frac{1}{2}\mathring{N}\left(e^0{}_{k;l} + e^0{}_{l;k} - e_{kl}{}^{;0}\right) - \frac{1}{2}(\mathring{N})^2 e^{00}\mathring{K}_{kl} \ , \tag{B.15}$$

where the relevant indices have been raised with the space-time background metric. This leads to the following formula for the linearised ADM momentum:

$$\delta P^l{}_k := \delta^l{}_k b^{mn}\delta K_{mn} - b^{ml}\delta K_{mk} + \left(b^{li}\mathring{K}^j{}_k - \delta^l{}_k\mathring{K}^{ij}\right)e_{ij} \ . \tag{B.16}$$

A rather lengthy calculation leads then to

$$\begin{aligned}
&\left[\sqrt{-g}(g^{l\alpha}\eta^{0\mu}-g^{0\alpha}\eta^{l\mu})-\sqrt{-\eta}(\eta^{l\alpha}\eta^{0\mu}-\eta^{0\alpha}\eta^{l\mu})\right]X_{\mu;\alpha}\\
&+16\pi V\mathbb{W}^{0l}{}_{\beta}n^{\beta}+16\pi\mathbb{W}^{0l}{}_{k}Y^{k}\approx\\
&\sqrt{b}Vb^{ij}b^{lm}(\mathring{D}_i e_{jm}-\mathring{D}_m e_{ji})+\sqrt{b}e_{mn}(b^{mn}b^{lk}-b^{ln}b^{mk})\mathring{D}_kV\\
&+\sqrt{b}\mathring{D}_k\left[\mathring{N}e^0{}_m(Y^lb^{km}-Y^kb^{lm})\right]+2\sqrt{b}(Y^lb^{km}-Y^kb^{lm})\delta K_{km}\\
&+\sqrt{b}e_{mn}\left[2Y^k\mathring{K}_k{}^mb^{ln}-Y^k\mathring{K}_k{}^lb^{mn}-Y^l\mathring{K}^{mn}\right]=\\
&\sqrt{b}Vb^{ij}b^{lm}(\mathring{D}_i e_{jm}-\mathring{D}_m e_{ji})+\sqrt{b}e_{mn}(b^{mn}b^{lk}-b^{ln}b^{mk})\mathring{D}_kV\\
&+\partial_k\left[\sqrt{b}\mathring{N}e^0{}_m(Y^lb^{km}-Y^kb^{lm})\right]+2\sqrt{b}Y^k\delta P^l{}_k\\
&+\sqrt{b}e_{mn}\left(\mathring{P}^l{}_kY^kb^{mn}-Y^l\mathring{P}^{mn}\right)\ .
\end{aligned}\tag{B.17}$$

The integral over a sphere of (B.17) coincides with the linearisation of the integral over a sphere of $\mathbb{U}^i+\mathbb{V}^i$, as the difference between those expressions is a complete divergence.

Let us finally show that the linearised expression above does *not* reproduce the Trautman-Bondi mass. It is most convenient to work directly in space-time Bondi coordinates (u,x,x^A) rather than in coordinates adapted to $\mathscr{S}$. We consider a metric of the Bondi form (4.1)–(4.2) with the asymptotic behavior (C.11)–(C.13); we have

$$e_{00}=1-xV+x^{-2}h_{AB}U^AU^B=2Mx+O_{\ln^* x}(x^2)\ ,$$

$$e_{03}=x^{-2}(e^{2\beta}-1)=O(1)\ ,$$

$$e_{0A}=-x^{-2}h_{AB}U^B=\frac{1}{2}\chi_A{}^B{}_{||B}+O(x)\ ,$$

$$e_{AB}=x^{-2}(h_{AB}-\breve{h}_{AB})=x^{-1}\chi_{AB}+O(1)\ ,$$

$$e_{33}=e_{3A}=0\ .$$

(Recall that $\breve{h}$ denotes the standard round metric on S^2.) For translational ${}^4b$-Killing vector fields X^μ we have $X_{\mu;\nu}=0$, hence

$$\mathbb{W}^{0l}=\mathbb{W}^{0l}{}_{\mu}X^\mu\ .$$

Now, taking into account (B.7) and (B.8), the linearised Freud superpotential takes the form:

$$\begin{aligned}
16\pi\mathbb{W}^{03}\ =\ &\sqrt{-\eta}\left[(e^A{}_A{}^{;3}-e^3{}_A{}^{;A})X^0+(e^0{}_A{}^{;A}-e^A{}_A{}^{;0})X^3\right.\\
&\left.+(e^3{}_A{}^{;0}-e^0{}_A{}^{;3})X^A\right]\ ,
\end{aligned}\tag{B.18}$$

The following formulae for the non-vanishing ${}^4B^\sigma{}_{\beta\gamma}$'s for the flat metric (C.5) are useful when working out (B.18):

$${}^4B^u{}_{AB}=x^{-1}\breve{h}_{AB},\quad {}^4B^x{}_{AB}=x\breve{h}_{AB},\quad {}^4B^A{}_{xB}=-x^{-1}\delta^A_B,\tag{B.19}$$

$$ {}^4B^x{}_{xx} = -2x^{-1}, \quad {}^4B^A{}_{BC} = \Gamma^A{}_{BC}(\check h) \ . \tag{B.20} $$

The Killing field corresponding to translations in spacetime can be characterised by a function κ which is a linear combination of the $\ell = 0$ and $\ell = 1$ spherical harmonics:

$$ X^0 = \kappa \ , \quad X^3 = -\frac{1}{2}x^2 \triangle \kappa \ , \quad X^A = -x\check h^{AB}\kappa_{,B} \ . $$

With some work one finds the following formula for the linearised superpotential for time translations ($\kappa = 1$)

$$ \begin{aligned} 16\pi \mathbb{W}^{03} &= \sqrt{-\eta}(e^A{}_A{}^{;3} - e^3{}_A{}^{;A}) \\ &= \sin\theta \left[4M - \frac{1}{2}\chi^{AB}{}_{||AB} + \chi^{CD}\partial_u \chi_{CD} + O_{\ln^* x}(x) \right] \ . \end{aligned} $$

The resulting integral reproduces the Trautman-Bondi mass if and only if the $\chi^{CD}\partial_u\chi_{CD}$ term above gives a zero contribution after being integrated upon; in general this will not be the case.

C Proof of Lemma 5.8

The object of this appendix is to calculate, for large R, the boundary integrand that appears in the integral identity (5.12), for a class of hyperboloidal initial data sets made precise in Theorem 5.4. We consider a conformally compactifiable polyhomogeneous initial data set $(\mathscr{S}, g, K)$, such that

$$ \mathrm{tr}_g K \text{ is constant to second order at } \partial\mathscr{S}. \tag{C.1} $$

In $(\mathscr{M}, {}^4g)$ we can always [18] introduce a Bondi coordinate system (u, x, x^A) such that $\mathscr{S}$ is given by an equation

$$ u = \alpha(x, x^A) \ , \quad \text{with} \quad \alpha(0, x^A) = 0 \ , \ \alpha_{,x}(0, x^A) > 0 \ , \tag{C.2} $$

where α is polyhomogeneous. It follows from [15, Equation (C.83)] that (C.1) is equivalent to

$$ \alpha_{,xx}|_{x=0} = 0 \ . \tag{C.3} $$

(Throughout this section we will make heavy use of the formulae of [15, Appendix C] without necessarily indicating this fact.) Polyhomogeneity implies then

$$ \alpha_{,A} = O_{\ln^* x}(x^3) \ , \tag{C.4} $$

where we use the symbol $f = O_{\ln^* x}(x^p)$ to denote the fact that there exists $N \in \mathbb{N}$ and a constant C such that

$$ |f| \le Cx^p(1 + |\ln x|^N) \ . $$

We also assume that this behaviour is preserved under differentiation in the obvious way:

$$ |\partial_x f| \le Cx^{p-1}(1 + |\ln x|^N) \ , \quad |\partial_A f| \le Cx^p(1 + |\ln x|^N) \ , $$

similarly for higher derivatives. The reason for imposing (C.1) is precisely Equation (C.3). For more general α's the expansions below acquire many further terms, and we have not attempted to carry through the (already scary) calculations below if (C.3) does not hold.

Alternatively, one can start from a space-time with a polyhomogeneous $\mathscr{I}^+$ and choose any space-like hypersurface $\mathscr{S}$ so that (C.2)–(C.3) hold. Such an approach can be used to study the positivity properties of the Trautman-Bondi mass, viewed as a function on the set of space-times rather than a function on the set of initial data sets.

We use the Bondi coordinates to define the background ${}^4b$:

$$ {}^4b = -\mathrm{d}u^2 + 2x^{-2}\mathrm{d}u\mathrm{d}x + x^{-2}\breve{h}_{AB}\mathrm{d}x^A\mathrm{d}x^B \;, \tag{C.5} $$

so that the components of the inverse metric read

$$ {}^4b^{ux} = x^2 \qquad {}^4b^{xx} = x^4 \qquad {}^4b^{AB} = x^2\breve{h}^{AB} \;. $$

The metric b induced on $\mathscr{S}$ takes thus the form

$$ \begin{aligned} b \;\; = \;\; & x^{-2}\Big[\Big(\breve{h}_{AB} + O_{\ln^* x}(x^8)\Big)\,\mathrm{d}x^A\mathrm{d}x^B + 2(1-x^2\alpha_{,x})\alpha_{,A}\mathrm{d}x\mathrm{d}x^A \\ & +2\alpha_{,x}\left(1-\frac{1}{2}x^2\alpha_{,x}\right)(\mathrm{d}x)^2\Big] \;. \end{aligned} \tag{C.6} $$

Let $\tilde{e}^a$ be a local orthonormal co-frame for the unit round metric $\breve{h}$ on S^2 (outside of the south and north pole one can, *e.g.*, use $\tilde{e}^1 = \mathrm{d}\theta$, $\tilde{e}^2 = \sin\theta\mathrm{d}\varphi$), we set

$$ e^a := x^{-1}\tilde{e}^a \;, \quad e^3 := x^{-1}\tilde{e}^3 \;, $$

where

$$ \tilde{e}^3 := \sqrt{\alpha_{,x}\left(2-x^2\alpha_{,x}\right)}\mathrm{d}x + \frac{1-x^2\alpha_{,x}}{\sqrt{\alpha_{,x}\left(2-x^2\alpha_{,x}\right)}}\alpha_{,A}\mathrm{d}x^A \;. $$

Assuming (C.3)-(C.4), it follows that the co-frame $\tilde{e}^i$ is close to being orthonormal for b:

$$ b = x^{-2}\left[\tilde{e}^1\tilde{e}^1 + \tilde{e}^2\tilde{e}^2 + \tilde{e}^3\tilde{e}^3 + O_{\ln^* x}(x^6)\right] = e^1e^1 + e^2e^2 + e^3e^3 + O_{\ln^* x}(x^4) \;. \tag{C.7} $$

Here and elsewhere, an equality $f = O_{\ln^* x}(x^p)$ *for a tensor field* f *means that the components of* f *in the coordinates* (x, x^A) *are* $O_{\ln^* x}(x^p)$.

Recall that in Bondi-Sachs coordinates (u, x, x^A) the space-time metric takes the form:

$$ {}^4g = -xV\mathrm{e}^{2\beta}\mathrm{d}u^2 + 2\mathrm{e}^{2\beta}x^{-2}\mathrm{d}u\mathrm{d}x + x^{-2}h_{AB}\left(\mathrm{d}x^A - U^A\mathrm{d}u\right)\left(\mathrm{d}x^B - U^B\mathrm{d}u\right) \;. \tag{C.8} $$

This leads to the following form of the metric g induced on $\mathscr{S}$

$$\begin{aligned} x^2 g &= 2\left[(h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta})\alpha_{,x}\alpha_{,C} + \mathrm{e}^{2\beta}\alpha_{,C} - h_{CB}U^B\alpha_{,x}\right]\mathrm{d}x^C\mathrm{d}x \\ &\quad + \left[(h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta})\alpha_{,D}\alpha_{,C} - 2h_{CB}U^B\alpha_{,D} + h_{CD}\right]\mathrm{d}x^C\mathrm{d}x^D \\ &\quad + \left[\left(h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta}\right)(\alpha_{,x})^2 + 2\mathrm{e}^{2\beta}\alpha_{,x}\right](\mathrm{d}x)^2 \end{aligned} \tag{C.9}$$

Let γ_{ij} be defined as

$$x^2 g = \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B + 2\gamma_{xA}\mathrm{d}x^A\mathrm{d}x + \gamma_{xx}(\mathrm{d}x)^2\ , \tag{C.10}$$

so that

$$\begin{aligned} \gamma_{CD} &:= h_{CD} - 2\alpha_{,(D}h_{C)B}U^B + (h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta})\alpha_{,D}\alpha_{,C}\ , \\ \gamma_{xC} &:= (h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta})\alpha_{,x}\alpha_{,C} + \mathrm{e}^{2\beta}\alpha_{,C} - h_{CB}U^B\alpha_{,x}\ , \\ \gamma_{xx} &:= \left[\left(h_{AB}U^AU^B\mathrm{e}^{-2\beta} - x^3V\right)\alpha_{,x} + 2\right]\mathrm{e}^{2\beta}\alpha_{,x}\ . \end{aligned}$$

If we assume that $(\mathscr{S}, g, K)$ is polyhomogeneous and conformally $C^1 \times C^0$-compactifiable, it follows that

$$h_{AB} = \breve{h}_{AB}(1 + \frac{x^2}{4}\chi^{CD}\chi_{CD}) + x\chi_{AB} + x^2\zeta_{AB} + x^3\xi_{AB} + O_{\ln^* x}(x^4)\ \ ,$$

where ζ_{AB} and ξ_{AB} are polynomials in $\ln x$ with coefficients which smoothly depend upon the x^A's. By definition of the Bondi coordinates we have $\det h = \det \breve{h}$, which implies

$$\breve{h}^{AB}\chi_{AB} = \breve{h}^{AB}\zeta_{AB} = 0\ .$$

Further,

$$\beta = -\frac{1}{32}\chi^{CD}\chi_{CD}x^2 + Bx^3 + O_{\ln^* x}(x^4)\ , \tag{C.11}$$

$$xV = 1 - 2Mx + O_{\ln^* x}(x^2)\ , \tag{C.12}$$

$$h_{AB}U^B = -\frac{1}{2}\chi_A{}^B{}_{||B}x^2 + u_Ax^3 + O_{\ln^* x}(x^4)\ , \tag{C.13}$$

where B and u_A are again polynomials in $\ln x$ with smooth coefficients depending upon the x^A's, while $||$ denotes covariant differentiation with respect to the metric $\breve{h}$. This leads to the following approximate formulae

$$(h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta})\alpha_{,D} = O_{\ln^* x}(x^5)\ , \quad h_{CB}U^B\alpha_{,D} = O_{\ln^* x}(x^5)\ ,$$

$$\gamma_{xA} = \alpha_{,A} + \alpha_{,x}\left[\frac{1}{2}\chi_A{}^B{}_{||B}x^2 - u_Ax^3\right] + O_{\ln^* x}(x^4)\ , \tag{C.14}$$

$$\sqrt{\gamma_{xx}} = \sqrt{2\alpha_{,x}}\left[1 - \frac{1}{4}(\alpha_{,x} + \frac{1}{8}\chi^{CD}\chi_{CD})x^2 + \left(\frac{1}{2}\alpha_{,x}M + B\right)x^3\right] + O_{\ln^* x}(x^4)\ . \tag{C.15}$$

Let

$$h_{ab} = h(\tilde{e}_a, \tilde{e}_b)\ ,$$

where $\tilde{e}_a$ is a basis of vectors tangent to S^2 dual to $\tilde{e}^a$, and let $\mu^a{}_b$ be the symmetric root of h_{ab},

$$\mu^a{}_c h_{ab} \mu^b{}_d = \delta_{cd} \,,$$

or, in matrix notation,

$$ {}^t\mu h \mu = id \,, \tag{C.16}$$

where ${}^t\mu^c{}_a = \mu^a{}_c$ stands for the transpose of μ. Let $\tilde{f}^a$, $a = 1, 2$, be the field of local orthonormal co-frames for the metric h_{AB} defined by the formula

$$\tilde{e}^a = \mu^a{}_b \tilde{f}^b \,. \tag{C.17}$$

We set

$$\tilde{f}^3 := \sqrt{\gamma_{xx}} \mathrm{d}x + \frac{\gamma_{xA}}{\sqrt{\gamma_{xx}}} \mathrm{d}x^A \,, \tag{C.18}$$

so that

$$x^2 g = \tilde{f}^1 \tilde{f}^1 + \tilde{f}^2 \tilde{f}^2 + \tilde{f}^3 \tilde{f}^3 + O_{\ln^* x}(x^4) \,. \tag{C.19}$$

There exists a matrix $M^k{}_l$ such that

$$\tilde{e}^k = M^k{}_l \tilde{f}^l \,.$$

Now,

$$\tilde{e}^3 = (1 - x^2 a) \tilde{f}^3 - x^2 b_a \mu^a{}_b \tilde{f}^b + O_{\ln^* x}(x^4) \,, \tag{C.20}$$

with

$$a := -\frac{1}{32} \chi^{CD} \chi_{CD} + \left(\frac{1}{2} \alpha_{,x} M + B \right) x \,,$$

$$b_a \tilde{e}^a = b_A dx^A := \sqrt{\frac{\alpha_{,x}}{2}} \left[\frac{1}{2} \chi_A{}^B{}_{||B} - u_A x \right] dx^A \,.$$

The matrix M is easily calculated to be

$$M^3{}_3 = 1 - x^2 a + O_{\ln^* x}(x^4) \,, \quad M^3{}_b = -x^2 b_a \mu^a{}_b + O_{\ln^* x}(x^4) \,, \tag{C.21}$$

$$M^a{}_3 = 0 \,, \quad M^a{}_b = \mu^a{}_b \,. \tag{C.22}$$

Since

$$e^k = x^{-1} \tilde{e}^k \,, \quad f^k = x^{-1} \tilde{f}^k \,,$$

it follows that

$$e^k = M^k{}_l f^l \,, \quad f_l = M^k{}_l e_k \,,$$

where f_l and e_k stand for frames dual to f^i and e^i. If we choose

$$\tilde{e}^1 = \mathrm{d}\theta \,, \quad \tilde{e}^2 = \sin\theta \mathrm{d}\varphi \,,$$

then

$$\tilde{e}^3 := \sqrt{\alpha_{,x} (2 - x^2 \alpha_{,x})} \mathrm{d}x + \frac{1 - x^2 \alpha_{,x}}{\sqrt{\alpha_{,x} (2 - x^2 \alpha_{,x})}} \alpha_{,A} \mathrm{d}x^A \,,$$

$$\tilde{e}_1 = \frac{\partial}{\partial\theta} - \alpha_{,\theta} \frac{1 - x^2 \alpha_{,x}}{\alpha_{,x} (2 - x^2 \alpha_{,x})} \frac{\partial}{\partial x} \,,$$

$$\tilde{e}_2 = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\varphi} - \alpha_{,\varphi}\frac{1-x^2\alpha_{,x}}{\alpha_{,x}\left(2-x^2\alpha_{,x}\right)}\frac{\partial}{\partial x}\right] ,$$

$$\tilde{e}_3 = \frac{1}{\sqrt{\alpha_{,x}\left(2-x^2\alpha_{,x}\right)}}\frac{\partial}{\partial x} ,$$

$$e_k = x\tilde{e}_k , \quad f_k = x\tilde{f}_k .$$

We also have the relation

$$e_i = (M^{-1})^k{}_i f_k ,$$

with

$$(M^{-1})^3{}_3 = \frac{1}{M^3{}_3} = 1 + x^2 a + O_{\ln^* x}(x^4) , \quad (M^{-1})^a{}_b = (\mu^{-1})^a{}_b , \tag{C.23}$$

$$(M^{-1})^a{}_3 = 0 , \quad (M^{-1})^3{}_b = -\frac{M^3{}_a(\mu^{-1})^a{}_b}{M^3{}_3} = x^2 b_b + O_{\ln^* x}(x^4) . \tag{C.24}$$

Consider, now, the integrand in (5.12):

$$B^3 = - < \psi, \nabla^3\psi + \gamma^3\gamma^i\nabla_i\psi > = - < \psi, \gamma^3\gamma^a\nabla_a\psi > ,$$

where the minus sign arises from the fact that we will use a g–orthonormal frame $\hat{f}_i$ in which $\hat{f}_3$ is *minus* the outer-directed normal to the boundary. Here ψ is assumed to be the restriction to $\mathscr{S}$ of a space-time covariantly constant spinor with respect to the background metric ${}^4b$:

$$\mathring{\nabla}\psi = 0 .$$

It follows that

$$\begin{aligned}\nabla_a\psi &= \hat{f}_a(\psi) + \left[-\frac{1}{2}K(\hat{f}_a,\hat{f}_j)\gamma^j\gamma_0 - \frac{1}{4}\omega_{ij}(\hat{f}_a)\gamma^i\gamma^j\right]\psi \qquad \text{(C.25)}\\ &= \left[-\frac{1}{2}\left(K(\hat{f}_a,\hat{f}_j) - \mathring{K}(\hat{f}_a,\hat{e}_j)\right)\gamma^j\gamma_0 + \frac{1}{4}\left(\mathring{\omega}_{ij}(\hat{f}_a) - \omega_{ij}(\hat{f}_a)\right)\gamma^i\gamma^j\right]\psi .\end{aligned}$$

This allows us to rewrite B^3 as

$$\begin{aligned}B^3 &= \frac{1}{2}\left(K(\hat{f}_a,\hat{f}_j) - \mathring{K}(\hat{f}_a,\hat{e}_j)\right) < \psi, \gamma^3\gamma^a\gamma^j\gamma_0\psi >\\ &\quad -\frac{1}{4}\left(\mathring{\omega}_{ij}(\hat{f}_a) - \omega_{ij}(\hat{f}_a)\right) < \psi, \gamma^3\gamma^a\gamma^i\gamma^j\psi >\\ &= \frac{1}{2}\left(K(\hat{f}_a,\hat{f}_j) - \mathring{K}(\hat{f}_a,\hat{e}_j)\right)\left(g^{j3}Y^a - g^{ja}Y^3\right)\\ &\quad +\frac{1}{2}\left(\mathring{\omega}_{3a}(\hat{f}_a) - \omega_{3a}(\hat{f}_a)\right) W ,\end{aligned} \tag{C.26}$$

where (W, Y^i) denotes the KID[8] associated to the spinor field ψ

$$W := < \psi, \psi > , \quad Y^k = < \psi, \gamma^k\gamma_0\psi > .$$

[8]To avoid a clash of notation with the Bondi function V we are using the symbol W for the normal component of the KID here.

We use the convention in which

$$\{\gamma^i, \gamma^j\} = -2\delta^{ij} \ ,$$

with γ^0 — anti-hermitian, and γ^i — hermitian. Because ψ is covariantly constant, W and Y^i satisfy the following equation

$$\partial_i W = K_{ij} Y^j \ .$$

Let $\hat{e}_i$ be an orthonormal frame for b; we will shortly see that we have the following asymptotic behaviors,

$$\mathring{\omega}_{3a}(\hat{f}_a) - \omega_{3a}(\hat{f}_a) = O(x^2) \ ,$$

$$K(\hat{f}_a, \hat{f}_j) - \mathring{K}(\hat{f}_a, \hat{e}_j) = O(x^2) \ ,$$
$$Y^k = O(x^{-1}) \ , \quad W = O(x^{-1}) \ ,$$

which determines the order to which various objects above have to be expanded when calculating B^3. In particular, these equations show that some non-obvious cancelations have to occur for the integral of B^3 to converge.

Since $\hat{e}_i$ is b–orthonormal, it holds that

$$2\mathring{\omega}_{klj} = b([\hat{e}_k, \hat{e}_l], \hat{e}_j) + b([\hat{e}_k, \hat{e}_j], \hat{e}_l) - b([\hat{e}_l, \hat{e}_j], \hat{e}_k) \ . \tag{C.27}$$

It follows from (C.7) that we can choose $\hat{e}_k$ so that

$$\hat{e}_k = (\delta_k^\ell + O_{\ln^* x}(x^6))e_\ell \ . \tag{C.28}$$

This choice leads to the following commutators:

$$[\hat{e}_1, \hat{e}_2] = -x\cot\theta\,\hat{e}_2 + O_{\ln^* x}(x^3)\hat{e}_1 + O_{\ln^* x}(x^3)\hat{e}_2 + O_{\ln^* x}(x^4)\hat{e}_3 \ , \tag{C.29}$$

$$\begin{aligned} [\hat{e}_a, \hat{e}_3] \quad &= \quad -\left[\alpha_{,x}(2 - x^2\alpha_{,x})\right]^{-1/2}\hat{e}_a - \left(\frac{\alpha_{,a}}{2\alpha_{,x}} + x\frac{\alpha_{,xa}}{4\alpha_x} + O_{\ln^* x}(x^4)\right)\hat{e}_3 \\ &\quad + O_{\ln^* x}(x^4)\hat{e}_1 + O_{\ln^* x}(x^4)\hat{e}_2 \ , \end{aligned} \tag{C.30}$$

$$[\hat{e}_i, \hat{e}_j] = c_{ij}{}^k \hat{e}_k \ , \tag{C.31}$$

$$c_{21}{}^1 = -c_{12}{}^1 = O_{\ln^* x}(x^3) \ , \tag{C.32}$$
$$c_{21}{}^2 = -c_{12}{}^2 = x\cot\theta + O_{\ln^* x}(x^3) \ , \tag{C.33}$$
$$c_{12}{}^3 = -c_{21}{}^3 = O_{\ln^* x}(x^4) \ , \tag{C.34}$$
$$c_{3a}{}^3 = -c_{a3}{}^3 = \tfrac{\alpha_{,a}}{2\alpha_{,x}} + x\tfrac{\alpha_{,xa}}{4\alpha_x} + O_{\ln^* x}(x^4) \ , \tag{C.35}$$
$$c_{3a}{}^b = -c_{a3}{}^b = \left[\alpha_{,x}(2 - x^2\alpha_{,x})\right]^{-1/2}\delta^b{}_a + O_{\ln^* x}(x^4) \ . \tag{C.36}$$

It follows from (C.19) that we can choose $\hat{f}_j$ so that

$$\hat{f}_j = (\delta_j^k + O_{\ln^* x}(x^4))f_k \ . \tag{C.37}$$

Let $\hat{M}$ be the transition matrix from the frame $\hat{e}_i$ to the frame $\hat{f}_j$,

$$\hat{f}_k = \hat{M}^\ell{}_k \hat{e}_\ell \ ,$$

if we define the functions $d_{ij}{}^k$ as

$$[\hat{f}_i, \hat{f}_j] = d_{ij}{}^k \hat{f}_k \ , \tag{C.38}$$

then we have

$$d_{ml}{}^k = \hat{M}^j{}_l \hat{M}^i{}_m c_{ij}{}^n (\hat{M}^{-1})^k{}_n + (\hat{M}^{-1})^k{}_j \hat{f}_m(\hat{M}^j{}_l) - (\hat{M}^{-1})^k{}_j \hat{f}_l(\hat{M}^j{}_m) \ . \tag{C.39}$$

Chasing through the definitions one also finds that

$$\hat{M}^i{}_j = M^i{}_j + O_{\ln^* x}(x^4) \ . \tag{C.40}$$

This leads to

$$d_{3aa} = M^3{}_3 c_{3aa} + O_{\ln^* x}(x^4) = 2\left[\alpha_{,x}(2 - x^2\alpha_{,x})\right]^{-1/2} M^3{}_3 + O_{\ln^* x}(x^4) \ .$$

Now,

$$2\mathring{\omega}_{klj} = c_{klj} + c_{kjl} - c_{ljk} \ , \tag{C.41}$$

$$2\omega_{klj} = d_{klj} + d_{kjl} - d_{ljk} \ , \tag{C.42}$$

$$\omega_{3ab} = d_{3(ab)} - \frac{1}{2} d_{ab3} \ , \tag{C.43}$$

$$\omega_{3aa} = d_{3aa} \ , \tag{C.44}$$

$$\mathring{\omega}_{3ab} = \left[\alpha_{,x}(2 - x^2\alpha_{,x})\right]^{-1/2} \delta_{ab} + O_{\ln^* x}(x^4) = O(1) \ , \tag{C.45}$$

$$\mathring{\omega}_{3a3} = O_{\ln^* x}(x^3) \ . \tag{C.46}$$

Using obvious matrix notation, from (C.16) we obtain

$$\mu = \mathrm{id} - \frac{x}{2}\chi + x^2 d + x^3 w + O_{\ln^* x}(x^4) \ , \tag{C.47}$$

where d and w are polynomials in $\ln x$ with coefficients which are x^A–dependent symmetric matrices. The condition that $\det \mu = 1$ leads to the relations

$$\mathrm{tr}\,\chi = 0 \ , \quad \mathrm{tr}\, d = \frac{1}{8}\mathrm{tr}\,\chi^2 \ , \quad \mathrm{tr}\, w = -\frac{1}{2}\mathrm{tr}\,(\chi d) \ .$$

Using $(\mu^{-1})^a{}_b \hat{f}_3(\mu^b{}_a) = \hat{f}_3(\det \mu) = 0$, the asymptotic expansions (C.33)-(C.36) together with (C.23)-(C.24), (C.28) and (C.37) one finds the following contribution to the first term $\frac{1}{2}\left(\mathring{\omega}_{3a}(\hat{f}_a) - \omega_{3a}(\hat{f}_a)\right) W$ in B^3:

$$\begin{aligned}
\mathring{\omega}_{3a}(\hat{f}_a) - \omega_{3a}(\hat{f}_a) &= \hat{M}^l{}_a \mathring{\omega}_{3al} - \omega_{3aa} \\
&= M^3{}_a \mathring{\omega}_{3a3} + (\mu^b{}_a - \delta^b{}_a)\mathring{\omega}_{3ab} + \mathring{\omega}_{3aa} - \omega_{3aa} + O_{\ln^* x}(x^4) \\
&= \frac{x^2}{\sqrt{2\alpha_{,x}}}\left[\underline{\frac{1}{16}\chi_{AB}\chi^{AB}} + x\left(M\alpha_{,x} + 2B - \frac{1}{2}\mathrm{tr}\,(\chi d)\right)\right] \\
&\quad + O_{\ln^* x}(x^4) \ .
\end{aligned} \tag{C.48}$$

The underlined term would give a diverging contribution to the integral of B^3 over the conformal boundary if it did not cancel out with an identical term from the K contribution, except when $\chi = 0$.

We choose now the Killing spinor ψ so that

$$W\mathring{n} + Y = \frac{\partial}{\partial t}$$

in the usual Minkowskian coordinates, where $\mathring{n}$ is the unit normal to $\mathscr{S}$. This leads to

$$W = \frac{x^{-1}}{\sqrt{2\alpha_{,x}}}\left[1 + O(x^2)\right] \;, \quad Y^x = \frac{1}{2\alpha_{,x}}\left[1 + O(x^2)\right] \;, \quad Y^A = O(x^2) \;,$$

$$Y^3 = \frac{x^{-1}}{\sqrt{2\alpha_{,x}}} + O(x) \;, \quad Y^a = O(x) \;.$$

In order to calculate the remaining terms in (C.26) we begin with

$$K(\hat{f}_a, \hat{f}_a) - \mathring{K}(\hat{f}_a, \hat{e}_a) = x^2 \mu^c{}_a \mu^d{}_a K(\tilde{e}_c, \tilde{e}_d) - x^2 \mu^c{}_a \mathring{K}(\tilde{e}_c, \tilde{e}_a) + O_{\ln^* x}(x^4) \;, \tag{C.49}$$

Using the formulae of [15, Appendix C.3] one finds

$$\mu^c{}_a \mu^d{}_a K(\tilde{e}_c, \tilde{e}_d) = h^{cd} K(\tilde{e}_c, \tilde{e}_d) = h^{AB} K_{AB} + O_{\ln^* x}(x^2) \;,$$

$$\Gamma^\omega{}_{AB} = \Gamma^u{}_{AB} - \alpha_{,x} \Gamma^x{}_{AB} + O_{\ln^* x}(x^3) \;, \tag{C.50}$$

$$\Gamma^u{}_{AB} = x^{-1} \mathrm{e}^{-2\beta}\left(h_{AB} - \frac{x}{2} h_{AB,x}\right) \;, \tag{C.51}$$

$$\Gamma^x{}_{AB} = -\frac{1}{2}\mathrm{e}^{-2\beta}\left(2\mathscr{D}_{(A}U_{B)} + \partial_u h_{AB} - 2Vx^2 h_{AB} + Vx^3 h_{AB,x}\right) \;, \tag{C.52}$$

where $\mathscr{D}_A$ denotes the covariant derivative with respect to the metric h. It follows that

$$\Gamma^\omega{}_{AB} h^{AB} = x^{-1}\mathrm{e}^{-2\beta}\left[2 + x\alpha_{,x}\left(U^A{}_{||A} - 2Vx^2\right)\right] + O_{\ln^* x}(x^3) \;. \tag{C.53}$$

Further

$$N = \frac{x^{-1}}{\sqrt{2\alpha_{,x}}}\left[1 + \beta + \frac{1}{4}x^3 \alpha_{,x} V + O(x^4)\right] \;.$$

Equation (C.50) specialised to Minkowski metric reads

$$\mathring{\Gamma}^\omega{}_{AB} = x^{-1}\breve{h}_{AB} - x\alpha_{,x}\breve{h}_{AB} + O_{\ln^* x}(x^3) \;, \tag{C.54}$$

yielding

$$\mathring{K}_{AB} \;=\; -\frac{x^{-2}}{\sqrt{2\alpha_{,x}}}\left[\breve{h}_{AB}\left(1 - \frac{3}{4}x^2\alpha_{,x}\right) + O_{\ln^* x}(x^4)\right] \;. \tag{C.55}$$

We therefore have

$$\begin{aligned} x^2 h^{AB} K_{AB} - x^2 \mu^b{}_a \mathring{K}_{ab} \;=\; & \frac{1}{\sqrt{2\alpha_{,x}}}\Big[2\beta + \frac{1}{8}x^2 \chi_{CD}\chi^{CD} - \frac{3}{2}x^2\alpha_{,x}(1 - xV) \\ & + x\alpha_{,x} U^A{}_{||A} - \frac{x^3}{2}\mathrm{tr}\,(\chi d)\Big] + O_{\ln^* x}(x^4) \;, \end{aligned} \tag{C.56}$$

which implies

$$\begin{aligned}\Big(K(\hat f_a,\hat f_a)-\mathring{K}(\hat f_a,\hat e_a)\Big)Y^3 &= \frac{x}{2\alpha_{,x}}\Bigg[\underline{\frac{1}{16}\chi_{AB}\chi^{AB}}+\\ &+x\left(2B-3M\alpha_{,x}+\frac{1}{2}\alpha_{,x}\chi^{AB}{}_{||AB}-\frac{1}{2}\mathrm{tr}\,(\chi d)\right)\Bigg]+O_{\ln^* x}(x^3)\ .\end{aligned} \tag{C.57}$$

Here we have again underlined a potentially divergent term. Using the formulae of [15, Appendix C] one further finds

$$\Big(K(\hat f_a,\hat f_3)-\mathring{K}(\hat f_a,\hat e_3)\Big)Y^a=O_{\ln^* x}(x^3)\ ,$$

which will give a vanishing contribution to B^3 in the limit. Collecting this together with (C.57) and (C.48) we finally obtain

$$B^3=\frac{1}{4}x^2\left(4M-\frac{1}{2}\chi^{AB}{}_{||AB}\right)+O_{\ln^* x}(x^3)\ , \tag{C.58}$$

We have thus shown that the integral of B^3 over the conformal boundary is proportional to the Trautman-Bondi mass, as desired.

D Proof of Lemma 5.9

From

$$\mathring{\nabla}\psi=0$$

we have

$$\begin{aligned}\gamma^\ell\nabla_\ell\psi &= \gamma^\ell\hat f_\ell(\psi)+\gamma^\ell\left[-\frac{1}{2}K(\hat f_\ell,\hat f_j)\gamma^j\gamma_0-\frac{1}{4}\omega_{ij}(\hat f_\ell)\gamma^i\gamma^j\right]\psi\\ &= \Bigg[-\frac{1}{2}\Big(K(\hat f_\ell,\hat f_j)-\mathring{K}(\hat f_\ell,\hat e_j)\Big)\gamma^\ell\gamma^j\gamma_0\\ &\quad+\frac{1}{4}\Big(\mathring{\omega}_{ij}(\hat f_\ell)-\omega_{ij}(\hat f_\ell)\Big)\gamma^\ell\gamma^i\gamma^j\Bigg]\psi\ .\end{aligned} \tag{D.1}$$

We start by showing that

$$\Big[\mathring{\omega}_{ij}(\hat f_l)-\omega_{ij}(\hat f_l)\Big]\gamma^l\gamma^i\gamma^j=O(x^2)\ .$$

In order to do that, note first

$$\gamma^i\gamma^j\gamma^k\Delta_{jki}=\varepsilon^{ijk}\Delta_{jki}\gamma^1\gamma^2\gamma^3-2g^{ij}\gamma^k\Delta_{jki}\ ,$$

where

$$\Delta_{jki}:=\mathring{\omega}_{jk}(\hat f_i)-\omega_{jk}(\hat f_i)=\hat M^l{}_i\mathring{\omega}_{jkl}-\omega_{jki}\ .$$

We claim that

$$\varepsilon^{ijk}\Delta_{jki}=O(x^2)\,,\quad \Delta_{jkk}=O(x^2)\ . \tag{D.2}$$

The intermediate calculations needed for this are as follows:

$$[\hat{e}_1, \hat{e}_2] = -x \cot\theta \hat{e}_2 + O_{\ln^* x}(x^4) \tag{D.3}$$

(which follows from (C.29)),

$$[\hat{e}_a, \hat{e}_3] = -[2\alpha_{,x}]^{-1/2} \hat{e}_a + O_{\ln^* x}(x^4) , \tag{D.4}$$

$$[\hat{f}_1, \hat{f}_2] = -x \cot\theta \hat{f}_2 + O(x^2) , \tag{D.5}$$

$$[\hat{f}_3, \hat{f}_a] = [2\alpha_{,x}]^{-1/2} \left(\hat{f}_a - \frac{1}{2} x \chi^b{}_a \hat{f}_b \right) + O(x^2) . \tag{D.6}$$

In order to calculate $\Delta_{123} + \Delta_{312} + \Delta_{231}$, we note that

$$\begin{aligned} \mathring{\omega}_{12}(\hat{f}_3) + \mathring{\omega}_{31}(\hat{f}_2) + \mathring{\omega}_{23}(\hat{f}_1) &= \mathring{\omega}_{123} + \mu_2{}^a \mathring{\omega}_{31a} + \mu_1{}^a \mathring{\omega}_{23a} + O(x^2) \\ &= (\mathring{\omega}_{123} + \mathring{\omega}_{312} + \mathring{\omega}_{231}) - \frac{1}{2} x \left(\chi_2{}^a \mathring{\omega}_{31a} + \chi_1{}^a \mathring{\omega}_{23a} \right) + O(x^2) . \end{aligned}$$

From $\mathring{\omega}_{3ab} = [2\alpha_{,x}]^{-1/2} \delta_{ab} + O_{\ln^* x}(x^3)$ the χ terms drops out. Equations (D.3)-(D.4) show that

$$2\mathring{\omega}_{123} + 2\mathring{\omega}_{312} + 2\mathring{\omega}_{231} = b([\hat{e}_2, \hat{e}_1], \hat{e}_3) + b([\hat{e}_3, \hat{e}_2], \hat{e}_1) + b([\hat{e}_1, \hat{e}_3], \hat{e}_2) = O(x^2) .$$

Similarly it follows from (D.5)-(D.6) that

$$2\omega_{123} + 2\omega_{312} + 2\omega_{231} = g([\hat{f}_2, \hat{f}_1], \hat{f}_3) + g([\hat{f}_3, \hat{f}_2], \hat{f}_1) + g([\hat{f}_1, \hat{f}_3], \hat{f}_2) = O(x^2) ,$$

yielding finally

$$\Delta_{123} + \Delta_{312} + \Delta_{231} = O(x^2) .$$

Next, $\Delta_{3kk} = \Delta_{3aa}$ is given by (C.48), and is thus $O(x^2)$. We continue with $\Delta_{akk} = \Delta_{abb} + \Delta_{a33}$. For $a = 1$ one finds

$$\omega_{122} + \omega_{133} = g([\hat{f}_1, \hat{f}_2], \hat{f}_2) + g([\hat{f}_1, \hat{f}_3], \hat{f}_3) = -x \cot\theta + O(x^2) .$$

Further,

$$\mathring{\omega}_{11}(\hat{f}_1) + \mathring{\omega}_{12}(\hat{f}_2) + \mathring{\omega}_{13}(\hat{f}_3) = \mathring{\omega}_{133} + \mu_a{}^b \mathring{\omega}_{1ab} + O(x^2) = \mathring{\omega}_{1kk} - \frac{1}{2} x \chi_a{}^b \mathring{\omega}_{1ab} + O(x^2) .$$

We have $\mathring{\omega}_{122} = -x \cot\theta + O_{\ln^* x}(x^3) = -x \cot\theta + O(x^2)$, while $\mathring{\omega}_{1ab} = O(x^2)$ as well, so that the χ's can be absorbed in the error terms, leading to

$$\Delta_{1kk} = \mathring{\omega}_{11}(\hat{f}_1) + \mathring{\omega}_{12}(\hat{f}_2) + \mathring{\omega}_{13}(\hat{f}_3) - (\omega_{122} + \omega_{133}) = O(x^2) .$$

For $a = 2$ we calculate

$$\omega_{211} + \omega_{233} = g([\hat{f}_2, \hat{f}_1], \hat{f}_1) + g([\hat{f}_2, \hat{f}_3], \hat{f}_3) = O(x^2) ,$$

and it is easy to check now that $\Delta_{2kk} = O(x^2)$. This establishes (D.2).

To estimate the contribution to (D.1) of the terms involving K we will need the following expansions

$$d_{3a}{}^{b} = \frac{1}{\sqrt{2\alpha_{,x}}}\left[\delta^{b}{}_{a} - \frac{1}{2}x\chi^{b}{}_{a}\right] + O_{\ln^* x}(x^2)\ , \tag{D.7}$$

$$d_{3a}{}^{3} = -\frac{x^2 b_a}{\sqrt{2\alpha_{,x}}} + O_{\ln^* x}(x^3) = O_{\ln^* x}(x^2)\ , \tag{D.8}$$

$$d_{ab}{}^{3} = x^2 c_{ab}{}^{c} b_c + O_{\ln^* x}(x^3) = O(x^2)\ . \tag{D.9}$$

Now, it follows from [15, Appendix C.3] that

$$tr_g K - \mathring{K}(\hat{f}_i, \hat{e}_i) = O_{\ln^* x}(x^2)\ .$$

Next, we claim that

$$\mathring{K}(\hat{f}_j, \hat{e}_k) - \mathring{K}(\hat{f}_k, \hat{e}_j) = O_{\ln^* x}(x^2)\ .$$

We have the following asymptotic formulae for the connection coefficients:

$$\Gamma^{\omega}{}_{xx} = 2x^{-1}\alpha_{,x} + O(x)\ , \quad \Gamma^{\omega}{}_{xA} = O(x)\ ,$$

$$\Gamma^{\omega}{}_{AB} = x^{-1}\breve{h}_{AB} + \frac{1}{2}\chi_{AB} + O(x)\ ,$$

and for the extrinsic curvature tensor:

$$K_{xx} = -N\Gamma^{\omega}{}_{xx} = -x^{-2}\left[\sqrt{2\alpha_{,x}} + O(x^2)\right] = \mathring{K}_{xx}\ ,$$

$$K_{xA} = -N\Gamma^{\omega}{}_{xA} = O(1)\ , \quad \mathring{K}_{xA} = O(x)\ ,$$

$$K_{AB} = -N\Gamma^{\omega}{}_{AB} \quad = \quad -\frac{x^{-2}}{\sqrt{2\alpha_{,x}}}\left[\breve{h}_{AB} + \frac{1}{2}x\chi_{AB} + O(x^2)\right]\ ,$$

$$\mathring{K}_{AB} \quad = \quad -\frac{x^{-2}}{\sqrt{2\alpha_{,x}}}\left[\breve{h}_{AB} + O(x^2)\right]\ . \tag{D.10}$$

From the above and [15, Appendix C.3] one obtains the following formulae

$$\mathring{K}(\hat{e}_k, \hat{e}_l) = -\frac{1}{\sqrt{2\alpha_{,x}}}\delta_{kl} + O_{\ln^* x}(x^2)\ , \tag{D.11}$$

$$K(\hat{e}_k, \hat{e}_l) = -\frac{1}{\sqrt{2\alpha_{,x}}}\left[\delta_{kl} + \frac{1}{2}x\chi_{kl}\right] + O_{\ln^* x}(x^2)\ , \tag{D.12}$$

where we have set $\chi_{3k} = 0$. Further

$$\hat{M}^{l}{}_{k} = \delta^{l}{}_{k} - \frac{1}{2}x\chi^{l}{}_{k} + O_{\ln^* x}(x^2)\ ,$$

$$\begin{aligned}\mathring{K}(\hat{f}_j,\hat{e}_k)-\mathring{K}(\hat{f}_k,\hat{e}_j) &= \hat{M}^l{}_j\mathring{K}(\hat{e}_l,\hat{e}_k)-\hat{M}^l{}_k\mathring{K}(\hat{e}_l,\hat{e}_j)\\ &= -\frac{1}{\sqrt{2\alpha_{,x}}}\left(\delta^l{}_j-\frac{1}{2}x\chi^l{}_j\right)\delta_{lk}+\frac{1}{\sqrt{2\alpha_{,x}}}\left(\delta^l{}_k-\frac{1}{2}x\chi^l{}_k\right)\delta_{lj}+O_{\ln^* x}(x^2)\\ &= O_{\ln^* x}(x^2)\ .\end{aligned}$$

This, together with [15, Equation (C.83)] yields

$$tr_g K-\mathring{K}(\hat{f}_i,\hat{e}_i)=-\frac{3}{\sqrt{2\alpha_{,x}}}+\frac{1}{\sqrt{2\alpha_{,x}}}\left(\delta^l{}_i-\frac{1}{2}x\chi^l{}_i\right)\delta_{li}+O_{\ln^* x}(x^2)=O_{\ln^* x}(x^2)\ .$$

We also have

$$\begin{aligned}\left[K(\hat{f}_l,\hat{f}_j)-\mathring{K}(\hat{f}_l,\hat{e}_j)\right]\gamma^l\gamma^j &= -K(\hat{f}_k,\hat{f}_k)-\mathring{K}(\hat{f}_l,\hat{e}_j)\gamma^l\gamma^j\\ &= -tr_g K+\mathring{K}(\hat{f}_k,\hat{e}_k)\\ &\quad +\frac{1}{2}\left(\mathring{K}(\hat{f}_j,\hat{e}_k)-\mathring{K}(\hat{f}_k,\hat{e}_j)\right)\gamma^k\gamma^j\ ,\end{aligned}$$

so that

$$\left[K(\hat{f}_l,\hat{f}_j)-\mathring{K}(\hat{f}_l,\hat{e}_j)\right]\gamma^l\gamma^j=O_{\ln^* x}(x^2)\ .$$

Since

$$\sqrt{\det g}=O(x^3)\,,\quad <\psi,\psi>=O(x^{-1})\ ,$$

we obtain

$$\sqrt{\det g}\left|\gamma^k\nabla_k\psi\right|^2=O_{\ln^* x}(1)\in L^1\ .$$

E Asymptotic expansions of objects on $\mathscr{S}$

Throughout this appendix coordinate indices are used.

E.1 Smooth case

Induced metric g. We write the spacetime metric in Bondi-Sachs coordinates, as in (C.8), and use the standard expansions for the coefficients of the metric (see *e.g.* [15, Equations (5.98)-(5.101)]). Let $\mathscr{S}$ be given by $\omega=$ const., where

$$\omega=u-\alpha(x,x^A)\ .$$

Two different coordinate systems will be used: (u,x,x^A) and (ω,x,x^A) — coordinates adapted to $\mathscr{S}$. To avoid ambiguity two different symbols for partial derivatives will be used: the comma stands for the derivative with $\omega=$ const. and $\wr$ stands for the derivative with $u=$ const. These two derivatives can be transformed into each other:

$$A_{,x}=A_{\wr x}+\alpha_{\wr x}\partial_u A\ ,$$

$$A_{,A}=A_{\wr A}+\alpha_{\wr A}\partial_u A\ .$$

For functions not depending on u (*e.g.,* α) the symbols mean the same. Derivations in covariant derivatives $_{||A}$ and $_{|i}$ are with $\omega=$ const.

Three-dimensional reciprocal metric. We have the following implicit formulae for the three dimensional inverse metric g^{ij}:

$$-\frac{g^{xA}}{g^{xx}} = {}^2g^{AB} g_{xB} \ ,$$

$$\frac{1}{g^{xx}} = g_{xx} + \frac{g^{xA}}{g^{xx}} g_{xA} \ ,$$

$$g^{AB} = {}^2g^{AB} + \frac{g^{xA}g^{xB}}{g^{xx}} \ ,$$

where ${}^2g^{AB}$ denotes the matrix inverse to (g_{AB}). The calculations get very complicated in general. To simplify them we will assume a particular form of α, *i.e.*,

$$\alpha = \text{const.} \cdot x + O(x^3) \ . \tag{E.1}$$

This choice is motivated by the form of α for standard hyperboloid $t^2 - r^2 = 1$ in Minkowski spacetime. In that case $\alpha = \frac{1}{x}\sqrt{1+x^2} - \frac{1}{x} = \frac{1}{2}x + O(x^3)$. It is further equivalent to the asymptotically CMC condition (4.3). With the above assumptions we have the following asymptotic expansions:

$${}^2g^{AB} = x^2 h^{AB} + O(x^7) \ ,$$

$$\begin{aligned}
-\frac{g^{xA}}{g^{xx}} &= \alpha^{,A} + x^2\alpha_{,x}\left(\frac{\chi^{AC}{}_{||C}}{2} - 2N^A x - \frac{(\chi^{CD}\chi_{CD})^{||A}}{16}x - \frac{\chi^{AB}\chi_B{}^C{}_{||C}}{2}x\right) \\
&\quad + O(x^4) \ ,
\end{aligned}$$

$$\frac{1}{g^{xx}} = \frac{2\alpha_{,x}}{x^2}\left[1 + 2\beta - \frac{x^2\alpha_{,x}}{2} + Mx^3\alpha_{,x} + O(x^4)\right] = g_{xx} \ ,$$

$$g^{xx} = \frac{x^2}{2\alpha_{,x}}\left[1 - 2\beta + \frac{x^2\alpha_{,x}}{2} - Mx^3\alpha_{,x} + O(x^4)\right] \ ,$$

$$\frac{1}{\sqrt{g^{xx}}} = \frac{\sqrt{2\alpha_{,x}}}{x}\left[1 + \beta - \frac{1}{4}x^2\alpha_{,x}(1 - 2Mx) + O(x^4)\right] \ ,$$

$$-g^{Ax} = \frac{x^4}{2}\left[\frac{\chi^{AC}{}_{||C}}{4} - 2xN^A - x\frac{(\chi^{CD}\chi_{CD})^{||A}}{16} - x\frac{\chi_C{}^A\chi^{CD}{}_{||D}}{2} + \frac{\alpha_{,A}}{x^2\alpha_{,x}} + O(x^2)\right] \ ,$$

$$g^{AB} = x^2 h^{AB} + O(x^6) \ .$$

Determinant of the induced metric ${}^2g$. The determinant of the 2-dimensional metric induced on the Bondi spheres $u = \text{const.}$, $x = \text{const.}$, equals

$$\det\left(\frac{1}{x^2}h\right) = \det\left(\frac{1}{x^2}\breve{h}\right) \ .$$

In our case the $\omega = \text{const.}$, $x = \text{const.}$ spheres are not exactly the Bondi ones, and one finds

$$\lambda := \sqrt{\det {}^2g} = \frac{\sqrt{\breve{h}}}{x^2} + O(x^3) \ , \quad \mathring{\lambda} := \sqrt{\det {}^2b} = \frac{\sqrt{\breve{h}}}{x^2} + O(x^6) \ .$$

We also note the following purely algebraic identities:

$$\lambda = \sqrt{g^{xx}} \cdot \sqrt{\det g} \ , \quad \mathring{\lambda} = \sqrt{b^{xx}} \cdot \sqrt{\det b} \ , \tag{E.2}$$

Lapse and shift. The function N (the *lapse*) and the vector S_i (the *shift*) can be calculated as follows:

$$N = \frac{1}{\sqrt{-{}^4g^{\omega\omega}}} \ ,$$

$$S_i = {}^4g_{\omega i} \ .$$

Hence

$$N = \frac{1}{x\sqrt{2\alpha_{,x}}}\left[1 + \beta + \frac{1}{4}x^2\alpha_{,x} - \frac{1}{2}x^3\alpha_{,x}M + O(x^4)\right] \ ,$$

$$S_x = \frac{1}{x^2}\left[1 + 2\beta - x^2\alpha_{,x} + 2x^3\alpha_{,x}M + O(x^4)\right] \ ,$$

$$S_A = \frac{1}{2}\chi^C{}_{A||C} - 2xN_A - \frac{1}{16}x(\chi^{CD}\chi_{CD})_{||A} + O(x^2) \ .$$

Christoffel coeficients. Using the induced metric (and the reciprocal metric) we can compute the coefficients ${}^3\Gamma^i{}_{jk}$. The results are:

$${}^3\Gamma^x{}_{xx} = -\frac{1}{x} + \frac{\alpha_{,xx}}{2\alpha_{,x}} + \beta_{\wr x} + \alpha_{,x}\left(\frac{3}{2}x^2M - \frac{1}{2}x + \partial_u\beta\right) + O(x^3) \ ,$$

$$\begin{aligned} {}^3\Gamma^x{}_{xA} \ &= \ \frac{1}{4}x\Big[\chi^C{}_{A||C} - 4xN_A - \frac{1}{8}x(\chi^{CD}\chi_{CD})_{||A} - \frac{1}{2}x\chi_{AC}\chi^{CD}{}_{||D} \\ &\quad + \frac{2\alpha_{,A}}{x^2\alpha_{,x}} + \frac{4\beta_{\wr A}}{x} + \frac{2\alpha_{,xA}}{x\alpha_{,x}} + O(x^2)\Big] \ , \end{aligned}$$

$${}^3\Gamma^x{}_{AB} = \frac{\breve{h}_{AB}}{2x\alpha_{,x}} + \frac{\chi_{AB}}{4\alpha_{,x}} + \frac{1}{4}x\breve{h}_{AB} - \frac{1}{4}x\partial_u\chi_{AB} - \frac{\beta\breve{h}_{AB}}{x\alpha_{,x}} + O(x^2) \ ,$$

$${}^3\Gamma^A{}_{xx} = \frac{1}{2}x\alpha_{,x}\chi^{AC}{}_{||C} + O(x^2) \ ,$$

$$\begin{aligned} {}^3\Gamma^A{}_{xB} \ &= \ -\frac{1}{x}\delta^A{}_B + \frac{1}{2}\chi^A{}_B + \frac{1}{2}x\alpha_{,x}\partial_u\chi^A{}_B + \frac{1}{2}x^2\left(3H^A{}_B - \frac{\chi^{CD}\chi_{CD}}{4}\chi^A{}_B\right) \\ &\quad + \frac{1}{4}x^2\alpha_{,x}\left(\chi_{BC}\partial_u\chi^{AC} - \chi^{AC}\partial_u\chi_{BC} + \chi^{AD}{}_{||DB} - \chi_{BD}{}^{||DA}\right) + O(x^3) \ , \end{aligned}$$

$${}^3\Gamma^A{}_{BC} = \Gamma(\breve{h})^A{}_{BC} + O(x) \ .$$

Extrinsic curvature. The tensor field K_{ij} can be computed as

$$K_{ij} = \frac{1}{2N}(S_{i|j} + S_{j|i} - \partial_u g_{ij}) \ ,$$

where by $A_{|i}$ we denote the covariant derivative of a quantity A with respect to g_{ij}. These covariant derivatives read:

$$\begin{aligned} S_{x|x} &= \frac{1}{x^3}\left[-1 - \frac{x\alpha_{,xx}}{2\alpha_{,x}} + x\beta_{\wr x} - 2\beta - \frac{1}{2}x^2\alpha_{,x} \right. \\ &\quad \left. + \frac{5}{2}x^3\alpha_{,x}M + x\alpha_{,x}\partial_u\beta + O(x^4)\right] \ , \end{aligned}$$

$$\begin{aligned} S_{x|A} &= \frac{\beta_{\wr A}}{x^2} - \frac{\alpha_{,A}}{2x^3\alpha_{,x}} - \frac{\alpha_{,xA}}{2x^2\alpha_{,x}} + \frac{1}{4x}\chi^C{}_{A||C} - N_A \\ &\quad - \frac{1}{32}(\chi^{CD}\chi_{CD})_{||A} - \frac{1}{8}\chi_{AC}\chi^{CD}{}_{||D} + O(x) \ , \end{aligned}$$

$$\begin{aligned} S_{A|x} &= -\frac{\beta_{\wr A}}{x^2} - \frac{\alpha_{,A}}{2x^3\alpha_{,x}} - \frac{\alpha_{,xA}}{2x^2\alpha_{,x}} + \frac{1}{4x}\chi^C{}_{A||C} - 3N_A \\ &\quad - \frac{3}{32}(\chi^{CD}\chi_{CD})_{||A} - \frac{1}{8}\chi_{AC}\chi^{CD}{}_{||D} \\ &\quad + \frac{1}{2}\alpha_{,x}\partial_u(\chi^C{}_{A||C}) + O(x) \ , \end{aligned}$$

$$S_{A|B} = -\frac{1}{x^2}\left[\frac{\check{h}_{AB}}{2x\alpha_{,x}} + \frac{\chi_{AB}}{4\alpha_{,x}} - \frac{1}{4}x\check{h}_{AB} - \frac{1}{4}x\partial_u\chi_{AB} + O(x^2)\right] \ .$$

Derivatives of g_{ij} with respect to u :

$$\partial_u g_{xx} = 4x^{-2}\alpha_{,x}\partial_u\beta + O(x) \ ,$$

$$\partial_u g_{xA} = \frac{1}{2}\alpha_{,x}\partial_u(\chi^C{}_{A||C}) + O(x) \ ,$$

$$\partial_u g_{AB} = x^{-1}\partial_u\chi_{AB} + O(1) \ .$$

Hence the extrinsic curvature:

$$\begin{aligned} K_{xx} &= \frac{\sqrt{2\alpha_{,x}}}{x^2}\left[-1 - \frac{x\alpha_{,xx}}{2\alpha_{,x}} + x\beta_{\wr x} - \beta - \frac{1}{4}x^2\alpha_{,x} \right. \\ &\quad \left. + 2x^3\alpha_{,x}M - x\alpha_{,x}\partial_u\beta + O(x^4)\right] \ , \end{aligned}$$

$$\begin{aligned} K_{xA} &= \frac{\sqrt{2\alpha_{,x}}}{2}\left[\frac{1}{2}\chi^C{}_{A||C} + \frac{\alpha_{,A}}{x^2\alpha_{,x}} - \frac{\alpha_{,xA}}{x\alpha_{,x}} - 4xN_A \right. \\ &\quad \left. - \frac{1}{8}x(\chi^{CD}\chi_{CD})_{||A} - \frac{1}{4}x\chi_{AC}\chi^{CD}{}_{||D} + O(x^2)\right] \ , \end{aligned}$$

$$K_{AB} = -\frac{\sqrt{2\alpha_{,x}}}{2x}\left[\frac{\check{h}_{AB}}{x\alpha_{,x}} - \frac{\beta\check{h}_{AB}}{x\alpha_{,x}} + \frac{\chi_{AB}}{2\alpha_{,x}} - \frac{3}{4}x\check{h}_{AB} + \frac{1}{2}x\partial_u\chi_{AB} + O(x^2)\right] \ .$$

The trace $\mathrm{tr}_g K = K^i{}_i$ can be easily calculated more precisely as

$$\mathrm{tr}_g K = \frac{1}{2N}[2S^i{}_{|i} - g^{ij}\partial_u g_{ij}] \ ,$$

or after multiplying by $\sqrt{\det g}$ and changing the covariant divergence to the ordinary one:

$$\sqrt{\det g}\,\mathrm{tr}_g K = \frac{1}{2N}\left[2(\sqrt{\det g}\,S^i)_{,i} - \frac{\partial_u(\det g)}{\sqrt{\det g}}\right] \ .$$

After some calculations we get

$$\begin{aligned}\mathrm{tr}_g K &= -\frac{1}{\sqrt{2\alpha_{,x}}} \left[3 - 3\beta - x\beta_{lx} + \frac{x\alpha_{,xx}}{2\alpha_{,x}} - \frac{3}{4}x^2\alpha_{,x}\right. \\ &\qquad \left. - \frac{1}{2}x^3\alpha_{,x}\chi^{CD}{}_{||CD} + x\alpha_{,x}\beta_{,u} + O(x^4)\right] \ .\end{aligned}$$

ADM momenta. The ADM momenta can be expressed in terms of the extrinsic curvature:

$$P^k{}_l = g^{ki}(g_{il}\mathrm{tr}_g K - K_{il}) \ .$$

Substituting the previously calculated $\mathrm{tr}_g K$ and K_{ij} we get

$$P^x{}_x = -\frac{1}{\sqrt{2\alpha_{,x}}}\left[2 - 2\beta - \frac{3}{2}x^2\alpha_{,x}(1-2Mx) - \frac{1}{2}x^3\alpha_{,x}\chi^{CD}{}_{||CD} + O(x^4)\right] \ ,$$

$$P^x{}_x - \mathring{P}^x{}_x = \frac{-1}{\sqrt{2\alpha_{,x}}}\left[-2\beta + 3x^3\alpha_{,x}M - \frac{1}{2}x^3\alpha_{,x}\chi^{CD}{}_{||CD} + O(x^4)\right] \ , \qquad \text{(E.3)}$$

$$\begin{aligned}P^A{}_B &= -\frac{1}{\sqrt{2\alpha_{,.x}}}\left[(2 - 2\beta - x\beta_{lx} + \frac{x\alpha_{,xx}}{2\alpha_{,x}})\delta^A{}_B + \frac{1}{2}x\chi^A{}_B - \frac{1}{2}x^2\alpha_{,x}\partial_u\chi^A{}_B\right] \\ &\quad + O(x^3) \ ,\end{aligned}$$

$$P^A{}_x = -\frac{1}{2}x^2\sqrt{2\alpha_{,x}}\chi^{AC}{}_{||C} + O(x^3) \ ,$$

$$\begin{aligned}P^x{}_A &= \frac{-x^3}{2\sqrt{2\alpha_{,x}}}\left[\frac{\chi^C{}_{A||C}}{x} - \frac{\alpha_{,xA}}{x^2\alpha_{,x}} - 6N_A - \frac{3(\chi^{CD}\chi_{CD})_{||A}}{16} - \frac{1}{2}\chi_{AC}\chi^{CD}{}_{||D}\right] \\ &\quad + O(x^4) \ ,\end{aligned}$$

$$\begin{aligned}P^x{}_A - \mathring{P}^x{}_A &= \frac{-x^3}{2\sqrt{2\alpha_{,x}}}\left[\frac{\chi^C{}_{A||C}}{x} - 6N_A - \frac{3(\chi^{CD}\chi_{CD})_{||A}}{16} - \frac{1}{2}\chi_{AC}\chi^{CD}{}_{||D}\right] \\ &\quad + O(x^4) \ . \qquad \text{(E.4)}\end{aligned}$$

The second fundamental form k_{AB}. The extrinsic curvature of the leaves of the "2+1 foliation" can be computed from the formula

$$k_{AB} = \frac{{}^3\Gamma^x{}_{AB}}{\sqrt{g^{xx}}} \; .$$

Hence

$$\begin{aligned} k_{AB} &= \frac{\sqrt{2\alpha_{,x}}}{2x}\left[\frac{1}{x\alpha_{,x}}\breve{h}_{AB} + \frac{1}{2\alpha_{,x}}\chi_{AB} \right. \\ &\quad \left. + \frac{1}{4}x\breve{h}_{AB} - \frac{1}{2}x\partial_u\chi_{AB} - \frac{\beta}{x\alpha_{,x}}\breve{h}_{AB} + O(x^2)\right] \; . \end{aligned}$$

We need a more accurate expansion of the trace $k = {}^2g^{AB}k_{AB}$. The formula

$$k_{AB} = \frac{\Gamma^x{}_{AB}}{\sqrt{g^{xx}}} - \frac{{}^4g^{x\omega}\Gamma^\omega{}_{AB}}{{}^4g^{\omega\omega}\sqrt{g^{xx}}} \; .$$

can be used. It is convenient to calculate ${}^2g^{AB}\Gamma^x{}_{AB}$ and ${}^2g^{AB}\Gamma^\omega{}_{AB}$ using the expressions for the Christoffel symbols given in [15, Appendix C]:

$$\Gamma^\omega{}_{AB} = x^{-1}\mathrm{e}^{-2\beta}(h_{AB} - \frac{1}{2}x h_{AB\wr x}) - \alpha_{,x}\Gamma^x{}_{AB} + O(x^3) \; ,$$

$$\Gamma^x{}_{AB} = -\frac{1}{2}\mathrm{e}^{-2\beta}(2\mathcal{D}(h)_{(A}U_{B)} + \partial_u h_{AB} - 2Vx^2 h_{AB} + Vx^3 h_{AB\wr x}) \; ,$$

where $\mathcal{D}(h)_A$ is a covariant derivative with respect to h_{AB} (we use the (u, x, x^A) coordinate system and all the differentiations with respect to x, x^A are at constant u). Hence:

$${}^2g^{AB}\Gamma^\omega{}_{AB} = 2x - 4\beta x + x^2\alpha_{,x}(U^A{}_{||A} - 2Vx^2) + O(x^5) \; ,$$

$${}^2g^{AB}\Gamma^x{}_{AB} = -x^2[U^A{}_{||A} - 2Vx^2 + O(x^3)] \; .$$

After substitution of relevant asymptotic expansions:

$$k = \frac{1}{\sqrt{g^{xx}}}\left[\frac{1}{4}x^4\chi^{CD}{}_{||CD} + \frac{1}{2}x^3(1-2Mx) + \frac{x}{\alpha_{,x}} - \frac{2\beta x}{\alpha_{,x}} + O(x^5)\right] \; ,$$

$$k = \sqrt{2\alpha_{,x}}\left[\frac{1}{4}x^3\chi^{CD}{}_{||CD} + \frac{1}{4}x^2(1-2Mx) + \frac{1}{\alpha_{,x}} - \frac{\beta}{\alpha_{,x}} + O(x^4)\right] \; ,$$

$$k - \mathring{k} = \sqrt{2\alpha_{,x}}\left[\frac{1}{4}x^3\chi^{CD}{}_{||CD} - \frac{1}{2}x^3 M - \frac{\beta}{\alpha_{,x}} + O(x^4)\right] \; . \qquad \text{(E.5)}$$

E.2 The polyhomogenous case

We give only the most important intermediate results which differ from the power-series case:

$$g^{xB} = -\frac{1}{4}x^4\chi^{AC}{}_{||C} + \frac{1}{2}x^5 W^A + O(x^5) \ ,$$

$$S_A = \frac{1}{2}\chi^C{}_{A||C} - xW_A + O(x) \ ,$$

$$K_{xA} = \sqrt{2\alpha_{,x}}[\frac{1}{4}\chi^C{}_{A||C} - xW_A - \frac{1}{2}x^2 W_{A\wr x} + O(x)] \ ,$$

$$P^x{}_A = -\frac{x^2}{2\sqrt{2\alpha_{,x}}}[\chi^C{}_{A||C} - 3xW_A - x^2 W_{A\wr x} + O(x)] \ .$$

F Decomposition of Poincaré group vectors into tangential and normal parts

The generators of Poincaré group are given after [15]:

$$X_{time} = \partial_\omega = \partial_u \ ,$$

$$X_{rot} = -\varepsilon^{AB}\alpha_{,A}v_{,B}\partial_\omega + \varepsilon^{AB}v_{,B}\partial_A \ ,$$

$$X_{trans} = (-v - x\alpha^{,A}v_{,A} + x^2 v\alpha_{,x})\partial_\omega - x^2 v\partial_x + xv^{,A}\partial_A \ ,$$

$$\begin{aligned} X_{boost} \ &= \ [xv((\alpha+\omega)x+1)\alpha_{,x} - (\alpha+\omega)v \\ &- \ \alpha^{,A}v_{,A}((\alpha+\omega)x+1)]\partial_\omega \\ &- \ xv[(\alpha+\omega)x+1]\partial_x + v^{,A}[(\alpha+\omega)x+1]\partial_A \ . \end{aligned}$$

The tensor ε^{AB} is defined as

$$\varepsilon^{AB} = \frac{1}{\sqrt{\breve{h}}}\{A,B\} \ ,$$

where $\{1,2\} = -\{2,1\} = 1$ and $\{1,1\} = \{2,2\} = 0$. By v we denote a function on the sphere which is a combination of $\ell = 1$ spherical harmonics. If we consider embedding of the sphere into $\mathbb{R}^3$, then

$$v(x^A) = v^i\frac{x_i}{r}$$

and we get a bijection between functions v and vectors $(v^i) \in \mathbb{R}^3$.

Let us now decompose a vector field X into parts tangent and normal to $\mathscr{S}$:

$$X = Y + Vn \ .$$

Y is a vector tangent to $\mathscr{S}$ and n is a unit ($n^2 = -1$), future-directed normal vector. Setting

$$\tau = 2\alpha_{,x} - \breve{h}^{AB}\alpha_{,A}\alpha_{,B} \ ,$$

we have the following decomposition of respective vectors :

$$V_{time} = \frac{1}{x\sqrt{\tau}}\left(1 - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{2\tau} + O(x^2)\right) \ ,$$

$$Y^x_{time} = \frac{1}{\tau}\left(1 - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{\tau} + O(x^2)\right) \ ,$$

$$Y^A_{time} = -\frac{1}{\tau}\left(a^{,A} - x\chi^{AC}\alpha_{,C} - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}\alpha^{,A}}{\tau} + O(x^2)\right) \ ,$$

$$V_{rot} = -\varepsilon^{AB}\alpha_{,A}v_{,B}\cdot\frac{1}{x\sqrt{\tau}}\left(1 - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{2\tau} + O(x^2)\right) \ ,$$

$$Y^x_{rot} = -\varepsilon^{AB}\alpha_{,A}v_{,B}\cdot\frac{1}{\tau}\left(1 - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{\tau} + O(x^2)\right) \ ,$$

$$\begin{aligned} Y^A_{rot} &= \varepsilon^{AB}v_{,B} \\ &\quad + \varepsilon^{CB}\alpha_{,C}v_{,B}\cdot\frac{1}{\tau}\left(\alpha^{,A} - x\chi^{AC}\alpha_{,C} - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}\alpha^{,A}}{\tau} + O(x^2)\right) \ , \end{aligned}$$

$$V_{trans} = \frac{1}{x\sqrt{\tau}}\left(-v - x\alpha^{,A}v_{,A} + v\frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{2\tau} + O(x^2)\right) \ ,$$

$$Y^x_{trans} = \frac{1}{\tau}\left(-v - x\alpha^{,A}v_{,A} + v\frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{\tau} + O(x^2)\right) \ ,$$

$$\begin{aligned} Y^A_{trans} &= xv^{,A} \\ &\quad - \frac{1}{\tau}\left(-v\alpha^{,A} + vx\chi^{AC}\alpha_{,C} + v\frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{\tau}\alpha^{,A} - x\alpha^{,C}v_{,C}\alpha^{,A} + O(x^2)\right) \ , \end{aligned}$$

$$\begin{aligned} V_{boost} &= \frac{1}{x\sqrt{\tau}}\Big[- (\omega+\alpha)v + xv\alpha_{,x} - v^{,C}\alpha_{,C}(\omega x + \alpha x + 1) \\ &\quad + \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{2\tau}((\omega+\alpha)v + v^{,C}\alpha_{,C}) + O(x^2)\Big] \ , \end{aligned}$$

$$\begin{aligned} Y^x_{boost} &= -xv + \frac{1}{\tau}\Big[- (\omega+\alpha)v + xv\alpha_{,x} - v^{,C}\alpha_{,C}(\omega x + \alpha x + 1) \\ &\quad + \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{\tau}\big((\omega+\alpha)v + v^{,C}\alpha_{,C}\big) + O(x^2)\Big] \ , \end{aligned}$$

$$\begin{aligned} Y^A_{boost} &= v^{,A}(\omega x + \alpha x + 1) \\ &\quad + \frac{1}{\tau}\Big[(\omega+\alpha)v\alpha^{,A} - xv\alpha_{,x}\alpha^{,A} + v^{,C}\alpha_{,C}\alpha^{,A}(\omega x + \alpha x + 1) \\ &\quad - x\big(\frac{\chi^{CD}\alpha_{,C}\alpha_{,D}\alpha^{,A}}{\tau} + \chi^{AC}\alpha_{,C}\big)\big((\omega+\alpha)v + v^{,C}\alpha_{,C}\big) + O(x^2)\Big] \ . \end{aligned}$$

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