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Bidimensional non-parametric estimation of well-being distribution and poverty index

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Abstract. In this paper, we introduce a kernel-type version for the bi-dimensional extension of the Foster, Greer, and Thorbecke index that was introduced by Duclos *et al.* (2006a) for the purpose of a dominance approach to multidimensional poverty. The measure they used in their dominance exercise is essentially a generalization, from one to two dimensions, of the FGT index with separate poverty aversion parameters for each dimension. Our estimator is constructed by using a bidimensional Parzen-Rosenblatt kernel of a probability density function (*pdf*). We next provide its complete asymptotic behaviour by establishing its almost-sure uniform and its uniform mean square consistencies. A simulation study shows that it performs well for small samples comparatively to the empirical plug-in estimator. Our results are also extensions of those of Dia (2008) and of Ciss *et al.* (2014) in one dimension.

Résumé. Dans ce papier, nous proposons un estimateur pour la version bidimensionnelle de l'indice de pauvreté de Foster, Greer et Thorbecke, introduit par Duclos *et al.* (2006a) pour l'étude de pauvreté dans un cadre multidimensionnel grâce à la dominance stochastique. La mesure qu'ils utilisent dans cet exercice est en fait une extension bidimensionnelle de l'indice FGT avec deux paramètres pour l'aversion de la pauvreté, un dans chaque direction. Dans le processus de construction de notre estimateur, nous utilisons l'estimateur bidimensionnel de la densité de probabilité de Parzen-Rosenblatt. La convergence uniforme presque sûre et la convergence uniforme en erreur quadratique moyenne sont ensuite établies. Des simulations numériques montrent que notre estimateur se comporte bien, même pour les échantillons de petites tailles, face à l'estimateur empirique. Nos résultats constituent aussi une extension au cas bidimensionnel de ceux de Dia (2008) et de Ciss *et al.* (2014).

Key words: Bi-dimensional extension of the FGT; Poverty frontier; Parzen-Rosenblatt kernel in 2D; Uniform almost-surely consistency; Rate of convergence.
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1. Introduction

Duclos *et al.* (2006a) considered a multidimensional extension of the FGT class of measures, to address robustness analysis of the choice of poverty indices and poverty lines. They used the dominance approach for poverty comparisons, as initially developed in Atkinson (1987), in Foster and Shorrocks (1988a), Foster and Shorrocks (1988b) and in Foster and Shorrocks (1988c). A major advantage of this approach is its ability in generating poverty orderings that are robust with respect to the determination of poverty lines. Then the sensivity of most of poverty measures to the poverty line makes this approach more important. Besides, it also ensures robustness with respect to the choice of a multidimensional poverty index over broad classes of them, as well as robustness over the manner in which multidimensional indicators interact between them, when describing overall individual well-being. Duclos *et al.* (2006b) also used the bivariate stochastic dominance techniques to investigate the incidence of poverty, measured in terms of household expenditures per capita and child height-for-age indicators.

Such important traits of this measure motivated us to have an asymptotic theory based on estimators constructed on random samples that would provide accurate approximations for small sizes. Their uniform asymptotic laws may lead to optimal choices of the two considered parameters. A number of different ways are possible to adopt, one of them is the use of the empirical plug-in estimator. We consider here, as a first study of such nature of this index, extensions in form of kernel-type statistics, that we already used in one-dimensional studies in Dia (2008), for $\alpha = 0$ and $\alpha \geq 1$ and, by Ciss *et al.* (2014) for $\alpha \in]0,1[$. By the way, the results exposed in these papers will be particular cases of results of the current paper. Our approach will demonstrate to give be better results than the one based on the empirical estimator.

To make the ideas more clear, let x and y be two indicators of individual well-being among, for example, income, expenditures, caloric consumption, life expectancy, height, body weight, extent of personal safety and freedom, etc. Throughout this paper (X, Y) stands for the value of (x, y) for a randomly selected individual of the population. Then (X, Y) is a random couple of nonnegative real numbers defined of a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whose cumulative distribution function (*cdf*) is denoted by $F(.,.)$ and we suppose that it admits a *pdf* $f(.,.)$. From now all the expections are done with respect to this probability space.

The bi-dimensional extension of the FGT Foster *et al.* (1984) class of poverty measures by Duclos *et al.* (2006a) is denoted by $P(z_1, z_2, \alpha_1, \alpha_2)$ and is defined as follows, for $(\alpha_1 \geq 0, \alpha_2 \geq 0)$:

$$= \begin{cases} \int_0^{z_1} \int_0^{z_2} \left(\frac{z_1 - x}{z_1}\right)^{\alpha_1} \left(\frac{z_2 - y}{z_2}\right)^{\alpha_2} f(x, y)\, dxdy & \text{if} \quad z_1 > 0 \text{ and } z_2 > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

where z_1 (resp. z_2) represents the poverty line for the dimension x (resp. y). This index is useful for ordinal robust comparisons of poverty, even when the measurements are made across the intersection of the two dimensions considered.

Now let us consider, for $n \geq 1$, a random sample $(X_1, Y_1), ..., (X_n, Y_n)$ from (X, Y), defined of the probability space defined above. The empirical plug-in estimator of 1 is given by

$$\widehat{P}_n(z_1, z_2, \alpha_1, \alpha_2) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(1-\frac{X_i}{z_1}\Big)_+^{\alpha_1}\Big(1-\frac{Y_j}{z_2}\Big)_+^{\alpha_2}$$

where $x_+ = \max(0, x)$.

From there, we use the Parzen (1962) kernel estimator of the density $f(x, y)$:

$$\hat{f}(x, y) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_1 h_2}K\Big(\frac{x-X_i}{h_1}\Big)K\Big(\frac{y-Y_i}{h_2}\Big). \tag{2}$$

Combining these two last facts, and based on Riemann sum, we are able to propose the following kernel estimator of the DSY index 1 : $P_n(z_1, z_2, \alpha_1, \alpha_2)$

$$= \frac{1}{n}\sum_{k=1}^{n}\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}K\Big(\frac{X_k-ih_1}{h_1}\Big)K\Big(\frac{Y_k-ih_2}{h_2}\Big), \tag{3}$$

where $[\frac{\cdot}{h_i}]$, is the integer part of $\frac{\cdot}{h_i}$, $i = 1, 2 \quad \alpha_i \geq 0$, $h_i = h_i(n)$, $i = 1, 2$ are positive nonrandom sequences of real numbers tending to zero as as n tends to infinity, and finally $K(x)$ is a Borel function satisfying the following hypotheses:

$$(\mathbf{H_1}) \sup_{-\infty<x<+\infty}|K(x)| < +\infty, (\mathbf{H_2}) \int_{\mathbb{R}} K(x)\, dx = 1, (\mathbf{H_3}) \lim_{x\to\pm\infty}|x||K(x)| = 0.$$

For readibiliy's sake, we expose our construction of the latter statistic in the appendix section 5.

As announced we are going to describe the complete asymptotic theory of (3). Our best achievement is establishing the almost-sure consitency with respect to the parameters $(\alpha_1, \alpha_2) \in (\mathbb{R}_+^*)^2$ as well as the uniform mean square efficiency. We have been able to conduct simulation studies that showed good performances for small sizes data, better that results from the empirical plug-in statistics.

The rest of the paper is organized as follows. In Section 2, will state full details of the results. In Section 3, we will provide simulation studies outcomes and relevant comments as well as a compairison with results from the empirical approach that was used until now. The complete proofs are then given in Section 4.

2. Convergence of the estimator

We will need a number of hypotheses and conditions for our theorems. We need to derive the following one $\mathbb{K}$,from the function K, defined on $\mathbb{R}^2$ by $\mathbb{K}(x, y) = K(x)K(y)$. This this latter inherits from K these two properties : $\sup_{-\infty<x,y<+\infty}|\mathbb{K}(x, y)| < +\infty$ and

$$\int\int_{\mathbb{R}^2}\mathbb{K}(x, y)\, dxdy = 1, \lim_{\|(x,y)\|\to\infty}\|(x, y)\||\mathbb{K}(x, y)| = 0.$$

Now additional hypotheses on $\mathbb{K}$ or K are the following :
($\mathbf{H}_4$) K is a bounded variation function on $\mathbb{R}$ and we denote by $V(\mathbb{R})$ its total variation.

(H$_5$) $\int\int_{\mathbb{R}^2} |uv||\mathbb{K}(u,v)| < +\infty$.
(H$_6$) There exists a non increasing function λ such that $\lambda(\frac{u}{h_1}, \frac{v}{h_2}) = O(h_1 h_2)$ on any bounded rectangle and for two couple of real numbers $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$|\mathbb{K}(x) - \mathbb{K}(y)| \leq \lambda \|x - y\| \quad \text{and} \quad \lambda(u,v) \longrightarrow 0, (u,v) \to (0,0), u \geq 0, v \geq 0,$$

where $\|\cdot\|$ stands for the Euclidean norm.

Finally, these condions depend of the *pdf* $f(x,y)$:
C$_1$: $f\ (x,y)$ is uniformly continuous.
C$_2$: $f(x,y)$ admits almost everywhere derivative $f^{'}(x,y) \in L_1(\mathbb{R} \times \mathbb{R})$.
We are now able to describe our results that we organize in subsections.

2.1. The uniform almost sure consistency and the behavior of the bias

Theorem 1. *Assume that the hypotheses H_4 and C_1 hold. Then, for all $b > 0$ $(i.e. b = (b_1, b_2), b_1 > 0$ and $b_2 > 0)$, the estimator $P_n(z_1, z_2, \alpha_1, \alpha_2)$ converges uniformly almost surely on $[0, b_1] \times [0, b_2]$ to $P(z_1, z_2, \alpha_1, \alpha_2)$ as $n \to +\infty$ i.e.*

$$P(\lim_{n \to +\infty} \sup_{(z_1,z_2) \in [0,b_1] \times [0,b_2]} |P_n(z_1, z_2, \alpha_1, \alpha_2) - P(z_1, z_2, \alpha_1, \alpha_2)| = 0) = 1,$$

provided $nh_1^2 h_2^2 (loglogn)^{-1} \to +\infty$ as $n \to +\infty$.

Theorem 2. *Assume that the hypotheses H_4, H_5 and C_2 hold, for all $b > 0$ $(b = (b_1, b_2)$, the estimator $P_n(z_1, z_2, \alpha_1, \alpha_2)$ converges uniformly almost surely on $[0, b_1] \times [0, b_2]$ to $P(z_1, z_2, \alpha_1, \alpha_2)$ as $n \to +\infty$ i.e.*

$$P(\lim_{n \to +\infty} \sup_{(z_1,z_2) \in [0,b_1] \times [0,b_2]} |P_n(z_1, z_2, \alpha_1, \alpha_2) - P(z_1, z_2, \alpha_1, \alpha_2)| = 0) = 1,$$

provided $nh_1^2 h_2^2 (loglogn)^{-1} \to +\infty$ as $n \to +\infty$.

Lemma 1. *If C_1 holds, then for all $b > 0$, we have*

$$\lim_{n \to +\infty} \sup_{(z_1,z_2) \in [0,b_1] \times [0,b_2]} |\mathbb{E}(P_n(z_1, z_2, \alpha_1, \alpha_2)) - P(z_1, z_2, \alpha_1, \alpha_2)| = 0.$$

Lemma 2. *Let $M = \sup_{(z_1,z_2) \in \mathbb{R}^2} \frac{F(z_1,z_2)}{z_1 z_2}$ and $A = \sup_{(x,y) \in \mathbb{R}^2} f(x,y)$, If H_5 and C_2 hold, then*

$$\begin{aligned} \sup_{(z_1,z_2) \in \mathbb{R}^2} |\mathbb{E}(P_n(z_1, z_2, \alpha_1, \alpha_2)) - P(z_1, z_2, \alpha_1, \alpha_2)| \leq h_1 h_2 \Big((\int_{\mathbb{R}^2} |f^{'}(x,y)|\, dxdy) \\ \times (\int_{\mathbb{R}^2} (|u|+1)(|v|+1) |\mathbb{K}(u,v)|\, dudv) \\ + 4(\alpha_1 \alpha_2 M + A h_1 h_2) \int_{-\infty}^{+\infty} |\mathbb{K}(u,v)|\, dudv \Big). \end{aligned}$$

Remark 1. If K satisfies the hypothesis H_5, then by using H_1, the kernel $\hat{\mathbb{K}} = \frac{\mathbb{K}^2}{\int_{\mathbb{R}^2} \mathbb{K}^2(y_1,y_2)\, dy_1 dy_2}$ also satisfies it.

From the two previous lemmas, we get the following corollaries

Corollary 1. *Under the assumptions of Lemma 1, we have uniformly on* $[0,b_1]\times[0,b_2]$

$$\lim_{n\to+\infty}\mathbb{E}\Big(\sum_{i=1}^{[\frac{z_1}{h_1}]}\sum_{j=1}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{2\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{2\alpha_2}K^2\Big(\frac{X_k-ih_1}{h_1}\Big)K^2\Big(\frac{Y_k-ih_2}{h_2}\Big)\Big)$$
$$=P(z_1,z_2,2\alpha_1,2\alpha_2)\int_{\mathbb{R}^2}\mathbb{K}^2(y_1,y_2)\,dy_1dy_2.$$

Corollary 2. *If the assumptions of Theorem 2 hold and if* $h_1h_2=O(n^{-1}\log\log n)^{1/4}$, *then for all* $b>0$, *we have almost-surely*

$$\sup_{(z_1,z_2)\in[0,b_1]\times[0,b_2]}|P_n(z_1,z_2,\alpha_1,\alpha_2)-P(z_1,z_2,\alpha_1,\alpha_2)|=O(n^{-1}\log\log n)^{1/4}.$$

2.2. The uniform mean square consistency

Theorem 3. *If* H_6 *and* C_1 *hold. Then*

1.

$$\lim_{n\to+\infty}n\mathbb{V}ar(P_n(z_1,z_2,\alpha_1,\alpha_2))=P(z_1,z_2,2\alpha_1,2\alpha_2)\int_{\mathbb{R}^2}\mathbb{K}^2(y_1,y_2)\,dy_1dy_2$$
$$-\Big(P(z_1,z_2,\alpha_1,\alpha_2)\Big)^2.$$

2. For all $b>0$,

$$\lim_{n\to+\infty}\sup_{(z_1,z_2)\in[0,b_1]\times[0,b_2]}\mathbb{E}\Big(P_n(z_1,z_2,\alpha_1,\alpha_2)-P(z_1,z_2,\alpha_1,\alpha_2)\Big)^2=0.$$

Theorem 4. *Assume that* H_6 *and* C_2 *hold. Then*

$$\lim_{n\to+\infty}nVar(P_n(z_1,z_2,\alpha_1,\alpha_2))=P(z_1,z_2,2\alpha_1,2\alpha_2)\int_{\mathbb{R}^2}\mathbb{K}^2(y_1,y_2)\,dy_1dy_2$$
$$-\Big(P(z_1,z_2,\alpha_1,\alpha_2)\Big)^2.$$

Moreover, if H_5 *holds we have for all* $b>0$,

$$\lim_{n\to+\infty}\sup_{(z_1,z_2)\in[0,b_1]\times[0,b_2]}\mathbb{E}\Big(P_n(z_1,z_2,\alpha_1,\alpha_2)-P(z_1,z_2,\alpha_1,\alpha_2)\Big)^2=0.$$

For the proof of this theorem, first we prove the Theorem 5 below using the

Lemma 3. *Let* $0\le\theta_i^j\le 1, i,j=1,2$. *Then for all* $x=(x_1,x_2), y=(y_1,y_2)$ *and* $x\ne y$ *we have*

$$\lim_{n\to+\infty}\sup_{(z_1,z_2)\in[0,1]\times[0,1]}\Big((h_1h_2)^{-2}\int\int_{-\infty}^{+\infty}|K\big(\frac{u_1-x_1+\theta_1^1}{h_1},\frac{u_2-x_2+\theta_1^2}{h_2}\big)$$
$$\times K\big(\frac{u_1-y_1+\theta_2^1}{h_1},\frac{u_2-y_2+\theta_2^2}{h_2}\big)|f(u_1,u_2)\,du_1du_2\Big)\ =\ 0.$$

Theorem 5. *Assume that the hypothesis H_6 holds. Then for all $b > 0$,*

$$\lim_{n\to+\infty} \sup_{(z_1,z_2)\in[0,b_1]\times[0,b_2]} \sum_{0\le i_1\ne j_1\le[\frac{z_1}{h_1}]} \sum_{0\le i_2\ne j_2\le[\frac{z_2}{h_2}]} \Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{j_1h_1}{z_1}\Big)^{\alpha_1}$$
$$\times\Big(1-\frac{i_2h_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{j_2h_2}{z_2}\Big)^{\alpha_2}$$
$$\times\int_{\mathbb{R}^2} K\Big(\frac{u_1-i_1h_1}{h_1}\Big)K\Big(\frac{u_1-j_1h_1}{h_1}\Big)K\Big(\frac{u_2-i_2h_2}{h_2}\Big)$$
$$K\Big(\frac{u_2-j_2h_2}{h_2}\Big)f(u_1,u_2)\,du_1du_2 \;=\; 0.$$

Remark 2. The estimator $P_n(z_1, z_2, \alpha_1, \alpha_2)$ has asymptotic efficiency with respect to $\widehat{P}_n(z_1, z_2, \alpha_1, \alpha_2)$,

$$e(z_1,z_2,\alpha_1,\alpha_2)=\frac{P(z_1,z_2,2\alpha_1,2\alpha_2)\int_{\mathbf{R}^2}\mathbb{K}^2(y_1,y_2)\,dy_1dy_2-(P(z_1,z_2,\alpha_1,\alpha_2))^2}{P(z_1,z_2,2\alpha_1,2\alpha_2)-(P(z_1,z_2,\alpha_1,\alpha_2))^2}.$$

The integral $\int_{\mathbf{R}^2} K^2(y_1, y_2)\,dy_1dy_2$ is strictly inferior to 1 for K product of two conventional kernels Parzen (1962) p.1068. Then we have in this case $e(z_1, z_2, \alpha_1, \alpha_2) < 1$. In Theorem 4, the rate of convergence in mean square is estimated to $O(\frac{1}{n})$ if h_1h_2 is estimated to $O(\frac{1}{\sqrt{n}})$.

3. Simulations

We conducted simulation studies based on of 75 replications of samples of size $n = 50$ and computed the value of our estimator and that of the empirical plug-in estimator. We considered Gaussian kernels that fullfil $H_i, i = 1, ..., 6$ and $h_j = 1/\sqrt{nlogn}, j = 1, 2$. As for the distribution, we used a couple of independent coordinates with first margin following a Pareto distribution type on $[0, 1]$ with parameters $x_0 = 0.02$ and $b = 0.2$, and a second margin following an exponential distribution with parameter $\lambda = 1$. For a fixed couple of poverty lines, for a fixed n, replications of samples give the following replicated values for the two estimators, our kernel-type one and the empirical plug-in estimator, in the respective replicated sequences :

$$(P_{n,1}(z_1,z_2,\alpha_1,\alpha_2),...,P_{n,75}(z_1,z_2,\alpha_1,\alpha_2))$$

and

$$(\widehat{P}_{n,1}(z_1,Z_2,\alpha_1,\alpha_2),...,\widehat{P}_{n,75}(z_1,z_2,\alpha_1,\alpha_2))$$

From each sequence, we compute the mean value mv_i, the mean square error msq_i and the variance σ_i, where $i = 1$ corresponds to the Kernel-type estimator and $i = 2$ to the empirical estimator. For the first case, this is :

$$mv_1=\overline{P_n(z_1,z_2,\alpha_1,\alpha_2)}=\frac{1}{75}\sum_{i=1}^{75}P_{n,i}(z_1,z_2,\alpha_1,\alpha_2),$$

$$msqe1=\frac{1}{75}\sum_{i=1}^{75}(P_{n,i}(z_1,z_2,\alpha_1,\alpha_2)-P(z_1,z_2,\alpha_1,\alpha_2))^2,$$

$$\sigma_1 = \frac{1}{74}\sum_{i=1}^{75}(P_{n,i}(z_1, z_2, \alpha_1, \alpha_2) - \overline{P_n(z_1, z_2, \alpha_1, \alpha_2)})^2.$$

For the second case, we similarly define mv_2, $msqe2$ and σ_2.

Next, we choose $(\alpha_1, \alpha_2) = (0, 0)$, $(\alpha_1, \alpha_2) = (1, 1)$ and $(\alpha_1, \alpha_2) = (2, 2)$. The outcomes are summerized in the table below.

(z_1, z_2)	$(0.1, 0.1)$	$(0.1, 0.4)$	$(0.1, 0.8)$	$(0.4, 0.4)$	$(0.4, 0.8)$
$(\alpha_1 = 0, \alpha_2 = 0); n = 50$					
mqse1	4.52376e-04	4.25607e-03	1.07638e-02	4.40944e-03	1.03522e-02
mqse2	0.871967	7.923746	23.95921	35.87961	104.3472
σ_1	9.91613e-06	9.84891e-05	2.46968e-04	7.26561e-04	1.5741e-03
σ_2	0.264786	1.283177	3.634951	1.867318	23.41582
$(\alpha_1 = 1, \alpha_2 = 1); n = 50$					
mqse1	2.486597	0.2256831	0.0739875	1.124062	0.342812
mqse2	1.993245	0.0701992	0.6961594	1.439354	10.39371
σ_1	6.54092e-08	5.20152e-04	5.26027e-06	6.83591e-05	1.80800e-04
σ_2	0.0125571	0.0639057	0.641102	1.507185	4.193532
$(\alpha_1 = 2, \alpha_2 = 2); n = 50$					
mqse1	1273.946	4.87621	0.168094	35.68207	0.342669
mqse2	1277.474	5.709271	0.507776	49.40013	4.335062
σ_1	4.31454e-10	7.37444e-08	2.16516e-07	1.40352e-05	4.40888e-05
σ_2	1.40887e-03	8.76890e-03	2.32100e-02	0.15877	0.323104

Table 1. Comparative table of results simulations for small samples

The studied cases $P(z_1, z_2, 0, 0), P(z_1, z_2, 1, 1), P(z_1, z_2, 2, 2)$ are commonly and respectively called the two-dimensional poverty rate intersection, the two-dimensional depth of poverty intersection and the two-dimensional severity of poverty intersection Duclos *et al.* (2006a).

The results speak themselves. The kernel-type estimator behave much better.

4. Proofs of the results.

Lemma 4. *Let* $0 \leq \theta_i < 1$, $i = 1, 2$. *If* $f(x, y)$ *is uniformly continuous and bounded, we have uniformly relatively to* $(\theta_i, y_i) \subset [0, 1] \times R$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} (h_{1,n}h_{2,n})^{-1}\mathbb{K}(\frac{u_1}{h_{1,n}}, \frac{u_2}{h_{2,n}})f(u_1 + y_1 - \theta_1 h_{1,n}, u_2 + y_2 - \theta_2 h_{2,n})\, du_1du_2$$
$$= f(y_1, y_2) \int_{\mathbb{R}^2} \mathbb{K}(u_1, u_2)\, du_1du_2.$$

Proof. Since $f(x, y)$ is uniformly continuous, then for $\varepsilon > 0$, there exists $\eta > 0$ such that for $\|\theta_i h_{i,n}\| < \eta$, $i = 1, 2$,

$$\begin{aligned}
&|\int_{\mathbb{R}^2} (h_{1,n}h_{2,n})^{-1}\mathbb{K}(\frac{u_1}{h_{1,n}}, \frac{u_2}{h_{2,n}})f(u_1 + y_1 - \theta_1 h_{1,n}, u_2 + y_2 - \theta_2 h_{2,n})\, du_1du_2 \\
&\quad - \int_{\mathbb{R}^2} (h_{1,n}h_{2,n})^{-1}\mathbb{K}(\frac{u_1}{h_{1,n}}, \frac{u_2}{h_{2,n}})f(u_1 + y_1, u_2 + y_2)\, du_1du_2| \\
&\quad \leq \int_{\mathbb{R}^2} (h_{1,n}h_{2,n})^{-1}|\mathbb{K}(\frac{u_1}{h_{1,n}}, \frac{u_2}{h_{2,n}})||f(u_1 + y_1 - \theta_1 h_{1,n}, u_2 + y_2 - \theta_2 h_{2,n}) \\
&\quad - f(u_1 + y_1, u_2 + y_2)|\, du_1du_2 \leq \varepsilon \int_{\mathbb{R}^2} (h_{1,n}h_{2,n})^{-1}|\mathbb{K}(\frac{u_1}{h_{1,n}}, \frac{u_2}{h_{2,n}})|\, du_1du_2.
\end{aligned}$$

For $y = (y_1, y_2)$, choose then n large enough such that $\|\theta h_n\| < \eta$, $\theta = (\theta_1, \theta_2)$ to get the inequality for all (θ_i, y_i), $i = 1, 2$. It comes up that

$$\begin{aligned}
&|\int_{\mathbb{R}^2} (h_{1,n}h_{2,n})^{-1}\mathbb{K}(\frac{u_1}{h_{1,n}}, \frac{u_2}{h_{2,n}})f(u_1 + y_1 - \theta_1 h_{1,n}, u_2 + y_2 - \theta_2 h_{2,n})\, du_1du_2 \\
&\qquad - f(y_1, y_2) \int_{\mathbb{R}^2} \mathbb{K}(u_1, u_2)\, du_1du_2| \\
&\qquad \leq |\int_{\mathbb{R}^2} (h_{1,n}h_{2,n})^{-1}\mathbb{K}(\frac{u_1}{h_{1,n}}, \frac{u_2}{h_{2,n}})f(u_1 + y_1, u_2 + y_2)\, du_1du_2 \\
&\qquad - \int_{\mathbb{R}^2} (h_{1,n}h_{2,n})^{-1}\mathbb{K}(\frac{u_1}{h_{1,n}}, \frac{u_2}{h_{2,n}})f(y_1, y_2)\, du_1du_2|.
\end{aligned}$$

Let $g(x, y)$ an uniformly continuous function verifying

$$\int\int_{-\infty}^{+\infty} |g(v_1, v_2)|\, dv_1dv_2 < \infty.$$

Let $h_n = (h_{1,n}, h_{2,n})$ be a sequence of positive constants satisfing

$$\lim_{n\to+\infty} h_n = 0.$$

Define

$$g_n(x_1, x_2) = \frac{1}{h_{1,n}h_{2,n}} \int\int_{-\infty}^{+\infty} \mathbb{K}(\frac{v_1}{h_{1,n}}, \frac{v_2}{h_{2,n}})g(x_1 - v_1, x_2 - v_2)\, dv_1dv_2.$$

Then

$$g_n(x_1, x_2) - g(x_1, x_2) \int\int_{-\infty}^{+\infty} \mathbb{K}(v_1, v_2)\, dv_1 dv_2$$
$$= \int_{\mathbb{R}^2} (h_{1,n}h_{2,n})^{-1}\mathbb{K}(\frac{v_1}{h_{1,n}}, \frac{v_2}{h_{2,n}})\, [g(x_1 - v_1, x_2 - v_2) - g(x_1, x_2)]\, dv_1 dv_2.$$

Let $\delta > 0$ and split the region of integration into two regions: $\|v\| \leq \delta$ and $\|v\| > \delta$. Then we have

$$\begin{aligned}
G &= |g_n(x_1, x_2) - g(x_1, x_2) \int\int_{-\infty}^{+\infty} \mathbb{K}(v_1, v_2)\, dv_1 dv_2| \\
&\leq \max_{\|v\|\leq\delta} |g(x_1 - v_1, x_2 - v_2) - g(x_1, x_2)| \int\int_{\|z\|\leq\frac{\delta}{\|h\|}} |\mathbb{K}(z_1, z_2)|\, dz_1 dz_2 \\
&+ \int\int_{\|v\|\geq\delta} \frac{|g(x_1 - v_1, x_2 - v_2)|}{\|v\|} \frac{\|v\|}{h_{1,n}h_{2,n}} \mathbb{K}(\frac{v_1}{h_{1,n}}, \frac{v_2}{h_{2,n}})\, dv_1 dv_2 \\
&+ |g(x_1, x_2)| \int\int_{\|v\|\geq\delta} (h_{1,n}h_{2,n})^{-1}\mathbb{K}(\frac{u_1}{h_{1,n}}, \frac{u_2}{h_{2,n}})\, dv_1 dv_2 \\
&\leq \max_{\|v\|\leq\delta} |g(x_1 - v_1, x_2 - v_2) - g(x_1, x_2)| \int\int_{-\infty}^{+\infty} |\mathbb{K}(z_1, z_2)|\, dz_1 dz_2 \\
&+ \frac{1}{\delta} \sup_{\|z\|\geq\frac{\delta}{h_{1,n}h_{2,n}}} \|(z_1, z_2)\|\, |\mathbb{K}(z_1, z_2)| \int\int_{\|z\|\geq\frac{\delta}{h_{1,n}h_{2,n}}} |g(z_1, z_2)|\, dz_1 dz_2 \\
&+ |g(x_1, x_2)| \int\int_{\|z\|\geq\frac{\delta}{h_{1,n}h_{2,n}}} |\mathbb{K}(z_1, z_2)|\, dz_1 dz_2.
\end{aligned}$$

As one lets n tends to ∞,by choosing a small δ, you may use the uniform continuity of g to prove that the fisrt term tends to 0. The second term tends to 0, from the hypotheses on $\mathbb{K}(x, y)$. The third term too, since $\mathbb{K}(x, y)$ has compact support. Then at every point (x_1, x_2), we have

$$\lim_{n\to+\infty} g_n(x_1, x_2) = g(x_1, x_2) \int\int_{-\infty}^{+\infty} \mathbb{K}(v_1, v_2)\, dv_1 dv_2.$$

Therefore

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} (h_{1,n}h_{2,n})^{-1}\mathbb{K}(\frac{u_1}{h_{1,n}}, \frac{u_2}{h_{2,n}}) f(u_1{+}y_1, u_2{+}y_2)\, du_1 du_2 = f(y_1, y_2) \int_{\mathbb{R}^2} \mathbb{K}(u_1, u_2)\, du_1 du_2$$

uniformly, the proof of the lemma is then complete. □

4.1. Convergence of the estimator. Proof of the Lemma 1

Observe first that

$$\lim_{(z_1,z_2)\to(0,0)} \frac{F(z_1, z_2) - F(z_1, 0) - F(0, z_2) + F(0, 0)}{z_1 z_2} = f(0, 0).$$

Indeed we have by the mean value theorem

$$F(z_1,z_2)-F(z_1,0)=(z_2-0)\frac{\partial F}{\partial z_2}(z_1,s_1) \quad \text{with} \quad s_1\in]0,z_2[,$$

and

$$F(0,0)-F(0,z_2)=-(z_2-0)\frac{\partial F}{\partial z_2}(0,s_1) \quad \text{with} \quad s_1\in]0,z_2[.$$

Therefore

$$\begin{aligned} F(z_1,z_2)-F(z_1,0)-F(0,z_2)+F(0,0) &= z_2(\frac{\partial F}{\partial z_2}(z_1,s_1)-\frac{\partial F}{\partial z_2}(0,s_1)) \\ &= z_2(z_1-0)\frac{\partial F}{\partial z_1\partial z_2}(s_1,s_2) \quad \text{with} \quad s_1\in]0,z_1[\\ &= z_1z_2\frac{\partial^2 F}{\partial z_1\partial z_2}(s_1,s_2). \end{aligned}$$

Hence

$$\frac{F(z_1,z_2)-F(z_1,0)-F(0,z_2)+F(0,0)}{z_1z_2}=\frac{\partial^2 F}{\partial z_1\partial z_2}(s_1,s_2),$$

and

$$\frac{\partial^2 F}{\partial z_1\partial z_2}(s_1,s_2)\to\frac{\partial^2 F}{\partial z_1\partial z_2}(0,0)=f(0,0) \quad \text{as} \quad (z_1,z_2)\to(0,0).$$

Therefore $\frac{\partial^2 F}{z_1 z_2}(z_1,z_2)$ is bounded.

Let $\bar{\Delta}_{h_1,i}=\Delta_{h_1,i}\cap[0,z_1]$; $\bar{\Delta}_{h_2,j}=\Delta_{h_2,j}\cap[0,z_2]$ and χ_B the indicator function of the set $B=B_1\times B_2$. For $(z_1,z_2)\in[0,b_1]\times[0,b_2]$. We have

$$\begin{aligned} \mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) = \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\int_{\mathbb{R}^2}\mathbb{K}(u_1,u_2) \\ \times f(u_1h_1-ih_1,u_2h_2-jh_2)\,du_1du_2, \end{aligned}$$

which can be written in the following form

$$\begin{aligned} \int_0^{z_1}\int_0^{z_2}\sum_{i=0}^{[\frac{z_1}{h_1}]}\chi_{\bar{\Delta}_{h_1,i}}(x)\sum_{j=0}^{[\frac{z_2}{h_2}]}\chi_{\bar{\Delta}_{h_2,j}}(y)\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2} \\ \times\int\int_{-\infty}^{+\infty}\mathbb{K}(u_1,u_2)f(u_1h_1-ih_1,u_2h_2-jh_2)\,du_1du_2dxdy \\ +(h_1([\frac{z_1}{h_1}]+1)-z_1)(h_2([\frac{z_2}{h_2}]+1)-z_2)\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2} \\ \times\int\int_{-\infty}^{+\infty}\mathbb{K}(u_1,u_2)f(u_1h_1+\frac{h_1[\frac{z_1}{h_1}]}{z_1},u_2h_2-\frac{h_2[\frac{z_2}{h_2}]}{z_2})\,du_1du_2. \end{aligned} \tag{4}$$

We have

$$\sup_{(z_1,z_2)\in\mathbb{R}^2} |(h_1([\frac{z_1}{h_1}]+1)-z_1)(h_2([\frac{z_2}{h_2}]+1)-z_2)\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2} \tag{5}$$

$$\times\int\int_{-\infty}^{+\infty}\mathbb{K}(u_1,u_2)f(u_1h_1+\frac{h_1[\frac{z_1}{h_1}]}{z_1},u_2h_2-\frac{h_2[\frac{z_2}{h_2}]}{z_2})\,du_1du_2|$$

$$\le h_1h_2\sup_{(x,y)\in\mathbb{R}^2}f(x,y)\int\int_{-\infty}^{+\infty}|\mathbb{K}(u_1,u_2)|\,du_1du_2,$$

since $|h_i([\frac{z_i}{h_i}]+1)-z_i|\le h_i \quad i=1,2.$

According to $\mathbf{H_2}$, we can write

$$P(z_1,z_2,\alpha_1,\alpha_2)=\int\int_{-\infty}^{+\infty}\int_0^{z_1}\int_0^{z_2}\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2}\mathbb{K}(u_1,u_2)\,du_1du_2f(x,y)\,dxdy. \tag{6}$$

Let $(x,y)\in\bar{\Delta}_{h_1,i}\times\bar{\Delta}_{h_2,j}$. By considering the terms in 4 and in 6 we get,

$$|\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\int\int_{-\infty}^{+\infty}\mathbb{K}(u_1,u_2)f(u_1h_1-ih_1,u_1h_2-jh_2)\,du_1du_2 \tag{7}$$

$$-\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2}\mathbb{K}(u_1,u_2)\,du_1du_2f(x,y)|$$

$$=|\int\int_{-\infty}^{+\infty}\Big[\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}f(u_1h_1-ih_1,u_2h_2-jh_2)$$

$$-\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2}f(x,y)\Big]\mathbb{K}(u_1,u_2)\,du_1du_2|$$

$$\le\int\int_{-\infty}^{+\infty}|\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}-\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\times$$

$$\Big(1-\frac{y}{z_2}\Big)^{\alpha_2}|f(x,y)|\mathbb{K}(u_1,u_2)|\,du_1du_2$$

$$+\int\int_{-\infty}^{+\infty}|\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}|\times$$

$$|f(u_1h_1-ih_1,u_1h_2-jh_2)-f(x,y)||\mathbb{K}(u_1,u_2)|\,du_1du_2.$$

For $(x,y)\in\Delta_{h_1,i}\times\Delta_{h_2,j}$, we have by the first order Taylor formula applied to the function

$$g(x,y)=\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2},$$

for $c_1\in]h_1i,x[$ and $c_2\in]h_2j,y[$,

$$\Big|\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}-\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2}\Big|$$

$$=\Big|\Big(1-\frac{c_1}{z_1}\Big)^{\alpha_1-1}\frac{\alpha_1}{z_1}\Big(1-\frac{c_2}{z_2}\Big)^{\alpha_2}\Big(ih_1-x\Big)$$

$$+\Big(1-\frac{c_1}{z_1}\Big)^{\alpha_1}\frac{\alpha_2}{z_2}\Big(1-\frac{c_2}{z_2}\Big)^{\alpha_2-1}\Big(jh_2-y\Big)\Big|$$

$$\le 2(\frac{\alpha_1h_1}{z_1}+\frac{\alpha_2h_2}{z_2})=2(\frac{z_2\alpha_1h_1+z_1\alpha_2h_2}{z_1z_2}).$$

Therefore, denoting by $I_1^{i,j}(x,y)$ the first integral of the right hand-side of (7) and

$$I_1(x,y) = \sum_{i=0}^{[\frac{z_1}{h_1}]} \chi_{\bar{\Delta}_{h_1,i}}(x) \sum_{j=0}^{[\frac{z_2}{h_2}]} \chi_{\bar{\Delta}_{h_2,j}}(y) I_1^{i,j}(x,y).$$

We have

$$\begin{aligned}\int_0^{z_1}\int_0^{z_2} I_1(x,y)\,dxdy &\leq \frac{2(\alpha_1 z_2+\alpha_2 z_1)\left\|\left(h_1,h_2\right)\right\|}{z_1 z_2}\times \\ &\int_0^{z_1}\int_0^{z_2}(\int\int_{-\infty}^{+\infty} f(x,y)|\mathbb{K}(u_1,u_2)|\,du_1du_2)\,dxdy \\ &= 2(\alpha_1 z_2+\alpha_2 z_1)\left\|\left(h_1,h_2\right)\right\|\times \\ &\left(\int\int_{-\infty}^{+\infty}|\mathbb{K}(u_1,u_2)|\,du_1du_2\right)\frac{F(z_1,z_2)}{z_1 z_2}.\end{aligned} \tag{8}$$

Denoting by $I_2^{i,j}(x,y)$ the second integral of the right hand-side of (7) and

$$I_2(x,y) = \sum_{i=0}^{[\frac{z_1}{h_1}]} \chi_{\bar{\Delta}_{h_1,i}}(x) \sum_{j=0}^{[\frac{z_2}{h_2}]} \chi_{\bar{\Delta}_{h_2,j}}(y) I_2^{i,j}(x,y).$$

We have

$$\begin{aligned}I_2^{i,j}(x,y) &\leq \int\int_{-\infty}^{+\infty}|f(u_1h_1-ih_1,u_2h_2-jh_2)-f(ih_1,jh_2)||\mathbb{K}(u_1,u_2)|\,du_1du_2 \\ &+\int\int_{-\infty}^{+\infty}|f(ih_1,jh_2)-f(x,y)||\mathbb{K}(u_1,u_2)|\,du_1du_2.\end{aligned}$$

Let $\varepsilon>0$, since $f(x,y)$ is uniformly continuous, there exists $\eta_0=\eta_0(z_1,z_2)>0$ such that $\|(ih_1,jh_2)-(x,y)\|<\eta_0$, hence if $\|h\|<\eta_0$, we have $|f(u_1h_1-ih_1,u_2h_2-jh_2)-f(x,y)|<\frac{\varepsilon}{\|b\|}$. Therefore

$$\begin{aligned}\int_0^{z_1}\int_0^{z_2} I_2(x,y)\,dxdy &\leq \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2\int\int_{-\infty}^{+\infty}|f(u_1h_1-ih_1,u_2h_2-jh_2)-f(ih_1,jh_2)| \\ &\times|\mathbb{K}(u_1,u_2)|\,du_1du_2+\varepsilon\int\int_{-\infty}^{+\infty}|\mathbb{K}(u_1,u_2)|\,du_1du_2dxdy.\end{aligned}$$

By the uniform continuity of $f(x,y)$ we have

$$\exists \quad \eta_1=\eta_1(z_1,z_2)>0, \|uh\|<\eta_1 \Rightarrow |f(u_1h_1-ih_1,u_2h_2-jh_2)-f(ih_1,jh_2)|<\frac{\varepsilon}{\|b\|}.$$

Hence

$$\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int\int_{-\infty}^{+\infty} |f(u_1h_1 - ih_1, u_2h_2 - jh_2) - f(ih_1, jh_2)||\mathbb{K}(u_1,u_2)|\, du_1du_2$$

$$\leq \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int\int_{\|uh\|<\eta_1} \frac{\varepsilon}{\|b\|}|\mathbb{K}(u_1,u_2)|\, du_1du_2$$

$$+ \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int\int_{\|uh\|\geq\eta_1} |f(u_1h_1 - ih_1, u_2h_2 - jh_2) - f(ih_1, jh_2)||\mathbb{K}(u_1,u_2)|\, du_1du_2$$

$$\leq \varepsilon \int\int_{-\infty}^{+\infty} |\mathbb{K}(u_1,u_2)|\, du_1du_2 + \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int\int_{\|uh\|\geq\eta_1} |f(u_1h_1 - ih_1, u_2h_2 - jh_2)|$$

$$\times |\mathbb{K}(u_1,u_2)|\, du_1du_2 + \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int\int_{\|uh\|\geq\eta_1} |f(ih_1, jh_2)||\mathbb{K}(u_1,u_2)|\, du_1du_2.$$

Since $f(x,y)$, is continuous, its is Riemann-integrable. Hence

$$\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2|f(ih_1, jh_2)| \to \int_0^{b_1}\int_0^{b_2} f(x,y)\, dxdy \qquad n \to +\infty.$$

Since

$$(h_1([\frac{b_1}{h_1}]+1) - b_1)(h_2([\frac{b_2}{h_2}]+1) - b_2) f(h_1[\frac{b_1}{h_1}], h_2[\frac{b_2}{h_2}]) \to 0, \quad n \to +\infty,$$

and

$$\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2|f(ih_1, jh_2)| \to \int_0^{b_1}\int_0^{b_2} f(x,y)\, dxdy.$$

This last sum is bounded, let A be its bound.

By the change of variables $v_i = u_ih_i \quad i = 1,2,$ we have

$$\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int\int_{\|uh\|\geq\eta_1} |f(u_1h_1 - ih_1, u_2h_2 - jh_2)||\mathbb{K}(u_1,u_2)|\, du_1du_2$$

$$+ \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int\int_{\|uh\|\geq\eta_1} |f(ih_1, jh_2)||\mathbb{K}(u_1,u_2)|\, du_1du_2$$

$$\leq \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int\int_{\|v\|\geq\eta_1} |f(v_1 - ih_1, v_2 - jh_2)| \frac{1}{h_1h_2}|\mathbb{K}(\frac{v_1}{h_1}, \frac{v_2}{h_2})|\, dv_1dv_2$$

$$+ A \int\int_{\|uh\|\geq\eta_1} |\mathbb{K}(u_1,u_2)|\, du_1du_2.$$

According to $(\mathbf{H_3})$ there exists $C > 0$ such that for $|\frac{v_1}{h_1}\frac{v_2}{h_2}| \geq C$, we have

$$|\frac{v_1}{h_1}\frac{v_2}{h_2}||\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})| \leq \frac{\eta_1\varepsilon}{\|b\|}.$$

Let $\eta = \inf(\eta_1, Ch_1h_2)$, $i = 1, 2$ being small enough, then

$$\begin{aligned}
&\frac{1}{\eta_1}\int\int_{\|v\|\geq\eta_1} |f(v_1 - ih_1, v_2 - jh_2)\frac{v_1}{h_1}\frac{v_2}{h_2}\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})|\, dv_1dv_2 \\
&\leq \frac{\varepsilon}{\|b\|}\int\int_{\|v\|\geq\eta_1} |f(v_1 - ih_1, v_2 - jh_2)|\, dv_1dv_2 \\
&\leq \frac{\varepsilon}{\|b\|}\int\int_{\mathbb{R}^2} |f(x,y)|\, dxdy = \frac{\varepsilon}{\|b\|}.
\end{aligned}$$

Where

$$\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2\int\int_{\|v\|\geq\eta_1} |f(v_1 - ih_1, v_2 - jh_2)|\frac{1}{h_1h_2}|\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})|\, dv_1dv_2 \leq \frac{z_1z_2\varepsilon}{\|b\|} \leq \varepsilon.$$

Since $A\int\int_{\|uh\|\geq\eta_1} |\mathbb{K}(u_1,u_2)|\, du_1du_2 \to 0, n \to +\infty$, we have together with 8 and 5

$$\begin{aligned}
\lim_{n\to+\infty}\ \sup_{(z_1,z_2)\in[0,b_1]\times[0,b_2]} &|\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) - P(z_1,z_2,\alpha_1,\alpha_2)| \\
&\leq 2\varepsilon\int\int_{-\infty}^{+\infty} \mathbb{K}(u_1,u_2)\, du_1du_2 + \varepsilon.
\end{aligned}$$

The proof of the lemma is complete. □

4.2. Proof of Lemma 2

For $(x,y) \in \bar{\Delta}_{h_1,i} \times \bar{\Delta}_{h_2,j}$ $\quad i = 1, ..., [\frac{z_1}{h_1}]; j = 1, ..., [\frac{z_2}{h_2}]$, we have

$$\begin{aligned}
&\int\int_{-\infty}^{+\infty} |\Big(1 - \frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1 - \frac{jh_2}{z_2}\Big)^{\alpha_2}||f(u_1h_1 - ih_1, u_1h_2 - jh_2)||\mathbb{K}(u_1,u_2)|\, du_1du_2 \\
&\leq (\int\int_{-\infty}^{+\infty}\int_x^{x+h_1(|u_1|+1)}\int_y^{y+h_2(|u_2|+1)} |f^{'}(t_1,t_2)||\mathbb{K}(u_1,u_2)|\, du_1du_2).
\end{aligned}$$

Hence $I_2(x,y)$ being as in the previous **Lemma 1**, the following inequality holds

$$\begin{aligned}
&\int_0^{z_1}\int_0^{z_2} I_2(x,y)\, dxdy \\
&\leq \int_0^{z_1}\int_0^{z_2}\Big(\int\int_{-\infty}^{+\infty}\int_x^{x+h_1(|u_1|+1)}\int_y^{y+h_2(|u_2|+1)} |f^{'}(t_1,t_2)||\mathbb{K}(u_1,u_2)|\, du_1du_2\Big)\, dxdy.
\end{aligned}$$

By the change of variables $t_1 = x + h_1(|u_1|+1)v_1$; $t_2 = y + h_2(|u_2|+1)v_2$, we have by Fubini's theorem

$$\int_0^{z_1}\int_0^{z_2} I_2(x,y)\,dxdy \le h_1h_2 \int\int_{\mathbb{R}^2} (|u_1|+1)(|u_2|+1)|\mathbb{K}(u_1,u_2)|$$
$$\times \Big(\int\int_{\mathbb{R}^2}\Big)|f^{'}(x+h_1(|u_1|+1)v_1, y+h_2(|u_2|+1)v_2)|\,dxdy\,du_1du_2 \int\int_0^1 dv_1dv_2$$
$$= \int\int_{\mathbb{R}^2} |f^{'}(x,y)|\,dxdy.$$

This inequality together with inequality 5 and 8 lead to the completion of the proof. □

4.3. Proof of Theorem 1 and Theorem 2

Let $\hat{F}_n$ be the empirical distribution of the sample $(X_1,Y_1),(X_2,Y_2),...,(X_n,Y_n)$ define by

$$F_n(l_1,l_2) = n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}(\chi_{X_i<l_1}\chi_{Y_j<l_2}),$$

where χ_A stands for the indicator function of a set A. We can write

$$P_n(z_1,z_2,\alpha_1,\alpha_2) = \int\int_{\mathbb{R}^2}\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathbb{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)\,d\hat{F}_n(l_1,l_2),$$

and

$$\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) = \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\int\int_{\mathbb{R}^2}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathbb{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)\,dF(l_1,l_2).$$

We have

$$|P_n(z_1,z_2,\alpha_1,\alpha_2) - \mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2))|$$
$$= |\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\int\int_{\mathbb{R}^2}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathbb{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)\,(d\hat{F}_n(l_1,l_2)-dF(l_1,l_2))|.$$

First we are going to apply the Fubini's theorem and then integrate by parts with respect to for each variable.

We obtain then

$$\begin{aligned}
&\int\int_{\mathbb{R}^2} \mathbb{K}(\frac{l_1 - ih_1}{h_1}, \frac{l_2 - jh_2}{h_2})\,(d\hat{F}_n(l_1,l_2) - dF(l_1,l_2)) \\
&\qquad = \int_{\mathbb{R}} K(\frac{l_1 - ih_1}{h_1}) \int_{\mathbb{R}} K(\frac{l_2 - jh_2}{h_2})\,(d\hat{F}_n(l_1,l_2) - dF(l_1,l_2)) \\
&\qquad = \int_{\mathbb{R}} K(\frac{l_1 - ih_1}{h_1})\Big[\int_{\mathbb{R}} K(\frac{l_2 - jh_2}{h_2})\frac{\partial^2(\hat{F}_n(l_1,l_2) - F(l_1,l_2))}{\partial l_1 \partial l_2} dl_1 dl_2\Big] \\
&\qquad = \int_{\mathbb{R}} K(\frac{l_1 - ih_1}{h_1})\Big\{\Big[K(\frac{l_2 - jh_2}{h_2})\frac{\partial(\hat{F}_n(l_1,l_2) - F(l_1,l_2))}{\partial l_1} dl_1\Big]_{-\infty}^{+\infty} \\
&\qquad\qquad - \int_{\mathbb{R}} \frac{\partial K(\frac{l_2 - jh_2}{h_2})}{\partial l_2} dl_2 \frac{\partial(\hat{F}_n(l_1,l_2) - F(l_1,l_2))}{\partial l_1} dl_1\Big\}
\end{aligned}$$

$$\begin{aligned}
&= -\int_{\mathbb{R}} K(\frac{l_1 - ih_1}{h_1})\frac{\partial(\hat{F}_n(l_1,l_2) - F(l_1,l_2))}{\partial l_1} dl_1 \int_{\mathbb{R}} \frac{\partial K(\frac{l_2 - jh_2}{h_2})}{\partial l_2} dl_2 \\
&= -\Big\{\Big[K(\frac{l_1 - ih_1}{h_1})(\hat{F}_n(l_1,l_2) - F(l_1,l_2))\Big]_{-\infty}^{+\infty} - \int_{\mathbb{R}} \frac{\partial K(\frac{l_1 - ih_1}{h_1})}{\partial l_1} dl_1 (\hat{F}_n(l_1,l_2) - F(l_1,l_2))\Big\} \\
&\qquad \times \int_{\mathbb{R}} \frac{\partial K(\frac{l_2 - jh_2}{h_2})}{\partial l_2} dl_2 \\
&= \int_{\mathbb{R}^2} dK(\frac{l_1 - ih_1}{h_1}) dK(\frac{l_2 - ih_2}{h_2})(\hat{F}_n(l_1,l_2) - F(l_1,l_2)),
\end{aligned}$$

where $d_{l_i}K(\cdot) = \dfrac{\partial K(\cdot)\, dl_i}{\partial l_i}$ denote the derivative of K with respect to l_i. Therefore

$$\begin{aligned}
&|P_n(z_1,z_2,\alpha_1,\alpha_2) - \mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2))| \\
&\le \sum_{i=0}^{[\frac{z_1}{h_1}]} \sum_{j=0}^{[\frac{z_2}{h_2}]} \int\int_{\mathbb{R}^2} |dK(\frac{l_1 - ih_1}{h_1}) dK(\frac{l_2 - ih_2}{h_2})| \sup_{(l_1,l_2)\in\mathbb{R}^2} |\hat{F}_n(l_1,l_2) - F(l_1,l_2)| \\
&\le \sum_{i=0}^{[\frac{z_1}{h_1}]} \sum_{j=0}^{[\frac{z_2}{h_2}]} \int_{\mathbb{R}} dV_{-\infty}^{\frac{l_1 - ih_1}{h_1}} K \int_{\mathbb{R}} dV_{-\infty}^{\frac{l_2 - ih_2}{h_2}} \sup_{(l_1,l_2)\in\mathbb{R}^2} |\hat{F}_n(l_1,l_2) - F(l_1,l_2)| \\
&\le [\frac{z_1}{h_1}][\frac{z_2}{h_2}] V(\mathbb{R}) \times V(\mathbb{R}) \sup_{(l_1,l_2)\in\mathbb{R}^2} |\hat{F}_n(l_1,l_2) - F(l_1,l_2)|.
\end{aligned}$$

Remarking that

$$\begin{aligned}
&|P_n(z_1,z_2,\alpha_1,\alpha_2) - (P_(z_1,z_2,\alpha_1,\alpha_2))| \\
&\quad = |P_n(z_1,z_2,\alpha_1,\alpha_2) - \mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) + \mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) - P(z_1,z_2,\alpha_1,\alpha_2)| \\
&\quad \le |P_n(z_1,z_2,\alpha_1,\alpha_2) - \mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2))| + |\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) - P(z_1,z_2,\alpha_1,\alpha_2)|.
\end{aligned}$$

By the Lemma 1 we have

$$|\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) - P(z_1,z_2,\alpha_1,\alpha_2)| \to 0 \qquad n \to +\infty,$$

and because of the previous inequality we have

$$|P_n(z_1,z_2,\alpha_1,\alpha_2) - \mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2))| \to 0 \qquad n \to +\infty.$$

□

4.4. Proof of Corollary 2

We have

$$\begin{aligned}&|P_n(z_1,z_2,\alpha_1,\alpha_2) - P(z_1,z_2,\alpha_1,\alpha_2)|\\ &= |\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) - P(z_1,z_2,\alpha_1,\alpha_2) + P_n(z_1,z_2,\alpha_1,\alpha_2) - \mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2))|.\end{aligned}$$

One other hand we have

$$\begin{aligned}&|P_n(z_1,z_2,\alpha_1,\alpha_2) - P(z_1,z_2,\alpha_1,\alpha_2)|\\ &\leq |\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) - P(z_1,z_2,\alpha_1,\alpha_2)| + |P_n(z_1,z_2,\alpha_1,\alpha_2) - \mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2))|.\end{aligned} \tag{9}$$

On the other hand by the Lemma 2 we have

$$\begin{aligned}&\sup_{(z_1,z_2)\in\mathbb{R}^2} |\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) - P(z_1,z_2,\alpha_1,\alpha_2)| \leq\\ &h_1h_2\Big(\big(\int_{\mathbb{R}^2} |f^{'}(x,y)|\,dxdy\big)\big(\int_{\mathbb{R}^2}(|u|+1)(|v|+1)|\mathbb{K}(u,v)|\,dudv\big)\\ &+ 4(\alpha_1\alpha_2 M + Ah_1h_2)\int_{-\infty}^{+\infty}|\mathbb{K}(u,v)|\,dudv\Big),\end{aligned}$$

and moreover

$$|P_n(z_1,z_2,\alpha_1,\alpha_2) - \mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2))| \leq [\frac{z_1}{h_1}][\frac{z_2}{h_2}]V(\mathbb{R})\times V(\mathbb{R}) \sup_{(l_1,l_2)\in\mathbb{R}^2} |\hat{F}_n(l_1,l_2) - F(l_1,l_2)|.$$

Therefore the second term of the right hand-side of inequality 9 tends to zero as $n \to +\infty$.The first term tends to $h_1h_2 = O(n^{-1}\log\log n)^{1/4}$ as $n \to +\infty$.

Hence the proof is complete. □

4.5. The uniform mean square consistency. Proof of Lemma 3

Let $\delta = (\delta_1,\delta_2) > 0$, and $\theta_1^i,\ \theta_2^i\quad i = 1,2$; i represent indices not powers.

Define

$$
\begin{aligned}
I_n(x_1,x_2,y_1,y_2) &= (h_1h_2)^{-2}\int\int_{-\infty}^{+\infty}|\mathbb{K}(\frac{u_1-x_1+\theta_1^1h_1}{h_1},\frac{u_2-x_2+\theta_1^2h_2}{h_2})\\
&\times \mathbb{K}(\frac{u_1-y_1+\theta_2^1h_1}{h_1},\frac{u_2-y_2+\theta_2^2h_2}{h_2})|f(u_1,u_2)\,du_1du_2\\
&= \int\int_{-\infty}^{+\infty}((h_1h_2)^{-1}\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2}))|((h_1h_2)^{-1}K(\frac{v_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1})\\
&\times K(\frac{v_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}))|f(x_1+v_1-\theta_1^1h_1,x_2+v_2-\theta_1^2h_2)\,du_1du_2\\
&= \int_{|v_1-\theta_1^1h_1|\le\delta_1}\int_{|v_2-\theta_1^2h_2|\le\delta_2}+\int_{|v_1-\theta_1^1h_1|>\delta_1}\int_{|v_2-\theta_1^2h_2|>\delta_2}\\
&+\int_{|v_1-\theta_1^1h_1|\le\delta_1}\int_{|v_2-\theta_1^2h_2|>\delta_2}+\int_{|v_1-\theta_1^1h_1|>\delta_1}\int_{|v_2-\theta_1^2h_2|\le\delta_2}.
\end{aligned}
$$

Since f is continuous, it is bounded on $I_1\times I_2=[x_1-\delta_1,x_1+\delta_1]\times[x_2-\delta_2,x_2+\delta_2]$. We assume that n is large enough such that $(x_1+v_1\pm\theta_1^1h_1,x_2+v_2\pm\theta_1^2h_2)\in I_1\times I_2$. Therefore

$$
\begin{aligned}
&\int_{|v_1-\theta_1^1h_1|\le\delta_1}\int_{|v_1-\theta_1^1h_1|\le\delta_2} \le \sup_{|v_1-\theta_1^1h_1|\le\delta_1}\ \sup_{|v_2-\theta_1^2h_2|\le\delta_2} f(x_1+v_1-\theta_1^1h_1,x_2+v_2-\theta_1^2h_2)\\
&\times\int_{-\frac{\delta_1}{h_1}+\theta_1^1\le u_1\le\frac{\delta_1}{h_1}+\theta_1^1}\int_{-\frac{\delta_1}{h_1}+\theta_1^1\le u_1\le\frac{\delta_1}{h_1}+\theta_1^1}|\mathbb{K}(u_1,u_2)|\\
&\times|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|\,du_1du_2
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&= \sup_{|v_1-\theta_1^1h_1|\le\delta_1}\ \sup_{|v_2-\theta_1^2h_2|\le\delta_2} f(x_1+v_1-\theta_1^1h_1,x_2+v_2-\theta_1^2h_2)\\
&\times\int\int_{-\infty}^{+\infty}\chi_{-\frac{\delta_1}{h_1}+\theta_1^1\le u_1\le\frac{\delta_1}{h_1}+\theta_1^1}(u_1)\chi_{-\frac{\delta_2}{h_2}+\theta_1^2\le u_2\le\frac{\delta_2}{h_2}+\theta_1^2}(u_2)|\mathbb{K}(u_1,u_2)|\\
&\times|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|\,du_1du_2.
\end{aligned}
$$

For all (u_1,u_2),

$$
\lim_{n\to+\infty}|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|=0.
$$

Write

$$
\begin{aligned}
&|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|\\
&=|(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1)(\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)K(\frac{x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1}+u_1)\\
&\times K(\frac{x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}+u_2)|\\
&\times|\frac{1}{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1+h_1u_1}\times\frac{1}{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2+h_2u_2}|.
\end{aligned}
$$

We have

$$|\frac{1}{x_i-\theta_1^i h_i - y_i + \theta_2^i h_i + h_i u_i}| = \frac{1}{|x_i-y_i||1-\frac{\theta_1^i-\theta_2^i-u_i}{x_i-y_i}h_i|} \qquad i=1,2.$$

Since $|u_i| \le \frac{\delta_i}{h_i}+\theta_1^i$, we may choose δ_i small enough such that for $n \ge n_0$, we have

$$|\frac{\theta_1^i-\theta_2^i-u_i}{x_i-y_i}h_i| \le \frac{3h_i+\delta_i}{|x_i-y_i|} = \eta_i < 1.$$

Therefore

$$|\frac{1}{x_i-\theta_1^i h_i - y_i + \theta_2^i h_i + h_i u_i}| \le \frac{1}{|x_i-y_i|(1-\eta_i)}. \tag{11}$$

Since $\mathbf{H_3}$ implies there exists B such that

$$\begin{aligned}|(&\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1)(\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)\\ &\times K(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1)K(\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)| \le B,\end{aligned}$$

$$\begin{aligned}|\mathbb{K}(&\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1, \frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|\\ &\le \frac{2B}{|x_1-y_1|(1-\eta_1)|x_2-y_2|(1-\eta_2)}.\end{aligned}$$

$|\mathbb{K}(u_1,u_2)|$ being integrable, by dominated convergence we get,

$$\int_{|v_1-\theta_1^1h_1|\le\delta_1}\int_{|v_1-\theta_1^1h_1|\le\delta_2} \to 0 \quad \text{as} \quad n\to+\infty.$$

Let $\int_{|v_1-\theta_1^1h_1|>\delta_1}\int_{|v_2-\theta_12h_2|>\delta_2}$ write it in the form

$$\begin{aligned}\int_{|v_1-\theta_1^1h_1|>\delta_1}\int_{|v_2-\theta_1^2h_2|>\delta_2} = &\int_{|v_1-\theta_1^1h_1|>\delta_1}\int_{|v_2-\theta_1^2h_2|>\delta_2} |v_1v_2(h_1h_2)^{-1}\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})((h_1h_2)^{-1}\\ &\times \mathbb{K}(\frac{v_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1},\frac{v_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}))\\ &\times \frac{f(x_1+v_1-\theta_1^1h_1, x_2+v_2-\theta_1^2h_2)}{v_1v_2}|\, dv_1dv_2.\end{aligned}$$

We get

$$\begin{aligned}\int_{|v_1-\theta_1^1h_1|>\delta_1}\int_{|v_2-\theta_1^2h_2|>\delta_2} \le &\frac{2}{\delta_1-\theta_1^1h_1}\times\frac{2}{\delta_2-\theta_1^2h_2}\sup_{|v_1-\theta_1^1h_1|>\delta_1}\sup_{|v_2-\theta_1^2h_2|>\delta_2}|\frac{v_1v_2}{h_1h_2}\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})|\\ &\times\int_{|v_1-\theta_1^1h_1|>\delta_1}\int_{|v_1-\theta_1^1h_1|>\delta_2}((h_1h_2)^{-1}\\ &\times \mathbb{K}(\frac{v_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1},\frac{v_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}))\\ &\times |f(x_1+v_1-\theta_1^1h_1, x_2+v_2-\theta_1^2h_2)|\, dv_1dv_2.\end{aligned}$$

Let the change of variables defined by

$$v_i + x_i - \theta_1^i h_i - y_i + \theta_2^i h_i = u_i, \quad i = 1, 2.$$

Then

$$\int_{|v_1-\theta_1^1 h_1|>\delta_1} \int_{|v_2-\theta_1^2 h_2|>\delta_2} \leq \frac{2}{\delta_1 - \theta_1^1 h_1} \times \frac{2}{\delta_2 - \theta_1^2 h_2} \sup_{|v_1-\theta_1^1 h_1|>\delta_1} \sup_{|v_2-\theta_1^2 h_2|>\delta_2} |\frac{v_1 v_2}{h_1 h_2} \mathbb{K}(\frac{v_1}{h_1}, \frac{v_2}{h_2})|$$
$$\times \int\int_{\mathbb{R}^2} |(h_1 h_2)^{-1} \mathbb{K}(\frac{u_1}{h_1}, \frac{u_2}{h_2})| f(u_1 + y_1 - \theta_2^1 h_1, u_2 + y_2 - \theta_2^2 h_2)\, du_1 du_2. \tag{12}$$

For Lemma 4 (replacing $\mathbb{K}$ by $|\mathbb{K}|$) and $(\mathbf{H_3})$ we have

$$|\int_{|v_1-\theta_1^1 h_1|>\delta_1} \int_{|v_1-\theta_1^1 h_1|>\delta_2} | \to 0 \quad \text{as} \quad n \to +\infty,$$

and the convergence is uniform in this case.

Consider the case $\int_{|v_1-\theta_1^1 h_1|\leq\delta_1} \int_{|v_1-\theta_1^1 h_1|>\delta_2} (\cdot)(\cdot)$, by Fubini's theorem we have

$$\int_{|v_1-\theta_1^1 h_1|\leq\delta_1} \int_{|v_1-\theta_1^1 h_1|>\delta_2} (\cdot)(\cdot) = \int_{|v_1-\theta_1^1 h_1|\leq\delta_1} (\cdot) \int_{|v_1-\theta_1^1 h_1|>\delta_2} (\cdot).$$

Using the two cases treated in one dimension (Lemma 2.3 Dia (2008) and Lemma 2.9 Ciss *et al.* (2014)) , we get

$$\int_{|v_1-\theta_1^1 h_1|\leq\delta_1} \to 0 \quad \text{and} \int_{|v_1-\theta_1^1 h_1|>\delta_2} \to 0 \quad \text{as} \quad n \to +\infty,$$

therefore

$$\int_{|v_1-\theta_1^1 h_1|\leq\delta_1} \int_{|v_2-\theta_1^2 h_2|>\delta_2} \to 0 \quad \text{as} \quad n \to +\infty.$$

Consider the case $\int_{|v_1-\theta_1^1 h_1|>\delta_1} \int_{|v_2-\theta_1^2 h_2|\leq\delta_2} (\cdot)(\cdot)$, similarly by Fubini's theorem we have

$$\int_{|v_1-\theta_1^1 h_1|>\delta_1} \int_{|v_2-\theta_1^2 h_2|\leq\delta_2} (\cdot)(\cdot) = \int_{|v_1-\theta_1^1 h_1|>\delta_1} (\cdot) \int_{|v_2-\theta_1^2 h_2|\leq\delta_2} (\cdot),$$

therefore

$$\int_{|v_1-\theta_1^1 h_1|>\delta_1} \int_{|v_2-\theta_1^2 h_2|\leq\delta_2} \to 0 \quad \text{as} \quad n \to +\infty.$$

This complete the proof of Lemma 3. □

Remark 3. According to the case considered, if C_2 is verified then for $\int_{|v_1-\theta_1^1 h_1|>\delta_1}\int_{|v_1-\theta_1^1 h_1|>\delta_2}$, the integral of the right hand-side of 12 becomes

$$\begin{aligned}
&\int\int_{\mathbb{R}^2}|(h_1h_2)^{-1}\mathbb{K}(\frac{u_1}{h_1},\frac{u_2}{h_2})||f(u_1+y_1-\theta_2^1h_1,u_2+y_2-\theta_2^2h_2)\\
&-f(\frac{u_1}{h_1},\frac{u_2}{h_2})+f(\frac{u_1}{h_1},\frac{u_2}{h_2})|du_1du_2\\
&\leq\int\int_{\mathbb{R}^2}|(h_1h_2)^{-1}\mathbb{K}(\frac{u_1}{h_1},\frac{u_2}{h_2})|\int_{\frac{u_1}{h_1}}^{u_1+y_1-\theta_2^1h_1}\int_{\frac{u_2}{h_2}}^{u_2+y_2-\theta_2^2h_2}|f^{'}(t_1,t_2)|\,dt_1dt_2du_1du_2\\
&+\int\int_{\mathbb{R}^2}f(\frac{u_1}{h_1},\frac{u_2}{h_2})|(h_1h_2)^{-1}\mathbb{K}(\frac{u_1}{h_1},\frac{u_2}{h_2})|\,du_1du_2\\
&\leq\int\int_{\mathbb{R}^2}|(h_1h_2)^{-1}\mathbb{K}(\frac{u_1}{h_1},\frac{u_2}{h_2})|\,du_1du_2\int\int_{\mathbb{R}^2}|f^{'}(t_1,t_2)|\,dt_1dt_2\\
&+\int\int_{\mathbb{R}^2}|\mathbb{K}(u_1,u_2)|f(u_1,u_2)\,du_1du_2.
\end{aligned}$$

The integrals of the right hand-side of this last inequality are bounded. Hence the theorem is valid under the hypothesis C_2.
For the cases $\displaystyle\int_{|v_1-\theta_1^1h_1|\leq\delta_1}\int_{|v_1-\theta_1^1h_1|>\delta_2}$ and $\displaystyle\int_{|v_1-\theta_1^1h_1|>\delta_1}\int_{|v_1-\theta_1^1h_1|\leq\delta_2}$ applying Fubini's theorem and the results of Lemma 2.3 Dia (2008) in the one-dimensional case we give us the result.

4.6. Proof of Theorem 5

We suppose condition $\mathbf{C_1}$ verified. Let $\Delta_1=[0,z_1]\times[0,z_1];\Delta_2=[0,z_2]\times[0,z_2]$. We can write

$$\begin{aligned}
&\sum_{0\leq i_1\neq j_1\leq[\frac{z_1}{h_1}]}\left(1-\frac{i_1h_1}{z_1}\right)^{\alpha_1}\left(1-\frac{j_1h_1}{z_1}\right)^{\alpha_1}\sum_{0\leq i_2\neq j_2\leq[\frac{z_2}{h_2}]}\left(1-\frac{i_2h_2}{z_2}\right)^{\alpha_2}\left(1-\frac{j_2h_2}{z_2}\right)^{\alpha_2}\\
&\times\int\int_{\mathbb{R}^2}|\mathbb{K}(\frac{u_1-i_1h_1}{h_1},\frac{u_1-j_1h_1}{h_1})\mathbb{K}(\frac{u_2-i_2h_2}{h_2},\frac{u_2-j_2h_2}{h_2})|f(u_1,u_2)\,du_1du_2)\\
&=\int\int_{\{(x_1,y_1)\in\Delta_1:|x_1-y_1|>0\}}\Phi_n(x_1,y_1)\,dx_1dy_1\int\int_{\{(x_2,y_2)\in\Delta_2:|x_2-y_2|>0\}}\Phi_n(x_2,y_2)\,dx_2dy_2,
\end{aligned}$$

where

$$\begin{aligned}
\Phi_n(x_1,y_1)=\frac{1}{h_1^2}\sum_{0\leq i_1\neq j_1\leq[\frac{z_1}{h_1}]}\chi_{\Delta_{h_1,i_1}\times\Delta_{h_1,j_1}}(x_1,y_1)\left(1-\frac{i_1h_1}{z_1}\right)^{\alpha_1}\left(1-\frac{j_1h_1}{z_1}\right)^{\alpha_1}\\
\times\int_{\mathbb{R}}|\mathbb{K}(\frac{u_1-i_1h_1}{h_1},\frac{u_1-j_1h_1}{h_1})|f_1(u_1)\,du_1,
\end{aligned}$$

and $\Phi_n(x_2,y_2)$ is obtained by replacing x_1 by x_2, y_1 by y_2 , u_1 by u_2 and f_1 by f_2.

Note $\Phi_{n_{12}}(x_1,y_1,x_2,y_2) := \Phi_n(x_1,y_1)\Phi_n(x_2,y_2)$. Therefore

$$\Phi_{n_{12}}(x_1,y_1,x_2,y_2) = \frac{1}{h_1^2h_2^2}\sum_{0\le i_1\ne j_1\le[\frac{z_1}{h_1}]}\chi_{\Delta_{h_1,i_1}\times\Delta_{h_1,j_1}}(x_1,y_1)\Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{j_1h_1}{z_1}\Big)^{\alpha_1}$$
$$\times\sum_{0\le i_2\ne j_2\le[\frac{z_2}{h_2}]}\chi_{\Delta_{h_2,i_2}\times\Delta_{h_2,j_2}}(x_2,y_2)\Big(1-\frac{i_2h_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{j_2h_2}{z_2}\Big)^{\alpha_2}$$
$$\times\int\int_{\mathbb{R}^2}|\mathbb{K}(\frac{u_1-i_1h_1}{h_1},\frac{u_1-j_1h_1}{h_1})\mathbb{K}(\frac{u_2-i_1h_2}{h_2},\frac{u_2-j_2h_2}{h_2})|f(u_1,u_2)\,du_1du_2.$$

If $(x_1,y_1)\in\Delta_{h_1,i_1}\times\Delta_{h_1,j_1}\quad i_1\ne j_1;\qquad (x_2,y_2)\in\Delta_{h_2,i_2}\times\Delta_{h_2,j_2}\quad i_2\ne j_2$ with the representations

$$x_1 = h_1i_1+\theta_1^1h_1,\quad y_1 = h_1j_1+\theta_2^1h_1\quad 0\le\theta_i^1<1,\quad i=1,2,$$

and

$$x_2 = h_2i_2+\theta_1^2h_2,\quad y_2 = h_2j_2+\theta_2^2h_2\quad 0\le\theta_i^2<1,\quad i=1,2.$$

We have

$$\frac{1}{h_1^2h_2^2}\Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{j_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{i_2h_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{j_2h_2}{z_2}\Big)^{\alpha_2}$$
$$\times\int\int_{\mathbb{R}^2}|\mathbb{K}(\frac{u_1-x_1+\theta_1^1h_1}{h_1},\frac{u_1-y_1+\theta_2^1h_1}{h_1})\mathbb{K}(\frac{u_2-x_2+\theta_1^2h_2}{h_2},\frac{u_2-y_2+\theta_2^2h_2}{h_2})|$$
$$\times f(u_1,u_2)\,du_1du_2$$
$$\le\frac{1}{h_1^2h_2^2}\int\int_{\mathbb{R}^2}|\mathbb{K}(\frac{u_1-x_1+\theta_1^1h_1}{h_1},\frac{u_1-y_1+\theta_2^1h_1}{h_1})\mathbb{K}(\frac{u_2-x_2+\theta_1^2h_2}{h_2},\frac{u_2-y_2+\theta_2^2h_2}{h_2})|$$
$$\times f(u_1,u_2)\,du_1du_2.$$

The right hand-side tends to zero as $n\to+\infty$ according to the Lemma 3.

Let $\delta_i=\frac{z_i}{2},\quad i=1,2$. Write

$$\frac{1}{h_1^2h_2^2}\int\int_{\mathbb{R}^2}|\mathbb{K}(\frac{u_1-x_1+\theta_1^1h_1}{h_1},\frac{u_1-y_1+\theta_2^1h_1}{h_1})\mathbb{K}(\frac{u_2-x_2+\theta_1^2h_2}{h_2},\frac{u_2-y_2+\theta_2^2h_2}{h_2})|$$
$$\times f(u_1,u_2)\,du_1du_2$$
$$=\int_{|v_1|\le\delta_1}\int_{|v_2|\le\delta_2}+\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2}+\int_{|v_1|\le\delta_1}\int_{|v_2|>\delta_2}+\int_{|v_1|>\delta_1}\int_{|v_2|\le\delta_2}.$$

Then, we have

$$\int\int_{\{(x_1,y_1)\in\Delta_1:|x_1-y_1|>0\}}\Phi_n(x_1,y_1)\,dx_1dy_1\int\int_{\{(x_2,y_2)\in\Delta_2:|x_2-y_2|>0\}}\Phi_n(x_2,y_2)\,dx_2dy_2$$
$$\le\int\int_{\{(x_1,y_1)\in\Delta_1:|x_1-y_1|>0\}}\sum_{0\le i_1\ne j_1\le[\frac{z_1}{h_1}]}\chi_{\Delta_{h_1,i_1}\times\Delta_{h_1,j_1}}(x_1,y_1)$$
$$\times\int\int_{\{(x_2,y_2)\in\Delta_2:|_x2-y_2|>0\}}\sum_{0\le i_2\ne j_2\le[\frac{z_2}{h_2}]}\chi_{\Delta_{h_2,i_2}\times\Delta_{h_2,j_2}}(x_2,y_2)$$
$$(\int_{|v_1|\le\delta_1}\int_{|v_2|\le\delta_2}+\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2}+\int_{|v_1|\le\delta_1}\int_{|v_2|>\delta_2}+\int_{|v_1|>\delta_1}\int_{|v_2|\le\delta_2}).$$

The proof of the remainder is conducted as follow:

First consider

$$\int\int_{\{(x_1,y_1)\in\Delta_1:|x_1-y_1|>0\}}\int\int_{\{(x_2,y_2)\in\Delta_2:|x_2-y_2|>0\}}\int_{|v_1|\leq\delta_1}\int_{|v_2|\leq\delta_2}.$$

Define $A=\sup_{(x,y)\in[0,z_1]\times[0,z_2]} f(x,y)$. The notations being as in the proof Lemma 3 with $\delta_i=\frac{z_i}{2},\quad i=1,2$, we have following inequality 10

$$\begin{aligned}&\int_{|v_1|\leq\delta_1}\int_{|v_2|\leq\delta_2}\leq A\int_{-\infty}^{+\infty}\chi_{-\frac{\delta_1}{h_1}\leq u_1\leq\frac{\delta_1}{h_1}}\int_{-\infty}^{+\infty}\chi_{-\frac{\delta_2}{h_2}\leq u_2\leq\frac{\delta_2}{h_2}}|\mathbb{K}(u_1,u_2)|\\ &\times|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|\,du_1du_2.\end{aligned}$$

For all (u_1,u_2),

$$\lim_{n\to+\infty}|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|=0.$$

Write

$$\begin{aligned}&|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|\\ &=|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)\\ &-\mathbb{K}(\frac{2z_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}+u_2)\\ &+\mathbb{K}(\frac{2z_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}+u_2)|(h_1h_2)^{-1}\\ &\leq(\lambda(\frac{2z_1}{h_1},\frac{2z_2}{h_2})+|\mathbb{K}(\frac{2z_1+x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}+u_2)|)\\ &\times(h_1h_2)^{-1}.\end{aligned}$$

Moreover,

$$\begin{aligned}&|\mathbb{K}(\frac{2z_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}+u_2)|(h_1h_2)^{-1}\\ &=|\frac{2z_1+x_1-y_1+h_1u_1}{h_1}||\frac{2z_2+x_2-y_2+h_2u_2}{h_2}|\\ &\times|\mathbb{K}(\frac{2z_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}+u_2)|\\ &\times\frac{1}{|2z_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}.\end{aligned}$$

We have

$$\begin{aligned}
TT &= |\frac{2z_1 + x_1 - y_1 + h_1 u_1}{h_1}||\frac{2z_2 + x_2 - y_2 + h_2 u_2}{h_2}| \\
&= |\frac{2z_1 + x_1 - y_1 + h_1 u_1 - \theta_1^1 h_1 + \theta_2^1 h_1}{h_1} + \frac{\theta_1^1 h_1 - \theta_2^1 h_1}{h_1}| \\
&\times |\frac{2z_2 + x_2 - y_2 + h_2 u_2 - \theta_1^2 h_2 + \theta_2^2 h_2}{h_2} + \frac{\theta_2^2 h_2 - \theta_2^2 h_2}{h_2}| \\
&\leq |\frac{2z_1 + x_1 - y_1 + h_1 u_1 - \theta_1^1 h_1 + \theta_2^1 h_1}{h_1}| + |\frac{\theta_1^1 h_1 - \theta_2^1 h_1}{h_1}| \\
&\times |\frac{2z_2 + x_2 - y_2 + h_2 u_2 - \theta_1^2 h_2 + \theta_2^2 h_2}{h_2}| + |\frac{\theta_1^2 h_2 - \theta_2^2 h_2}{h_2})| \\
&\leq |\frac{2z_1 + x_1 - y_1 + h_1 u_1 - \theta_1^1 h_1 + \theta_2^1 h_1}{h_1}||\frac{2z_2 + x_2 - y_2 + h_2 u_2 - \theta_1^2 h_2 + \theta_2^2 h_2}{h_2}| \\
&+ |\frac{2z_1 + x_1 - y_1 + h_1 u_1 - \theta_1^1 h_1 + \theta_2^1 h_1}{h_1}| + |\frac{2z_2 + x_2 - y_2 + h_2 u_2 - \theta_1^2 h_2 + \theta_2^2 h_2}{h_2}| + 1.
\end{aligned}$$

Let $B = \sup_{y \in \mathbb{R}} |y||K(y)|$ and $C = \sup_{y \in \mathbb{R}} |K(y)|$, then we have

$$\begin{aligned}
&|\frac{2z_1 + x_1 - y_1 + h_1 u_1}{h_1}||\frac{2z_2 + x_2 - y_2 + h_2 u_2}{h_2}||K(\frac{2z_1 + x_1 - \theta_1^1 - y_1 + \theta_2^1 h_1}{h_1} + u_1) \\
&\qquad \times K(\frac{z_2 + x_2 - \theta_1^2 - y_2 + \theta_2^2 h_2}{h_2} + u_2)| \\
&\qquad \leq B^2 + 2BC + C
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&|K(\frac{2z_1 + x_1 - \theta_1^1 - y_1 + \theta_2^1 h_1}{h_1} + u_1)K(\frac{z_2 + x_2 - \theta_1^2 - y_2 + \theta_2^2 h_2}{h_2} + u_2)|(h - 1h_2)^{-1} \\
&\qquad \leq \frac{B^2 + 2BC + C}{|2z_1 + x_1 - y_1 + h_1 u_1||2z_2 + x_2 - y_2 + h_2 u_2|},
\end{aligned}$$

hence

$$\begin{aligned}
&|\mathbb{K}(\frac{x_1 - \theta_1^1 h_1 - y_1 + \theta_2^1 h_1}{h_1} + u_1, \frac{x_2 - \theta_1^2 h_2 - y_2 + \theta_2^2 h_2}{h_2} + u_2)(h_1 h_2)^{-1}| \\
&\qquad \leq (\lambda(\frac{2z_1}{h_1}, \frac{2z_2}{h_2}) + \frac{B^2 + 2BC + C}{|2z_1 + x_1 - y_1 + h_1 u_1||2z_2 + x_2 - y_2 + h_2 u_2|}.
\end{aligned}$$

We conclude that for h_i, $i = 1, 2$, small enough

$$\begin{aligned}\int_{|v_1|\le\delta_1}\int_{|v_2|\le\delta_2} &\le \frac{A}{|2z_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}\\ &\times\int\int_{\mathbb{R}^2}|\mathbb{K}(u_1,u_2)|(B^2+2BC+C)\,du_1du_2\\ &< \frac{AD}{|2z_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}\\ &\le \frac{AD}{|2z_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}\\ &\le \frac{AD}{(2z_1+x_1-y_1+h_1u_1)(2z_2+x_2-y_2+h_2u_2)},\end{aligned}$$

where D being the finite bound of $\int\int_{\mathbb{R}^2}|\mathbb{K}(u_1,u_2)|(B^2+2BC+4C)\,du_1du_2$.

Finally, we have

$$\int_{|v_1|\le\delta_1}\int_{|v_2|\le\delta_2} \le \frac{AD}{(2z_1+x_1-y_1+h_1u_1)(2z_2+x_2-y_2+h_2u_2)} + O(h_1h_2).$$

Since $-\delta_i \le h_1u_1 \le \delta_i$, we have $\frac{z_i}{2} \le 2z_i + x_i - y_i + h_iu_i \le \frac{7z_i}{2}$, $i = 1, 2$, we get

$$\int_{|v_1|\le\delta_1}\int_{|v_2|\le\delta_2} \le \frac{2AD}{z_1z_2} + O(h_1h_2).$$

Therefore by Lebesgue-dominated convergence theorem, we have

$$\begin{aligned}&\lim_{n\to+\infty}\int\int_{\Delta_1}\int\int_{\Delta_2}\Big(\int_{|v_1|\le\delta_1}\int_{|v_2|\le\delta_2}\Big)\,dx_1dx_2dy_1dy_2\\ &\qquad= \int\int_{\Delta_1}\int\int_{\Delta_2}\lim_{n\to+\infty}\Big(\int_{|v_1|\le\delta_1}\int_{|v_2|\le\delta_2}\Big)\,dx_1dx_2dy_1dy_2 = 0.\end{aligned}$$

Consider then $\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2}$.

We use the second part, by analogous reasoning, of the proof of Lemma 3

$$\begin{aligned}\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2} &\le \frac{2}{\delta_1}\times\frac{2}{\delta_2}\sup_{|v_1|>\delta_1}\sup_{|v_2|>\delta_2}\Big|\frac{v_1}{h_1}\frac{v_2}{h_2}\Big|\mathbb{K}\Big(\frac{v_1}{h_1},\frac{v_2}{h_2}\Big)\Big|\\ &\times\int\int_{\mathbb{R}^2}|(h_1h_2)^{-1}\mathbb{K}\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)|f(u_1+y_1-\theta_2^1h_1, u_2+y_2-\theta_2^2h_2)\,du_1du_2,\end{aligned} \tag{13}$$

hence

$$\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2} \to 0, \qquad n\to+\infty$$

uniformly.

Consider $\int_{|v_1|\leq\delta_1}\int_{|v_2|>\delta_2}(\cdot)$ by Fubini's theorem we get

$$\int_{|v_1|\leq\delta_1}\int_{|v_2|>\delta_2}(\cdot)=\int_{|v_1|\leq\delta_1}(\cdot)\int_{|v_2|>\delta_2}(\cdot)\to 0*0=0,\quad n\to+\infty,$$

according to the Lemma 2.3 for the one-dimensional case.

Similarly we have

$$\int_{|v_1|>\delta_1}\int_{|v_2|\leq\delta_2}(\cdot)=\int_{|v_1|>\delta_1}(\cdot)\int_{|v_2|\leq\delta_2}(\cdot)\to 0\times 0=0,\quad n\to+\infty.$$

Consequently

$$\lim_{n\to+\infty}\int\int_{\Delta_1}\int\int_{\Delta_2}\int\int_{\mathbb{R}^2}\to 0,\quad n\to+\infty,$$

since Δ_1 and Δ_2 are bounded. The proof of the lemma is complete. □

Remark 4. If C_2 is verified then the Theorem is again valid. Indeed it suffices to apply Remark 3 to inequality 13 in the case $\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2}$.

For the case $\int_{|v_1|\leq\delta_1}\int_{|v_2|>\delta_2}$ (resp. $\int_{|v_1|>\delta_1}\int_{|v_2|\leq\delta_2}$), apply Remark 3 to $\int_{|v_2|>\delta_2}$ (resp. $\int_{|v_1|>\delta_1}$).

4.7. Proof of Theorem 3

We suppose condition $(\mathbf{C_1})$ satisfied, then

$$\begin{aligned}n\mathbb{V}ar(P_n(z_1,z_2,\alpha_1,\alpha_2))&=\mathbb{E}(\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}K(\frac{X_k-ih_1}{h_1})K(\frac{Y_k-jh_2}{h_2}))^2\\&-\mathbb{E}^2\Big(\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}K(\frac{X_k-ih_1}{h_1})K(\frac{Y_k-jh_2}{h_2})\Big).\end{aligned}$$

Since

$$\mathbb{E}(\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}K(\frac{X_k-ih_1}{h_1})K(\frac{Y_k-jh_2}{h_2}))^2$$
$$=\mathbb{E}\Big[(\sum_{i=0}^{[\frac{z_1}{h_1}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}K(\frac{X_k-ih_1}{h_1}))^2(\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}K(\frac{Y_k-jh_2}{h_2}))^2\Big]$$
$$=\mathbb{E}\Big[\{\sum_{i=0}^{[\frac{z_1}{h_1}]}\Big(1-\frac{ih_1}{z_1}\Big)^{2\alpha_1}K^2(\frac{X_k-ih_1}{h_1})$$
$$+\sum_{i\neq l_1}^{[\frac{z_1}{h_1}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{l_1h_1}{z_1}\Big)^{\alpha_1}K(\frac{X_k-ih_1}{h_1})K(\frac{X_k-l_1h_1}{h_1})\}$$
$$\times\{\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{jh_2}{z_2}\Big)^{2\alpha_2}K^2(\frac{Y_k-jh_2}{h_2})$$
$$+\sum_{j\neq l_2}^{[\frac{z_2}{h_2}]}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{l_2h_2}{z_2}\Big)^{\alpha_2}K(\frac{Y_k-jh_2}{h_2})K(\frac{Y_k-l_2h_2}{h_2})\}\Big]$$
$$=\mathbb{E}\Big[m_1m_3+m_1m_4+m_2m_3+m_2m_4\Big].$$

where

$$m_1:=\sum_{i=0}^{[\frac{z_1}{h_1}]}\Big(1-\frac{ih_1}{z_1}\Big)^{2\alpha_1}K^2(\frac{X_k-ih_1}{h_1}),$$
$$m_2:=\sum_{i\neq l_1}^{[\frac{z_1}{h_1}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{l_1h_1}{z_1}\Big)^{\alpha_1}K(\frac{X_k-ih_1}{h_1})K(\frac{X_k-l_1h_1}{h_1}),$$
$$m_3:=\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{jh_2}{z_2}\Big)^{2\alpha_2}K^2(\frac{Y_k-jh_2}{h_2}),$$
$$m_4:=\sum_{j\neq l_2}^{[\frac{z_2}{h_2}]}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{l_2h_2}{z_2}\Big)^{\alpha_2}K(\frac{Y_k-jh_2}{h_2})K(\frac{Y_k-l_2h_2}{h_2}).$$

We have, by the Theorem 5 $\mathbb{E}(m_2m_4)\to 0,\quad$ as $n\to+\infty$.

According to Fubini, and the Theorem 2.5 Dia (2008) and Theorem 2.5 Ciss *et al.* (2014) in the unidimensional case, $\mathbb{E}(m_1m_4)\to 0$, and $\mathbb{E}(m_2m_3)\to 0,\quad$ as $n\to+\infty$.

On the other hand by Corollary 1,

$$\mathbb{E}(m_1m_3)\to P(z_1,z_2,2\alpha_1,2\alpha_2)\int_{\mathbb{R}^2}\mathbb{K}^2(y_1,y_2),$$

therefore

$$n\mathbb{V}ar(P_n(z_1,z_2,\alpha_1,\alpha_2)) \to P(z_1,z_2,\alpha_1,\alpha_2)\int_{\mathbb{R}^2}\mathbb{K}^2(y_1,y_2) - P^2(z_1,z_2,\alpha_1,\alpha_2).$$

According to the Lemma 1 we have

$$\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) \to P(z_1,z_2,\alpha_1,\alpha_2) \quad \text{as } n\to+\infty,$$

therefore

$$\mathbb{E}^2(P_n(z_1,z_2,\alpha_1,\alpha_2)) \to P^2(z_1,z_2,\alpha_1,\alpha_2) \quad \text{as } n\to+\infty.$$

Define

$$biais(P_n(z_1,z_2,\alpha_1,\alpha_2)) = \mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) - P(z_1,z_2,\alpha_1,\alpha_2).$$

We have

$$\begin{aligned}&\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2) - P_(z_1,z_2,\alpha_1,\alpha_2))^2\\ &= biais^2(P_n(z_1,z_2,\alpha_1,\alpha_2)) + \mathbb{V}ar(P_n(z_1,z_2,\alpha_1,\alpha_2)),\end{aligned}$$

and

$$|\int_{\mathbb{R}^2}\mathbb{K}^2(y_1,y_2)P(z_1,z_2,\alpha_1,\alpha_2) - P^2(z_1,z_2,\alpha_1,\alpha_2)| \le \int_{\mathbb{R}^2}\mathbb{K}^2(y_1,y_2)+1.$$

Hence

$$\mathbb{V}ar(P_n(z_1,z_2,\alpha_1,\alpha_2)) = O(\frac{1}{n}).$$

By Lemma 1, we have

$$|\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)) - P(z_1,z_2,\alpha_1,\alpha_2)| \to 0, \quad n\to+\infty,$$

therefore

$$biais^2(P_n(z_1,z_2,\alpha_1,\alpha_2)) \to 0, \quad n\to+\infty,$$

hence

$$\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2) - P(z_1,z_2,\alpha_1,\alpha_2))^2 \to 0, \quad n\to+\infty.$$

If condition C_2 is satisfied, the theorem is valid again, by Corollary 1, of Lemma 2 and using Remark 4 of Theorem 5 and by Theorem 2, we obtain the result of Theorem 4.

5. Appendix : Construction of the estimator based on the Riemann sums

For $z_1>0, z_2>0$, and h_1 (resp. h_2) the length of the subdivisions for $[0,z_1]$ (resp. $[0,z_2]$), let $\Delta_{h_1,i} = [h_1 i, h_1(i+1)[$, $i=0,...,[\frac{z_1}{h_1}]-1$, (resp. $\Delta_{h_2,j} = [h_2 j, h_2(j+1)[$, $j=0,...,[\frac{z_2}{h_2}]-1$) be a partition of $[0,z_1]$, (resp. $[0,z_2]$). By the Riemann sum definition of the integral, we have that the limit of the following Riemann sum, say $\mathbf{I}_n$, over the two-dimensional rectangle $[0,z_1]\times[0,z_2]$:

$$\mathbf{I}_n = \frac{1}{n}\sum_{k=1}^{n}\sum_{i=0}^{[\frac{z_1}{h_1}]-1}\sum_{j=0}^{[\frac{z_2}{h_2}]-1}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_1}{z_1}\Big)^{\alpha_2}\times$$
$$K\Big(\frac{X_k-ih_1}{h_1}\Big)K\Big(\frac{Y_k-ih_2}{h_2}\Big)+\Big(z_1-h_1[\frac{z_1}{h_1}]\Big)\times$$
$$\Big(z_2-h_2[\frac{z_2}{h_2}]\Big)\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}$$
$$\times\frac{1}{n}\sum_{k=1}^{n}\frac{1}{h_1h_2}K(\frac{X_k-[\frac{z_1}{h_1}]h_1}{h_1})K(\frac{Y_k-[\frac{z_1}{h_1}]h_2}{h_2})$$
$$+\frac{1}{nh_1}\sum_{k=1}^{n}\sum_{j=0}^{[\frac{z_2}{h_2}]-1}\Big(z_1-h_1[\frac{z_1}{h_1}]\Big)\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\times$$
$$\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}K(\frac{X_k-[\frac{z_1}{h_1}]h_1}{h_1})K(\frac{Y_k-jh_2}{h_2})$$
$$+\frac{1}{nh_2}\sum_{k=1}^{n}\sum_{i=0}^{[\frac{z_1}{h_1}]-1}\Big(z_2-h_2[\frac{z_2}{h_2}]\Big)\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\times$$
$$\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}K(\frac{X_k-ih_1}{h_1})K(\frac{Y_k-[\frac{z_2}{h_2}h_2}{h_2})$$

corresponding to the integral, say $\mathbf{I}$:

$$\mathbf{I}=\int_0^{z_1}\int_0^{z_2}\Big(\frac{z_1-x}{z_1}\Big)^{\alpha_1}\Big(\frac{z_2-y}{z_2}\Big)^{\alpha_2}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_1h_2}K(\frac{x-X_i}{h_1})K(\frac{x-Y_i}{h_2})\,dxdy,$$

i.e., $\mathbf{I}=\lim_{n\to\infty}\mathbf{I_n}$. This sum can be written as:

$$\mathbf{I}_n=\frac{1}{n}\sum_{k=1}^{n}\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_2}{z_2}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\times$$
$$K\Big(\frac{X_k-ih_1}{h_1}\Big)K\Big(\frac{Y_k-ih_2}{h_2}\Big)+\mathcal{V}_n(z_1,z_2)+\mathcal{W}_n(z_1,z_2)+\mathcal{U}_n(z_1,z_2),$$

where

$$\mathcal{V}_n(z_1,z_2)=\frac{1}{n}\frac{\Big(z_1-h_1[\frac{z_1}{h_1}]\Big)\Big(z_2-h_2[\frac{z_2}{h_2}]\Big)-h_1h_2}{h_1h_2}\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}$$
$$\times\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\sum_{k=1}^{n}K(\frac{X_k-[\frac{z_1}{h_1}]h_1}{h_1})K(\frac{Y_k-[\frac{z_1}{h_1}]h_2}{h_2}),$$

$$\mathcal{W}_n(z_1,z_2)=\frac{1}{n}\sum_{k=1}^{n}\sum_{j=0}^{[\frac{z_2}{h_2}]-1}\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}$$
$$\times K(\frac{X_k-[\frac{z_1}{h_1}]h_1}{h_1})K(\frac{Y_k-jh_2}{h_2})\Big(\frac{1}{h_1}\Big(z_1-h_1[\frac{z_1}{h_1}]\Big)-1\Big),$$

$$\mathcal{U}_n(z_1,z_2) = \frac{1}{n}\sum_{k=1}^{n}\sum_{0=1}^{[\frac{z_1}{h_1}]-1}\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}$$
$$\times K(\frac{Y_k-[\frac{z_2}{h_2}]h_2}{h_2})K(\frac{X_k-ih_1}{h_1})\Big(\frac{1}{h_2}\Big(z_2-h_2[\frac{z_2}{h_2}]\Big)-1\Big).$$

Now it comes up that

$$\mathcal{V}_n \longrightarrow 0, \qquad \mathcal{W}_n \longrightarrow 0, \qquad \mathcal{U}_n \longrightarrow 0, \qquad \text{almost-surely} \quad \text{as} \qquad n \to +\infty.$$

We have

$$|\mathcal{V}_n(z_1,z_2)| \leq \begin{cases} \Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{h_y[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\sup_{(x,y)\in\mathbb{R}^2}|\mathbb{K}(x,y)| & \text{if} \quad (\alpha_1>0,\alpha_2>0) \\ \frac{2}{n}\sum_{k=1}^{n}K(\frac{X_k-[\frac{z_1}{h_1}]h_1}{h_1})K(\frac{Y_k-[\frac{z_1}{h_1}]h_2}{h_2}) & \text{if} \quad (\alpha_1=0,\alpha_2=0) \\ \Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\sup_{(x,y)\in\mathbb{R}^2}|\mathbb{K}(x,y)| & \text{if} \quad (\alpha_1>0,\alpha_2=0) \\ \Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\sup_{(x,y)\in\mathbb{R}^2}|\mathbb{K}(x,y)| & \text{if} \quad (\alpha_1=0,\alpha_2>0). \end{cases}$$

Since

$$\begin{aligned} |\frac{\Big(z_1-h_1[\frac{z_1}{h_1}]\Big)\Big(z_2-h_2[\frac{z_2}{h_2}]\Big)-h_1h_2}{h_1h_2}| &= |\frac{z_1z_2-z_1h_2[\frac{z_2}{h_1}]-z_2h_1[\frac{z_1}{h_1}]+h_1h_2[\frac{z_1}{h_1}][\frac{z_2}{h_2}]-h_1h_2}{h_1h_2}| \\ &= |\frac{z_1z_2}{h_1h_2}+[\frac{z_1}{h_1}][\frac{z_2}{h_2}]-\frac{z_1}{h_1}[\frac{z_2}{h_2}]-\frac{z_2}{h_2}[\frac{z_1}{h_1}]-1| \\ &= |\frac{z_2}{h_2}(\frac{z_2}{h_1}-[\frac{z_1}{h_1}])+[\frac{z_2}{h_2}]([\frac{z_1}{h_1}]-\frac{z_1}{h_1})-1| \\ &= |([\frac{z_2}{h_2}]-\frac{z_2}{h_2})([\frac{z_1}{h_1}]-\frac{z_1}{h_1})-1| \leq 2. \end{aligned}$$

We obtain for each of the following cases: $(\alpha_1 > 0,\alpha_2 > 0),(\alpha_1 > 0,\alpha_2 = 0)$ and $(\alpha_1 = 0,\alpha_2 > 0)$

$$\mathcal{V}_n(z_1,z_2) \longrightarrow 0 \quad p.s. \qquad \text{as} \quad n \to +\infty.$$

For $(\alpha_1 = 0,\alpha_2 = 0)$, we have also the same result since

$$\begin{aligned} &\Big|\mathbb{E}\Big[\frac{1}{n}\sum_{k=1}^{n}K(\frac{X_k-[\frac{z_1}{h_1}]h_1}{h_1})K(\frac{Y_k-[\frac{z_1}{h_1}]h_2}{h_2})\Big]\Big| \\ &= \Big|h_1h_2\int\int\mathbb{K}(u,v)f(x,y)\,dudv\Big| \\ &\leq h_1h_2\sup_{(u,v)\in\mathbb{R}^2}\int\int|\mathbb{K}(u,v)|\,dudv = O(h_1h_2). \end{aligned}$$

From $\mathbb{V}ar(Z) = \mathbb{E}(Z^2) - \mathbb{E}^2(Z) \leq \mathbb{E}(Z^2)$, we have

$$\mathbb{E}\Big[K^2(\frac{X_k-[\frac{z_1}{h_1}]h_1}{h_1})K^2(\frac{Y_k-[\frac{z_1}{h_1}]h_2}{h_2})\Big] \leq h_1h_2\sup_{(x,y)\in\mathbb{R}^2}\int\int|\mathbb{K}^2(u,v)|\,dudv = O(h_1h_2).$$

Them by applying the strong Kolmogorov's theorem of the large numbers,

$$\frac{2}{n}\sum_{i=1}^{n}K(\frac{X_k-[\frac{z_1}{h_1}]h_1}{h_1})K(\frac{Y_k-[\frac{z_1}{h_1}]h_2}{h_2}) \longrightarrow 0 \quad \text{as} \quad n \to +\infty.$$

Since K being bounded, there exists $M \in \mathbb{R}_+$ such that, $|K| \leq M$, therefore have

$$|\mathcal{W}_n(z_1, z_2)| \leq \begin{cases} \left(1 - \frac{h_1 \left[\frac{z_1}{h_1}\right]}{z_1}\right)^{\alpha_1} \sup_{(x,y)\in\mathbb{R}^2} |\mathbb{K}(x,y)| & \text{if} \quad (\alpha_1 > 0, \alpha_2 > 0) \\ \frac{1}{n} M \left[\frac{z_2}{h_2}\right] \sum_{k=1}^{n} K\left(\frac{X_k - \left[\frac{z_1}{h_1}\right] h_1}{h_1}\right) & \text{if} \quad (\alpha_1 = 0, \alpha_2 = 0) \\ \left[\frac{z_2}{h_2}\right] \sup_{(x,y)\in\mathbb{R}^2} |\mathbb{K}(x,y)| & \text{if} \quad (\alpha_1 > 0, \alpha_2 = 0) \\ \left[\frac{z_2}{h_2}\right] \sup_{(x,y)\in\mathbb{R}^2} |\mathbb{K}(x,y)| & \text{if} \quad (\alpha_1 = 0, \alpha_2 > 0), \end{cases}$$

then

$$\mathcal{W}_n(z_1, z_2) \longrightarrow 0 \quad p.s. \qquad \text{as} \quad n \to +\infty.$$

For $\mathcal{U}_n(z_1, z_2)$ it suffices from $\mathcal{W}_n(z_1, z_2)$ to effect a change of role between i and j, x and y, X and Y.

Note that

$$\left| \frac{z_1 - h_1 \left[\frac{z_1}{h_1}\right] - h_1}{h_1} \right| = \left| \frac{z_1}{h_1} - \left[\frac{z_1}{h_1}\right] - 1 \right| \leq 1.$$

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References

Atkinson, A.B., 1987. On the Measurement of Poverty, *Econometrica*. 55 (4), 749-764.

Ciss, Y. Dia, G. and Diakhaby, A., 2014. Non-parametric estimation of income distribution and poverty index in the unidimensional context with $\alpha \in]0,1[$. *Submitted*.

Dia, G., 2008. Estimation nonparamètrique de la distribution des revenus et de l'indice de pauvreté. *C. R. Acad.Sci. Paris*, Ser. I, **346**, 907-912.

Duclos, J.-Y. Sahn, D.E. and Younger, S.D., 2006. Robust multidimensional poverty comparison. *Economic Journal*, **113**, 943-968.

Duclos, J.-Y. Sahn, D.E. and Younger, S.D., 2006. Robust multidimensional spatial poverty comparisons in Ghana, Madagascar, and Uganda. *World Bank Econ. Rev.*, **20**(1), 91-113.

Foster, J.E., Greer, J. and Thorbecke, E., 2010. The Foster-Greer-Thorbecke (FGT) Poverty Measures: Twenty Five Years Later. *Journal of Economic Inequality*, **8**(4), 491-524.

Foster, J. E., Greer, J. and Thorbecke, E., 1984. A class of decomposable poverty measures. *Econometrica*, **52**, 761-776.

Foster, J.E. and Shorrocks, A.F., 1988. Poverty Orderings. *Econometrica*, **56**, 173-177.

Foster, J.E. and Shorrocks, A.F., 1988. Poverty Orderings and Welfare Dominance. *Social Choice Welfare*, 5, 179-198.

Foster, J.E. and Shorrocks, A.F., 1988. Inequality and Poverty Orderings. *European Economic Review*, **32**, 654-662.

Parzen, E., 1962. On estimation of a probability density function and mode. *Annals of Mathematical Statistics*, **33**(3), 1065-1076.