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CORRECTION

GROWTH PROFILE AND INVARIANT MEASURES FOR THE WEAKLY SUPERCRITICAL CONTACT PROCESS ON A HOMOGENEOUS TREE

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The proof of Theorem 3 in [1] is incorrect, as it relies on a faulty use of the strong Markov property. The theorem asserts that $\beta = \beta(\lambda)$ is strictly increasing in λ for $\lambda < \lambda_2$, where λ is the infection rate parameter for the contact process, λ_2 is the upper critical value (at the transition from weak to strong survival), and $\beta = \lim_{n\to\infty} u_n^{1/n}$ where $u_n =$ probability that a vertex x_n at distance n from the root is ever infected, given that only the root is infected at time $t = 0$. In this note we shall prove the following slightly weaker result.

THEOREM 3′. *If $\beta(\lambda) < 1/\sqrt{d}$ and $\lambda_* < \lambda$ then $\beta(\lambda_*) < \beta(\lambda)$.*

This leaves open the possibility that $\beta(\lambda) = 1/\sqrt{d}$ on an interval $[\lambda_3, \lambda_2]$ of positive length. The proof of Theorem 3′ below relies on the following estimate proved by Schonmann (Theorem 2) in [4]: If $\beta(\lambda) < 1/\sqrt{d}$ then for some constant $0 < C < \infty$ and every integer $n \geq 1$,

$$\frac{\beta(\lambda)^n}{Cn} \leq u_n. \tag{0.1}$$

Schonmann's argument makes essential use of the fact, proved in [1], that $\beta < 1/\sqrt{d}$ implies $\eta < 1$, where $\eta = \lim_{t\to\infty} P\{\text{root} \in A_t\}^{1/t}$.

The proof of Theorem 3′ also uses the following elementary results.

LEMMA 1. *Let $X, X_1, X_2, \dots$ be independent, identically distributed, positive integer-valued random variables, and let N be a geometrically distributed random variable independent of the random variables $X_1, X_2, \dots$. Suppose that the probability generating function $\varphi(z) := Ez^X$ is finite for $1 \leq z < R$ and infinite at $z = R$. Then*

$$\limsup_{n\to\infty} P\left\{\sum_{i=1}^{N} X_i > n\right\}^{1/n} > 1/R. \tag{0.2}$$

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PROOF. Let $S_N = \sum_{i=1}^{N} X_i$. If N has geometric distribution with parameter $0 < p < 1$, that is, if $P\{N = k\} = pq^k$ for all $k = 0, 1, 2, \dots$, where $q = 1 - p$, then the probability generating function of S_N is $Ez^{S_N} = p/(1 - q\varphi(z))$. Since $\varphi(z) \to \infty$ as $z \to R-$, there exists $1 < \rho < R$ such that $\varphi(\rho) = 1/q$. Thus, the smallest positive singularity of the probability generating function Ez^{S_N} is no larger than ρ. The result (0.2) now follows from Pringsheim's theorem. □

LEMMA 2. *Let X be a positive-integer valued random variable with probability generating function $\varphi(z) = Ez^X$. If, for some $R > 1$,*

$$\lim_{n\to\infty} P\{X \geq n\}^{1/n} = 1/R \tag{0.3}$$

and

$$\sum_{n\geq 1} R^n P\{X \geq n\} = \infty \tag{0.4}$$

then $\varphi(z)$ is finite for $1 \leq z < R$ and is infinite at $z = R$.

PROOF. This follows by Pringsheim's theorem from the identity

$$\sum_{n=1}^{\infty} z^n P\{X \geq n\} = z\frac{Ez^X - 1}{z - 1}.$$

□

To prove Theorem 3′ we shall compare the growth of the contact processes A_t and A_t^* with infection rates λ and λ_*, respectively, by constructing an auxiliary process B_t whose growth rate is strictly greater than that of A_t^* and is such that $B_t \subseteq A_t$. The construction of these processes uses the augmented percolation structure $\mathcal{P}$ described in Section 6 of [1]. Recall that $\mathcal{P}$ consists of mutually independent Poisson processes of recovery marks and infection arrows attached to vertices and neighboring pairs of vertices, respectively; the recovery mark processes have intensity 1 and the infection arrow processes have intensity λ. In addition, $\mathcal{P}$ attaches to each infection arrow α a Bernoulli-p random variable ξ_α, where $\lambda_* = \lambda q/(1 + p)$ and $q = 1 - p$. The contact process A_t is constructed from the system of infection arrows and recovery marks of $\mathcal{P}$ in the usual way (see [3]).

The contact process $A_t^* = A'_{t/(1+p)}$ is constructed in a similar fashion, but using the modified percolation structure $\mathcal{P}'$ obtained from $\mathcal{P}$ by changing all infection arrows α such that $\xi_\alpha = 1$ to recovery marks at their base points. The percolation structure $\mathcal{P}'$ again consists of independent Poisson processes of recovery marks and infection arrows; the recovery mark processes have intensity $1 + p$ and the infection arrow processes have intensity λq. The contact process A'_t is built using the arrows and recovery marks of $\mathcal{P}'$ in the usual way. Clearly, if A_t and $A_{t'}$ have common initial state $A_0 = A'_0 = \{e\}$ then $A_t \subseteq A'_t$ for all $t \geq 0$.

The auxiliary process B_t is constructed using a sequence of independent *severed* contact processes $C^1, C^2, \dots$. A contact process C_t is *severed* across an edge ε of the tree if it is constructed in the usual manner using the percolation structure $\mathcal{P}'$ from which all infection arrows across the edge ε have been removed. Let $x_0, x_1, x_2, \dots$ be the successive vertices along a geodesic ray γ emanating from $x_0 = e$, and let x_{-1} be a vertex at distance one from e distinct from x_1. Define C_t^1 to be the contact process with initial state $C_0^1 = \{e\}$ that is severed across the edge joining x_{-1} and e. By construction, $C_t^1 \subseteq A_t'$. Since A_t' does not survive strongly, there is a maximal integer $\nu(1)$ such that vertex $x_{\nu(1)}$ is ever infected by the severed contact process, that is,

$$x_{\nu(1)} \in \bigcup_{t \geq 0} C_t^1.$$

Observe that, since $x_{\nu(1)}$ is the farthest vertex along the geodesic ray γ ever infected by the severed contact process C^1, there must be a recovery mark ρ_1 at $x_{\nu(1)}$ terminating the last period of infection at $x_{\nu(1)}$.

Define $C^2, C^3, \dots$ inductively as follows: First, adjoin to the vertex $x_{\nu(1)+1}$, beginning at the time τ_1 of the recovery ρ_1, a contact process C^2 that is severed across the edge joining $x_{\nu(1)}$ to $x_{\nu(1)+1}$. Define $\nu(2) > \nu(1)$ to be the index of the last vertex $x_{\nu(2)}$ along the geodesic ray γ that is ever infected by C^2, and let ρ_2 be the recovery mark terminating the last epoch of infection at $x_{\nu(2)}$. Adjoin to $x_{\nu(2)+1}$, starting at the time τ_2 of the recovery mark ρ_2, a contact process C^3 that is severed across the edge joining $x_{\nu(2)}$ to $x_{\nu(2)+1}$. Continue this process indefinitely. Observe that the severed contact processes $C^1, C^2, C^3, \dots$ are, modulo translation in time and "space," independent and identically distributed, since each C^n is built in the part of the percolation structure not used by the preceding contact processes $C^1, C^2, \dots, C^{n-1}$. Thus, the random variables $\nu(1), \nu(2) - \nu(1), \dots$ are also independent and identically distributed.

Each of the severed contact processes C^n reaches a rightmost point $x_{\nu(n)}$ along the geodesic ray γ, and the last epoch of infection is terminated by a recovery mark ρ_n. Recovery marks in the percolation structure $\mathcal{P}'$ are of two types: those that were recovery marks in the percolation structure $\mathcal{P}$, and those that were infection arrows in $\mathcal{P}$. Let N be the first of the contact processes C^n whose terminal recovery mark ρ_n is *not* an infection arrow in $\mathcal{P}$ pointing from $x_{\nu(n)}$ to $x_{\nu(n)+1}$. Note that N has the geometric distribution with parameter p. Define B_t to be the process obtained by concatenating the processes $C^1, C^2, \dots, C^N$. Since the concatenations are all across infection arrows in $\mathcal{P}$, the process $B(t)$ is dominated by $A(t)$, that is, $B(t) \subseteq A(t)$ for all $t \geq 0$. Consequently, the vertex $x_{\nu(N)}$ is among the vertices infected by A_t, and so, in particular,

$$u_n := P\{x_n \in A_t \text{ for some } t \geq 0\} \geq P\{\nu(N) \geq n\}. \tag{0.5}$$

Since $\beta(\lambda) = \lim_{n \to \infty} u_n^{1/n}$, it now follows, by Lemmas 1 and 2, that to prove Theorem 3′ it suffices to prove the following.

PROPOSITION 3. *The random variable* $\nu := \nu(1)$ *is such that*

$$\lim_{n\to\infty} P\{\nu \geq n\}^{1/n} = \beta(\lambda_*) \tag{0.6}$$

and

$$\sum_{n=1}^{\infty} P\{\nu \geq n\}/\beta(\lambda_*)^n = \infty. \tag{0.7}$$

PROOF. Relation (0.6) was proved in [2]. The proof of relation (0.7) will be based on Schonmann's inequality (0.1) and the fact that $\beta(\lambda) < \sqrt{d}$ implies $\eta(\lambda) < 1$. Set $v_n = P\{\nu \geq n\}$; this is the probability that the contact process C^1 started at the root and severed across the edge joining x_{-1} and $x_0 = e$ ever infects the vertex x_n. Consider the event F_n that A_t ever infects x_n. On this event, there must be an infection trail from e to x_n, and this trail must visit x_{-1}, if at all, for a last time, and so, after passing through an infection arrow from x_{-1} to e for a last time, must then avoid the edge joining x_{-1} to e altogether. Let $T(1), T(2), \ldots$ be the times of the infection arrows from x_{-1} to e in the percolation structure $\mathcal{P}$, and let $T(0) = 0$. Since these are stopping times for the contact process, and since they mark the occurrence times of a rate-λ Poisson process, the strong Markov property implies that

$$u_n \leq \sum_{k=0}^{\infty} v_n P\{x_{-1} \in A_{T(k)}\} \leq v_n \lambda \int_{t=0}^{\infty} P\{x_{-1} \in A_t\}\,dt.$$

Conditional on the event $x_{-1} \in A_t$, the probability that $x_0 \in A_{t+1}$ is at least $\varepsilon(\lambda) := (1 - e^{-\lambda})e^{-1}$, this being the probability that there is at least one infection arrow from x_{-1} to x_0 and no recovery marks at x_0 in a time interval of duration 1. Consequently,

$$\begin{aligned} \lambda \int_{t=0}^{\infty} P\{x_{-1} \in A_t\}\,dt &\leq \lambda\varepsilon(\lambda)^{-1} \int_{t=1}^{\infty} P\{x_0 \in A_t\}\,dt \\ &\leq \lambda\varepsilon(\lambda)^{-1} \int_{t=1}^{\infty} \eta^t\,dt \\ &:= C' < \infty. \end{aligned}$$

Thus, by (0.1),

$$v_n \geq u_n/C' \geq \beta(\lambda)^n/C'Cn,$$

and inequality (0.7) now follows. □

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