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STABLE LOGARITHMIC MAPS TO DELIGNE–FALTINGS PAIRS II*

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Abstract. We make an observation which enables one to deduce the existence of an algebraic stack of logarithmic maps for all generalized Deligne–Faltings logarithmic structures (in particular simple normal crossings divisors) from the simplest case with characteristic generated by $\mathbb{N}$ (essentially the smooth divisor case).

Key words. Moduli spaces, logarithmic structures.

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1. Introduction. The idea of stable logarithmic maps was introduced in a legendary lecture by Bernd Siebert in 2001 [Sie01]. However, the program has been on hold for a while, since Mark Gross and Bernd Siebert were working on other projects in mirror symmetry. Only recently they have taken up the unfinished project of Siebert jointly [GS]. The central object is a stack $\mathcal{K}_\Gamma(Y)$ parameterizing what one calls *stable logarithmic maps* of log-smooth curves into a logarithmic scheme Y with Γ indicating the relevant numerical data, such as genus, marked points, curve class and other indicators (contact orders) related to the logarithmic structure. One needs to show $\mathcal{K}_\Gamma(Y)$ is algebraic and proper. Gross and Siebert's approach builds on insights from tropical geometry, obtained by probing the stack of logarithmic maps using the standard logarithmic point. It covers the case of targets with Zariski logarithmic structures, under assumptions spelled out in [GS, Definition 3.3].

In [Chen], the second author considers another combinatorial construction of the stack $\mathcal{K}_\Gamma(Y)$ when the logarithmic structure Y on the underlying scheme $\underline{Y}$ is associated to the choice of a line bundle with a section. The motivating case is that of a pair $(\underline{Y}, \underline{D})$, where $\underline{D}$ is a smooth divisor in the smooth locus of the scheme $\underline{Y}$ underlying Y. It should be pointed out that these stable logarithmic maps are not identical to those of Kim [Kim09], though they are closely related.

Our point is that based solely on this special case, one can give a "pure thought" proof of algebraicity and properness of the stack $\mathcal{K}_\Gamma(Y)$ whenever Y is a so called *Deligne–Faltings* logarithmic structure, see Theorem 2.6. By saying Y is a generalized Deligne–Faltings logarithmic structure we mean that there is a fine saturated sharp monoid P and a sheaf homomorphism $P \to \overline{\mathcal{M}}_Y$ which locally lifts to a chart $P \to \mathcal{M}_Y$; the slightly simpler *Deligne–Faltings logarithmic structure* is the case where $P = \mathbb{N}^k$. This in turn covers many of the cases of interest, such as a variety with a simple normal crossings divisor, or a simple normal crossings degeneration of a variety with a simple normal crossings divisor. We generalize it a bit further in Theorem 3.15, and deduce the case of a family $X \to B$ in Theorem 5.7. Our present results do not cover the case of a normal crossings divisor which is not simple, but we expect to cover this case using descent arguments.

The purpose of this note is to set up a general categorical framework which enables us to make this construction. This general setup is of use not only for $\mathcal{K}_\Gamma(Y)$.

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In particular we have applications, pursued elsewhere [ACGM], to constructing the target of *evaluation maps* of logarithmic Gromov-Witten theory.

All logarithmic schemes in this note are assumed to be fine and saturated logarithmic schemes - abbreviated fs logarithmic schemes - unless indicated otherwise. Following Ogus [Ogu06], logarithmic schemes are denoted by roman letters, and their underlying schemes indicated by underlines.

We work over the field of complex numbers $\mathbb{C}$; more general base schemes are certainly possible but would require additional care.

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2. Logarithmic maps: a tale of two categories.

2.1. Stable maps. Recall that an *n-pointed prestable curve* $(C, p_1, \cdots, p_n)$ over an algebraically closed field is a proper connected curve with at most nodes as singularities, along with n ordered smooth distinct closed points on C. An n-pointed family of prestable curve over S is a flat family of curves over S, along with n sections, such that every geometric fiber is an n-pointed prestable curve.

Let $\underline{Y}$ be a projective scheme over $\mathbb{C}$. The stack of stable maps to $\underline{Y}$ is defined as follows: one fixes discrete data (g, n, β) where g, n are non-negative integers standing for genus and number of marked points, and $\beta \in H_2(\underline{Y}, \mathbb{Z})$ is the homology class of an algebraic curve on $\underline{Y}$. A *pre-stable* map to $\underline{Y}$ over a scheme $\underline{S}$ is a diagram

$$\begin{tikzcd} \underline{C} \arrow[r] \arrow[d] & \underline{Y} \\ \underline{S} & \end{tikzcd}$$

where $\underline{C} \to \underline{S}$ is a proper flat family of n-pointed prestable curves, and $\underline{C} \to \underline{Y}$ a morphism. The prestable map is *stable* if on the fibers the groups $\mathrm{Aut}_{\underline{Y}}(\underline{C}_s)$ are finite. Morphisms of prestable maps are defined as cartesian diagrams

$$\begin{tikzcd} \underline{C}' \arrow[r] \arrow[d] & \underline{C} \arrow[r] \arrow[d] & \underline{Y} \\ \underline{S}' \arrow[r] & \underline{S}. & \end{tikzcd}$$

One easily sees that prestable maps form a category fibered in groupoids over the category of schemes. It is an important theorem that this fibered category is an algebraic stack $\underline{K}^{pre}(\underline{Y})$, and the substack $\underline{K}_{g,n}(\underline{Y}, \beta)$ of stable maps of type (g, n, β) is proper with projective coarse moduli space [Kon95]. When $\underline{Y}$ is smooth, there is a perfect obstruction theory, giving rise to a virtual fundamental class $[\underline{K}_{g,n}(\underline{Y}, \beta)]^{\mathrm{virt}}$ underlying the usual algebraic treatment of Gromov–Witten theory [LT98, BF97].

The main result of this section, Theorem 2.6, is an analogue of the following evident result: assume $\underline{Y} = \underline{Y}_1 \times_{\underline{Y}_2} \underline{Y}_3$. Then

$$\underline{K}^{pre}(\underline{Y}) = \underline{K}^{pre}(\underline{Y}_1) \underset{\underline{K}^{pre}(\underline{Y}_2)}{\times} \underline{K}^{pre}(\underline{Y}_3).$$

2.2. Stable logarithmic maps as a stack over $\mathfrak{LogSch}$. Let Y be a fs logarithmic scheme with projective underlying scheme $\underline{Y}$. One can repeat the construction above, replacing prestable curves by proper *logarithmically smooth* curves [Kat00], and replacing all morphisms of schemes by morphisms of logarithmic schemes: a *pre-stable logarithmic map over* S is a diagram of *logarithmic* schemes

$$\begin{tikzcd} C \arrow[r] \arrow[d] & Y \\ S & \end{tikzcd}$$

such that $C \to S$ is a logarithmically smooth curve, where the underlying curve $\underline{C} \to \underline{S}$ is a family of usual proper pre-stable curves with markings.

Given the above pre-stable logarithmic map over S, one obtains a family of usual pre-stable maps $\underline{C} \to \underline{Y}$ over $\underline{S}$ by removing the logarithmic structures, but keeping all the marked points. We define a pre-stable logarithmic map to be *stable* if the associated underlying prestable map is stable; this is equivalent to requiring the sheaf $\Omega_{C/S}$ of logarithmic differentials to be ample relative to $\underline{Y} \times \underline{S}$. Arrows are again defined using cartesian diagrams:

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & Y \\ S' \arrow[r] & S. & \end{tikzcd}$$

Again this is evidently a category fibered in groupoids, but this time *over the category* $\mathfrak{LogSch}$ *of fs logarithmic schemes.* It is proven in [Chen] when the logarithmic structure Y is given by a line bundle with a section, and more generally in [GS], that this category is represented by an *fs logarithmic algebraic stack* $\mathcal{K}_{g,n}(Y,\beta) = (\underline{\mathcal{K}}, \mathcal{M}_{\underline{\mathcal{K}}})$: there is an algebraic stack $\underline{\mathcal{K}}$, along with an fs logarithmic structure $\mathcal{M}_{\underline{\mathcal{K}}}$, such that stable logarithmic maps over S are equivalent to logarithmic morphisms $S \to \mathcal{K}_{g,n}(Y,\beta)$. We refer to [Ols03b, Section 5] for more details on logarithmic structures on stacks. Denote the universal logarithmically smooth curve by $\mathfrak{C} \to \mathcal{K}_{g,n}(Y,\beta)$.

It is natural to search for general criteria for algebraicity of such logarithmic moduli analogous to Artin's work [Art74]. We do not address this general question here.

2.3. Stable logarithmic maps as a stack over $\mathfrak{Sch}$. The existence of $\mathcal{K} := \mathcal{K}_{g,n}(Y,\beta)$ has immediate strong implications on the structure of stable logarithmic maps. Objects of the underlying stack $\underline{\mathcal{K}}$ over a scheme $\underline{S}$ can be understood as follows: an object is after all an arrow $\underline{S} \to \underline{\mathcal{K}}$. It automatically gives rise to a cartesian diagram

$$\begin{tikzcd} C^{\min} \arrow[r] \arrow[d] & \mathfrak{C} \arrow[r] \arrow[d] & Y \\ S^{\min} \arrow[r] \arrow[d] & \mathcal{K} \arrow[d] & \\ \underline{S} \arrow[r] & \underline{\mathcal{K}}, & \end{tikzcd}$$

in particular an object $S^{\min} \to \mathcal{K}$, but here *the logarithmic structure* $S^{\min}$ *is pulled back from* $\mathcal{K}$. Moreover, *every* stable logarithmic map factors uniquely through one of this type: Given a stable logarithmic map over S we have a morphism $S \to \mathcal{K}$ by definition, giving rise to an extended cartesian diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & C^{\min} \arrow[r] \arrow[d] & \mathfrak{C} \arrow[r] \arrow[d] & Y \\
S \arrow[r] & S^{\min} \arrow[r] \arrow[d] & \mathcal{K} \arrow[d] & \\
& \underline{S} \arrow[r] & \underline{\mathcal{K}}. &
\end{tikzcd}$$

Following B. Kim [Kim09] we call a logarithmic map over S *minimal* (not to be confused with logarithmic minimal models of the minimal model program) if $S \to \mathcal{K}$ is strict, namely the logarithmic structure on S is the pullback of the logarithmic structure on $\mathcal{K}$. It follows tautologically that the underlying stack $\underline{\mathcal{K}}$ precisely parametrizes logarithmic maps with minimal logarithmic structure.

In fact this thought process is reversible: the construction of [Chen] in the case of a Deligne–Faltings logarithmic structure of rank-1 goes by way of constructing a proposed minimal logarithmic structure associated to any logarithmic map, and verifying that logarithmic maps where the logarithmic structure is the proposed minimal one are indeed minimal (every object maps uniquely to a minimal one), and form an algebraic stack over $\mathfrak{Sch}$ carrying a logarithmic structure. In short, the second categorical interpretation, of $\underline{\mathcal{K}}$ as a stack over $\mathfrak{Sch}$, takes precedence here.

One is tempted to try to mimic the same construction in general. *This is not the route taken here.* In fact we use the universality of the category $\mathcal{K}$ over $\mathfrak{LogSch}$ for given Y to deduce its algebraicity from cases of simpler Y. Minimal object are obtained as an afterthought in Proposition 3.18 and described combinatorially in Section 4.1. It is worthwhile setting this up in general.

2.4. The general setup. Consider a commutative diagram

$$\begin{tikzcd}
X \arrow[dd] \arrow[dr] & & W \arrow[dd] \arrow[dl] \\
& B \arrow[dd] & \\
\underline{X} \arrow[rr, "\underline{f}"] \arrow[dr] & & \underline{W} \arrow[dl] \\
& \underline{B} &
\end{tikzcd}$$

where $X \to B$ and $W \to B$ are morphisms of fs logarithmic schemes or algebraic spaces, the bottom triangle consists of morphisms of usual schemes, and the vertical arrows are the canonical maps from a logarithmic scheme to its underlying scheme. We define a contravariant functor

$$\mathrm{Lift}_{\underline{f}} : \mathfrak{LogSch} \to \mathfrak{Sets}$$

as follows: an element of $\mathrm{Lift}_{\underline{f}}(S)$ over an fs logarithmic scheme S is a pair

$$(S \to B, f_S : X_S \to W_S)$$

where $S \to B$ is a morphism, and $f_S : X_S \to W_S$ is an S-morphism which makes the diagram

$$\begin{tikzcd} X_S \arrow[r, "f_S"] \arrow[d] & W_S \arrow[d] \\ \underline{X}_{\underline{S}} \arrow[r, "\underline{f}_{\underline{S}}"] & \underline{W}_{\underline{S}} \end{tikzcd}$$

commutative. Given a morphism $g : S' \to S$ we define $\mathrm{Lift}_{\underline{f}}(S) \to \mathrm{Lift}_{\underline{f}}(S')$ by taking pullbacks.

We can ask the following:

PROBLEM 2.5. Is the functor $\mathrm{Lift}_{\underline{f}}$ represented by an fs logarithmic algebraic space $\mathcal{L}ift_{\underline{f}}$? Under what conditions is it proper?

Problem 2.5 was solved when $X \to B$ is a family of log smooth curves and $W = Y \times B$ with Y a rank-1 DF logarithmic scheme in [Chen]. A much more general situation is covered in [GS]. In this paper we use a reduction step given in Theorem 2.6 to deduce the case where W is a generalized DF logarithmic scheme from the case covered in [Chen]. The case where $\underline{X} \to \underline{B}$ is an isomorphism is investigated in [ACGM] using similar methods.

Reformulation using a space of sections. Following the techniques of Olsson [Ols06], we remove the geometry of W entirely from the picture as follows: consider $\overline{W} = \underline{W} \times_{\underline{B}} B$ and set $Z = X \times_{\overline{W}} W$. Then Z is a logarithmic scheme over X. The functor $\mathrm{Lift}_{\underline{f}}$ is evidently isomorphic to the functor of sections $\mathrm{Sec}_{X/B}(Z/X)$, which assigns to S the set of pairs $(S \to B, X_S \to Z_S)$, where $S \to B$ is a logarithmic morphism and $X_S \to Z_S$ is a section of $Z_S \to X_S$.

The key observation is the following:

THEOREM 2.6. *Let $\Delta = (Z_\alpha, \pi_{\alpha\beta} : Z_\alpha \to Z_\beta)$ be a finite diagram of fs logarithmic schemes over X. Assume*

$$Z = \varprojlim(\Delta)$$

in the category of fs logarithmic schemes. Then the canonical map

$$Sec_{X/B}(Z/X)(S) \to \varprojlim Sec_{X/B}(Z_\alpha/X)(S)$$

is bijective for any fs logarithmic schemes S over B.

In particular, if $Sec_{X/B}(Z_\alpha/X)$ are represented by logarithmic schemes, then $Sec_{X/B}(Z/X)$ is represented by their limit.

If the reader finds these limits a bit off-putting, we point out that the result is equivalent to the case of fiber products.

The existence of such limits in the category of fs logarithmic schemes is proven in [Kat89]: the case of arbitrary logarithmic structures is treated in (1.6), coherent logarithmic structures in (2.6), and fine logarithmic structures follow from (2.7); the case of fs logarithmic structures follows from the analogous adjoint functor with P^{int} replaced by P^{sat} ([Ogu06, Chapter I, Proposition 1.2.3]).

Proof of the theorem. An element of $\mathrm{Sec}_{X/B}(Z/X)$ over an arrow $S \to B$ is by definition a section $s : X_S \to Z$, and composing with the canonical maps $Z \to Z_\alpha$ we get sections $s_\alpha : X_S \to Z_\alpha$ such that for each arrow $\pi_{\alpha\beta} : Z_\alpha \to Z_\beta$ in Δ we have $\pi_{\alpha\beta} \circ s_\alpha = s_\beta$. This in particular gives us a diagram of elements $s_\alpha \in \mathrm{Sec}_{X/B}(Z_\alpha/X)(S)$ with $\pi_{\alpha\beta}(s_\alpha) = s_\beta$, namely an element of $\varprojlim \mathrm{Sec}_{X/B}(Z_i/X)(S)$.

The process is completely reversible, hence the bijection. ◻

Setup with stacks. Below we use a slight generalization: we replace B, X, W by logarithmic algebraic stacks, keeping the morphisms $X \to B$ and $W \to B$ representable by schemes. In this situation we prefer to define a *category* $\widetilde{\mathrm{Lift}}_{\underline{f}}$, fibered in groupoids over $\mathfrak{LogSch}$, whose objects over S are pairs $(S \to B, f_S : X_S \to W_S)$ as above and arrows are defined by cartesian diagrams.

The construction of Z remains the same, and in this situation $\mathrm{Sec}_{X/B}(Z/X)$ is naturally a category fibered in groupoids over $\mathfrak{LogSch}$. We obtain the following:

Corollary 2.7. *In this situation $Sec_{X/B}(Z/X)(S) \to \varprojlim Sec_{X/B}(Z_\alpha/X)(S)$ is an equivalence. In particular, if $Sec_{X/B}(Z_\alpha/X)$ are represented by logarithmic algebraic stacks which are representable over B, then $Sec_{X/B}(Z/X)(S)$ is represented by their limit.*

Proof. This can be tested after a base change $B' \to B$ where B' is a logarithmic scheme, where the Theorem 2.6 applies. ◻

3. The stacks of stable logarithmic maps. Theorem 2.6 applies directly to the category $\mathrm{Lift}_{\underline{f}}$ and Problem 2.5. In the present section we make this as explicit as possible in case $X \to B$ is a family of logarithmically smooth curves, including a discussion of *contact orders*, the deformation-invariant numerical data encoded in the logarithmic structure.

Recall that a *toric sharp monoid* is a finitely generated, torsion free and saturated monoid whose only invertible element is the origin.

3.1. The case of Deligne–Faltings pairs: setup.

Definition 3.2. A logarithmic structure $\mathcal{M}_Y$ on $\underline{Y}$ is called a *Deligne–Faltings logarithmic structure*, if there exists a finitely generated free monoid P and a map $P \to \overline{\mathcal{M}}_Y$, which locally lifts to a chart. The logarithmic scheme $Y = (\underline{Y}, \mathcal{M}_Y)$ is called a *Deligne–Faltings (DF) pair.*

If P is only assumed to be a toric sharp monoid instead of free, then $\mathcal{M}_Y$ is a *generalized Deligne–Faltings logarithmic structure*, and $Y = (\underline{Y}, \mathcal{M}_Y)$ is called a *generalized Deligne–Faltings pair.*

An even more general situation is considered in Section 3.14.

Remark 3.3. Let Y be a fs logarithmic scheme over $\mathbb{C}$, and $x \in Y$ a point. Then étale locally near x one can choose a chart using the characteristic monoid: $\overline{\mathcal{M}}_{Y,x} \to \mathcal{M}_Y$. We refer to [Ols03b, Section 2] for more discussions on charts of logarithmic structures.

Let the target Y be a DF pair. We first break up the logarithmic structure $\mathcal{M}_Y$ into cases covered in [Chen]. We write $P \cong \mathbb{N}^m$, and index the m copies of $\mathbb{N}$ by the set $\{1, 2, \cdots, m\}$. We denote by $\mathbb{N}_i \hookrightarrow P$ the i-th copy of $\mathbb{N}$. Note that we have the following composition

$$\mathbb{N}_i \to P \to \overline{\mathcal{M}}_Y$$

which induces a pre-logarithmic structure by the following composition:

$$\mathbb{N}_i \times_{\overline{\mathcal{M}}_Y} \mathcal{M}_Y \to \mathcal{M}_Y \to \mathcal{O}_Y.$$

Let $\mathcal{M}_i$ be the logarithmic structure associated to the above pre-logarithmic structure. Note that $\mathcal{M}_i$ is a rank-1 DF-logarithmic structure on $\underline{Y}$. Furthermore, the inclusion $\mathbb{N}_i \to P$ identifies $\mathcal{M}_i$ as a sub-logarithmic structure of $\mathcal{M}_Y$, and the following decomposition holds:

$$\mathcal{M}_Y = \mathcal{M}_1 \oplus_{\mathcal{O}_Y^*} \mathcal{M}_2 \oplus_{\mathcal{O}_Y^*} \cdots \oplus_{\mathcal{O}_Y^*} \mathcal{M}_m. \tag{3.3.1}$$

Denote $Y_i = (\underline{Y}, \mathcal{M}_i)$ for $i = 1, \ldots, m$. Viewing $\underline{Y} = Y_0$ as a logarithmic scheme with trivial logarithmic structure, the above decomposition is equivalent to

$$Y = Y_1 \times_{Y_0} Y_2 \times_{Y_0} \cdots \times_{Y_0} Y_m, \tag{3.3.2}$$

which can be written as

$$Y = \varprojlim Y_i.$$

3.4. Review of contact orders in the rank-1 case. Consider a morphism $f : C \to Y_i$ from a logarithmically smooth curve to Y_i, and a marked point p_j in C with local parameter x. In [Chen, Definition 3.2.3] one defines the *contact order* c_i^j of f at p_j. The characteristic monoid of C at p_j is of the form $\overline{\mathcal{M}}_S \oplus \mathbb{N}$, where the second factor is generated by $\log x$. The morphism f induces a homomorphism $\bar{f}^\flat : \mathbb{N} \to \overline{\mathcal{M}}_S \oplus \mathbb{N}$, and $\bar{f}^\flat(\delta) = e + c_i^j$ with the generator $\delta \in \mathbb{N}$, an element $e \in \overline{\mathcal{M}}_S$, and $c_i^j \in \mathbb{N}$. The integer c_i^j is the required contact order of f at p_j.

Note that the logarithmic structure $\mathcal{M}_i$ corresponds to a pair (L_i, s_i), consisting of a line bundle L_i on $\underline{Y}$, and a section $s_i \in H^0(L_i^\vee)$. The contact orders satisfy the following relation:

$$\deg f^* L_i^\vee = \sum_{j=1}^n c_i^j.$$

3.5. Numerical data and the associated moduli. We can now introduce notation for the numerical data of a logarithmic map $f : C \to Y$:

NOTATION 3.6. Denote $\Gamma = (g, n, \beta, \mathbf{c})$ where

1. $g \in \mathbb{Z}_{\geq 0}$ denotes the genus of the source curve;
2. $n \in \mathbb{Z}_{\geq 0}$ denotes the number of marked points;
3. $\beta \in H_2(\underline{Y}, \mathbb{Z})$ denotes the class of an algebraic curve;
4. $\mathbf{c} = \{(c_i^j)_{i=1}^m\}_{j=1}^n$ where $c_i^j \in \mathbb{Z}_{\geq 0}$ denotes the contact order of the j-th marking with respect to $\mathcal{M}_i$. We require that

   $$\beta \cdot c_1(L_i^\vee) = \sum_{j=1}^n c_i^j,$$

   where $c_1(L_i^\vee)$ is the first chern class of $L_i^\vee$.

Definition 3.7. Denote by $\mathcal{K}_\Gamma(Y)$ (respectively $\mathcal{K}_\Gamma^{pre}(Y)$) the category fibered over $\mathfrak{LogSch}$, which parametrizes stable logarithmic maps (respectively pre-stable logarithmic maps) with the discrete data Γ in Notation 3.6.

Note that the set $\mathbf{c}$, hence $\mathcal{K}_\Gamma(Y)$, depends on the choice of P and its decomposition. For example, consider the union of two disjoint divisors $D = D_1 \cup D_2$ in $\underline{Y}$. Denote by I_i the ideal sheaf of D_i, and $s_i : I_i \to \mathcal{O}_{\underline{Y}}$ the natural injection. Let $\mathcal{M}_Y$ be the standard logarithmic structure associated to D on $\underline{Y}$, and Y the logarithmic scheme $(\underline{Y}, \mathcal{M}_Y)$. Now we have two maps of sheaves of monoids

$$\mathbb{N} \to \overline{\mathcal{M}}_Y, \quad \text{and} \quad \mathbb{N}^2 \to \overline{\mathcal{M}}_Y, \tag{3.7.1}$$

where the first one is given by the ideal sheaf I_D of D with the natural injection $s_D : I_D \to \mathcal{O}_Y$, and the second one is given by the two pairs (I_i, s_i) for $i = 1, 2$. Note that both maps of monoids in (3.7.1) can be lifted locally to charts of $\mathcal{M}_Y$.

Consider a marked point p with assigned contact order c under the choice $P = \mathbb{N}$. This includes the following two cases: p can have contact orders c with D_1 and 0 with D_2, or it can have contact order c with D_2 and 0 with D_1. However if we choose the second case where $P = \mathbb{N}^2$, then the contact orders along both D_1 and D_2 will be specified. Thus, for different choices of P we obtain different $\mathbf{c}$, resulting in different stacks $\mathcal{K}_\Gamma(Y)$.

The category $\mathcal{K}_{g,n}(Y, \beta)$ introduced in 2.2 parametrizes minimal stable logarithmic maps without restricting the contact orders. It does not depend on the choice of the monoid P. For any choice of P and Γ, the category $\mathcal{K}_\Gamma(Y)$ is open and closed in $\mathcal{K}_{g,n}(Y, \beta)$. We clearly have

$$\mathcal{K}_{g,n}(Y, \beta) = \coprod_{\mathbf{c}} \mathcal{K}_{(g,n,\beta,\mathbf{c})}(Y),$$

where $(g, n, \beta, \mathbf{c})$ runs through all possible choices of $\mathbf{c}$.

3.8. Canonical contact orders. A further refined and entirely canonical formalism of contact orders follows from [ACGM]: given any fs logarithmic scheme Y one defines an Artin stack $\wedge Y$, locally of finite type over $\underline{Y}$, parameterizing standard logarithmic points in Y. We call it *the evaluation space of* Y. Given a stable logarithmic map $f : C \to Y$ and an integer j with $1 \leq j \leq n$, the restriction $f_{\Sigma_j} : \Sigma_j \to Y$ of f to the j-th marking, is an element of $\wedge Y$. This defines the j-th evaluation map $\mathcal{K}_{g,n}(Y, \beta) \to \wedge Y$.

Definition 3.9. A *logarithmic sector* of Y is an element $c \in \pi_0(\wedge Y)$, namely a connected component of the evaluation space of Y. A stable map f is said to have canonical contact order $\mathbf{c} = (c^1, \ldots, c^n)$ if the j-th evaluation map lands in the logarithmic sector $c^j \in \pi_0(\wedge Y)$. Given $\Gamma = (g, n, \beta, \mathbf{c})$ with $\mathbf{c} = (c^1, \ldots, c^n)$ and $c^i \in \pi_0(\wedge Y)$ we define $\mathcal{K}_\Gamma(Y)$ precisely as in Definition 3.7.

In [GS, Formula (1.6)], Gross and Siebert introduce numerical data denoted u_p. This data is closely related to our canonical contact orders; the relationship is made more precise in [ACGM].

The discussion which follows works equally well with the explicit and down-to-earth contact orders of Definition 3.6, as with the canonical contact orders of Definition 3.9.

3.10. The case of Deligne–Faltings pairs: the stacks. Fix numerical data $\Gamma = (g, n, \beta, \mathbf{c})$, with $\mathbf{c}$ as in 3.6. For each $i = 1, \ldots, m$, we have the contact orders $\mathbf{c}_i = \{c_i^j\}_{j=1}^n$. Consider the discrete data $\Gamma_i = (g, n, \beta, \mathbf{c}_i)$. Then by [Chen] we have the logarithmic Deligne–Mumford stack $\mathcal{K}_{\Gamma_i}(Y_i)$, parametrizing minimal stable logarithmic maps with discrete data Γ_i. If instead we use canonical contact orders of 3.9, then we can take $\mathbf{c}_i$ to be the image of $\mathbf{c}$ under the canonical map $\pi_0(\wedge Y) \to \pi_0(\wedge Y_i)$.

Consider the stack $\mathcal{K}_{g,n}(\underline{Y}, \beta)$ of usual stable maps. Since the universal curve is prestable, it has a canonical logarithmically smooth structure coming from $\mathfrak{M}_{g,n}$; for stable curves this is F. Kato's theorem [Kat00], see [Ols03a, Theorem 1.2] for the general result in arbitrary dimension. We put this in the setup of $\mathrm{Sec}_{X/B}(Z/X)$ by setting B to be $\mathcal{K}_{g,n}(\underline{Y}, \beta)$ or $\mathcal{K}_{g,n}^{pre}(\underline{Y}, \beta)$ with the pull-back logarithmic structure coming from $\mathfrak{M}_{g,n}$; X to be the universal curve with its canonical logarithmic structure; and $W = Y \times B$. Consider $Z = X \times_{(\underline{W} \times_{\underline{B}} B)} W$ as before. Denote by $W_i = Y_i \times B$, and $Z_i = X \times_{(\underline{W} \times_{\underline{B}} B)} W_i$ for $i = 1, \cdots, m$. Let $Z_0 = X \times_{(\underline{W} \times_{\underline{B}} B)} \underline{W}$. Then by (3.3.2), we have

$$Z \cong Z_1 \times_{Z_0} Z_2 \times_{Z_0} \cdots \times_{Z_0} Z_m,$$

or equivalently

$$Z = \varprojlim Z_i.$$

COROLLARY 3.11. *The fibered categories $\mathcal{K}_\Gamma(Y)$ and $\mathcal{K}_\Gamma^{pre}(Y)$ are represented by algebraic stacks equipped with fs logarithmic structures. Furthermore, $\mathcal{K}_\Gamma(Y)$ is representable and finite over $\mathcal{K}_{g,n}(\underline{Y}, \beta)$.*

Proof. For the first statement, it follows from Theorem 2.6 and Corollary 2.7 that

$$\mathcal{K}_\Gamma^{pre}(Y) \cong \mathcal{K}_{\Gamma_1}^{pre}(Y_1) \times_{\mathcal{K}_{g,n}^{pre}(\underline{Y},\beta)} \cdots \times_{\mathcal{K}_{g,n}^{pre}(\underline{Y},\beta)} \mathcal{K}_{\Gamma_m}^{pre}(Y_m), \tag{3.11.1}$$

where the fibered product are taken in the category of fs logarithmic stacks. It was shown in [Chen] that the first statement holds for $\mathcal{K}_{\Gamma_i}^{pre}(Y_i)$. Note that

$$\mathcal{K}_\Gamma(Y) \cong \mathcal{K}_\Gamma^{pre}(Y) \times_{\mathcal{K}_{g,n}^{pre}(\underline{Y},\beta)} \mathcal{K}_{g,n}(\underline{Y}, \beta).$$

Thus, the first statement follows.

Similarly, we have a fiber product of fs logarithmic stacks

$$\mathcal{K}_\Gamma(Y) \cong \mathcal{K}_{\Gamma_1}(Y_1) \times_{\mathcal{K}_{g,n}(\underline{Y},\beta)} \cdots \times_{\mathcal{K}_{g,n}(\underline{Y},\beta)} \mathcal{K}_{\Gamma_m}(Y_m). \tag{3.11.2}$$

Now the second statement follows from the finiteness of products in the category of fs logarithmic schemes, see [Ogu06, Chapter 2, 2.4.5]. ∎

3.12. The case of generalized Deligne–Faltings pairs. Consider the target Y with generalized DF logarithmic structure $\mathcal{M}_Y$, with a fixed map $P \to \overline{\mathcal{M}}_Y$ as in Definition 3.2. Again we have $B = \mathcal{K}_{g,n}(\underline{Y}, \beta)$ with the pull-back logarithmic structure coming from $\mathfrak{M}_{g,n}$; X is the universal curve of B with its canonical logarithmic structure; $W = Y \times B$; and $Z = X \times_{(\underline{W} \times_{\underline{B}} B)} W$ as before.

By [Ogu06, Chapter 1, 2.1.9(7)], we have the following coequalizer diagram of monoids

$$\mathbb{N}^{n_2} \underset{v_2}{\overset{v_1}{\rightrightarrows}} \mathbb{N}^{n_1} \xrightarrow{q} P, \tag{3.12.1}$$

where n_1 and n_2 are non-negative integers. This is equivalent to say that

$$P = \lim_{\to}(\mathbb{N}^{n_2} \rightrightarrows \mathbb{N}^{n_1}).$$

Thus we have the following push-out diagram of fs monoids:

$$\begin{tikzcd} \mathbb{N}^{n_2} \oplus \mathbb{N}^{n_2} \arrow[r, "id\oplus id"] \arrow[d, "v_1\oplus v_2"'] & \mathbb{N}^{n_2} \arrow[d] \\ \mathbb{N}^{n_1} \arrow[r] & P. \end{tikzcd} \tag{3.12.2}$$

Consider the composition $\mathbb{N}^{n_i} \to P \to \overline{\mathcal{M}}_Y$. Again, we can construct a logarithmic structure $\mathcal{M}_i$ associated to the pre-logarithmic structure

$$\mathbb{N}^{n_i} \times_{\overline{\mathcal{M}}_Y} \mathcal{M}_Y \to \mathcal{M}_Y \to \mathcal{O}_Y.$$

Thus, we obtain two DF logarithmic structures $\mathcal{M}_1$ and $\mathcal{M}_2$, with the following coequalizer diagram of logarithmic structures:

$$\mathcal{M}_2 \underset{v_2}{\overset{v_1}{\rightrightarrows}} \mathcal{M}_1 \xrightarrow{q} \mathcal{M}_Y. \tag{3.12.3}$$

Denote by $\mathcal{M}_3 = \mathcal{M}_2 \oplus_{\mathcal{O}_Y^*} \mathcal{M}_2$. Since $\mathcal{M}_2$ is DF, the natural map

$$\mathbb{N}^{n_3} := \mathbb{N}^{n_2} \oplus \mathbb{N}^{n_2} \to \overline{\mathcal{M}}_3$$

locally lifts to a chart. Thus $\mathcal{M}_3$ is also a DF logarithmic structure. The coequalizer diagram (3.12.3) is equivalent to the following push-out diagram:

$$\begin{tikzcd} \mathcal{M}_3 \arrow[r, "id\oplus id"] \arrow[d, "v_1\oplus v_2"'] & \mathcal{M}_2 \arrow[d] \\ \mathcal{M}_1 \arrow[r] & \mathcal{M}_Y. \end{tikzcd} \tag{3.12.4}$$

Denote by $Y_i = (\underline{Y}, \mathcal{M}_i)$, for $i = 1, 2$, and 3. Then (3.12.3) induces an equalizer of logarithmic schemes:

$$Y \to Y_1 \rightrightarrows Y_2,$$

or equivalently

$$Y = \lim_{\longleftarrow} Y_i. \tag{3.12.5}$$

By construction, the map $\mathbb{N}^{n_i} \to \overline{\mathcal{M}}_i$ locally lifts to a chart. Hence each Y_i is a DF pair.

The diagram (3.12.4) implies that (3.12.5) is equivalent to the following cartesian diagram of fs logarithmic schemes:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y_2 \arrow[d] \\ Y_1 \arrow[r] & Y_3. \end{tikzcd} \tag{3.12.6}$$

Denote by $W_i = Y_i \times B$, and $Z_i = X \times_{(\underline{W_i} \times_{\underline{B}} B)} W_i$. Then (3.12.5) and (3.12.6) implies that

$$Z = \varprojlim Z_i, \tag{3.12.7}$$

or equivalently the cartesian diagram of fs logarithmic schemes:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z_2 \arrow[d] \\ Z_1 \arrow[r] & Z_3. \end{tikzcd} \tag{3.12.8}$$

We can define discrete data using canonical contact orders $\mathbf{c}$ as in Definition 3.9. Alternatively, we can define a concrete and down-to-earth version using $Irr(P)$, the set of irreducible elements in P. Since the monoid P is toric and sharp, $Irr(P)$ forms a finite set of generators of P. For each element $\alpha \in Irr(P)$, there is a free sub-monoid $\mathbb{N}_\alpha \cong \mathbb{N} \hookrightarrow P$, which induces a rank-1 DF sub-logarithmic structure $\mathcal{M}_\alpha \subset \mathcal{M}_Y$. Denote by (L_α, s_α) to be the pair of line bundles and sections corresponding to $\mathcal{M}_\alpha$. Again, we fix the numerical data $\Gamma = (g, n, \beta, \mathbf{c})$ such that

1. the data g, n, β denote the genus, number of marked points, and curve class as in Notation 3.6;
2. $\mathbf{c} = \{(c^j_\alpha)_{\alpha \in Irr(P)}\}_{j=1}^n$ is a set of tuples, where c^j_α denotes the contact order of the j-th marking with respect to $\mathcal{M}_\alpha$ satisfying

$$\beta \cdot c_1(L_\alpha^\vee) = \sum_{j=1}^n c^j_\alpha.$$

For any stable logarithmic map $f : C \to Y$ over a geometric fiber with discrete data Γ, the composition

$$f' : C \to Y \to Y_i$$

induces a stable logarithmic map to Y_i, with discrete data Γ_i given as follows.

In fact, Γ and Γ_i have the same data g, n, β. To determine Γ_i we only need to consider the contact orders. By the construction of $\mathcal{M}_i$, we have a map from the constant sheaf of free monoid $\mathbb{N}^{n_i} \to \overline{\mathcal{M}}_i$, which locally lifts to a chart. Denote by δ' the image of δ in P given by (3.12.2). Then we have a decomposition:

$$\delta' = \sum_{\alpha \in Irr(P)} k_\alpha \cdot \alpha \tag{3.12.9}$$

where k_α is a non-negative integer. Thus, for the j-th marking, the contact order of δ in Γ_i is given by

$$c^j_\delta := \sum_{\alpha \in Irr(P)} k_\alpha \cdot c^j_\alpha.$$

Note that the decomposition in (3.12.9) is not necessarily unique. But one can check that the above c^j_δ does not depend on a choice of (3.12.9). In fact, consider the composition of morphisms of monoids at the j-th marking:

$$c^j : P \to f^*\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C \cong \overline{\mathcal{M}}_S \oplus \mathbb{N} \to \mathbb{N}$$

where the last arrow is the projection to the second factor, and S is the base fs log scheme of the geometric fiber. Then we check that c^j is given by the correspondence $\delta \mapsto c^j_\delta$ for any $\delta \in P$. Thus, we see that Γ_i depend uniquely on Γ and $\mathcal{M}_i$.

COROLLARY 3.13. *Assume that the logarithmic structure of Y is a generalized Deligne–Faltings logarithmic structure. Then the categories $\mathcal{K}_\Gamma(Y)$ and $\mathcal{K}^{pre}_\Gamma(Y)$ are algebraic stacks with fs logarithmic structures. Furthermore, the stack $\mathcal{K}_\Gamma(Y)$ is representable and finite over $\mathcal{K}_{g,n}(\underline{Y},\beta)$.*

Proof. Again Theorem 2.6 implies that

$$\mathcal{K}^{pre}_\Gamma(Y) \cong \mathcal{K}^{pre}_{\Gamma_1}(Y_1) \times_{\mathcal{K}^{pre}_{\Gamma_3}(Y_3)} \mathcal{K}^{pre}_{\Gamma_2}(Y_2) \tag{3.13.1}$$

and

$$\mathcal{K}_\Gamma(Y) \cong \mathcal{K}_{\Gamma_1}(Y_1) \times_{\mathcal{K}_{\Gamma_3}(Y_3)} \mathcal{K}_{\Gamma_2}(Y_2) \tag{3.13.2}$$

where this is a fibered product of fs logarithmic stacks. Now the statement follows from the same argument as in Corollary 3.11. ∎

As before, we have $\mathcal{K}_{g,n}(Y,\beta) = \coprod_{\Gamma\in\Lambda} \mathcal{K}_\Gamma(Y)$, where Λ is the set of all possible $\Gamma = (g,n,\beta,\mathbf{c})$ with fixed g,n and β. Similarly, we have $\mathcal{K}^{pre}_{g,n}(Y,\beta) = \coprod_{\Gamma\in\Lambda} \mathcal{K}^{pre}_\Gamma(Y)$.

3.14. A further generalization. We can weaken the Deligne–Faltings assumption in Corollary 3.13 as follows. Consider a fine and saturated logarithmic scheme Y with a surjective homomorphism of sheaves of monoids $\mathbb{N}^k_Y \to \overline{\mathcal{M}}_Y$. For example, if Y is a projective toric variety with log structure given by its toric boundary, then the above surjection exists. In fact, it is proved in [CS, Appendix A] that in this case $\mathcal{M}_Y$ is a generalized Deligne-Faltings log structure. In other cases, including toroidal cases, it is easier to check the natural condition $\mathbb{N}^k_Y \twoheadrightarrow \overline{\mathcal{M}}_Y$ imposed here than to construct a monoid exhibiting the generalized Deligne-Faltings property.

Again, we can take $\mathcal{M}_{Y'}$ to be the logarithmic structure associated to the following pre-logarithmic structure:

$$\mathbb{N}^k_Y \times_{\overline{\mathcal{M}}_Y} \mathcal{M}_Y \to \mathcal{M}_Y \to \mathcal{O}_Y.$$

Thus, $\mathcal{M}_{Y'}$ is a DF logarithmic structure with a natural map of logarithmic structures over $\underline{Y}$:

$$\mathcal{M}_{Y'} \to \mathcal{M}_Y.$$

Consider the new logarithmic scheme $Y' = (\underline{Y}, \mathcal{M}_{Y'})$. We have a natural map $\psi : Y \to Y'$ with $\underline{\psi} = id_{\underline{Y}}$.

As in Section 2.2, we have a fibered category $\mathcal{K}_{g,n}(Y,\beta)$ parameterizing genus g, n-pointed stable logarithmic maps with curve class β to Y over $\mathfrak{LogSch}$. One can also introduce canonical contact orders as in Definition 3.9. Note that the map ψ induces a natural map of the fibered categories

$$\phi : \mathcal{K}_{g,n}(Y,\beta) \to \mathcal{K}_{g,n}(Y',\beta). \tag{3.14.1}$$

THEOREM 3.15. *The fibered category $\mathcal{K}_{g,n}(Y,\beta)$ is a logarithmic algebraic stack. Furthermore, the underlying morphism $\underline{\phi}$ is representable and finite, and the map of logarithmic structures $\phi^*\mathcal{M}_{\mathcal{K}_{g,n}(Y',\beta)} \to \mathcal{M}_{\mathcal{K}_{g,n}(Y,\beta)}$ is surjective.*

Proof. By Corollary 3.13, we have that $\mathcal{K}_{g,n}(Y',\beta)$ is a proper logarithmic algebraic stack. It therefore suffices to prove the second statement.

We construct $\mathcal{K}_{g,n}(Y,\beta)$ locally over $\mathcal{K}_{g,n}(Y',\beta)$. We may assume we have a fs logarithmic scheme of finite type B' and a stable logarithmic map $(C'/B', C' \to Y')$. We claim that $B := B' \times_{\mathcal{K}_\Gamma(Y')} \mathcal{K}_\Gamma(Y) \to B'$ is a finite map of the underlying schemes, and the logarithmic structure of B' surjects to that of B.

We take $Z = Y \times_{Y'} C'$ to be the fiber product in the category of fine logarithmic schemes. Then the canonical projection $Z \to C'$ induces a closed embedding $\underline{Z} \to \underline{C'}$. By [SGA64, VIII Theorem 6.4] and the reduction argument [Abr94, Theorem 6(3)], there is a universal closed sub-scheme $\underline{W} \subset \underline{B'}$ with the following property: for any $\underline{T} \to \underline{B'}$, if $\underline{Z} \times_{\underline{B'}} \underline{T} \to \underline{C'} \times_{\underline{B'}} \underline{T}$ is an isomorphism, then the map $\underline{T} \to \underline{B'}$ factors through $\underline{W}$ uniquely. Note that any element of B over some fs logarithmic scheme T is a commutative diagram

$$\begin{tikzcd} Z_T \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Y \arrow[d] \\ C'_T \arrow[r] \arrow[d] \arrow[u, bend left, "s_T"] & C' \arrow[r] \arrow[d] & Y' \\ T \arrow[r] & B'. \end{tikzcd} \tag{3.15.1}$$

where the two upper squares are cartesian squares of fine logarithmic schemes, and the bottom one is a cartesian square of fs logarithmic schemes. The map s_T is a section of $Z_T \to C'_T$. Thus we have an isomorphism of the underlying schemes $\underline{Z_T} \cong \underline{C'_T}$. This implies that the map $\underline{T} \to \underline{B'}$, hence $\underline{B} \to \underline{B'}$ factors through $\underline{W}$. We replace B' by $\underline{W}$, with the pullback logarithmic structure.

Since the problem is local, we may assume that there are sections $\sigma_i : B' \to C'$ for $i = 1, 2, \cdots, n$ landing in the generic locus, and meeting every component of every fiber. Let $B'_0 = B'$, $Z_0 = Z$, and $C'_0 = C'$. We construct B by induction.

Denote by $B'_1 = B'_0 \times_{C'_0} Z_0$, where the product is taken via σ_1 in the category of fs logarithmic schemes. It follows that the underlying map of the first projection $h_1 : B'_1 \to B'_0$ is finite, and the map of logarithmic structures $h_1^\flat : \mathcal{M}_{B'_0} \to \mathcal{M}_{B'_1}$ is surjective. Let Z_1 and C'_1 be the pull-back of Z_0 and C'_0 via h_1. By restricting to the universal closed sub-scheme, we might assume that the fibers Z_1 and C'_1 are fine logarithmic schemes with isomorphic underlying schemes. We claim that the map $B \to B'$ factors through B'_1 uniquely. To see this, we pick a commutative diagram as in (3.15.1). This induces a commutative diagram

$$\begin{tikzcd} Z_{0,T} \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ C'_{0,T} \arrow[r] \arrow[d] \arrow[u, bend left, "s_T"] & C'_0 \arrow[d] \\ T \arrow[r] \arrow[u, bend left, "(\sigma_i)_T"] & B'_0, \arrow[u, bend right, "\sigma_i"'] \end{tikzcd}$$

hence a commutative diagram

$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & B'_1 \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ & B'_0 \arrow[r, "\sigma_1"] & C'_0 \end{tikzcd}$$

where the dashed arrow is induced by the universal property of fibered product. This proves the claim.

Replacing B'_{i-1} by B'_i one at a time, and repeating the previous step, we obtain a sequence of maps $\{h_i : B'_i \to B'_{i-1}\}_i$ such that

1. the underlying map $\underline{h}_i$ is finite;
2. the map of logarithmic structures $h_i^\flat : h_i^* \mathcal{M}_{B'_{i-1}} \to \mathcal{M}_{B'_i}$ is surjective;
3. There is a canonical map $B \to B'_{i-1}$ of fibered categories over $\mathfrak{LogSch}$ which factors through B'_i uniquely.

Denote by $Z_n \to C'_n \to B'_n$ the pull-back of $Z \to C' \to B'$ via $B'_n \to B'$. By the reduction procedure using the universal closed subscheme argument, we have that the fibers Z_n and C'_n have the same underlying structure.

By [Ols03b, Lemma 3.5], the characteristic monoid of any fine logarithmic structures are constructible, and the characteristic of a fine logarithmic structure at a point determines the characteristic in a neighborhood. Thus, after taking finitely many sections σ_i, we may assume $Z_n \to C'_n \to B'_n$ has the property that the map $Z_n \to C'_n$ gives an isomorphism of characteristics along the smooth non-marked locus of each fiber, hence is isomorphic along the smooth non-marked locus as a logarithmic scheme.

We claim that $Z_n \to C'_n$ is an isomorphism of logarithmic schemes. Since the underlying map is an isomorphism, it is enough to show that the induced map of characteristics $\bar{l} : \overline{\mathcal{M}}_{C'_n} \to \overline{\mathcal{M}}_{Z_n}$ is an isomorphism. Note that we have a cartesian diagram of fine logarithmic schemes:

$$\begin{tikzcd} Z_n \arrow[r] \arrow[d] & Y \arrow[d] \\ C'_n \arrow[r] & Y'. \end{tikzcd}$$

Since the map of logarithmic structures $\mathcal{M}_{Y'} \to \mathcal{M}_Y$ is surjective, the map of monoids $\bar{l}$ is also surjective.

Denote by $\underline{f} : \underline{C'_n} \to \underline{Y}$ the underlying stable map, and $\pi_n : C'_n \to B'_n$ the map of logarithmic curves. Locally at a marked point $p \in C'_n$, we have a push-out diagram of fine monoids:

$$\begin{tikzcd} \underline{f}^*\overline{\mathcal{M}}_{Y'} \arrow[r] \arrow[d, two heads] & \overline{\mathcal{N}}_p \oplus \pi_n^*\overline{\mathcal{M}}_{B'_n} \arrow[d, two heads, "\bar{l}"] \\ \underline{f}^*\overline{\mathcal{M}}_Y \arrow[r] & \overline{\mathcal{M}}_{Z_n} \end{tikzcd}$$

where $\overline{\mathcal{N}}_p \cong \mathbb{N} = \langle \delta \rangle$ is the characteristic of the standard logarithmic structure given by the marking at p. Pick two elements $c_1\delta + e_1,\ c_2\delta + e_2 \in \overline{\mathcal{N}}_p \oplus \pi_n^*\overline{\mathcal{M}}_{B'_n}$ locally at p, where $e_1, e_2 \in \pi_n^*\overline{\mathcal{M}}_{B'_n}$, and c_1, c_2 are two non-negative integers. Assume that $\bar{l}(c_1 \cdot \delta + e_1) = \bar{l}(c_2 \cdot \delta + e_2)$. By generalizing to a nearby smooth non-marked point of p, we have $\bar{l}(e_1) = \bar{l}(e_2)$. Since $\bar{l}$ is an isomorphism at the generic smooth non-marked points, this implies that $e_1 = e_2$. Since Z_n is fine, we have $\bar{l}(c_1\delta) = \bar{l}(c_2\delta)$. Note that $\bar{l}(c\delta) = 0$ implies $c = 0$, since if $\bar{l}(c\delta)$ comes from an element of $\mathcal{O}^*$, so does $c\delta$. So if $c_1 \neq c_2$, then since Z_n is fine there exists another positive integer c_3 such that $\bar{l}(c_3\delta) = 0$, which implies that $c_3 = 0$, which gives a contradiction. Therefore, we have $c_1 \cdot \delta + e_1 = c_2 \cdot \delta + e_2$. This implies that $\bar{l}$ is injective, hence an isomorphism at each marked point. A similar argument implies that $\bar{l}$ is also an isomorphism at each node. Therefore, the map $Z_n \to C'_n$ is an isomorphism of logarithmic schemes.

Note that we have an element of B over B'_n, which is given by the identity map $id : C'_n \to Z_n$. This induces a map $B'_n \to B$. We claim that this map is an isomorphism of fibered categories over $\mathfrak{LogSch}$. Given an element of B over T as in (3.15.1), it is equivalent to having a commutative diagram

$$\begin{tikzcd} Z_T \arrow[r] \arrow[d] & Z_n \arrow[d] \\ C'_T \arrow[r] \arrow[d] \arrow[u, bend left, "s_T"] & C'_n \arrow[d] \\ T \arrow[r] & B'_n, \end{tikzcd}$$

where the two squares are cartesian squares of logarithmic schemes. Note that having s_T is equivalent to have a commutative diagram

$$\begin{tikzcd} & \underline{f}_T^*\mathcal{M}_{Y'} \arrow[dl] \arrow[dr] & \\ \underline{f}_T^*\mathcal{M}_Y \arrow[rr] & & \mathcal{M}_{C'_T}. \end{tikzcd}$$

Since the morphism $\underline{f}_T^*\mathcal{M}_{Y'} \to \underline{f}_T^*\mathcal{M}_Y$ is a surjection, the section s_T if exists, is unique. Hence, it is the pull-back of $id : C'_n \to Z_n$ via $T \to B'_n$. This proves that $B = B'_n$. $\square$

3.16. Minimal objects. Consider the universal family of stable logarithmic maps with discrete data Γ:

(3.16.1)
$$\begin{tikzcd} \mathfrak{C} \arrow[r] \arrow[d] & Y \\ \mathcal{K}_\Gamma(Y). & \end{tikzcd}$$

This brings us to the definition of minimal stable logarithmic maps:

Definition 3.17. A stable logarithmic map $f : C \to Y$ over a logarithmic scheme S is called *minimal*, if there exists a map $g : \underline{S} \to \underline{\mathcal{K}_\Gamma}(Y)$ of the underlying structure, such that f is obtained by strict pull-back of (3.16.1) via g.

PROPOSITION 3.18. *Given any stable logarithmic maps $f : C \to Y$ over a logarithmic scheme S, there exists (up to a unique isomorphism) a unique minimal stable logarithmic map $f^{\min} : C^{\min} \to Y$ over $S^{\min}$, and a logarithmic map $S \to S^{\min}$ such that*

1. *The logarithmic map f is given by the pull-back of $f^{\min}$ via $S \to S^{\min}$.*
2. *The underlying map $\underline{S} \to \underline{S}^{\min}$ is the identity.*

Proof. This follows from the universal property of $\mathcal{K}_\Gamma(Y)$ as a fibered category over $\mathfrak{LogSch}$. ∎

4. The combinatorial description of minimality.

4.1. The case of Deligne–Faltings pairs. The main result of [Chen] gives more than just Corollary 3.13 for the case $P = \mathbb{N}$: we have an explicit combinatorial description of the minimal logarithmic structure associated to a given logarithmic map, in terms of marked graphs. A similar description is possible in general. We first present here the case $P = \mathbb{N}^m$.

4.1.1. The graph. Fix a logarithmic map $f : C \to Y$ over S, such that $\underline{S}$ is a geometric point. Recall that we have a decomposition of Y as in (3.3.2). Denote by f_i the following composition

$$C \xrightarrow{f} Y \to Y_i, \quad i = 1, \ldots, m. \tag{4.1.1}$$

In analogy to Definition 3.3.2 and Construction 3.4.1 of [Chen], we associate to the logarithmic map f an *m-marked graph* G, formed out of the dual graph of the curve C with the following extra data:

1. m partitions of the vertices of G in two types $V(G) = V_0^{(i)}(G) \sqcup V_1^{(i)}(G)$, where a vertex $v \in V_0^{(i)}(G)$ if and only if the associated component $C_v \subset C$ is non-degenerate with respect to the map $C \to Y_i$.
2. m integer weights $c_l^{(i)} \geq 0, i = 1, \ldots, m$ on the edges $l \in E(G)$, such that $c_l^{(i)}$ is the contact order of f_i along l ([Chen, Definition 3.2.6]).
3. m orientations of edges: whenever an edge l has extremities v, v' and $c_l^{(i)} > 0$, we choose one orientation $v >_i v'$ or $v' >_i v$ of the edge l, given by f_i as in [Chen, Definition 3.2.7].

4.1.2. The monoid. We introduce a variable e_l for each edge $l \in E(G)$, and m variables $e_v^{(i)}$, $i = 1, 2, \cdots, m$ for each vertex $v \in V(G)$. Denote by $h_{l,i}$ the edge equations $e_{v'}^{(i)} = e_v^{(i)} + c_l^{(i)} e_l$ for every edge l with extremities $v \leq_i v'$; and $h_{v,i}$ the vertex equations $e_v^{(i)} = 0$, for $v \in V_0^{(i)}(G)$. Consider the monoid

$$M(G) = \left\langle \; e_v^{(i)}, e_l \;\middle|\; v \in V(G),\ l \in E(G) \;\right\rangle \Big/ \left\langle\; h_{l,i}, h_{v,i} \;\middle|\; l \in E(G),\ v \in V_0^{(i)}(G) \;\right\rangle.$$

Denote by $T(G)$ the torsion part of $M(G)^{gp}$. Then we have the following composition

$$M(G) \to M(G)^{gp} \to M(G)^{gp}/T(G).$$

Denote by $N(G)$ the image of $M(G)$ in $M(G)^{gp}/T(G)$, and $\overline{\mathcal{M}}(G)$ the saturation of $N(G)$ in $M(G)^{gp}/T(G)$.

REMARK 4.2. When $m = 1$, the description in 4.1 is identical to the case in [Chen].

Proposition 4.3. *There is a canonical morphism $\overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}_S$. A logarithmic structure is minimal if and only if this morphism is an isomorphism.*

Proof. Consider the minimal stable logarithmic map $f^{\min} : C^{\min} \to Y$ over $S^{\min}$, and a logarithmic map $S \to S^{\min}$, which satisfy the conditions in Proposition 3.18. Denote by f_i the stable logarithmic map given by the composition (4.1.1). Consider the minimal stable logarithmic map $f_i^{\min} : C_i^{\min} \to Y_i$ over $S_i^{\min}$ and the logarithmic map $S^{\min} \to S_i^{\min}$ given by Proposition 3.18. By (3.11.2), we have a fiber product of logarithmic schemes:

$$S^{\min} \cong S_1^{\min} \times_{\underline{S}} \cdots \times_{\underline{S}} S_m^{\min}. \tag{4.3.1}$$

Note that the characteristic monoid of the right hand side of (4.3.1) is given by $\overline{\mathcal{M}}(G)$. Thus, we obtain the map

$$\overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}_S.$$

Assuming the logarithmic map f is minimal, this is equivalent to the map $S \to S^{\min}$ being an isomorphism, which is equivalent to the map on the level of characteristic $\overline{\mathcal{M}}_{S^{\min}} \to \overline{\mathcal{M}}_S$ being an isomorphism of monoid. This proves the statement. ∎

4.4. The case of generalized Deligne–Faltings pairs. Consider a logarithmic map $f : C \to Y$ over S, such that $\underline{S}$ is a geometric point. We now assume that Y is a generalized DF pair with a fixed global presentation $P \to \mathcal{M}_Y$. Thus by (3.12.6), we have a cartesian diagram of logarithmic schemes:

$$\begin{tikzcd} Y \arrow[r, "h_2"] \arrow[d, "h_1"'] & Y_2 \arrow[d, "u_1"] \\ Y_1 \arrow[r, "u_2"'] & Y_3 \end{tikzcd} \tag{4.4.1}$$

where Y_i are DF pairs for $i = 1, 2, 3$. Let f_i be the following composition

$$C \xrightarrow{f} Y \to Y_i, \quad \text{for } i = 1, 2, 3. \tag{4.4.2}$$

Then we obtain the marked graph G_i for the stable logarithmic map f_i as in Section 4.1. Note that by removing all the weights and orientations, the underlying graph $\underline{G_i}$ is the dual graph $\underline{G}$ of the underlying curve C over S, for all i. Denote by $f_i^{\min} : C_i^{\min} \to Y_i$ over S_i the associated minimal logarithmic maps of f_i, and by $f^{\min} : C^{\min} \to Y$ over $S^{\min}$ the associated minimal logarithmic map of f. By (3.13.2), we obtain a cartesian diagram of logarithmic schemes:

$$\begin{tikzcd} S^{\min} \arrow[r] \arrow[d] & S_2 \arrow[d] \\ S_1 \arrow[r] & S_3. \end{tikzcd} \tag{4.4.3}$$

Let $\overline{\mathcal{M}}$ be the push-out of the following diagram in the category of toric sharp monoids:

$$\begin{tikzcd} \overline{\mathcal{M}}(G_3) \arrow[r] \arrow[d] & \overline{\mathcal{M}}(G_2) \\ \overline{\mathcal{M}}(G_1). & \end{tikzcd} \tag{4.4.4}$$

Then we have:

PROPOSITION 4.5. *There is a canonical map $\phi : \overline{\mathcal{M}} \to \overline{\mathcal{M}}_S$. The logarithmic map f is minimal if and only if ϕ is an isomorphism.*

Proof. The statement can be proved similarly to the argument of Proposition 4.3, using the fact that (4.4.3) is cartesian. □

REMARK 4.6. We feel that the expression of $\overline{\mathcal{M}}$ using the push-out diagram (4.4.4) is good enough for the purpose of calculations. But it is possible to write down an explicit formation of the base monoids as in Section 4.1.2. We leave it for interested readers.

4.7. Compatibility with Gross-Siebert's construction. As mentioned in the introduction, another construction of the stack of stable logarithmic maps is given by Gross and Siebert using so called basic stable log maps when the target is equipped with Zariski log structures Satisfying a certain global generation property. In fact, when our condition in Theorem 3.15 applies to the logarithmic structures on the targets, the notions of basic log maps and minimal logarithmic maps are identical.

PROPOSITION 4.8. *A stable logarithmic map is minimal if and only if it is basic in the sense of [GS, Definition 1.17].*

Proof. This follows since basic logarithmic maps and minimal logarithmic maps satisfy the same universal property: by Proposition 3.18, any given logarithmic map is uniquely the pullback of a unique minimal logarithmic map on the same underlying base scheme. The same universal property holds for basic logarithmic maps by [GS, Proposition 1.20]. □

REMARK 4.9. Both minimality and basicness can be defined by putting constraints on the characteristic monoids of the bases. Thus, Proposition 4.8 can be proved directly by comparing the base monoids of both minimal and basic stable log maps. We provide an argument when the logarithmic structure on the target Y is Deligne–Faltings. The general situation can be proved similarly, but with more complicated notation and combinatorics.

We assume that there is a map $\mathbb{N}^k \to \overline{\mathcal{M}}_Y$ locally lifting to a chart of $\mathcal{M}_Y$. Consider the logarithmic scheme Y_i as in Section 4.1. Let D_i be the locus with nontrivial logarithmic structure in Y_i. Consider a stable logarithmic map $f : C \to Y$ over a logarithmic scheme S. Since both minimality and basicness can be defined fiber-wise, we might assume that $\underline{S}$ is a geometric point.

Consider a generic point $\eta \in \underline{C}$ with associated irreducible component $v \in V(\underline{G})$, where G is the marked graph of f. In [GS, Construction 1.15], the factor P_η can be viewed as the free monoid generated by the degeneracy $e_{v,i}$ of v for $i = 1, 2, \cdots, k$ with the condition that $e_{v,i} = 0$ if and only if the component v does not map into D_i, or equivalently $v \in V^i_{nd}(G)$. Consider the monoid $\prod_{q \in \underline{C}} \mathbb{N}$ appearing as a factor in the expression [GS, (1.15)], where $q \in C$ denotes the nodes. It can be viewed as the free monoid generated by the elements e_l for each $l \in E(G)$. Note that the condition $a_q(m)$ in [GS, Construction 1.15] is exactly the edge condition $h_{l,i}$ in Section 4.1.2. Thus, the description of P_η and $\prod_{q \in \underline{C}} \mathbb{N}$ using the elements associated to vertices and edges induces a natural isomorphism $Q \to \overline{\mathcal{M}}(G)$, where Q is the monoid defined in [GS, (1.14)]. In fact, the object $(Q, \overline{\mathcal{M}}_C, \psi, \phi)$ defined in [GS, Construction 1.15] is equivalent to the data of a marked graph G. This provides an explicit derivation of Proposition 4.8 in this case.

5. The case of a degeneration.

5.1. Stable logarithmic maps relative to a base. Consider a family of projective logarithmic schemes $\pi : X \to B$, such that $\mathcal{M}_X$ and $\mathcal{M}_B$ are generalized DF logarithmic structures. We defined in [Chen, Definition 2.1.2] *a family of pre-stable logarithmic maps over S with target X/B* as a commutative diagram of logarithmic schemes:

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{\phi} & B, \end{array} \tag{5.1.1}$$

such that the family $C \to S$ is a pre-stable logarithmically smooth curve. For simplicity, we denote it by $\xi = (C \to S, f, \phi)$, and omit the target $X \to B$, if there is no danger of confusion. The logarithmic map ξ is called *stable*, if the underlying map $\underline{\xi}$ is stable in the usual sense.

Consider two pre-stable logarithmic maps $\xi_1 = (C_1 \to S_1, f_1, \phi_1)$ and $\xi_2 = (C_2 \to S_2, f_2, \phi_2)$. An *arrow* $\xi_1 \to \xi_2$ is given by a pair $(g : C_1 \to C_2, h : S_1 \to S_2)$ which fits in the following commutative diagram:

$$\begin{array}{ccccc} C_1 & \xrightarrow{g} & C_2 & \xrightarrow{f_2} & X \\ \downarrow & & \downarrow & & \downarrow \\ S_1 & \xrightarrow{h} & S_2 & \xrightarrow{\phi_2} & B, \end{array}$$

such that the square on the left is a cartesian square of logarithmic schemes, and $f_1 = f_2 \circ g$ and $\phi_1 = \phi_2 \circ h$.

We fix a curve class β on the fiber of π. Define $\mathcal{K}_{g,n}(X/B, \beta)$ (respectively $\mathcal{K}^{pre}_{g,n}(X/B, \beta)$) to be the fibered category over $\mathfrak{LogSch}$, parameterizing stable (respectively pre-stable) logarithmic maps to X/B with genus g, n marked points and curve class β on the fiber.

5.2. The case when $\pi : X \to B$ is strict. Note that we have a natural map $\mathcal{K}_{g,n}(X/B, \beta) \to \mathcal{K}_{g,n}(\underline{X}/\underline{B}, \beta)$ by removing all logarithmic structures from the target. Thus, the stack $\mathcal{K}_{g,n}(\underline{X}/\underline{B}, \beta)$ is the stack of usual stable maps to $\underline{X}/\underline{B}$ with the canonical logarithmic structure associated to the underlying curves as in [Kat00] and [Ols03a, Theorem 1.2]. By [FP97] or [AO01, Theorem 2.8], we know that the stack $\mathcal{K}_{g,n}(\underline{X}/\underline{B}, \beta)$ is proper over $\underline{B}$. Similarly, we have the natural map $\mathcal{K}^{pre}_{g,n}(X/B, \beta) \to \mathcal{K}^{pre}_{g,n}(\underline{X}/\underline{B}, \beta)$, where $\mathcal{K}^{pre}_{g,n}(\underline{X}/\underline{B}, \beta)$ is the stack parameterizing usual pre-stable maps to $\underline{X}/\underline{B}$ with the canonical logarithmic structure associated to the underlying curves. We first consider the strict case:

LEMMA 5.3. *Assume that the map $\pi : X \to B$ is strict. Then there is a canonical isomorphism of logarithmic stacks*

$$\mathcal{K}^{pre}_{g,n}(X/B, \beta) \cong \mathcal{K}^{pre}_{g,n}(\underline{X}/\underline{B}, \beta) \times_{\underline{B}} B.$$

In particular, by requiring the stability conditions, we have

$$\mathcal{K}_{g,n}(X/B, \beta) \cong \mathcal{K}_{g,n}(\underline{X}/\underline{B}, \beta) \times_{\underline{B}} B.$$

Proof. Consider the natural commutative diagram

$$\begin{tikzcd} \mathcal{K}^{pre}_{g,n}(X/B,\beta) \arrow[r] \arrow[d] & B \arrow[d, "\pi"] \\ \mathcal{K}^{pre}_{g,n}(\underline{X}/\underline{B},\beta) \arrow[r] & \underline{B}. \end{tikzcd} \tag{5.3.1}$$

We need to prove the above diagram is cartesian. Consider an object

$$\xi = (p : C \to S, f : C \to X, \phi : S \to B) \in \mathcal{K}^{pre}_{g,n}(X/B,\beta).$$

This is equivalent to the data of

$$\xi' = (p : C \to S, \underline{f} : C \to \underline{X}, \underline{\phi} : S \to \underline{B}) \in \mathcal{K}^{pre}_{g,n}(\underline{X}/\underline{B},\beta)$$

together with maps of logarithmic structures $f^\flat : \underline{f}^*\mathcal{M}_X \to \mathcal{M}_C$ and $\phi^\flat : \phi^*\mathcal{M}_B \to \mathcal{M}_S$ such that the following diagram of logarithmic structures is commutative:

$$\begin{tikzcd} \underline{f}^* \circ \pi^*\mathcal{M}_B \arrow[r] \arrow[d] & \underline{f}^*\mathcal{M}_X \arrow[d] \\ p^*\mathcal{M}_S \arrow[r] & \mathcal{M}_C. \end{tikzcd} \tag{5.3.2}$$

Since the map π is strict, the data of $f^\flat$, $\phi^\flat$, satisfying (5.3.2) is equivalent to giving a map of logarithmic schemes $\phi : S \to B$, whose underlying structure is compatible with $\underline{\phi}$ in ξ'. This proves that (5.3.1) is cartesian.

The second statement follows since

$$\mathcal{K}_{g,n}(X/B,\beta) \cong \mathcal{K}_{g,n}(\underline{X}/\underline{B},\beta) \times_{\mathcal{K}^{pre}_{g,n}(\underline{X}/\underline{B},\beta)} \mathcal{K}^{pre}_{g,n}(X/B,\beta)$$

∎

5.4. Stack parameterizing logarithmic sources and targets. Denote by $\mathfrak{M}_{g,n}$ the stack of pre-stable curves with its canonical logarithmic structure associated to the family of curves. It is proved in [Kat00, Theorem 4.1] in the stable case, and further developed in [Ols07], that the logarithmic stack $\mathfrak{M}_{g,n}$ represents the category of all genus g, n-marked pre-stable logarithmically smooth curves over $\mathfrak{LogSch}$. Consider the stack

$$\mathfrak{B} = B \times \mathfrak{M}_{g,n}.$$

It represents a fibered category over $\mathfrak{LogSch}$, such that for each logarithmic scheme S, it associates the groupoid of diagrams of the following form:

$$\begin{tikzcd} C \arrow[d] & \\ S \arrow[r, "\phi"] & B, \end{tikzcd} \tag{5.4.1}$$

where $C \to S$ is a logarithmically smooth curve. Denote (5.4.1) by $\zeta = (C/S, \phi)$. Given two objects $\zeta_1 = (C_1/S_1, \phi_1)$ and $\zeta_2 = (C_2/S_2, \phi_2)$, an arrow $\zeta_2 \to \zeta_1$ is given by the following commutative diagram:

$$\begin{tikzcd} C_1 \arrow[r, "g"] \arrow[d] & C_2 \arrow[d] & \\ S_1 \arrow[r, "h"] & S_2 \arrow[r, "\phi_2"] & B, \end{tikzcd} \tag{5.4.2}$$

such that the square is cartesian of logarithmic schemes, and $\phi_1 = \phi_2 \circ h$.

LEMMA 5.5. *We have the following canonical isomorphism of logarithmic stacks:*

$$\mathfrak{B} \cong \mathcal{K}^{pre}_{g,n}(B/B, 0).$$

Proof. By Lemma 5.3, we have

$$\mathcal{K}^{pre}_{g,n}(B/B, 0) \cong \mathcal{K}^{pre}_{g,n}(\underline{B}/\underline{B}, 0) \times_{\underline{B}} B \cong (\mathfrak{M}_{g,n} \times \underline{B}) \times_{\underline{B}} B \cong \mathfrak{B}.$$

$\square$

5.6. Construction of $\mathcal{K}_{g,n}(X/B, \beta)$. We have the following:

THEOREM 5.7. *The fibered categories $\mathcal{K}_{g,n}(X/B, \beta)$ and $\mathcal{K}^{pre}_{g,n}(X/B, \beta)$ are represented by algebraic stacks with a natural fs logarithmic structures. Furthermore, the underlying stack of $\mathcal{K}_{g,n}(X/B, \beta)$ is a DM-stack of finite type.*

Proof. Consider the stack $\mathcal{K}^{pre}_{g,n}(B, 0)$, and the natural map $\mathcal{K}^{pre}_{g,n}(X, \beta) \to \mathcal{K}^{pre}_{g,n}(B, 0)$ induced by $\pi : X \to B$ as $\pi_* \beta = 0$. We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{K}^{pre}_{g,n}(X/B, \beta) \arrow[r] \arrow[d] & \mathcal{K}^{pre}_{g,n}(X, \beta) \arrow[d] \\ \mathfrak{B} \arrow[r] & \mathcal{K}^{pre}_{g,n}(B, 0), \end{tikzcd} \tag{5.7.1}$$

where the right arrow is induced by π, the left and top arrow is obtained by removing the maps to X and to B in (5.1.1) respectively, and the bottom arrow is obtained by the composition in (5.4.1).

In fact, giving an object $\xi = (C/S, f, \phi) \in \mathcal{K}^{pre}_{g,n}(X/B, \beta)$ over S is equivalent to giving an object $\zeta = (C/S, \phi)$, and a pre-stable logarithmic map $f : C \to X$, which induce the same map $C \to B$. Thus (5.7.1) is a fibered diagram of fs logarithmic stacks. This proves that $\mathcal{K}^{pre}_{g,n}(X/B, \beta)$ is an algebraic stack with a natural fs logarithmic structure. Note that with the stability condition, $\mathcal{K}_{g,n}(X/B, \beta)$ forms an open substack of $\mathcal{K}^{pre}_{g,n}(X/B, \beta)$, hence is also algebraic.

Note that the image of the $\mathcal{K}_{g,n}(X/B, \beta)$ in $\mathfrak{B}$ is contained in an open substack of finite type. Therefore, the stack $\mathcal{K}_{g,n}(X/B, \beta)$ is of finite type.

Finally, since $\mathfrak{B} \cong \mathcal{K}^{pre}_{g,n}(B/B, 0)$ by Lemma 5.5, it follows that the bottom arrow of (5.7.1) is representable. Since the underlying stack of $\mathcal{K}_{g,n}(X, \beta)$ is a DM-stack, the finiteness of saturation implies that the underlying stack of $\mathcal{K}_{g,n}(X/B, \beta)$ is also a DM-stack.

This finishes the proof of the statement. $\square$

PROPOSITION 5.8. *The stack $\mathcal{K}_{g,n}(X/B, \beta)$ is proper over $\underline{B}$.*

Proof. Denote by $\overline{X} = \underline{X} \times_{\underline{B}} B$. Thus we have a canonical map $X \to \overline{X}$, and a strict map $\pi' : \overline{X} \to B$. Note also that $\overline{X}$ is a generalized Deligne–Faltings logarithmic structure. By Theorem 5.7 and (5.7.1), we have the following cartesian diagram:

$$\begin{tikzcd} \mathcal{K}_{g,n}(X/B,\beta) \arrow[r] \arrow[d] & \mathcal{K}_{g,n}(X,\beta) \arrow[d] \\ \mathcal{K}_{g,n}(\overline{X}/B,\beta) \arrow[r] \arrow[d] & \mathcal{K}_{g,n}(\overline{X},\beta) \arrow[d] \\ \mathfrak{B} \arrow[r] & \mathcal{K}^{pre}_{g,n}(B,\pi_*\beta). \end{tikzcd} \tag{5.8.1}$$

Note that by Corollary 3.13 both $\mathcal{K}_{g,n}(X,\beta)$ and $\mathcal{K}_{g,n}(\overline{X},\beta)$ are representable and finite over $\mathcal{K}_{g,n}(\underline{X},\beta)$. It follows from [LMB00, Lemma 3.12] that the canonical map $\mathcal{K}_{g,n}(X,\beta) \to \mathcal{K}_{g,n}(\overline{X},\beta)$ is representable and finite, hence is proper. By Lemma 5.3, the stack $\mathcal{K}_{g,n}(\overline{X}/B,\beta)$ is proper over B. Thus the statement of the proposition follows. ∎

5.9. Minimal objects in the degeneration case. By (5.7.1), we have a universal diagram of stable logarithmic maps:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathcal{K}_{g,n}(X/B,\beta) \arrow[r] & B. \end{tikzcd} \tag{5.9.1}$$

Definition 5.10. Consider a stable logarithmic map $\xi = (C \to S, f, \phi)$ as in (5.1.1). It is called *minimal* if there is a map of underlying structures $g : \underline{S} \to \underline{\mathcal{K}}_{g,n}(X/B,\beta)$, such that ξ is obtained by the strict pull-back of (5.9.1) via g.

Corollary 5.11. *Given a stable logarithmic map $\xi = (C \to S, f, \phi)$, there exists a minimal stable logarithmic map $\xi^{\min} = (C^{\min} \to S^{\min}, f^{\min}, \phi^{\min})$, and a logarithmic map $g : S \to S^{\min}$ such that*

1. *ξ is obtained by pull-back $\xi^{\min}$ via g.*
2. *The underlying map $\underline{g}$ is an identity.*

Furthermore, the pair $(\xi^{\min}, g)$ is unique up to a unique isomorphism.

Proof. By Theorem 5.7, the logarithmic map ξ is equivalent to a logarithmic map $S \to \mathcal{K}_{g,n}(X/B,\beta)$. Define $\xi^{\min}$ to be the minimal logarithmic map given by the underlying map $\underline{S} \to \underline{\mathcal{K}}_{g,n}(X/B,\beta)$. This proves the statement. ∎

5.12. Compatibility of minimality. We show that minimality in the degeneration case is equivalent to minimality of the map to the total space. First a lemma:

Lemma 5.13. *Given a stable logarithmic map $\xi = (C \to S, f, \phi) \in \mathcal{K}_{g,n}(X/B,\beta)$, its image in $\mathcal{K}^{pre}_{g,n}(B,0)$ is a logarithmic map with zero contact orders.*

Proof. Note that the map $\mathcal{K}_{g,n}(X/B,\beta) \to \mathcal{K}^{pre}_{g,n}(B,\pi_*\beta)$ factors through $\mathfrak{B} = B \times \mathfrak{M}_{g,n}$, which induces stable logarithmic maps with only zero contact orders. ∎

Proposition 5.14. *Consider $\xi = (C \to S, f, \phi) \in \mathcal{K}_{g,n}(X/B,\beta)$ a stable logarithmic map over S. It is minimal in the sense of Definition 5.10, if and only if the* *induced stable logarithmic map* $f : C \to X$ *over* S *in* $\mathcal{K}_{g,n}(X, \beta)$ *is minimal in the sense of Definition 3.17.*

Proof. In fact, we will prove that the top arrow in (5.7.1) is strict. This is equivalent to showing that the bottom arrow in (5.7.1) is strict. This can be checked directly using the construction in the push-out diagram (4.4.4).

Consider a diagram

$$\begin{tikzcd} C \arrow[d] & \\ S \arrow[r, "\phi"] & B, \end{tikzcd}$$

induced by a strict map $S \to \mathfrak{B}$. It is enough to consider the case that $\underline{S}$ is a geometric point. Then we obtain an induced logarithmic map $h : C \to B$ over S. Consider the minimal logarithmic map $h' : C' \to B$ over S' with the following commutative diagram

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d] & C' \arrow[r, "h'"] \arrow[d, "p"] & B \\ S \arrow[r, "k"] & S' \arrow[ur, dashed, "\underline{\phi}"'] & \end{tikzcd} \tag{5.14.1}$$

such that $h = h' \circ j$, the underlying map $\underline{k}$ is the identity, the square is cartesian of fs logarithmic schemes, and $\underline{\phi}$ is the underlying map of ϕ. It is enough to show that the map on the level of characteristics $\bar{k} : \overline{\mathcal{M}}_{S'} \to \overline{\mathcal{M}}_S$ is an isomorphism.

Note that the map h' has only zero contact orders. Otherwise, the composition $h = h' \circ j$ will have non-zero contact orders, which violates Lemma 5.13. Since all nodes of C are non-distinguished, all the edge equations in Section 4.1.2 are trivial. Hence the construction in Section 4.4 implies a natural splitting $\overline{\mathcal{M}}_{S'} = \overline{\mathcal{M}}_{\underline{S}}^{C/S} \oplus \overline{\mathcal{M}}$, where $\overline{\mathcal{M}}_{\underline{S}}^{C/S}$ can be viewed as the submonoid in $\overline{\mathcal{M}}_{S'}$ generated by the elements associated to edges, and $\overline{\mathcal{M}}$ is generated by the element associated to vertices.

On the other hand, we have $\mathcal{M}_S = \overline{\mathcal{M}}_{\underline{S}}^{C/S} \oplus \underline{\phi}^* \overline{\mathcal{M}}_B$. The map $\bar{k}$ induces a map $g : \overline{\mathcal{M}} \to \underline{\phi}^* \overline{\mathcal{M}}_B$. Using the fact that all contact orders are zero and the construction in (4.4.4), we can check that g is an isomorphism. This implies that $\bar{k}$ is an isomorphism. ∎

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