CANONICAL MAPS OF SURFACES DEFINED BY ABELIAN COVERS[*]

RONG DU[†] AND YUN GAO[‡]

Abstract. In this paper, we classified the surfaces whose canonical maps are abelian covers over $\mathbb{P}^2$. Moveover, we give defining equations for Perssson's surface and Tan's surfaces with odd canonical degrees explicitly.

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1. Introduction. Let X be a minimal surface of general type over $\mathbb{C}$. As usual, let p_g, q, $\chi(\mathcal{O}_X)$, K_X^2 be the numerical invariants of X,

$$\varphi_{K_X} : X \to \Sigma \subset \mathbb{P}^{p_g - 1}$$

be the canonical map of X and $d = \deg \varphi_{K_X}$ be the canonical degree of X. It is well known that the behavior of φ_{K_X} is quite complicated, however the degree of it can not be too large. In 1979, Beauville ([Bea]) proved that when φ_{K_X} is a generically finite map, the degree is at most 36. Furthermore, $\deg \varphi_{K_X} = 36$ if and only if $K_X^2 = 36$, $p_g = 3$, $q = 0$, $\Sigma = \mathbb{P}^2$ and $|K_X|$ is base point free. Later, Xiao also found some restrictions on surfaces with high canonical degrees ([Xiao]). For surfaces of general type, since the canonical degree is bounded above, it is interesting to know which positive integers d's occur as the degree of the canonical map. Among all known surfaces with highest canonical degree less than 36 is the surface with canonical degree 16 which was constructed by Persson([Per]) as follows. let $\pi : X \to Y$ be a double cover of a Campedelli surface Y branched along $2K_Y$, where the Campedelli surface is the surface of general type with $p_g = q = 0$, $K_Y^2 = 2$, and $|2K_Y|$ is base point free. He found that $\varphi_{K_X} = \pi \circ \varphi_{K_Y}$, hence φ_{K_X} is degree 16 over $\mathbb{P}^2$. We see that his construction is based on the the existence of Campedelli surface, however the construction of a Campedelli surface is not an easy work ([Pet]). The remaining problem that whether there exists a surface of general type over $\mathbb{P}^2$ with canonical degree of the gap between 17 and 35 is still open.

If Σ is a canonical surface, there are plenty of examples (see [Bea], [Cat1], [V-Z]) with canonical degrees being 2. For $d = 3$ and $d = 5$, Tan ([Tan]) and Pardini ([Par1]) constructed several surfaces independently. When $p_g(\Sigma) = 0$, Beauville ([Bea]) constructed surfaces with $\chi(\mathcal{O}_X)$ arbitrarily large and the canonical degrees $2, 4, 6$ and 8. For $d = 9$, Tan also constructed a surface in [Tan]. However, as far as we know that it is still not clear which positive integers d's occur as the degree of the canonical map even when $\Sigma = \mathbb{P}^2$.

In this paper, we show that if the canonical map is an abelian cover over $\mathbb{P}^2$ then the only possible canonical degrees of a surface of general type are $2, 3, 4, 6, 8, 9, 16$ by explicit constructions. Moreover, by using abelian cover, we list the defining equations for Perssson's surface and Tan's surfaces with odd canonical degrees.

We list our main theorems as follows.

THEOREM 1.1. *Assume* $\varphi : X \longrightarrow \mathbb{P}^2$ *is an abelian cover.*
(1) *Let* $\deg \varphi = 16$, *then* $\varphi = \varphi_{K_X}$ *if and only if* X *is defined by*

$$z_1^2 = \ell_1\ell_3\ell_4\ell_7, \quad z_2^2 = \ell_1\ell_2\ell_4\ell_5, \quad z_3^2 = \ell_1\ell_2\ell_3\ell_6, \quad z_4^2 = \ell_2\ell_5\ell_6\ell_8.$$

   In particular, the surface defined by the first three equations is a Campedelli surface.
(2) *Let* $\deg \varphi = 9$, *then* $\varphi = \varphi_{K_X}$ *if and only if* X *is defined by* $z_1^3 = a_1a_2^2, z_2^3 = a_1a_2a_3$.
(3) *Let* $\deg \varphi = 8$, *then* $\varphi = \varphi_{K_X}$ *if and only if* X *is defined by either* $z_1^2 = \ell_1\ell_2\ell_7\ell_8, z_2^2 = \ell_3\ell_4\ell_7\ell_8, z_3^2 = \ell_5\ell_6\ell_7\ell_8$, *or* $z_1^2 = a_1a_4, z_2^2 = a_2a_4, z_3^2 = a_3a_4$, *or* $z_1^2 = a_1a_2, z_2^4 = a_2^3a_3$.
(4) *Let* $\deg \varphi = 6$, *then* $\varphi = \varphi_{K_X}$ *if and only if* X *is defined by* $z_1^2 = a_1a_2, z_2^3 = a_2a_3^2$.
(5) *Let* $\deg \varphi = 4$, *then* $\varphi = \varphi_{K_X}$ *if and only if* X *is defined by either* $z^4 = a_1^2b_2^3$, *or* $z_1^2 = b_1, z_2^2 = b_2$.
(6) *Let* $\deg \varphi = 3$, *then* $\varphi = \varphi_{K_X}$ *if and only if* X *is defined by either* $z^3 = c_1^2$, *or* $z^3 = c_2$.
(7) *Let* $\deg \varphi = 2$, *then* $\varphi = \varphi_{K_X}$ *if and only if* X *is* $z^2 = h$.
Here ℓ_i*'s define different lines and there are at most three lines among them passing through one point, for* $i = 1 \cdots 8$*;* a_i*'s are reduced of degree 2;* b_i*'s are reduced of degree 4;* c_1, c_2 *are reduced of degree 6;* h *is reduced of degree 8 and all the irreducible components of* a_i*'s ,* b_i*'s,* c_1, c_2 *and* h *are simply normal crossing.*

THEOREM 1.2. *If*

$$\varphi = \varphi_{K_X} : X \longrightarrow \mathbb{P}^2,$$

is an abelian cover, then d *is equal to* $2, 3, 4, 6, 8, 9$, *or* 16. *In particular, if the canonical degree of* X *is* 36, *then* φ_{K_X} *can not be an abelian cover.*

In section 2. we present the main facts on abelian covers which are the key points for solving our problem. In section 3, we prove our main theorems. The defining equations of Tan's examples will be given in section 4.

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2. Abelain covers. The theory of cyclic covers of algebraic surfaces was studied first by Comessatti in [Com]. Then F. Catanese ([Cat2]) studied smooth abelian covers in the case $(\mathbb{Z}_2)^{\oplus 2}$ and R. Pardini ([Par2]) analyzed the general case. In this section, we shall recall some basic definitions and results for abelian covers which will facilitate our subsequent discussion. Since our point of view is to find the defining equations for covering surfaces by explicit calculation, we use the expressions and notations appearing in [Gao].

Let $\varphi : X \to Y$ is an abelian cover associated to abelian group $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, i.e., function field $\mathbb{C}(X)$ of X is an abelian extension of the rational function field $\mathbb{C}(Y)$ with Galois group G. Without lose of generality, we can assume $n_1|n_2\cdots|n_k$.

DEFINITION 2.1. *The dates of abelian cover over Y with group G are k effective divisors $D_1, \cdots, D_k$ and k linear equivalent relations*

$$D_1 \sim n_1 L_1, \cdots, D_k \sim n_k L_k.$$

Let $\mathscr{L}_i = \mathscr{O}_Y(L_i)$ and f_i be the defining equation of D_i, i.e., $D_i = \text{div}(f_i)$, where $f_i \in H^0(Y, \mathscr{L}_i^{n_i})$. Denote $\mathbf{V}(\mathscr{L}_i) = \mathbf{Spec} S(\mathscr{L}_i)$ to be the line bundle corresponding to $\mathscr{L}_i$, where $S(\mathscr{L}_i)$ is the sheaf of symmetric $\mathscr{O}_Y$ algebra. Let z_i be the fiber coordinate of $\mathbf{V}(\mathscr{L}_i)$. Then the abelian cover can be realized by the normalizing of surface V defined by the system of equations

$$z_1^{n_1} = f_1, \cdots, z_k^{n_k} = f_k.$$

So we have the following diagram:
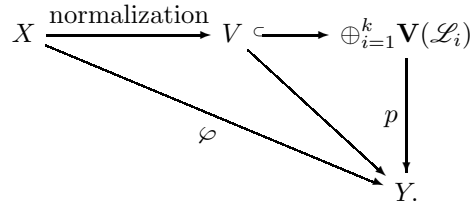
Sometimes we call X is defined by these equations if there is no confusions in the context.

We summerize our main results as follows.

THEOREM 2.2. *(See [Gao]) Denote by $[Z]$ the integral part of a $\mathbb{Q}$-divisor Z,* $L_g = -\sum\limits_{i=1}^{k} g_i L_i + \left[\sum\limits_{i=1}^{k} \frac{g_i}{n_i} D_i\right]. Then$

$$(2.1) \qquad\qquad \varphi_* \mathscr{O}_X = \bigoplus_{g \in G} \mathscr{O}_Y(-L_g).$$

where $g = (g_1, \cdots, g_k) \in G$.

So the decomposition of $\varphi_* \mathscr{O}_X$ is totally determined by the abelian cover.
By Theorem 2.2 we can get following corollary.

COROLLARY 2.3. *If X is non-singular, D is the divisor on Y, then*

$$h^i(X, \varphi^* \mathscr{O}_Y(D)) = \sum_{g \in G} h^i(Y, \mathscr{O}_Y(D - L_g))$$

The following result will be used to calculate the ramification index.

THEOREM 2.4. *Let P be an irreducible and reduced hypersurface in Y, let $\bar{P} = \pi^{-1}(P)$ be the reduced preimage of P in X, and let a_i be the multiplicity of P in $D_i = div(f_i)$. Then*

$$\pi^* P = \frac{|G|}{d_P} \bar{P},$$

where

$$d_P = \gcd\left(|G|, \ |G|\frac{a_1}{n_1}, \ \cdots, \ |G|\frac{a_k}{n_k}\right)$$

is the number of points in the preimage of a generic point on P.

3. Canonical map. Let X be a surface of general type whose canonical map φ_{K_X} is a generically finite cover of a surface in $\mathbb{P}^2$.

LEMMA 3.1. *Assume $\varphi : X \to \mathbb{P}^2$ is a finite cover of degree 36, $\varphi_{K_X} = \varphi$ if and only if*

$$\varphi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus E \oplus \mathcal{O}_{\mathbb{P}^2}(-4),$$

where E is a rank 34 bundle satisfying $E^\vee \cong E(4)$ and $H^0(E(1)) = 0$.

Proof. If φ is a finite cover, we take E_0 as the trace free part of $\varphi_* \mathcal{O}_X$, so $\varphi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus E_0$. By relative duality,

$$\varphi_* \omega_X \cong (\varphi_* \mathcal{O}_X)^\vee \otimes \omega_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3) \oplus E_0^\vee(-3).$$

Since $\omega_X = \varphi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, by projection formula, $\varphi_* \omega_X = \varphi_* \varphi^*(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}_{\mathbb{P}^2}(1) \otimes \varphi_* \mathcal{O}_X$, so

$$\mathcal{O}_{\mathbb{P}^2}(1) \oplus E_0(1) \cong \mathcal{O}_{\mathbb{P}^2}(-3) \oplus E_0^\vee(-3).$$

We see that $E_0 = E \oplus \mathcal{O}_{\mathbb{P}^2}(-4)$, and $E^\vee = E(4)$.

On the other hand, if $\varphi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus E \oplus \mathcal{O}_{\mathbb{P}^2}(-4)$, then

$$\begin{aligned}
\varphi_*(\omega_X \otimes \varphi^* \mathcal{O}_{\mathbb{P}^2}(-1)) \quad &= \varphi_* \omega_X \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \\
&= (\varphi_* \mathcal{O}_X)^\vee \otimes \omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \\
&= (\mathcal{O}_{\mathbb{P}^2} \oplus E \oplus \mathcal{O}_{\mathbb{P}^2}(-4))^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(-4) \\
&= (\mathcal{O}_{\mathbb{P}^2} \oplus E^\vee \oplus \mathcal{O}_{\mathbb{P}^2}(4)) \otimes \mathcal{O}_{\mathbb{P}^2}(-4) \\
&= \mathcal{O}_{\mathbb{P}^2}(-4) \oplus E \oplus \mathcal{O}_{\mathbb{P}^2},
\end{aligned}$$

$\mathcal{O}_{\mathbb{P}^2}(-4) \oplus E \oplus \mathcal{O}_{\mathbb{P}^2}$ admits a non zero global section, so $\omega_X \otimes \varphi^* \mathcal{O}_{\mathbb{P}^2}(-1)$ admits also non zero global section. Namely there is an effective divisor Z such that $K_X = \varphi^* H + Z$, where H is the divisor of the hyperplane. Because

$$p_g(X) = h^2(\varphi_* \mathcal{O}_X) = h^2(\mathcal{O}_{\mathbb{P}^2} \oplus E \oplus \mathcal{O}_{\mathbb{P}^2}(-4)) = h^0(E(1)) + 3 = 3.$$

Z is the fixed part of $|K_X|$. It implies that

$$K_X^2 \geqslant (\varphi^* H)^2 = 36 = 9(p_g(X) + 1) \geqslant 9\chi(\mathcal{O}_X),$$

by Miyaoka-Yau inequality, $K_X^2 = 36$, $Z = 0$ and $q(X) = 0$. So $\varphi_{K_X} = \varphi$. $\square$

LEMMA 3.2. *If $\varphi = \varphi_{K_X}$ is a finite abelian cover of degree d over $\mathbb{P}^2$, then $\varphi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus d-2} \oplus \mathcal{O}_{\mathbb{P}^2}(-4)$.*

Proof. Because φ is an abelian cover, $\varphi_* \mathcal{O}_X$ is a direct sum of the line bundle.

$$\varphi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^2}(-l_i).$$

Assume $0 < l_1 \leqslant l_2 \leqslant \cdots \leqslant l_{d-1}$. Since $K_X = \varphi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, for any $m \geqslant 1$,

$$p_m(X) = h^0(mK_X) = h^0(\varphi^*(\mathcal{O}_{\mathbb{P}^2}(m))) = h^0(\mathcal{O}_{\mathbb{P}^2}(m)) + \sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^2}(m - l_i)).$$

Because $p_g(X) = p_1(X) = h^0(\varphi^*(\mathcal{O}_{\mathbb{P}^2}(1))) = h^0(\mathcal{O}_{\mathbb{P}^2}(1)) = 3$, we see that

$$h^0(\mathcal{O}_{\mathbb{P}^2}(1 - l_i)) = 0, \forall i$$

So $l_i \geqslant 2$.

And $h^2(\varphi_* \mathcal{O}_X) = h^2(\mathcal{O}_{\mathbb{P}^2}) + \sum_{i=1}^{d-1} h^2(\mathcal{O}_{\mathbb{P}^2}(-l_i))$, so $3 = \sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^2}(l_i - 3))$, then $l_i \leqslant 4$.

Then there are two cases:

(1) $l_1 = \cdots = l_{d-2} = 2$, $\quad l_{d-1} = 4$, and

(2) $l_1 = \cdots = l_{d-4} = 2$, $\quad l_{d-3} = l_{d-2} = l_{d-1} = 3$.

On the other hand, if $m = 2$, we have the second plurigenus $p_2(X) = K_X^2 + \chi(\mathcal{O}_X) = d + 4$. So

$$d + 4 = h^0(\mathcal{O}_{\mathbb{P}^2}(2)) + \sum_{i=1}^{d-1} h^0\big(\mathcal{O}_{\mathbb{P}^2}(2 - l_i)\big).$$

The second case does not satisfy the equation. So the lemma is proved. ∎

Now suppose $\varphi : X \to \mathbb{P}^2$ be an abelian cover associated to an abelian group $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$. Then X is the normalization of the surface defined by

$$z_1^{n_1} = f_1 = \prod_\alpha p_\alpha^{\alpha_1}, \cdots, z_k^{n_k} = f_k = \prod_\alpha p_\alpha^{\alpha_k},$$

where p_α's are prime factors and $\alpha = (\alpha_1, \cdots, \alpha_k) \in G$. Denote x_α to be the degree of p_α, $e_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in G$, $1 \leq i \leq k$, and l_g be the degree of L_g. So x_α and l_g are all integers. Then

$$(3.1) \qquad\qquad n_i l_{e_i} = \sum_\alpha \alpha_i x_\alpha \quad i = 1, \cdots k,$$

$$(3.2) \qquad\qquad l_g = \sum_{i=1}^{k} g_i l_{e_i} - \sum_\alpha \left[\sum_{i=1}^{k} \frac{g_i \alpha_i}{n_i}\right] x_\alpha.$$

LEMMA 3.3. *Using the notation as above, if $\varphi = \varphi_{K_X}$, then there exists $g' = (g_1', \cdots, g_k') \in G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, such that x_α satisfies the following equations*

$$(*) \qquad \begin{cases} n_i l_{e_i} = \sum\limits_\alpha \alpha_i x_\alpha \\ l_{g'} = \sum\limits_{i=1}^{k} g_i' l_{e_i} - \sum\limits_\alpha \left[\sum\limits_{i=1}^{k} \frac{g_i' \alpha_i}{n_i}\right] x_\alpha = 4 \\ l_g = \sum\limits_{i=1}^{k} g_i l_{e_i} - \sum\limits_\alpha \left[\sum\limits_{i=1}^{k} \frac{g_i \alpha_i}{n_i}\right] x_\alpha = 2, \qquad g \neq g', g \in G \end{cases}$$

Proof. It comes from Lemma 3.2 directly. ∎

By the above lemma, finding surfaces whose canonical map are abelian covers over $\mathbb{P}^2$ is equivalent to finding the integral roots $\{x_\alpha\}$ of the above equations.

THEOREM 3.4. *If the canonical degree of X is 36, then φ_{K_X} can not be an abelian cover.*

Proof. From the famous theorem of Beauville([Bea]), we know the image of $\varphi_{K_X}(X)$ is $\mathbb{P}^2$. We assume that $\varphi_{K_X} : X \to \mathbb{P}^2$ is an abelian cover associated to abelian group G. Then $|G| = 36$.

If $G = \mathbb{Z}_{36}$, then there exists some $g' \in \mathbb{Z}_{36}$ and corresponding l such that $\{x_\alpha\}$ must be satisfied the following equations:

$$\begin{cases} l_{g'} = g'l_1 - \sum_{\alpha=1}^{35}[\frac{g'\alpha}{36}]x_\alpha = 4, \\ l_g = gl_1 - \sum_{\alpha=1}^{35}[\frac{g\alpha}{36}]x_\alpha = 2, \qquad g \neq g', \quad 1 \le g \le 35 \\ 36l_1 = \sum_{\alpha=1}^{35}\alpha x_\alpha. \end{cases}$$

After computation, we find that there is no non-negative integral solution for the above equations for any $1 \le g' \le 35$.

Using the same method, we find there is no non-negative integral solution for $(*)$ as $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ and $n_1|n_2$.

So φ_{K_X} can not be abelian. $\square$

THEOREM 3.5. *Assume $\varphi : X \longrightarrow \mathbb{P}^2$ is an abelian cover.*
(1) Let $\deg\varphi = 16$, then $\varphi = \varphi_{K_X}$ if and only if X is defined by

$$(3.3) \qquad z_1^2 = \ell_1\ell_3\ell_4\ell_7, \quad z_2^2 = \ell_1\ell_2\ell_4\ell_5, \quad z_3^2 = \ell_1\ell_2\ell_3\ell_6, \quad z_4^2 = \ell_2\ell_5\ell_6\ell_8.$$

In particular, the surface defined by the first three equations is a Campedelli surface.
(2) Let $\deg\varphi = 9$, then $\varphi = \varphi_{K_X}$ if and only if X is defined by $z_1^3 = a_1a_2^2, z_2^3 = a_1a_2a_3$.
(3) Let $\deg\varphi = 8$, then $\varphi = \varphi_{K_X}$ if and only if X is defined by either $z_1^2 = a_1a_4, z_2^2 = a_2a_4, z_3^2 = a_3a_4$, or $z_1^2 = a_1a_2, z_2^4 = a_2^3a_3$.
(4) Let $\deg\varphi = 6$, then $\varphi = \varphi_{K_X}$ if and only if X is defined by $z_1^2 = a_1a_2, z_2^3 = a_2a_3^2$.
(5) Let $\deg\varphi = 4$, then $\varphi = \varphi_{K_X}$ if and only if X is defined by either $z^4 = a_1^2b_2^3$, or $z_1^2 = b_1, z_2^2 = b_2$.
(6) Let $\deg\varphi = 3$, then $\varphi = \varphi_{K_X}$ if and only if X is defined by either $z^3 = c_1^2$, or $z^3 = c_2$.
(7) Let $\deg\varphi = 2$, then $\varphi = \varphi_{K_X}$ if and only if X is $z^2 = h$.
Here ℓ_i's define different lines and there are at most three lines among them passing through one point, for $i = 1\cdots 8$; a_i's are reduced of degree 2; b_i's are reduced of degree 4; c_1, c_2 are reduced of degree 6; h is reduced of degree 8 and all the irreducible components of a_i's, b_i's, c_1, c_2 and h are simply normal crossing.

Proof. If $\varphi = \varphi_{K_X}$, then $K_X = \varphi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, $|K_X|$ has no fixed part and is base point free.

Let $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$. Since the arguments are similar, we only prove the most complicated case when $|G| = 16$. By Lemma 3.3, we only need to find the integral solution of the equations $(*)$. After computation, we find if and only if $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, there are integral solutions.
  (1) $l_{(1,0,0,0)} = 4$, $x_{(1,1,0,1)} = x_{(1,1,1,0)} = x_{(1,0,1,1)} = x_{(1,0,1,0)} = x_{(1,0,0,1)} = x_{(1,0,0,0)} = x_{(1,1,1,1)} = x_{(1,1,0,0)} = 1$, $x_{(0,1,0,0)} = x_{(0,0,0,1)} = x_{(0,0,1,0)} = x_{(0,0,1,1)} = x_{(0,1,1,0)} = x_{(0,1,0,1)} = x_{(0,1,1,1)} = 0$,

(2) $l_{(1,1,0,0)} = 4$, $x_{(0,1,1,1)} = x_{(1,0,1,1)} = x_{(0,1,1,0)} = x_{(1,0,1,0)} = x_{(1,0,0,1)} = x_{(0,1,0,0)} = x_{(1,0,0,0)} = x_{(0,1,0,1)} = 1$, $x_{(0,0,1,1)} = x_{(1,1,0,0)} = x_{(1,1,1,1)} = x_{(1,1,0,1)} = x_{(1,1,1,0)} = x_{(0,0,0,1)} = x_{(0,0,1,0)} = 0$,

(3) $l_{(1,1,1,0)} = 4$, $x_{(1,1,1,0)} = x_{(0,1,0,1)} = x_{(1,0,0,1)} = x_{(0,1,0,0)} = x_{(1,0,0,0)} = x_{(1,1,1,1)} = x_{(0,0,1,0)} = x_{(0,0,1,1)} = 1$, $x_{(0,1,0,1)} = x_{(1,1,0,0)} = x_{(0,1,1,0)} = x_{(0,1,1,1)} = x_{(1,1,0,1)} = x_{(1,0,1,1)} = x_{(0,0,0,1)} = 0$,

(4) $l_{(1,1,1,1)} = 4$, $x_{(0,1,1,1)} = x_{(1,1,0,1)} = x_{(1,1,1,0)} = x_{(1,0,1,1)} = x_{(0,1,0,0)} = x_{(1,0,0,0)} = x_{(0,0,1,0)} = x_{(0,0,0,1)} = 1$, $x_{(0,0,1,1)} = x_{(1,0,1,0)} = x_{(1,1,0,0)} = x_{(1,0,0,1)} = x_{(0,1,1,0)} = x_{(0,1,0,1)} = x_{(1,1,1,1)} = 0$.

Although there are 4 sets of solution, these defining equations are equivalent after quotient the group $sl(4, \mathbb{Z}_2)$ which means that they are same surfaces. So the surface X is defined by (3.3). Next we are going to discuss the configurations of ℓ_i's as following three cases.

- If ℓ_i's are in general position, i.e. they are simply normal crossing. Now we prove X is smooth.
  Let p_{ij} be the intersection point of ℓ_i and ℓ_j. The cover is locally defined by

$$z_1^2 = x^{a_{11}} y^{a_{12}}, \quad z_2^2 = x^{a_{21}} y^{a_{22}}, \quad z_3^2 = x^{a_{31}} y^{a_{32}}, \quad z_4^2 = x^{a_{41}} y^{a_{42}},$$

  where $a_{ij} = 0$ or 1 for all i, j.
  It is easy to check that $\{(a_{i1}, a_{i2})\} \not\subseteq \{(1,1), (0,0)\}$ i.e., at least one pair $\{(a_{i1}, a_{i2})\} = \{(1,0)\}$ or $\{(0,1)\}$. Without lose of generality, we can assume $(a_{11}, a_{21}) = (1,0)$, i.e. $z_1^2 = x$. The cover is branched along the smooth line. So the surface is smooth.
  From Theorem 2.4, the ramification index of $H_i = \mathrm{Div}(\ell_i)$ is 2. Thus $K_X = \varphi^*(\mathcal{O}_{\mathbb{P}^2}(-3) + \frac{1}{2} \sum_{i=1}^8 H_i) = \varphi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, which means $\varphi = \varphi_{K_X}$.

- If there are three of ℓ_i's intersecting at a point, we blow up these triple points at first. Let $\sigma : P \to \mathbb{P}^2$ be the blowing-ups of these triple points with $\{E_s\}$ the except curves and $\pi : \Sigma \to P$ be the corresponding abelian cover, i.e., the pull back of φ by σ. Then it is easy to check that E_s's are in the branch locus of π and the ramification indices are also 2. Similarly, we can show that Σ is smooth. Then the canonical divosor

$$K_\Sigma = \pi^*(\sigma^*(-3H) + \sum_s E_s + \frac{1}{2}(\sigma^*(8H) - 3\sum_s E_s) + \frac{1}{2}\sum_s E_s)$$
$$= \pi^*\sigma^*(H),$$

  where H is a hyperplane on $\mathbb{P}^2$. So the surface is minimal and the degree of the canonical map is 16. In this case, X has only A_1 type singularities.

- If there are d lines among ℓ_i's passing through a point, $d \geq 4$. We blow up these points such that the branch locus are normal crossing. Let $\sigma : P \to \mathbb{P}^2$ be the blowing-ups of these points with $\{E_s | s \in S\}$ the except curves and $\pi : \Sigma \to P$ be the corresponding abelian cover. Then Σ is smooth. $\{E_s | s \in S\}$ decomposes into 2 parts: $\{E_s | s \in S_1\}$ are in the branch locus of

π and $\{E_s | s \in S_2\}$ are not . Then the canonical divisor

$$K_\Sigma = \pi^*(\sigma^*(-3H) + \sum_{s \in S} E_s + \frac{1}{2}(\sigma^*(8H) - d\sum_{s \in S} E_s) + \frac{1}{2}\sum_{s \in S_1} E_s)$$

$$= \pi^*(\sigma^*(H) + \frac{3-d}{2}\sum_{s \in S_1} E_s + \frac{2-d}{2}\sum_{s \in S_2} E_s),$$

where H is a hyperplane on $\mathbb{P}^2$. Obviously, $K_\Sigma^2 < 16$. There is no -1 curves on Σ by resolution and the singularities on X are not rational double points.

Therefore the configuration of ℓ_i has at most triple points.

Let X_1 be the surface defined by the first three equations of (3.3) and suppose $\pi' : X_1 \to \mathbb{P}^2$. When ℓ_i's are in general position, we can show that X_1 is smooth and $K_{X_1}^2 = 2$. By Theorem 2.2, $L_g \sim 2H$ for $g \neq 0$, where H is a hyperplane on $\mathbb{P}^2$. So $h^0(-L_g) = h^0(K_{\mathbb{P}^2} + L_g) = 0$ for all nonzero $g \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Due to Corollary 2.3, $p_g(X_1) = q(X_1) = 0$. Therefore the surface X_1 is a Campedelli surface.

Now we compute the fundamental group of the surface X_1.

According to the result of the canonical resolution of double cover, the preimage of each $L_i = \text{Div}(\ell_i)$ by π' is irreducible. Then assume $\pi'^*(L_i) = 2A_i$, we can get $A_i^2 = 2$.

Due to $2A_i \sim \pi'^*(H)$, $2(A_i - A_j) \sim 0$. Assume $A_i \sim A_j$, $i \neq j$, then $h^0(A_i) \leq 2$ since $h^0(A_i + A_j) \geq h^0(A_i) + h^0(A_j) - 1$. And the equation holds if and only if A_i and A_j do not intersect, which is contradict to $A_i^2 = 2$. So $h^0(A_i) = 1$, which means that A_i is not linear equivalent to A_j.

Therefore each $A_i - A_j$ is a 2-torsion element in X_1. So X_1 has at least 6 2-torsion elements.

Denote $B = \sum_{i=1}^7 L_i$ to be the branch locus in $\mathbb{P}^2$. $\pi_1(\mathbb{P}^2 - B) = \mathbb{Z}^{\oplus 6}$ is an abelian group. We know that $X_1 - \pi'^*(B)$ is an étale cover over $\mathbb{P}^2 - B$. So $\pi_1(X_1 - \pi'^*(B))$ is a subgroup of $\pi_1(\mathbb{P}^2 - B)$ which is abelian and $\pi_1(X_1)$ is the quotient group of $\pi_1(X_1 - \pi'^*(B))$, then $\pi_1(X_1)$ is also an abelian group, which means $\pi_1(X_1) = H_1(X_1, \mathbb{Z})$.

As $q = 0$, $H^1(X_1, \mathbb{C}) = 0$. Using the universal coefficient theorem for cohomology, we know $H^1(X_1, \mathbb{C}) = H_1(X_1, \mathbb{C}) = 0$. Since $H_1(X_1, \mathbb{C}) = H^1(X_1, \mathbb{Z}) \bigotimes \mathbb{C} = 0$, $H^1(X_1, \mathbb{Z})$ only has torsion parts.

Due to the following exact sequence

$$H^1(X_1, \mathcal{O}_{X_1}) \longrightarrow H^1(X_1, \mathcal{O}_{X_1}^*) \longrightarrow H^2(X_1, \mathbb{Z}) \longrightarrow H^2(X_1, \mathcal{O}_{X_1})$$

and $p_g(X_1) = q(X_1) = 0$, we know $\text{Pic} X_1 = H^1(X_1, \mathcal{O}_{X_1}^*) = H^2(X_1, \mathbb{Z})$.

From the following sequence

$$0 \longrightarrow \text{Ext}(H_1(X_1, \mathbb{Z}), \mathbb{Z}) \longrightarrow H^2(X_1, \mathbb{Z}) \longrightarrow \text{Hom}(H_2(X_1, \mathbb{Z}), \mathbb{Z}) \longrightarrow 0,$$

we have $H_1(X_1, \mathbb{Z}) = \text{Tor}(H^2(X_1, \mathbb{Z}))$.

For any Campedelli surface X, it is well known that $|\pi_1(X)| \leqslant 9$([Rei]). So the torsion group of the Picard group of X_1 is $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, i.e., $\pi_1(X_1) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. $\square$

REMARK 3.6.
(1) From the above theorem, we know the defining equations of Persson's surface.
(2) The Campedelli surface constructed in the proof is as same as that given by Kulikov ([Kul]). In fact, Campedelli surfaces with fundamental group of order 8 are completely classified in [ML-P-R].

COROLLARY 3.7. *If*

$$\varphi = \varphi_{K_X} : X \longrightarrow \mathbb{P}^2,$$

is an abelian cover, then d is equal to $2, 3, 4, 6, 8, 9$ *or* 16.

4. Defining equations of Tan's examples. In [Tan], Tan constructed a series of nice surfaces whose canonical maps are of odd degrees by using a so-called conception $\mathbb{Z}_p$-set of the singularities.

Let N be a set of singular points of surface Σ, and let $\sigma : Y \to \Sigma$ be the minimal resolution of Σ with $\Gamma_1, \cdots, \Gamma_n$ the components of $\sigma^{-1}(N)$. If there are positive integers $a_i < p$ and a divisor D on Y such that $\sum_{i=1}^{n} a_i \Gamma \sim pD$ then N is called a $\mathbb{Z}_p$-set of singularities on Σ. A p fold cyclic cover can be determined by the $\mathbb{Z}_p$-set.

Take 3 points q_1, q_2, q_3 in general position on $\mathbb{P}^2$, and every 3 lines passing through each point. This configuration of 9 lines has $n + 3$ triple points and $27 - 3n$ double points. And he assumed that all lines passing through q_1, q_2 contain no 4 triple points. Blowing up the $n + 3$ triple points, we suppose each A_i to be the strict transforms of the 3 lines passing through q_i. Then a Galois triple cover π_1 with branch locus $A_1 + A_2 + A_3$ can be defined and $N = \pi_1^{-1}(\bigcup_{i<j} A_i \bigcap A_j)$ is a $\mathbb{Z}_3$-set. His example X is the 3 fold cyclic cover determined by N. In order to show the canonical map of X is of degree 3, Tan calculated the invariants of such surface. But the computation of the invariants of the surface is pretty complicated. The main part of [Tan] is the computation of $p_g(X)$.

We can reconstruct the Tan's example by explicitly equations. Let $\varphi : X \to \mathbb{P}^2$ be an abelian cover branched over the line configuration described by Tan. Assume ℓ_1, ℓ_2, ℓ_3 are the 3 lines through q_1, ℓ_4, ℓ_5, ℓ_6 are the 3 lines through q_2 and ℓ_7, ℓ_8, ℓ_9 are the 3 lines through q_3. φ is defined by the following equations.

$$\pi : \quad z_1^3 = \ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7 \ell_8 \ell_9, \quad z_2^3 = \ell_1 \ell_2 \ell_3 \ell_4^2 \ell_5^2 \ell_6^2.$$

Let $\sigma : P \to \mathbb{P}^2$ be blowing-ups of q_1, q_2, q_3. $\pi : X \to P$ is the pull back of φ by σ. Let Σ be the minimal resolution of $\pi' : Y \to \mathbb{P}^2$ which is a cyclic cover defined by $z^3 = \ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7 \ell_8 \ell_9$. It is clear that X and Y have at most A_1-type singularities. We calculate invariants $q(\Sigma) = q(X) = 0$ and $p_g(X) = p_g(\Sigma) = 8 - n$ by Theorem 2.2. The canonical map of X factorizes through that of Σ, so it is of degree 3.

In [Tan], the author also constructed a surface whose canonical map is of degree 5 by using $\mathbb{Z}_5$ set. We can also reconstruct it as follows.

Take five lines $\ell_1, \cdots, \ell_5$ in general position on $\mathbb{P}^2$. $\pi : X \to \mathbb{P}^2$ is an abelian cover defined by

$$z_1^5 = \ell_1 \ell_2 \ell_3 \ell_4 \ell_5, \qquad z_2^5 = \ell_1 \ell_2^2 \ell_3^3 \ell_4^4.$$

The 5 fold covering surface Σ over $\mathbb{P}^2$ is defined by the first equation. Similarly, we get $K_X = 25$, $K_\Sigma = 5$, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_\Sigma) = 5$, $p_g(\mathcal{O}_X) = p_g(\mathcal{O}_\Sigma)$. Therefore, the canonical map of X factorizes through that of Σ. This means that the degree of the canonical map of X is 5.

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