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EIGENVALUES OF HECKE OPERATORS ON HILBERT MODULAR GROUPS*

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Abstract. Let F be a totally real field, let I be a nonzero ideal of the ring of integers $\mathcal{O}_F$ of F, let $\Gamma_0(I)$ be the congruence subgroup of Hecke type of $G = \prod_{j=1}^d \mathrm{SL}_2(\mathbb{R})$ embedded diagonally in G, and let χ be a character of $\Gamma_0(I)$ of the form $\chi\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) = \chi(d)$, where $d \mapsto \chi(d)$ is a character of $\mathcal{O}_F$ modulo I.

For a finite subset P of prime ideals $\mathfrak{p}$ not dividing I, we consider the ring $\mathcal{H}^I$, generated by the Hecke operators $T(\mathfrak{p}^2)$, $\mathfrak{p} \in P$ (see §3.2) acting on (Γ, χ)-automorphic forms on G.

Given the cuspidal space $L^{2,\mathrm{cusp}}_\xi\bigl(\Gamma_0(I)\backslash G, \chi\bigr)$, we let V_ϖ run through an orthogonal system of irreducible G-invariant subspaces so that each V_ϖ is invariant under $\mathcal{H}^I$. For each $1 \leq j \leq d$, let $\lambda_\varpi = (\lambda_{\varpi,j})$ be the vector formed by the eigenvalues of the Casimir operators of the d factors of G on V_ϖ, and for each $\mathfrak{p} \in P$, we take $\lambda_{\varpi,\mathfrak{p}} \in \mathcal{J}_\mathfrak{p} := [0, 1+\mathrm{N}(\mathfrak{p})) \cup i\bigl(0, \sqrt{1+\mathrm{N}(\mathfrak{p})^2}\bigr]$) so that $\lambda^2_{\varpi,\mathfrak{p}} - \mathrm{N}(\mathfrak{p})$ is the eigenvalue on V_ϖ of the Hecke operator $T(\mathfrak{p}^2)$. If for some prime $\mathfrak{p}$ the Hecke operator $T(\mathfrak{p})$ can be defined then its eigenvalue on V_ϖ is real and equal to $\lambda_{\varpi,\mathfrak{p}}$ or $-\lambda_{\varpi,\mathfrak{p}}$.

For each family of expanding boxes $t \mapsto \Omega_t$, as in (3) in $\mathbb{R}^d$, and fixed interval $J_\mathfrak{p}$ in $\mathcal{J}_\mathfrak{p}$, for each $\mathfrak{p} \in P$, we consider the counting function

$$N(\Omega_t;\, (J_\mathfrak{p})_{\mathfrak{p}\in P}) := \sum_{\varpi,\, \lambda_\varpi \in \Omega_t\,:\, \lambda_{\varpi,\mathfrak{p}} \in J_\mathfrak{p}\ ,\forall \mathfrak{p}\in P} |c^r(\varpi)|^2 .$$

Here $c^r(\varpi)$ denotes the normalized Fourier coefficient of order r at ∞ for the elements of V_ϖ, with $r \in \mathcal{O}'_F \smallsetminus \mathfrak{p}\,\mathcal{O}'_F$ for every $\mathfrak{p} \in P$.

In the main result in this paper, Theorem 1.1, we give, under some mild conditions on the Ω_t, the asymptotic distribution of the function $N(\Omega_t;\, (J_\mathfrak{p})_{\mathfrak{p}\in P})$, as $t \to \infty$. We show that at the finite places outside I the eigenvalues of the Hecke operator $T(\mathfrak{p}^2)$ are equidistributed compatibly with the Sato-Tate measure, whereas at the archimedean places the eigenvalues λ_ϖ are equidistributed with respect to the Plancherel measure.

As a consequence, if we pick an infinite place l and we prescribe $\lambda_{\varpi,j} \in \Omega_j$ for all infinite places $j \neq l$ and $\lambda_{\varpi,\mathfrak{p}} \in J_\mathfrak{p}$ for all finite places $\mathfrak{p}$ in P for fixed sets Ω_j and fixed intervals $J_\mathfrak{p} \subset \mathcal{J}_\mathfrak{p}$ with positive measure and then allow $\lambda_{\varpi,l}$ to run over larger and larger regions, then there are infinitely many representations ϖ in such a set, and their positive density is as described in Theorem 1.1.

Key words. Automorphic representations, Hecke operators, Hilbert modular group, Plancherel measure, Sato-Tate measure.

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1. Introduction and discussion of main results. We work with a totally real number field F of degree d, the Lie group $G = \mathrm{SL}_2(\mathbb{R})^d$ considered as the product of $\mathrm{SL}_2(F_j)$ for all archimedean completions $F_j \cong \mathbb{R}$ of F. The group $\mathrm{SL}_2(F)$ is diagonally embedded in G. We consider the congruence subgroup $\Gamma = \Gamma_0(I)$ with I a nonzero ideal in the ring of integers $\mathcal{O}_F = \mathcal{O}$ of F, a character χ of $(\mathcal{O}_F/I)^*$ inducing a character $\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)$ of Γ, and a compatible central character determined by $\xi \in \{0,1\}^d$.

$L^2_\xi(\Gamma\backslash G, \chi)$ is the Hilbert space of classes of square integrable functions transforming on the left by Γ according to the character χ, and transforming by the center Z of G according to the central character determined by ξ. We work with a maximal

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orthogonal system $\{V_\varpi\}_\varpi$ of irreducible subspaces in the cuspidal Hilbert subspace of $L^2_\xi(\Gamma\backslash G,\chi)$. For each ϖ, there is an eigenvalue vector $\lambda_\varpi = (\lambda_{\varpi,j})_j \in \mathbb{R}^d$, where $\lambda_{\varpi,j}$ is the eigenvalue of the Casimir operator of the factor $\mathrm{SL}_2(\mathbb{R})$ at place j in the product $G = \mathrm{SL}_2(\mathbb{R})^d$.

We normalize the Fourier terms of automorphic forms at the cusp ∞ as discussed in §2.5. With this normalization, one obtains Fourier coefficients $c^r(\varpi)$ that are the same for all automorphic forms in V_ϖ. The order r of the Fourier terms runs through the inverse different $\mathcal{O}'_F$ of $\mathcal{O}_F$.

In [9] (Theorem 4.5, Proposition 4.6 and Theorem 5.3) we prove that, under some mild conditions on the family of compact sets $t \mapsto \Omega_t$ in $\mathbb{R}^d$:

$$(1)\qquad \sum_{\varpi,\,\lambda_\varpi\in\Omega_t} |c^r(\varpi)|^2 = \frac{2\sqrt{|D_F|}\,\mathrm{Vol}\,(\Gamma\backslash G)}{(2\pi)^d}\,\mathrm{Pl}(\Omega_t)\,\big(1+o(1)\big) \qquad (t\to\infty)\,.$$

(In [9] the factor $\mathrm{Vol}\,(\Gamma\backslash G)$ is not present. It is due to a different normalization of measures discussed in §2.5.) The factor D_F is the discriminant of F over $\mathbb{Q}$, the $c^r(\varpi)$ are Fourier coefficients, and Pl denotes the Plancherel measure given by $\mathrm{Pl} = \otimes_j \mathrm{Pl}_{\xi_j}$ on $\mathbb{R}^d$, where

$$(2)\qquad \begin{aligned}\mathrm{Pl}_0(f) &= \int_{1/4}^\infty f(\lambda)\tanh\pi\sqrt{\lambda-\tfrac14}\,d\lambda \\ &\quad+ \sum_{b\geq 2,\, b\equiv 0 \bmod 2} (b-1)\, f\left(\tfrac b2\left(1-\tfrac b2\right)\right),\\ \mathrm{Pl}_1(f) &= \int_{1/4}^\infty f(\lambda)\coth\pi\sqrt{\lambda-\tfrac14}\,d\lambda \\ &\quad+ \sum_{b\geq 3,\, b\equiv 1 \bmod 2} (b-1)\, f\left(\tfrac b2\left(1-\tfrac b2\right)\right).\end{aligned}$$

In this paper we work with a family of the following type:

$$(3)\qquad \Omega_t = [-t,t]^Q \times \prod_{j\in E} [A_j, B_j]\,,$$

where $\{1,\dots,d\} = Q \sqcup E$ is a partition of the archimedean places of F for which $Q \neq \emptyset$. The end points A_j and B_j of the fixed interval with $j\in E$ are not allowed to lie in the set $\left\{\frac b2(1-\frac b2)\ :\ b\equiv \xi_j\,,\ b>1\right\}$, of discrete series eigenvalues. The variable t tends to infinity. The set E can be empty, but the set Q has to contain at least one place.

The asymptotic formula (1) shows that if the box $\prod_{j\in E}[A_j,B_j]$ has positive Plancherel measure $\otimes_{j\neq l}\mathrm{Pl}_{\xi_j}$, then there are infinitely many eigenvalue vectors λ_ϖ that project to this box.

If one of the factors $[A_j,B_j]$ has zero density for Pl_{ξ_j}, which happens in particular if $[A_j,B_j]\subset \left(0,\frac14\right)$, then the asymptotic formula has no meaning. For such a situation it is better to use the following formulation:

$$(4)\qquad \sum_{\varpi,\,\lambda_\varpi\in\Omega_t} |c^r(\varpi)|^2 = \frac{2\sqrt{|D_F|}\,\mathrm{Vol}\,(\Gamma\backslash G)}{(2\pi)^d}\mathrm{Pl}(\Omega_t) + o\big(V_1(\Omega_t)\big) \qquad (t\to\infty)\,,$$

where the reference measure V_1 has product structure $V_1 = \otimes_j V_{1,\xi_j}$ with

$$\begin{aligned}(5)\qquad \int h\, dV_{1,0} &= \frac{1}{2}\int_{5/4}^{\infty} h(\lambda)\, d\lambda + \frac{1}{2}\int_0^{5/4} |\lambda - 1/4|^{-1/2}\, d\lambda \\ &\quad + \sum_{\beta>0\,,\ \beta\equiv\frac{1}{2}(1)} \beta\, h(1/4-\beta^2)\,, \\ \int h\, dV_{1,1} &= \frac{1}{2}\int_{5/4}^{\infty} h(\lambda)\, d\lambda + \frac{1}{2}\int_{1/4}^{5/4} |\lambda - 1/4|^{-1/2}\, d\lambda \\ &\quad + \sum_{\beta>0\,,\ \beta\equiv 0(1)} \beta\, h(1/4-\beta^2)\,.\end{aligned}$$

This measure V_1 is positive on all sets in which eigenvalue vectors can occur, and is comparable to Pl near points λ for which all coordinates stay a positive distance away from $\left(0, \frac{1}{4}\right)$. So the asymptotic formula does not exclude exceptional $\lambda_{\varpi,j}$, but only limits their density.

For congruence groups $\Gamma_0(I)$ over totally real number fields it is impossible, in general, to define Hecke operators $T(\mathfrak{p})$ corresponding to the Hecke operators T_p for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. However, one can consider Hecke operators of the form $T(\mathfrak{p}^2)$ for primes $\mathfrak{p}$ not dividing I. If the prime ideal $\mathfrak{p} = \pi_\mathfrak{p}\mathcal{O}$ is principal, then the action of the Hecke operator $T(\mathfrak{p}^2)$ on (Γ,χ)-automorphic functions f on G (i.e., transforming on the left according to the character χ of Γ) is given by

$$\begin{aligned}(6)\qquad f|T(\mathfrak{p}^2)(g) &= \chi(\pi_\mathfrak{p}) f\left(\begin{pmatrix}\pi_\mathfrak{p} & 0\\ 0 & 1/\pi_\mathfrak{p}\end{pmatrix} g\right) + \sum_{b\in\mathcal{O}/\mathfrak{p}} f\left(\begin{pmatrix}1 & b/\pi_\mathfrak{p}\\ 0 & 1\end{pmatrix} g\right) \\ &\quad + \sum_{b\in\mathcal{O}/\mathfrak{p}^2} \chi(\pi_\mathfrak{p})^{-1} f\left(\begin{pmatrix}1/\pi_\mathfrak{p} & b/\pi_\mathfrak{p}\\ 0 & \pi_\mathfrak{p}\end{pmatrix} g\right)\,.\end{aligned}$$

This does not depend on the choice of the generator $\pi_\mathfrak{p}$. The Hecke operator in (6) preserves (Γ,χ)-automorphy, square integrability and cuspidality. For non-principal $\mathfrak{p}\nmid I$, we will show that there are also Hecke operators $T(\mathfrak{p}^2)$ with similar properties (see §3.2). They generate a commutative algebra of symmetric bounded operators on $L^{2,\mathrm{cusp}}_\xi(\Gamma\backslash G,\chi)$. Furthermore, the orthogonal system $\{V_\varpi\}$ can be chosen in such a way that each operator acts on V_ϖ by multiplication by a fixed scalar. One can show that the eigenvalue $\lambda_{\varpi,\mathfrak{p}^2}$ of $T(\mathfrak{p}^2)$ on V_ϖ is a real number with absolute value at most $\mathrm{N}(\mathfrak{p})^2 + \mathrm{N}(\mathfrak{p}) + 1$. (See §4.2.) Hence there is

$$(7)\qquad \lambda_{\varpi,\mathfrak{p}} \in \mathcal{J}_\mathfrak{p} \;:=\; \left[0, 1+\mathrm{N}(\mathfrak{p})\right] \,\cup\, i\left(0, \sqrt{1+\mathrm{N}(\mathfrak{p})^2}\right]$$

such that $\lambda^2_{\varpi,\mathfrak{p}} - \mathrm{N}(\mathfrak{p})$ is the eigenvalue of $T(\mathfrak{p}^2)$ in V_ϖ.

If the prime $\mathfrak{p}\nmid I$ is of the form $\mathfrak{p} = \mathcal{O}\pi_\mathfrak{p}$ with totally positive $\pi_\mathfrak{p}$ for which $\chi(\pi_\mathfrak{p}) = 1$, then one can define the Hecke operator $T(\mathfrak{p})$ by the formula

$$(8)\qquad (f|T(\mathfrak{p})(g) \;=\; f\left(\begin{pmatrix}\sqrt{\pi_\mathfrak{p}} & 0\\ 0 & \frac{1}{\sqrt{\pi_\mathfrak{p}}}\end{pmatrix} g\right) + \sum_{b\in\mathcal{O}/\mathfrak{p}} f\left(\begin{pmatrix}\frac{1}{\sqrt{\pi_\mathfrak{p}}} & \frac{b}{\sqrt{\pi_\mathfrak{p}}}\\ 0 & \sqrt{\pi_\mathfrak{p}}\end{pmatrix} g\right),$$

and $T(\mathfrak{p})$ satisfies $T(\mathfrak{p})^2 = T(\mathfrak{p}^2) + \mathrm{N}(\mathfrak{p})$. In such a situation we can further arrange the system $\{V_\varpi\}$ to be such that $T(\mathfrak{p})$ has eigenvalue in $\left[-1-\mathrm{N}(\mathfrak{p}), 1+\mathrm{N}(\mathfrak{p})\right]$ on V_ϖ.

This eigenvalue is equal to $\lambda_{\varpi,\mathfrak{p}}$ or $-\lambda_{\varpi,\mathfrak{p}}$, with $\lambda_{\varpi,\mathfrak{p}}$ as chosen above. In general, one cannot define $T(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$.

The *Sato-Tate measure* is the measure on $\mathbb{R}$ given by

$$\begin{aligned} f &\mapsto \frac{2}{\pi}\int_0^\pi f\big(2\sqrt{\mathrm{N}(\mathfrak{p})}\,\cos\vartheta\big)\,\sin^2\vartheta\,d\vartheta \\ &= \frac{1}{2\pi\sqrt{\mathrm{N}(\mathfrak{p})}}\int_{-2\sqrt{\mathrm{N}(\mathfrak{p})}}^{2\sqrt{\mathrm{N}(\mathfrak{p})}} f(\lambda)\,\sqrt{4\mathrm{N}(\mathfrak{p})-\lambda^2}\,d\lambda\,. \end{aligned} \tag{9}$$

See p. 106–107 of [12]. This is the measure one expects to describe the distribution of the eigenvalues of the operators $T(\mathfrak{p})$ if they are defined. We have to deal with the parameter $\lambda_{\varpi,\mathfrak{p}} \in \mathcal{J}_\mathfrak{p}$ in (7), and use the following measure $\Phi_\mathfrak{p}$ on $\mathcal{J}_\mathfrak{p}$, which is related to the Sato-Tate measure:

$$\Phi_\mathfrak{p}(f) \;=\; \frac{1}{\pi\sqrt{\mathrm{N}(\mathfrak{p})}}\int_0^{2\sqrt{\mathrm{N}(\mathfrak{p})}} f(\lambda)\,\sqrt{4\mathrm{N}(\mathfrak{p})-\lambda^2}\,d\lambda\,. \tag{10}$$

The main goal of this paper is to prove the following distribution result:

THEOREM 1.1. *Let $t\mapsto\Omega_t$ be a family of compact sets in $\mathbb{R}^d$ as in* (3) *and let P be a finite set of prime ideals in $\mathfrak{p}\in\mathcal{O}_F$, $\mathfrak{p}\nmid I$. For each $\mathfrak{p}\in P$ let $J_\mathfrak{p}$ be an interval in $[0,\infty)\cup i(0,\infty)$. Then for any $r\in\mathcal{O}_F'$ such that $r\notin\mathfrak{p}\,\mathcal{O}_F'$ for every $\mathfrak{p}\in P$, we have:*

$$\begin{aligned} &\sum_{\varpi,\,\lambda_\varpi\in\Omega_t\,:\,\lambda_{\varpi,\mathfrak{p}}\in J_\mathfrak{p}\ ,\forall\mathfrak{p}\in P} |c^r(\varpi)|^2 \\ &\quad= \frac{2\sqrt{|D_F|}\,\mathrm{Vol}\,(\Gamma\backslash G)}{(2\pi)^d}\left(\mathrm{Pl}(\Omega_t)\prod_{\mathfrak{p}\in P}\Phi_\mathfrak{p}(J_\mathfrak{p}) + o\big(V_1(\Omega_t)\big)\right). \end{aligned} \tag{11}$$

(By an interval in $[0,\infty)\cup i(0,\infty)$ we mean a connected subset.)

The theorem shows that in the Hilbert modular case the parameters $\lambda_{\varpi,\mathfrak{p}}$ with $\mathfrak{p}\nmid I$ are equidistributed with respect to the measure $\Phi_\mathfrak{p}$. We recall that $\lambda_{\varpi,\mathfrak{p}^2} = \lambda_{\varpi,\mathfrak{p}}^2 - \mathrm{N}(\mathfrak{p})$ is the eigenvalue of $T(\mathfrak{p}^2)$ in the space $V_\mathfrak{p}$. For those $\mathfrak{p}\in P$ for which $T(\mathfrak{p})$ can be defined, by the same method one can prove the result in Theorem 11 using the eigenvalues $\pm\lambda_{\varpi,\mathfrak{p}}$ of $T(\mathfrak{p})$, with $\Phi_\mathfrak{p}$ replaced with the Sato-Tate measure at these places. In the case $F=\mathbb{Q}$ and $I=\mathbb{Z}$ that is Proposition 4.10 in [4]. (Since $d=1$, the set E has to be empty in this case.)

The theorem stays true for more general families $t\mapsto\Omega_t$, as discussed in §B.1 in the appendix.

COROLLARY 1.2. *Let E be a set of archimedean places, with $0\le\#E<d$, and let S be a finite set of finite places outside I. Suppose that $J_v\subset\mathbb{R}$ is a bounded interval for each $v\in E\cup S$. Suppose that the J_j with $j\in E$ satisfy the condition on the end points mentioned below* (3). *If*

$$\prod_{j\in E}\mathrm{Pl}_{\xi_j}(J_j)\prod_{\mathfrak{p}\in S}\Phi_\mathfrak{p}(J_\mathfrak{p})>0$$

then there are infinitely many representations ϖ such that $\lambda_{\varpi,v}\in J_v$ for all $v\in S$.

According to the *Ramanujan-Petersson* conjecture one should have that

$$\lambda_{\varpi,j} \notin (0,1/4),\ \forall j\ , \quad \lambda_{\varpi,\mathfrak{p}} \in [0, 2\sqrt{\mathrm{N}(\mathfrak{p})}],\ \forall \mathfrak{p} \nmid I\ . \tag{12}$$

In the language of representation theory this means that local factors ϖ_v of ϖ cannot be in the complementary series for any place v of F outside I. Theorem 1.1 gives support to this conjecture: If $v = j$ is an archimedean place, then we take $E = \{j\}$ and $[A_j, B_j] \subset \left(0, \frac{1}{4}\right)$. If $v = \mathfrak{p}$ is a finite place not dividing I, we take $S = \{\mathfrak{p}\}$ and $J_{\mathfrak{p}}$ such that $J_{\mathfrak{p}} \cap [-\mathrm{N}(\mathfrak{p}), 3\mathrm{N}(\mathfrak{p})] = \emptyset$. We conclude from the theorem that exceptional eigenvalues such that $\lambda_{\varpi,j} \in [A_j, B_j]$ or $\lambda_{\varpi,\mathfrak{p}} \in J_{\mathfrak{p}}$ are relatively scarce: the sum in the left hand side of (11) is $o\big(V_1(\Omega_t)\big)$ as $t \to \infty$. The theorem gives more: the $\lambda_{\varpi,v}$ are distributed according to the Plancherel measure if v is an archimedean place, and according to the measure $\Phi_{\mathfrak{p}}$, compatible with the Sato-Tate distribution if v is a finite place outside I.

In these results we fix one finite unramified place (or finitely many), and consider the distribution of the eigenvalues $\lambda_{\varpi,\mathfrak{p}}$ of the Hecke operators for this place $\mathfrak{p}$ (or the joint distribution of the Hecke eigenvalues for the finitely many fixed places), averaging over an infinite set of automorphic representations ϖ. That point of view is in some sense orthogonal to results like those in [1], where an automorphic representation ϖ is fixed, and it is shown that the distribution of Hecke eigenvalues $\lambda_{\varpi,\mathfrak{p}}$ where $\mathfrak{p}$ runs over the (unramified finite) places, is given by the Sato-Tate distribution.

The paper is organized as follows. In §2 we introduce some notations and recall some facts on automorphic forms. To handle Hecke operators it is convenient to work in an adelic context recalled in §2.6. In §3 we discuss the Hecke algebra $\mathcal{H}^I$ and the relation of its eigenvalues to Fourier coefficients. In §4 we give the proof of the main result, Theorem 1.1. To this end, we need to generalize the asymptotic result (4) to a sum in which $|c^r(\varpi)|^2$ is replaced by a product of Fourier coefficients of possibly different order and at possibly different cusps. This generalized asymptotic result is Theorem B.1, proved by using the sum formula given in Theorem A.2. We have included this auxiliary material in two Appendices to avoid interrupting the flow of proof of the main result.

2. Preliminaries. In this section we will introduce some notations and recall known facts on automorphic forms and Hecke operators in our context.

2.1. Notations. For any ring R we denote:

$$n(x) \ = \ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ for } x \in R, \qquad h(t) \ = \ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \text{ for } t \in R^*\ . \tag{13}$$

In the case $R = \mathbb{R}^d$ we also use

$$a(y) \ = \ \left(\begin{pmatrix} \sqrt{y_1} & 0 \\ 0 & 1/\sqrt{y_1} \end{pmatrix}, \ldots, \begin{pmatrix} \sqrt{y_d} & 0 \\ 0 & 1/\sqrt{y_d} \end{pmatrix} \right) \qquad \text{for } y \in (0,\infty)^d \tag{14}$$

$$k(\vartheta) \ = \ \left(\begin{pmatrix} \cos\vartheta_1 & \sin\vartheta_1 \\ -\sin\vartheta_1 & \cos\vartheta_1 \end{pmatrix}, \ldots, \begin{pmatrix} \cos\vartheta_d & \sin\vartheta_d \\ -\sin\vartheta_d & \cos\vartheta_d \end{pmatrix} \right) \qquad \text{for } \vartheta \in \mathbb{R}^d$$

and write

$$\begin{aligned} N \ &= \ \left\{ n(x) \ : \ x \in \mathbb{R}^d \right\}, \qquad A \ = \ \left\{ a(y) \ : \ y \in (0,\infty)^d \right\}, \\ K \ &= \ \left\{ k(\vartheta) \ : \ \vartheta \in \mathbb{R}^d \right\}. \end{aligned} \tag{15}$$

The center Z of G is the set of $h(\zeta)$ with $\zeta \in \{1,-1\}^d \subset \mathbb{R}^d$.

The function $S : \mathbb{R}^d \to \mathbb{R}$ given by $S(x) = \sum_j x_j$ extends $\mathrm{Tr}_{F/\mathbb{Q}} : F \to \mathbb{Q}$.

2.2. Cusps. As in §2.1.4 of [8] we choose a set $\mathcal{P}$ of representatives of the Γ-equivalence classes of cusps. We take ∞ as representative of $\Gamma\,\infty$. For each $\kappa \in \mathcal{P}$ we choose $g_\kappa \in \mathrm{SL}_2(F)$ such that $\kappa = g_\kappa\,\infty$. For $\kappa = \infty$ we can and do take $g_\infty = 1$. We use the notations $N^\kappa = g_\kappa N g_\kappa^{-1}$ and $P^\kappa = g_\kappa NAZ g_\kappa^{-1}$, and we put $\Gamma_{P^\kappa} := \Gamma \cap P^\kappa$ and $\Gamma_{N^\kappa} := \Gamma \cap N^\kappa$. So Γ_{P^κ} equals the subgroup of Γ fixing κ, and has Γ_{N^κ} as a normal commutative subgroup of the form $\Gamma_{N^\kappa} = \left\{ g_\kappa n(\xi) g_\kappa^{-1}\ \xi \in M_\kappa \right\}$ for some fractional ideal M_κ of F, which is a lattice in $\mathbb{R}^d$, under the embedding of F in $\mathbb{R}^d$. There is the dual lattice M'_κ consisting of all $r \in \mathbb{R}^d$ such that $S(rx) \in \mathbb{Z}$ for all $x \in M_\kappa$. So M'_κ is the fractional ideal $\mathcal{O}'\, M_\kappa^{-1}$, where $\mathcal{O}'$ is the inverse of the different ideal. Note that M_κ and M'_κ depend on the choice of g_κ, but Γ_{P^κ} and Γ_{N^κ} do not. For the cusp ∞ we have taken $g_\infty = 1$, so $M_\infty = \mathcal{O}$ and $M'_\infty = \mathcal{O}'$.

$\mathcal{P}_\chi$ is the subset of $\kappa \in \mathcal{P}$ for which the fixed character χ of Γ is trivial on Γ_{N^κ}. For $\kappa \in \mathcal{P}_\chi$, the dual lattice M'_κ describes the characters of $g_\kappa N g_\kappa^{-1}$ that are trivial on Γ_{N^κ}. For general $\kappa \in \mathcal{P}$ we put

$$\tilde{M}'_\kappa := \left\{ r \in \mathbb{R}^d \ :\ \chi(g_\kappa n(x) g_\kappa^{-1}) \ =\ e^{2\pi i S(rx)} \text{ for every } x \in M_\kappa \right\}. \tag{16}$$

Thus, $\tilde{M}'_\kappa$ is a shift of M'_κ inside $\mathbb{R}^d$, and $\kappa \in \mathcal{P}_\chi$ if and only if $0 \in \tilde{M}'_\kappa$.

2.3. Automorphic functions and Fourier terms. By a (Γ,χ)-automorphic function on G we mean any function on G that satisfies $f(\gamma g) = \chi(\gamma) f(g)$ for all $\gamma \in \Gamma$ and $g \in G$. We call a (Γ,χ)-automorphic function an *automorphic form* if f is an eigenfunction of the d Casimir operators of the factors of $G = \mathrm{SL}_2(\mathbb{R})^d$, and if it has a *weight* $q \in \mathbb{Z}^d$, i.e., $f(gk(\vartheta)) = f(g) e^{iS(q\vartheta)}$ for any $\vartheta \in \mathbb{R}^d$, where $S(q,\vartheta) = \sum_j q_j \vartheta_j$. All automorphic forms on G are real-analytic functions.

Let f be a continuous (Γ,χ)-automorphic function on G. For each $\kappa \in \mathcal{P}$ the function $x \mapsto f\bigl(g_\kappa n(x) g\bigr)$ on $\mathbb{R}^d$ transforms according to a character $\xi \mapsto e^{2\pi i S(r\xi)}$ of M_κ for some $r \in \tilde{M}'_\kappa$. So it has an absolutely convergent Fourier expansion

$$\begin{aligned} f(g_\kappa g) \ &=\ \sum_{r \in \tilde{M}'_\kappa} F_{\kappa,r} f(g)\,, \\ (F_{\kappa,r} f)(g) \ &=\ \frac{1}{\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)} \int_{\mathbb{R}^d/M_\kappa} e^{-2\pi i S(rx)} f(g_\kappa n(x) g)\, dx\,. \end{aligned} \tag{17}$$

In [8] and [9] we considered only the Fourier expansion at the cusp ∞. For the relation with Hecke operators we need to consider in this paper Fourier expansions at other cusps as well.

We call an (Γ,χ)-automorphic function *cuspidal* is all its Fourier terms of order zero vanish. Since $0 \in \tilde{M}'_\kappa$ only if $\kappa \in \mathcal{P}_\chi$, the function f is cuspidal if $F_{\kappa,0} f = 0$ for all $\kappa \in \mathcal{P}_\chi$.

2.4. Automorphic representations. Let $L^2(\Gamma_0(I)\backslash G,\chi)$ denote the Hilbert space of classes of functions transforming according to χ i.e. $f(\gamma g) = \chi(\gamma) f(g)$ for any $\gamma \in \Gamma_0(I)$ and $g \in G$. The group G acts unitarily on this Hilbert space by right translation. This space is split up according to central characters, indicated by $\xi \in \{0,1\}^d$. By $L^2_\xi(\Gamma_0(I)\backslash G,\chi)$ we mean the subspace on which the center acts by

$$\left(\begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_1 \end{pmatrix}, \ldots, \begin{pmatrix} \zeta_d & 0 \\ 0 & \zeta_d \end{pmatrix} \right) \mapsto \prod_j \zeta_j^{\xi_j},$$

where $\zeta_j \in \{1,-1\}$. This subspace can be non-zero only if the following compatibility condition holds:

$$\chi(-1) = \prod_j (-1)^{\xi_j}. \tag{18}$$

We assume this throughout this paper.

There is an orthogonal decomposition

$$L^2_\xi(\Gamma_0(I)\backslash G,\chi) = L^{2,\mathrm{cont}}_\xi(\Gamma_0(I)\backslash G,\chi) \oplus L^{2,\mathrm{discr}}_\xi(\Gamma_0(I)\backslash G,\chi). \tag{19}$$

The G-invariant subspace $L^{2,\mathrm{cont}}_\xi(\Gamma_0(I)\backslash G,\chi)$ can be described by integrals of Eisenstein series and the orthogonal complement $L^{2,\mathrm{discr}}_\xi(\Gamma_0(I)\backslash G,\chi)$ is a direct sum of closed irreducible G-invariant subspaces. If $\chi = 1$, the constant functions form an invariant subspace. All other irreducible invariant subspaces have infinite dimension. They are cuspidal and span the space $L^{2,\mathrm{cusp}}_\xi(\Gamma_0(I)\backslash G,\chi)$, the orthogonal complement of the constant functions in $L^{2,\mathrm{discr}}_\xi(\Gamma_0(I)\backslash G,\chi)$.

We fix a maximal orthogonal system $\{V_\varpi\}_\varpi$ of irreducible subspaces in the Hilbert space $L^{2,\mathrm{cusp}}_\xi(\Gamma_0(I)\backslash G,\chi)$. This system is unique if all ϖ are inequivalent. In general, there might be multiplicities, due to oldforms.

In §3.2 we will discuss Hecke operators $T(\mathfrak{a}^2)$, which act in $L^2_\xi(\Gamma_0(I)\backslash G,\chi)$ as symmetric bounded operators. These operators commute with the Casimir operators of the factors $\mathrm{SL}_2(\mathbb{R})$ of G, and leave invariant the space $L^{2,\mathrm{cusp}}_\xi(\Gamma_0(I)\backslash G,\chi)$. (See §4.2.) Then we can and will assume that the V_ϖ are invariant under these Hecke operators $T(\mathfrak{a}^2)$.

Each irreducible automorphic representation ϖ of $G = \prod_j \mathrm{SL}_2(\mathbb{R})$ is the tensor product $\otimes_j \varpi_j$ of irreducible representations of $\mathrm{SL}_2(\mathbb{R})$. Here and in the sequel, j is supposed to run over the d archimedean places of F.

The factor ϖ_j can (almost) be characterized by the eigenvalue $\lambda_{\varpi,j}$ of the Casimir operator of $\mathrm{SL}_2(\mathbb{R})$, and the central character, which is indicated by ξ_j.

The eigenvalues $\lambda_{\varpi,j}$ are known to be in the following subsets of $\mathbb{R}$ depending on the central character ξ_j:

$$\begin{aligned} &\left\{\tfrac{b}{2}-\tfrac{b^2}{4} \;:\; b\geq 2 \text{ even}\right\} \,\cup\, \left[\lambda_0,\infty\right) && \text{if } \xi_j = 0\,,\\ &\left\{\tfrac{b}{2}-\tfrac{b^2}{4} \;:\; b\geq 3 \text{ odd}\right\} \,\cup\, \left[\tfrac14,\infty\right) && \text{if } \xi_j = 1\,, \end{aligned} \tag{20}$$

where $\lambda_0 \in \left(0,\frac14\right]$. By a conjecture due to Selberg in the spherical case, it is expected that one can take $\lambda_0 = \frac14$. The result $\frac14 \geq \lambda_0 \geq \frac14 - \left(\frac19\right)^2$, due to Kim-Shahidi and Kim-Shahidi-Sarnak, [11], has recently been improved to $\lambda_0 \geq \frac14 - \left(\frac{7}{64}\right)^2$ by Blomer and Brumley, [2]. We call $\lambda_\varpi = (\lambda_{\varpi,j}) \in \mathbb{R}^d$ the *eigenvalue vector* of the representation ϖ. As discussed in §1.1 of [9], spectral theory shows that the set of eigenvalue vectors $\{\lambda_\varpi\}$ is discrete in $\mathbb{R}^d$.

The correspondence between values of $\lambda = \lambda_{\varpi,j}$ and equivalence classes of unitary representations of $\mathrm{SL}_2(\mathbb{R})$ of infinite dimension is one-to-one if $\lambda > 0$ for $\xi = \xi_j = 0$, and if $\lambda > \frac14$ if $\xi = 1$. In the other cases, $\lambda = \frac{b}{2} - \frac{b^2}{4}$ with $b \in \mathbb{Z}_{\geq 1}$, $b \equiv \xi \bmod 2$. In these cases, there are two equivalence classes, one with lowest weight b (holomorphic type), and one with highest weight $-b$ (antiholomorphic type). If $b \geq 2$, representations of these classes occur discretely in $L^2\big(\mathrm{SL}_2(\mathbb{R})\big)$, and are called *discrete series representations.* The representations in the case $b = 1$ are sometimes called *mock discrete series.* They do not occur discretely in $L^2\big(\mathrm{SL}_2(\mathbb{R})\big)$. All these representations, discrete series or not, may occur as an irreducible component of $L^{2,\mathrm{cusp}}_\xi(\Gamma_0(I)\backslash G, \chi)$.

2.5. Fourier coefficients. In the version in [9] of the asymptotic formula (1) the factor $\mathrm{Vol}\,(\Gamma\backslash G)$ is not present, due to a different choice of the norm in the Hilbert space $L^2_\xi(\Gamma\backslash G, \chi)$. There we used $\|f\|^2 = \int_{\Gamma\backslash G} |f|^2\, dg$, with the Haar measure dg indicated in §2.1 of [8]. Here we prefer to use the norm

$$\|f\| = \Big(\frac{1}{\mathrm{Vol}\,(\Gamma\backslash G)} \int_{\Gamma\backslash G} |f|^2\, dg\Big)^{1/2}. \tag{21}$$

With this choice the norm $\|f\|$ of $f \in L^2_\xi(\Gamma\backslash G, \chi)$ does not change if we consider f as an element of $L^2_\xi(\Gamma_1\backslash G, \chi)$ for a subgroup Γ_1 of finite index in Γ.

As discussed in §2.3.4 in [8], the Fourier expansion of one automorphic form in V_ϖ determines the Fourier expansion of any automorphic form in V_ϖ. This is valid at all cusps, not only at the cusp ∞ considered in [8] and [9]. We review the choice of the Fourier coefficients of an automorphic representation V_ϖ, indicating the differences in comparison with [8]. We put references to formulas and sections in [8] in italics.

Like in *(2.27)*, we determine the Fourier coefficients $c^{\kappa,r}(\varpi)$, for $\kappa \in \mathcal{P}$ and $r \in \tilde{M}'_\kappa$, by

$$F_{\kappa,r}\psi_{\varpi,q} \;=\; c^{\kappa,r}(\varpi)\, d^r(q,\nu_\varpi)\, W_q(r,\nu_\varpi)\,. \tag{22}$$

Here $d^r(q,\nu_\varpi)$ is the factor with exponentials and gamma functions in *(2.28)*, with a choice of $\nu_\varpi = (\nu_{\varpi,j})_{1\le j\le d} \in \big(i[0,\infty) \cup (0,\infty)\big)^d$ such that $\lambda_{\varpi,j} = \frac14 - \nu^2_{\varpi,j}$. The "Whittaker function" $W_q(r,\nu_\varpi)$ on G is a standard function on G, given in *(2.12)*, with weight q and the right transformation behavior for translations by elements of N on the left. The weight q in (22) runs over all weights that occur in V_ϖ. The $\psi_{\varpi,q}$ form a basis of the space of K-finite vectors in V_ϖ. They are related by the action of the Lie algebra of G and have norms as indicated in *(2.26)*. Then this determines the Fourier coefficients $c^{\kappa,r}(\varpi)$, independent of the weight q. The difference with [8] is the choice of the norm in (21), which can be summarized as follows:

$$\begin{aligned}
&f \in L^2_\xi(\Gamma\backslash G,\chi): && \|f\|_{\mathrm{here}} &&= \frac{1}{\sqrt{\mathrm{Vol}\,(\Gamma\backslash G)}}\,\|f\|_{\mathrm{there}}\,,\\
&\psi_{\varpi,q} \text{ in (2.26) of [8]:} && \psi^{\mathrm{here}}_{\varpi,q} &&= \sqrt{\mathrm{Vol}\,(\Gamma\backslash G)}\,\psi^{\mathrm{there}}_{\varpi,q}\,,\\
&c^r(\varpi) \text{ in (2.27) of [8]:} && c^{\infty,r}_{\mathrm{here}}(\varpi) &&= \sqrt{\mathrm{Vol}\,(\Gamma\backslash G)}\,c^r_{\mathrm{there}}(\varpi)\,.
\end{aligned}$$

As a consequence, we use now Fourier coefficients $c^r(\varpi) = c^{\infty,r}(\varpi)$ that are equal to $\sqrt{\mathrm{Vol}\,(\Gamma\backslash G)}$ times the corresponding coefficients in [8] and [9]. This causes the factor $\mathrm{Vol}\,(\Gamma\backslash G)$ in (1), which stayed hidden in the normalization used in [9]. The wish to make the dependence on the ideal I in $\Gamma = \Gamma_0(I)$ explicit is our motivation to go over to the norm in (21).

In the choice of the $c^{\kappa,r}(\varpi)$ there is for each ϖ the freedom of a complex factor with absolute value one. In the result we work with absolute values, so these choices do not influence the results of this paper.

In the case $F = \mathbb{Q}$ and $\nu_\varpi = \nu \in i(0,\infty)$ the function $u : z = x+iy \mapsto \psi_{\varpi,0}\begin{pmatrix}\sqrt{y} & \frac{x}{\sqrt{y}}\\ 0 & \frac{1}{\sqrt{y}}\end{pmatrix}$ is a Maass cusp form of weight zero. Each cusp κ of $\Gamma_0(N)$ can be described as $\kappa = g_\kappa \infty$ with now $g_\kappa \in \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{Q})$, such that the subgroup of $\Gamma_0(N)$ fixing κ is generated by $g_\kappa \left(\begin{smallmatrix}1 & w\\ 0 & 1\end{smallmatrix}\right) g_\kappa^{-1}$ for some $w \in \mathbb{N}$. Then $M_\kappa = w\mathbb{Z}$, and $M'_\kappa = \frac{1}{w}\mathbb{Z}$. There is $\alpha_\kappa \in \mathbb{Q}$ depending on χ such that $\tilde{M}'_\kappa = \alpha_\kappa + \frac{1}{w}\mathbb{Z}$. The Maass form u has at the cusp κ the Fourier expansion

$$u\big(g_\kappa z\big) \;=\; \sum_{r \equiv \alpha_\kappa \bmod 1/w} \frac{c^{\kappa,r}(\varpi)}{\sqrt{2|r|}\,\Gamma(\frac{1}{2}+\nu)}\, e^{2\pi i r x}\, W_{0,\nu}(4\pi|r|y)\,,$$

where $W_{k,m}$ is a Whittaker function. (If $\alpha_\kappa \in \frac{1}{w}\mathbb{Z}$ the term with $r = 0$ should be omitted.) If we apply to u the Maass weight raising and weight lowering operators we get Maass forms of even weights q that are explicit multiples of $\psi_{\varpi,q}$, with Fourier expansions containing $W_{q\,\mathrm{Sign}\,(r)/2,\nu}(4\pi|r|y)$. The normalizing factors $d^r(q,\nu)$ are such that the same coefficients $c^{\kappa,r}(\varpi)$ occur in all these Fourier expansions.

2.6. Automorphic forms on the adele group. We now consider the embedding of $\Gamma = \Gamma_0(I) \subset \mathrm{SL}_2(F)$ in $\mathrm{SL}_2(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the group of finite F-adeles.

We recall that $\mathbb{A}_f$ is the subset of $\prod_{\mathfrak{p}} F_{\mathfrak{p}}$ of those $a = (a_{\mathfrak{p}})_{\mathfrak{p}} \in \prod_{\mathfrak{p}} F_{\mathfrak{p}}$ such that $a_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$ for all but finitely many primes $\mathfrak{p}$ in $\mathcal{O}$. By $\mathcal{O}_{\mathfrak{p}}$ we denote the closure of $\mathcal{O}$ in the completion $F_{\mathfrak{p}}$ of F at place $\mathfrak{p}$. The ring $\mathbb{A}_f$ contains the subring $\bar{\mathcal{O}} = \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}$, which is the completion of $\mathcal{O}$ for the topology in which the system of neighborhoods of 0 is generated by the non-zero ideals in $\mathcal{O}$.

The local ring $\mathcal{O}_{\mathfrak{p}}$ has maximal ideal $\bar{\mathfrak{p}}$, which is the closure of $\mathfrak{p} \subset \mathcal{O}$ in $\mathcal{O}_{\mathfrak{p}}$. The closure $\bar{I}_{\mathfrak{p}}$ of I in $\mathcal{O}_{\mathfrak{p}}$ is equal to $\mathcal{O}_{\mathfrak{p}}$ if $\mathfrak{p} \nmid I$, and is a nonzero ideal contained in $\bar{\mathfrak{p}}$ if $\mathfrak{p} \mid I$.

For each $c \in \mathrm{SL}_2(F)$ the group $\Gamma \cap c\Gamma c^{-1}$ has finite index in $\Gamma = \Gamma_0(I)$. This system of subgroups is cofinal with the system of all congruence subgroups of Γ. The closure $\bar{\Gamma}$ of Γ in $\mathrm{SL}_2(\mathbb{A}_f)$ is the completion of Γ in the topology for which the system of neighborhoods of the unit element is generated by the congruence subgroups. There is a decomposition as a direct product

$$\bar{\Gamma} = \prod_{\mathfrak{p}} \bar{\Gamma}_{\mathfrak{p}}, \qquad \text{where } \bar{\Gamma}_{\mathfrak{p}} = \begin{cases} \mathrm{SL}_2(\mathcal{O}_{\mathfrak{p}}) & \text{if } \mathfrak{p} \nmid I,\\ \left\{ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_{\mathfrak{p}}) \;:\; c \in \bar{I}_{\mathfrak{p}} \right\} & \text{if } \mathfrak{p} \mid I. \end{cases}$$

The isomorphism $(\mathcal{O}/I)^* \cong \prod_{\mathfrak{p}|I} (\mathcal{O}/\mathfrak{p}^{v_{\mathfrak{p}}(I)})^*$ implies that the character χ mod I can be written uniquely as a product $\chi(x) = \prod_{\mathfrak{p}|I} \chi_{\mathfrak{p}}(x)$ for $x \in \mathcal{O}$ relatively prime to I, where $\chi_{\mathfrak{p}}$ is a character of $(\mathcal{O}/\mathfrak{p}^{v_{\mathfrak{p}}(I)})^* \cong (\bar{\mathcal{O}}_{\mathfrak{p}}/\bar{I}_{\mathfrak{p}})^*$. In particular, $\chi_{\mathfrak{p}}$ induces a character of $\mathcal{O}_{\mathfrak{p}}^*$.

We now define a character $\hat{\chi}$ on the subgroup

$$\{u \in \mathbb{A}_f^* : u_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^*, \text{ if } \mathfrak{p} \mid I, \text{ and } u_{\mathfrak{p}} \in F_{\mathfrak{p}}^* \text{ if } \mathfrak{p} \nmid I\},$$

by setting $\hat{\chi}(u) = \prod_{\mathfrak{p}|I} \chi_{\mathfrak{p}}(u_{\mathfrak{p}})$. If $x \in \mathcal{O}$ is relatively prime to I then, for the diagonal embedding $\mathcal{O} \subset \mathbb{A}_f^*$, we have $\chi(x) = \hat{\chi}(x)$. Thus we have extended the character χ of $(\mathcal{O}/I)^*$.

Recall that we use the symbol χ also to denote the character $\chi \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) = \chi(d)$ of Γ. We may extend this character χ to the subgroup $\{g \in \mathrm{SL}_2(\mathbb{A}_f) : g_{\mathfrak{p}} \in \bar{\Gamma}_{\mathfrak{p}}, \text{ for } \mathfrak{p} \mid I\}$ of $\mathrm{SL}_2(\mathbb{A}_f)$ by

$$\hat{\chi}\begin{pmatrix} a & b\\ c & d\end{pmatrix} \;=\; \prod_{\mathfrak{p}|I} \chi_{\mathfrak{p}}(d) \;=\; \hat{\chi}(d)\,. \tag{23}$$

This character is trivial on $\mathrm{SL}_2(F_\mathfrak{p})$ for all $\mathfrak{p} \nmid I$. For all $\mathfrak{p} \mid I$, we denote by $\hat\chi_\mathfrak{p}$ the restriction of $\hat\chi$ to $\bar\Gamma_\mathfrak{p}$.

The adele ring $\mathbb{A}$ of F is the direct product $\mathbb{A}_\infty \times \mathbb{A}_f$, with $\mathbb{A}_\infty = \mathbb{R}^d$. So $G = \mathrm{SL}_2(\mathbb{R})^d = \mathrm{SL}_2(\mathbb{A}_\infty)$. The group $\mathrm{SL}_2(F)$ can be viewed as embedded discretely in $\mathrm{SL}_2(\mathbb{A})$. Any (Γ,χ)-automorphic function f on G determines uniquely a corresponding function f_a on $\mathrm{SL}_2(\mathbb{A})$ by

$$f_a(c(g_\infty,u)) \;=\; \hat\chi(u)^{-1} f(g_\infty) \qquad g_\infty \in \mathrm{SL}_2(\mathbb{A}_\infty),\ u \in \bar\Gamma,\ c \in \mathrm{SL}_2(F)\,. \tag{24}$$

Here we use that $\mathrm{SL}_2(\mathbb{A}_f) = \mathrm{SL}_2(F)\bar\Gamma$, a consequence of strong approximation for SL_2. So f_a transforms on the right according to the character $\hat\chi^{-1}$ of $\bar\Gamma$, and is left-invariant under $\mathrm{SL}_2(F)$. Square integrability of f on $\Gamma\backslash G$ is equivalent to square integrability of f_a on $\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})$. At the finite places we normalize the Haar measure on $\mathrm{SL}_2(F_\mathfrak{p})$ so that $\bar\Gamma_\mathfrak{p}$ has volume 1.

3. Hecke algebra. One of the advantages of adele groups is the possibility to view the classical Hecke operators as convolution operators. In this section we study the structure of a ring of Hecke operators, locally in §3.1, globally in §3.2.

For $\Gamma = \Gamma_0(I)$ and the character χ, consider the convolution algebra $\mathcal{H}$ of compactly supported functions ψ on $\mathrm{SL}_2(\mathbb{A}_f)$ satisfying $\psi(u_1 g u_2) = \hat\chi(u_1)^{-1}\,\psi(g)\,\hat\chi(u_2)^{-1}$ for $u_1, u_2, \in \bar\Gamma$ and $g \in \mathrm{SL}_2(\mathbb{A}_f)$. Convolution gives not only a multiplication on $\mathcal{H}$, but also an action of $\mathcal{H}$ on (Γ,χ)-automorphic functions f_a:

$$f_a * \psi(g) \;=\; \int_{\mathrm{SL}_2(\mathbb{A}_f)} f_a(gx^{-1})\,\psi(x)\,dx \;=\; \int_{\mathrm{SL}_2(\mathbb{A}_f)} f_a(g_\infty, x)\,\psi(x^{-1}g_f)\,dx\,, \tag{25}$$

with $g = (g_\infty, g_f) \in \mathrm{SL}_2(\mathbb{A}) = G \times \mathrm{SL}_2(\mathbb{A}_f)$. The support of a non-zero $\psi \in \mathcal{H}$ is a non-empty compact subset of $\mathrm{SL}_2(\mathbb{A}_f)$ that satisfies $\bar\Gamma\,\mathrm{supp}\,(\psi)\,\bar\Gamma = \mathrm{supp}\,(\psi)$. By compactness, the set $\bar\Gamma\backslash\mathrm{supp}\,(\psi)$ is finite. Strong approximation allows us to pick representatives of $\bar\Gamma\backslash\mathrm{supp}\,\psi$ in $\mathrm{SL}_2(F)$. We normalize the Haar measure on $\mathrm{SL}_2(\mathbb{A}_f)$ by giving $\bar\Gamma$ volume 1. Then we have for $g_\infty \in G$:

$$\begin{aligned} f_a * \psi(g_\infty, 1) \;&=\; \sum_{\xi \in \bar\Gamma\backslash\mathrm{supp}\,\psi} f_a(g_\infty, \xi^{-1})\,\psi(\xi) \\ &=\; \sum_{\xi \in \Gamma\backslash(\mathrm{SL}_2(F)\cap\mathrm{supp}\,\psi)} f_a\big(\xi^{-1}(\xi g_\infty, 1)\big)\,\psi(\xi) \\ &=\; \sum_{\xi \in \Gamma\backslash(\mathrm{SL}_2(F)\cap\mathrm{supp}\,\psi)} f(\xi g_\infty)\,\psi(\xi)\,. \end{aligned} \tag{26}$$

Thus we obtain the action of ψ in terms of a finite sum of left translates of the (Γ,χ)-automorphic function f.

If $f_a \in L^2\big(\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})\big)$, then so is $F_\xi : g \mapsto f_a\big(g(1,\xi^{-1})\big)\,\psi(\xi)$ for all $\xi \in \mathrm{SL}_2(\mathbb{A}_f)$, and $\|F_\xi\|_2 = \|f_a\|_2\ |\psi(\xi)|$. Hence the convolution operator $f_a \mapsto f_a * \psi$ defines a bounded operator on the Hilbert space $L^2\big(\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})\big)$ with norm at most

$$\|\psi\|_\infty \cdot \#\big(\bar\Gamma\backslash\mathrm{supp}\,\psi\big)\,. \tag{27}$$

For each $x \in \mathrm{SL}_2(\mathbb{A}_f)$, right translation R_x given by $(R_x f)(g) = f(gx)$ is a unitary operator in $L^2\big(\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})\big)$. We put $\psi^*(x) = \overline{\psi(x^{-1})}$. If ψ as above satisfies $\psi^* = \psi$, then convolution by ψ defines a symmetric bounded operator on $L^2\big(\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})\big)$.

3.1. Local Hecke algebra. Before defining the subalgebra of $\mathcal{H}$ with which we will work, we will consider some pertinent local convolution algebras, for primes outside I. We recall that the character $\hat{\chi}$ is trivial on $\mathrm{SL}_2(F_\mathfrak{p})$ for all $\mathfrak{p} \nmid I$.

Definition 3.1. For each finite prime $\mathfrak{p}$ not dividing I we denote by $\mathcal{H}_\mathfrak{p}$ the convolution algebra of compactly supported locally constant functions on $\mathrm{SL}_2(F_\mathfrak{p})$ that are left and right invariant under translation by elements of $\mathrm{SL}_2(\mathcal{O}_\mathfrak{p})$.

We recall the well known description of the structure of $\mathcal{H}_\mathfrak{p}$. Let $\hat{\pi}_\mathfrak{p}$ be a generator of the maximal ideal $\bar{\mathfrak{p}}$ of $\mathcal{O}_\mathfrak{p}$.

The unit element of $\mathcal{H}_\mathfrak{p}$ is the characteristic function $\varepsilon_\mathfrak{p}$ of $\bar{\Gamma}_\mathfrak{p} = \mathrm{SL}_2(\mathcal{O}_\mathfrak{p})$.

A basis of $\mathcal{H}_\mathfrak{p}$ is given by the characteristic functions $T(\mathfrak{p}^{2k})$ of the sets

$$\Delta(\mathfrak{p}^{2k}) \;=\; \left\{ \hat{\pi}_\mathfrak{p}^{-k} g \;:\; g \in M_2(\mathcal{O}_\mathfrak{p}),\, \det g \;=\; \hat{\pi}_\mathfrak{p}^{2k} \right\}, \tag{28}$$

for $k \in \mathbb{N}_0$. So $\Delta(\mathfrak{p}^0) = \varepsilon_\mathfrak{p}$. We have

$$\begin{aligned} \Delta(\mathfrak{p}^{2k}) \;&=\; \bigsqcup_{l=0}^{2k} \bigsqcup_{b \in \mathcal{O}_\mathfrak{p}/\bar{\mathfrak{p}}^l} \bar{\Gamma}_\mathfrak{p} \begin{pmatrix} \hat{\pi}_\mathfrak{p}^{k-l} & b\hat{\pi}_\mathfrak{p}^{-k} \\ 0 & \hat{\pi}_\mathfrak{p}^{l-k} \end{pmatrix} \\ &=\; \bigsqcup_{l=0}^{2k} \bigsqcup_{\beta \in \bar{\mathfrak{p}}^{l-2k}/\bar{\mathfrak{p}}^{2l-2k}} \bar{\Gamma}_\mathfrak{p} h\big(\hat{\pi}_\mathfrak{p}^{k-l}\big) n(\beta), \end{aligned} \tag{29}$$

in the notations of §2.1. For $k \geq 1$ we have the relation:

$$T(\mathfrak{p}^{2k}) * T(\mathfrak{p}^2) \;=\; T(\mathfrak{p}^{2k+2}) + \mathrm{N}(\mathfrak{p})\, T(\mathfrak{p}^{2k}) + \mathrm{N}(\mathfrak{p})^2\, T(\mathfrak{p}^{2k-2}). \tag{30}$$

This implies that $\mathcal{H}_\mathfrak{p}$ is a polynomial ring in the variable $T(\mathfrak{p}^2)$. Furthermore, (30) allows us to check that $\mathcal{H}_\mathfrak{p}$ is isomorphic to the subring of the ring $\mathbb{C}[X_\mathfrak{p}^2, X_\mathfrak{p}^{-2}]$ of even Laurent polynomials that are invariant under $X_\mathfrak{p}^2 \mapsto X_\mathfrak{p}^{-2}$, by sending $T(\mathfrak{p}^{2k})$ to

$$\mathrm{N}(\mathfrak{p})^k \sum_{j=0}^{2k} X_\mathfrak{p}^{2k-2j}. \tag{31}$$

Each character of $\mathcal{H}_\mathfrak{p}$ corresponds to a map $X_\mathfrak{p} \mapsto \mathrm{N}(\mathfrak{p})^{\nu_\mathfrak{p}}$, where the parameter

$$\nu_\mathfrak{p} \in \mathbb{C} \bmod \frac{2\pi i}{\log \mathrm{N}(\mathfrak{p})}\mathbb{Z}, \tag{32}$$

is determined up to $\nu_\mathfrak{p} \leftrightarrow -\nu_\mathfrak{p}$. The quantity $\nu_\mathfrak{p}$ is related (but not equal) to the usual Satake parameter.

Thus we have described the structure of $\mathcal{H}_\mathfrak{p}$ completely for primes $\mathfrak{p} \nmid I$. For primes $\mathfrak{p}$ dividing I the algebra $\mathcal{H}_\mathfrak{p}$ is the convolution algebra of compactly supported functions on $\mathrm{SL}_2(F_\mathfrak{p})$ that transform on the left and on the right by the character $\hat{\chi}_\mathfrak{p}^{-1}$ of $\bar{\Gamma}_\mathfrak{p}$. We do not go into its structure, and only note that its unit element $\varepsilon_\mathfrak{p}$ is equal to $\hat{\chi}_\mathfrak{p}^{-1}$ on $\bar{\Gamma}_\mathfrak{p}$ and is zero outside $\bar{\Gamma}_\mathfrak{p}$.

3.2. Global Hecke algebra. We shall work with the following subring of $\mathcal{H}$:

Definition 3.2. We denote by $\mathcal{H}^I$ the convolution subalgebra of $\mathcal{H}$ of compactly supported bi-$\hat{\chi}^{-1}$-equivariant functions on $\mathrm{SL}_2(\mathbb{A}_f)$ spanned by

$$\bigotimes_{\mathfrak{p} \nmid I} \psi_\mathfrak{p} \;\otimes\; \bigotimes_{\mathfrak{p} | I} \varepsilon_\mathfrak{p},$$

where $\psi_\mathfrak{p} \in \mathcal{H}_\mathfrak{p}$ for all $\mathfrak{p} \nmid I$, and $\psi_\mathfrak{p} = \varepsilon_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$.

We denote by $\otimes_\mathfrak{p}\psi_\mathfrak{p}$ the function $g \mapsto \prod_\mathfrak{p} \psi_\mathfrak{p}(g_\mathfrak{p})$ on the group $\mathrm{SL}_2(\mathbb{A}_f)$. The condition that $\psi_\mathfrak{p} = \varepsilon_\mathfrak{p}$ for almost all $\mathfrak{p}$ ensures that $\otimes_\mathfrak{p}\psi_\mathfrak{p}$ makes sense. The algebra $\mathcal{H}^I$ is isomorphic to the (restricted) tensor product of the algebras $\mathcal{H}_\mathfrak{p}$ with $\mathfrak{p} \nmid I$. It is a commutative algebra. In this way we restrict our attention to the places where the local Hecke algebra is unramified.

Next we build elements of $\mathcal{H}$ from elements of $\mathcal{H}_\mathfrak{p}$ with $\mathfrak{p} \nmid I$. Since the character $\hat\chi$ may be non-trivial this requires some care. We also describe the action of elements of $\mathcal{H}$ on automorphic forms for $\Gamma_0(I)$.

We consider a nonzero ideal $\mathfrak{a}$ in $\mathcal{O}$ prime to I. It has the form $\mathfrak{a} = \prod_{\mathfrak{p}\in P} \mathfrak{p}^{k_\mathfrak{p}}$, with P a finite set of prime ideals not dividing I, and positive $k_\mathfrak{p}$. We define

$$T(\mathfrak{a}^2) \;=\; \bigotimes_{\mathfrak{p}\in P} T(\mathfrak{p}^{2k_\mathfrak{p}}) \otimes \bigotimes_{\mathfrak{p}\notin P} \varepsilon_\mathfrak{p}\,. \tag{33}$$

Thus defined, convolution by $T(\mathfrak{a}^2)$ satisfies $T(\mathfrak{a}^2)^* \;=\; T(\mathfrak{a}^2)$.

With $\mathfrak{a}$ and P as above, we consider the action of $T(\mathfrak{a}^2)$ on automorphic forms, first under the assumption that all $\mathfrak{p} \in P$ are principal ideals in $\mathcal{O}$.

So $\mathfrak{p} = (\pi_\mathfrak{p})$. Then $\pi_\mathfrak{p} \in \hat\pi_\mathfrak{p}\mathcal{O}_\mathfrak{p}^*$, and $\pi_\mathfrak{p} \in \mathcal{O}_\mathfrak{q}^*$ at all other finite places $\mathfrak{q}$. The element $\hat\pi_\mathfrak{p}$ embedded in $\mathbb{A}_f^*$ by taking 1 at all places not equal to $\mathfrak{p}$, satisfies $\hat\chi\big(\hat\pi_\mathfrak{p}\big) = 1$. However, the representative $\pi_\mathfrak{p} \in F^*$ need not be in the kernel of χ, so we may have $\hat\chi(\pi_\mathfrak{p}) \neq 0$.

The Iwasawa decomposition of $\mathrm{SL}_2(F_\mathfrak{p})$ shows that we can choose representatives $\hat x$ of $\bar\Gamma\backslash\operatorname{supp} T(\mathfrak{a}^2)$ of the form $\hat x = \begin{pmatrix} q & 0 \\ 0 & 1/q\end{pmatrix}\begin{pmatrix} 1 & \hat\beta \\ 0 & 1\end{pmatrix} \in \mathrm{SL}_2(\mathbb{A}_f)$, with

$$\begin{aligned} q_\mathfrak{p} \;&=\; \hat\pi_\mathfrak{p}^{k_\mathfrak{p}-l_\mathfrak{p}}\,, \quad \hat\beta_\mathfrak{p} \in \bar{\mathfrak{p}}^{l_\mathfrak{p}-2k_\mathfrak{p}} \qquad &&\text{if } \mathfrak{p}\in P\,,\\ q_\mathfrak{q} \;&=\; 1\,, \qquad\quad \hat\beta_\mathfrak{q} \;=\; 0 &&\text{if } \mathfrak{q}\notin P\,, \end{aligned} \tag{34}$$

where each $l_\mathfrak{p}$ runs from 0 to $2k_\mathfrak{p}$, and where $\hat\beta_\mathfrak{p}$ runs through a system of representatives of $\bar{\mathfrak{p}}^{l_\mathfrak{p}-2k_\mathfrak{p}}/\bar{\mathfrak{p}}^{2l_\mathfrak{p}-2k_\mathfrak{p}}$. The representatives $\hat x$ need not be in $\mathrm{SL}_2(F)$. We take

$$x \;=\; \begin{pmatrix} q_\mathfrak{b} & 0 \\ 0 & 1/q_\mathfrak{b}\end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1\end{pmatrix} \in \mathrm{SL}_2(F)$$

such that $x \in \bar\Gamma\hat x$ with $\mathfrak{b} = \prod_{\mathfrak{p}\in P}\mathfrak{p}^{l_\mathfrak{p}}$, $q_\mathfrak{b} = \prod_{\mathfrak{p}\in P}\pi_\mathfrak{p}^{k_\mathfrak{p}-l_\mathfrak{p}}$, and β in a system of representatives of $\mathfrak{b}\,\mathfrak{a}^{-2}/\mathfrak{b}^2\,\mathfrak{a}^{-2}$. In this way for $g \in \mathrm{SL}_2(\mathbb{A})$

$$\big(f_a * T(\mathfrak{a}^2)\big)(g) \;=\; \sum_{\mathfrak{b}|\mathfrak{a}^2}\;\sum_{\beta\in\mathfrak{b}\mathfrak{a}^{-2}/\mathfrak{b}^2\mathfrak{a}^{-2}} \hat\chi(q_\mathfrak{b}) f_a\left(g\left(1, n(\beta)^{-1}h(q_\mathfrak{b})^{-1}\right)\right)\,, \tag{35}$$

which becomes in terms of (Γ,χ)-automorphic functions on $\Gamma\backslash G$:

$$\big(f|T(\mathfrak{a}^2)\big)(g_\infty) \;=\; \sum_{\mathfrak{b}|\mathfrak{a}^2}\;\sum_{\beta\in\mathfrak{b}\mathfrak{a}^{-2}/\mathfrak{b}^2\mathfrak{a}^{-2}} \hat\chi(q_\mathfrak{b}) f\left(h(q_\mathfrak{b})n(\beta)g_\infty\right) \qquad (g_\infty \in G)\,. \tag{36}$$

If we can choose $\pi_\mathfrak{p}$ totally positive for all $\mathfrak{p} \in P$, a description of $T(\mathfrak{a})$ similar to the latter formula is possible for all ideals $\mathfrak{a}$ built with prime ideals in P.

In general, not all prime ideals $\mathfrak{p} \in P$ are principal, so we proceed as follows. Taking $\beta \in \mathfrak{b}\mathfrak{a}^{-2} \subset F$, $\beta \in \hat{\beta} + \mathfrak{b}^2\mathfrak{a}^{-2}$, we have in the notations of (34):

$$h(q)n(\hat{\beta}) \in \bar{\Gamma} h(q)n(\beta)\,.$$

By strong approximation there exists $g_{\mathfrak{b}} \in \mathrm{SL}_2(F)$ such that

$$h(q)n(\hat{\beta}) \in \bar{\Gamma} g_{\mathfrak{b}} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}. \tag{37}$$

Indeed, for $\mathfrak{b}$ such that $\mathfrak{a}\,\mathfrak{b}^{-1}$ is principal, we may take $g_{\mathfrak{b}}$ to be a diagonal matrix, in particular, $g_{\mathcal{O}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For other $\mathfrak{b}$ we use that in any field we have

$$\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (t \neq 0)\,.$$

For any ideal J there are elements $\xi, \eta \in F$ and $u, v \in \bar{J}$, such that $q = \xi + u$, $q^{-1} = \eta + v$ for q as in (34). If we take J as the product of sufficiently high (depending on q) powers of primes in P and I, we have

$$h(q) = \begin{pmatrix} 1 & \xi + u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\eta - v & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi + u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \bar{\Gamma} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\eta & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that $h(q) \in \bar{\Gamma} \cdot \mathrm{SL}_2(F)$. This leads to the following description of convolution by $T(\mathfrak{a}^2)$:

$$\big(f_a * T(\mathfrak{a}^2)\big)(g) = \sum_{\mathfrak{b}|\mathfrak{a}^2} \hat{\chi}(g_{\mathfrak{b}})^{-1} \sum_{\beta \in \mathfrak{b}\mathfrak{a}^{-2}/\mathfrak{b}^2\mathfrak{a}^{-2}} f_a\left(g, \left(1, n(\beta)^{-1} g_{\mathfrak{b}}^{-1}\right)\right), \tag{38}$$

for any $g \in \mathrm{SL}_2(\mathbb{A})$. Since the $g_{\mathfrak{b}} n(\beta)$ in this sum are elements of $\mathrm{SL}_2(F)$, we can go over to the corresponding function on G, and obtain:

Proposition 3.3. *Let $\mathfrak{a}$ be a non-zero integral ideal in $\mathcal{O}$ prime to I. For each integral ideal $\mathfrak{b}$ dividing $\mathfrak{a}$ there exists $g_{\mathfrak{b}} \in \mathrm{SL}_2(F)$ such that for any (Γ, χ)-automorphic function f on G, and for any $g \in G$:*

$$\big(f | T(\mathfrak{a}^2)\big)(g) = \sum_{\mathfrak{b}|\mathfrak{a}^2} \hat{\chi}(g_{\mathfrak{b}})^{-1} \sum_{\beta \in \mathfrak{b}\mathfrak{a}^{-2}/\mathfrak{b}^2\mathfrak{a}^{-2}} f\left(g_{\mathfrak{b}} n(\beta) g\right). \tag{39}$$

In the sum, β runs over elements of $\mathfrak{b}\mathfrak{a}^{-2} \subset F$ representing the classes modulo $\mathfrak{b}^2\mathfrak{a}^2$. The elements $g_{\mathfrak{b}}$ are in $\mathrm{SL}_2(F)$. For $\mathfrak{b} = \mathcal{O}$ we can take $g_{\mathfrak{b}} = \mathrm{Id}$. If all prime ideals dividing $\mathfrak{a}$ are principal we can take all $g_{\mathfrak{b}}$ to be diagonal matrices satisfying (37).

In particular, the Hecke operators act in the space $L^2_\xi(\Gamma\backslash G, \chi)$ as bounded operators. (See (27).) The relation $T(\mathfrak{a}^2)^* = T(\mathfrak{a}^2)$ implies that $T(\mathfrak{a}^2)$ acts as a symmetric bounded operator. Its norm is bounded by the number of terms in (39).

4. Distribution of eigenvalues of Hecke operators. It will be critical for this paper to work out the relation between the eigenvalues of Hecke operators and the Fourier coefficients of automorphic forms. In the first subsection we discuss the action of Hecke operators on Fourier expansions (Propositions 4.3 and 4.4) and then we apply these results to give an expression for the eigenvalues of Hecke operators on automorphic cusp forms (Theorem 4.5). In the final subsection we give a proof of our main result, Theorem 1.1 in several steps. The main tools in the proof are Theorems B.1 and 4.5.

4.1. Hecke operators and Fourier expansion.

LEMMA 4.1. *Let $\mathfrak{a}$ be a non-zero integral ideal in $\mathcal{O}$ relatively prime to I, and let $\mathfrak{b} \mid \mathfrak{a}^2$. Then there is a unique $\kappa \in \mathcal{P}$ such that the element $g_{\mathfrak{b}}$ in Proposition 3.3 has the form $g_{\mathfrak{b}} = \gamma g_\kappa p$ with $\gamma \in \Gamma$ and $p = n(b)\,h(a) \in \mathrm{SL}_2(F)$.*

For each (Γ,χ)-automorphic function f on G the function

$$f_{\mathfrak{b}} : g \mapsto \sum_{\beta \in \mathfrak{b}\mathfrak{a}^{-2}/\mathfrak{b}^2\mathfrak{a}^{-2}} f(g_{\mathfrak{b}}\, n(\beta)\, g) \tag{40}$$

is left-invariant under Γ_{N^∞}. The Fourier terms of $f_{\mathfrak{b}}$ are given by

$$F_{\infty,r} f_{\mathfrak{b}}(g) \;=\; \begin{cases} \mathrm{N}(\mathfrak{b})\, \chi(\gamma)\, F_{\kappa,a^{-2}r} f(pg) & \text{for } r \in \mathfrak{b}^{-1}\mathfrak{a}^2\mathcal{O}'\,, \\ 0 & \text{for other } r \in \mathcal{O}'\,. \end{cases} \tag{41}$$

Proof. The element $g_{\mathfrak{b}} \in \mathrm{SL}_2(F)$ is in $\bar\Gamma\, h(q)$ with q as in (34). Since $g_{\mathfrak{b}}\,\infty$ is a cusp, it is of the form $\gamma\kappa$ with $\gamma \in \Gamma$, for a unique $\kappa \in \mathcal{P}$. Then $g_\kappa^{-1}\gamma^{-1}g_{\mathfrak{b}} \in \mathrm{SL}_2(F)$ fixes ∞, and is hence of the form $p = n(b)h(a)$ with $b \in F$, $a \in F^*$.

We claim that

$$a^2\mathfrak{b}^2\mathfrak{a}^{-2} \;=\; M_\kappa\,. \tag{42}$$

Indeed, the construction of the system of representatives in §3.2 implies that for different β in a system of representatives of $\mathfrak{b}\mathfrak{a}^{-2}/\mathfrak{b}^2\mathfrak{a}^{-2}$ the sets $\Gamma\gamma_{\mathfrak{b}} n(\beta)$ are disjoint, and furthermore $g_{\mathfrak{b}} n(\beta) g_{\mathfrak{b}}^{-1} \in \Gamma$ if $\beta \in \mathfrak{b}^2\mathfrak{a}^{-2}$. Hence, for all $\beta \in \mathfrak{b}^2\mathfrak{a}^{-2}$ we have $g_\kappa p n(\beta) p^{-1} g_\kappa^{-1} \in \Gamma$, which implies that $n(a^2\beta) \in g_\kappa^{-1}\Gamma_{N^\kappa} g_\kappa$ for all such β. This shows that $a^2\mathfrak{b}^2\mathfrak{a}^{-2} \subset M_\kappa$.

Conversely, if $x \in M_\kappa$, then $g_\kappa n(x) g_\kappa^{-1} \in \Gamma$, and $g_{\mathfrak{b}} n(a^{-2}x) g_{\mathfrak{b}}^{-1} \in \gamma^{-1}\Gamma\gamma = \Gamma$. We have $g_{\mathfrak{b}} = uh(q)$, with $u \in \bar\Gamma$. Hence $h(q)n(a^{-2}x)h(q)^{-1} \in \bar\Gamma$, and $q^2a^{-2}x \in \bar{\mathcal{O}}$. Since $q\bar{\mathcal{O}} = \bar{\mathfrak{a}}\hat{\mathfrak{b}}^{-1}$, we have $x \in a^2\hat{\mathfrak{b}}^2\bar{\mathfrak{a}}^{-2} \cap F = a^2\mathfrak{b}^2\mathfrak{a}^{-2}$.

Moreover, for $\beta \in \mathfrak{b}^2\mathfrak{a}^{-2}$, we have

$$\chi\left(g_\kappa n(a^2\beta) g_\kappa^{-1}\right) = \hat\chi(g_\kappa p)\hat\chi\big(n(\beta)\hat\chi(g_\kappa p)^{-1} = 1\,.$$

So χ is trivial on Γ_{N^κ} and $\tilde M'_\kappa$ as defined in (16) is equal to $M'_\kappa = a^{-2}\mathfrak{b}^{-2}\mathfrak{a}^2\mathcal{O}'$.

The set $\bigsqcup_{\beta \in \mathfrak{b}\mathfrak{a}^{-2}/\mathfrak{b}^2\mathfrak{a}^{-2}} \Gamma g_{\mathfrak{b}} n(\beta)$ is right-invariant under the group $\{n(\omega) : \omega \in \mathfrak{b}\mathfrak{a}^{-2}\}$, which contains Γ_{N^∞}. Hence $f_{\mathfrak{b}}$ in (40) is left-invariant under Γ_{N^∞}. It has a Fourier expansion with terms of order $r \in \mathcal{O}'$, and the terms with r outside $(\mathfrak{b}\mathfrak{a}^{-2})' = \mathfrak{b}^{-1}\mathfrak{a}^2\mathcal{O}'$ vanish. For $r \in \mathfrak{b}^{-1}\mathfrak{a}^2\mathcal{O}'$:

$$\begin{aligned} F_{\infty,r} f_{\mathfrak{b}}(g) \;&=\; \frac{1}{\mathrm{Vol}\,(\mathbb{R}^d/\mathfrak{b}\mathfrak{a}^{-2})} \int_{\mathbb{R}^d/\mathfrak{b}\mathfrak{a}^{-2}} e^{-2\pi i S(rx)}\, f_{\mathfrak{b}}\,(n(x)g)\; dx \\ &=\; \frac{1}{\mathrm{Vol}\,(\mathbb{R}^d/\mathfrak{b}\mathfrak{a}^{-2})} \int_{\mathbb{R}^d/\mathfrak{b}\mathfrak{a}^{-2}} \sum_{\beta \in \mathfrak{b}\mathfrak{a}^{-2}/\mathfrak{b}^2\mathfrak{a}^{-2}} e^{-2\pi i S\left(r(x+\beta)\right)}\, f\,(\gamma g_\kappa p n(x+\beta)g)\; dx\,, \end{aligned}$$

where we have written $g_{\mathfrak{b}} = \gamma g_\kappa p$, and have used that $S(r\beta) \in \mathbb{Z}$. The function $x \mapsto e^{-2\pi i S(rx)} f\,(\gamma g_\kappa p n(x) g)$ is a function on $\mathbb{R}^d/\mathfrak{b}^2\mathfrak{a}^{-2}$. Hence the integration over $x \in \mathbb{R}^d/\mathfrak{b}\mathfrak{a}^{-2}$ and the summation over $\beta \in \beta\mathfrak{a}^{-2}/\mathfrak{b}^2\mathfrak{a}^{-2}$ is replaced by integration over $x \in \mathbb{R}^d/\mathfrak{b}^2\mathfrak{a}^{-2}$:

$$F_{\infty,r} f_{\mathfrak{b}}(g) \;=\; \frac{1}{\mathrm{Vol}\,(\mathbb{R}^d/\mathfrak{b}\mathfrak{a}^{-2})} \int_{\mathbb{R}^d/\mathfrak{b}^2\mathfrak{a}^{-2}} e^{-2\pi i S(rx)}\, f\,(\gamma g_\kappa p n(x) g)\; dx\,.$$

Using also $p = n(b)h(a)$ and the fact that f is (Γ,χ)-automorphic we obtain

$$\begin{aligned} F_{\infty,r} f_{\mathfrak{b}}(g) &= \frac{\chi(\gamma)}{\mathrm{Vol}\,(\mathbb{R}^d/\mathfrak{b}\mathfrak{a}^{-2})} \int_{\mathbb{R}^d/\mathfrak{b}^2\mathfrak{a}^{-2}} e^{-2\pi i S(rx)}\, f\,(g_\kappa n(b)h(a)n(x)g)\;dx \\ &= \frac{\chi(\gamma)}{\mathrm{Vol}\,(\mathbb{R}^d/\mathfrak{b}\mathfrak{a}^{-2})} \int_{\mathbb{R}^d/\mathfrak{b}^2\mathfrak{a}^{-2}} e^{-2\pi i S(rx)}\, f\,\big(g_\kappa n(a^2x)pg\big)\;dx\,. \end{aligned}$$

Next we carry out the substitution $x \mapsto a^{-2}x$:

$$F_{\infty,r} f_{\mathfrak{b}}(g) = \frac{\chi(\gamma)}{|\mathrm{N}(a)|^2\ \mathrm{Vol}\,(\mathbb{R}^d/\mathfrak{b}\mathfrak{a}^{-2})} \int_{\mathbb{R}^d/a^2\mathfrak{b}^2\mathfrak{a}^{-2}} e^{-2\pi i S(ra^{-2}x)}\, f\,(g_\kappa n(x)pg)\;.$$

We have seen in (42) that $a^2\mathfrak{b}^2\mathfrak{a}^{-2} = M_\kappa$. So we obtain the formula in (41) when we show that

$$\frac{1}{\big|\mathrm{N}(a)\big|^2\,\mathrm{Vol}\,(\mathbb{R}^d/\mathfrak{b}\mathfrak{a}^{-2})} = \frac{\mathrm{N}(\mathfrak{b})}{\mathrm{Vol}\,(\mathbb{R}^2/M_\kappa)}\,.$$

This follows from the facts that $\mathrm{Vol}\,\big(\mathbb{R}^d/\mathfrak{b}\mathfrak{c}\big) = \mathrm{N}(\mathfrak{b})\,\mathrm{Vol}\,\big(\mathbb{R}^d/\mathfrak{c}\big)$ and $\mathrm{Vol}\,\big(\mathbb{R}^d/a\Lambda\big) = |\mathrm{N}(a)|\,\mathrm{Vol}\,\big(\mathbb{R}^d/\Lambda\big)$, for all fractional ideals $\mathfrak{b}$ and $\mathfrak{c}$, all lattices Λ, and all $a \in F^*$. □

LEMMA 4.2. *Suppose that $\kappa = \infty$ in the situation of Lemma 4.1. Then we can choose $g_{\mathfrak{b}} = h(a)$, with $a \in F^*$, and $\gamma = 1$. The Fourier terms of $f_{\mathfrak{b}}$ are given by*

$$F_{\infty,r} f_{\mathfrak{b}}(g) = \begin{cases} \mathrm{N}(\mathfrak{b})\; F_{\infty,a^{-2}r} f\big(h(a)g\big) & \text{if } r \in \mathfrak{b}^{-1}\mathfrak{a}^2\mathcal{O}'\,, \\ 0 & \text{for other } r \in \mathcal{O}'\,. \end{cases} \tag{43}$$

If $\mathfrak{b} = \mathfrak{a}$, then we can further take $a = 1$, while if $\mathfrak{b} \neq \mathfrak{a}$ then $a \in F^* \smallsetminus \mathcal{O}^*$.

Proof. We have $g_{\mathfrak{b}} = \gamma n(b)h(a) \in \bar{\Gamma}h(q)$ with q as in (34). This implies that $a = uq$ with $u \in \bar{\mathcal{O}}^*$, and hence the ideal $q\bar{\mathcal{O}} = \mathfrak{a}\mathfrak{b}^{-1}\bar{\mathcal{O}}$ is generated by $a \in F^*$. So $\mathfrak{a}\mathfrak{b}^{-1}$ is a principal ideal, a is a generator of $\mathfrak{a}\mathfrak{b}^{-1}$, and we can choose $g_{\mathfrak{b}} = h(a)$, $\gamma = 1$. So $p = h(a)$ and the formulas for the Fourier terms follow from (41) in the previous lemma. □

PROPOSITION 4.3. *Let $\mathfrak{a}$ be a non-zero integral ideal in $\mathcal{O}$ relatively prime to I, and let f be a (Γ,χ)-automorphic function on G. For each non-zero integral ideal $\mathfrak{b}$ dividing $\mathfrak{a}$ there exist unique $\kappa(\mathfrak{b}) \in \mathcal{P}$, $p(\mathfrak{b}) = n\big(b(\mathfrak{b})\big)\,h\big(a(\mathfrak{b})\big) \in \mathrm{SL}_2(F)$, and $\gamma(\mathfrak{b}) \in \Gamma$ such that for all $r \in \mathcal{O}'$*

$$\begin{aligned} &F_{\infty,r}\big(f|T(\mathfrak{a}^2)\big)(g) \\ &\quad= \sum_{\mathfrak{b}|\mathfrak{a}^2\,,\ r\in\mathfrak{b}^{-1}\mathfrak{a}^2\mathcal{O}'} \mathrm{N}(\mathfrak{b})\;\hat{\chi}\big(g_{\kappa(\mathfrak{b})}p(\mathfrak{b})\big)^{-1}\;F_{\kappa(\mathfrak{b}),a(\mathfrak{b})^{-2}r} f\big(p(\mathfrak{b})g\big)\,. \end{aligned} \tag{44}$$

If $\kappa(\mathfrak{b}) = \infty$, then either $\mathfrak{b} = \mathfrak{a}$, $a(\mathfrak{b}) = 1$, $b(\mathfrak{b}) = 0$, and $\gamma(\mathfrak{b}) = 1$, or $\mathfrak{b} \neq \mathfrak{a}$, and $a(\mathfrak{b}) \in F^* \smallsetminus \mathcal{O}^*$.

Proof. Replacing the inner sum on the right hand side of (39) in Proposition 3.3 by $f_{\mathfrak{b}}$ in (40) in Lemma 4.1, we obtain

$$\big(f|T(\mathfrak{a}^2)\big) = \sum_{\mathfrak{b}|\mathfrak{a}^2} \hat{\chi}(g_{\mathfrak{b}})^{-1}\; f_{\mathfrak{b}}(g)\,.$$

Applying Lemma 4.1 for each $\mathfrak{b} \mid \mathfrak{a}^2$ we obtain $\kappa(\mathfrak{b})$, $p(\mathfrak{b})$ and $\gamma(\mathfrak{b})$, now explicitly depending on $\mathfrak{b}$. For a given $\mathfrak{b}$ we obtain a non-zero contribution to the Fourier term of order $r \in \mathcal{O}'$ only if $r \in \mathfrak{b}^{-1}\mathfrak{a}^2\mathcal{O}'$ (see (41) in Lemma 4.1), and that contribution is

$$\begin{aligned}\hat{\chi}(g_{\mathfrak{b}})^{-1}\, \mathrm{N}(\mathfrak{b})\, \chi\bigl(\gamma(\mathfrak{b})\bigr)\, F_{\kappa(\mathfrak{b}),a(\mathfrak{b})^{-2}r} f\bigl(p(\mathfrak{b})g\bigr) \\ = \mathrm{N}(\mathfrak{b})\, \hat{\chi}\bigl(g_{\kappa(\mathfrak{b})}p(\mathfrak{b})\bigr)^{-1} F_{\kappa(\mathfrak{b}),a(\mathfrak{b})^{-2}r} f\bigl(p(\mathfrak{b})g\bigr)\,.\end{aligned}$$

The other statements follow from Lemma 4.2. □

We recall that we defined cuspidality, after Proposition 3.3, as the vanishing of all Fourier terms $F_{\kappa,0}$ with $\kappa \in \mathcal{P}_\chi$. So the following proposition tells us that the Hecke operators $T(\mathfrak{a}^2)$ preserve cuspidality.

PROPOSITION 4.4. *Let f be a (Γ,χ)-automorphic continuous function on G for which $F_{\kappa,0}f = 0$ for all $\kappa \in \mathcal{P}_\chi$. Then*

$$F_{\kappa,0}\bigl(f|T(\mathfrak{a}^2)\bigr) = 0 \qquad \text{for all } \kappa \in \mathcal{P}_\chi\,,$$

for each non-zero ideal $\mathfrak{a}$ in $\mathcal{O}$ that is relatively prime to I.

Proof. Let f and $\mathfrak{a}$ be as in the proposition. Consider $\kappa \in \mathcal{P}_\chi$. Then

$$\begin{aligned}F_{\kappa,0}\bigl(f|T(\mathfrak{a}^2)\bigr)(g) &= \sum_{\mathfrak{b}|\mathfrak{a}} \frac{\hat{\chi}(g_{\mathfrak{b}})^{-1}}{\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)} \\ &\quad \cdot \int_{\mathbb{R}^d/M_\kappa} \sum_{\beta \in \mathfrak{b}\mathfrak{a}^{-2}/\mathfrak{b}^2\mathfrak{a}^{-2}} f(g_{\mathfrak{b}} n(\beta) g_\kappa n(x) g)\, dx \\ &= \int_{\mathbb{R}^d/M_\kappa} \sum_j c_j\, f(h_j n(x)\, g)\, dx\,,\end{aligned}$$

where h_j runs over a finite set of elements of $\mathrm{SL}_2(F)$, and $c_j \in \mathbb{C}$. We note that the function $x \mapsto \sum_j c_j f(h_j n(x)\, g)$ on $\mathbb{R}^d$ is M_κ-periodic, but the individual terms may not be M_κ-periodic. Since the finitely many h_j are all in $\mathrm{SL}_2(F)$, there is a fractional ideal Λ_0 with finite index in M_κ such that the individual terms are Λ_0-periodic.

Write $h_j = \gamma_j g_{\kappa_j} n(b_j) h(a_j)$ with $g_j \in \Gamma$, $\kappa_j \in \mathcal{P}$ and $p_j = \begin{pmatrix} a_j & b_j \\ 0 & 1/a_j \end{pmatrix} \in \mathrm{SL}_2(F)$. Then for all fractional ideals Λ in Λ_0 we have

$$\begin{aligned}\int_{\mathbb{R}^d/\Lambda} f\bigl(h_j n(x) g\bigr)\, dx &= \chi(\gamma_j) \int_{\mathbb{R}^d/\Lambda} f\bigl(g_{\kappa_j} n(b_j) h(a_j) n(x) g\bigr)\, dx \\ &= \chi(\gamma_j)\, \bigl|\mathrm{N}(a_j)\bigr| \int_{\mathbb{R}^d/a_j\Lambda} f\bigl(g_{\kappa_j} n(x) h(a_j) g\bigr)\, dx\,.\end{aligned} \tag{45}$$

For each $\kappa' \in \mathcal{P}$ the fractional ideal $M_{\kappa'}$ contains a fractional ideal $\hat{M}_{\kappa'}$ on which χ is trivial. (For $\kappa' \in \mathcal{P}_\chi$ we can take $\hat{M}_{\kappa'} = M_{\kappa'}$.) The assumptions of the proposition imply that for all $g \in G$:

$$\int_{\mathbb{R}^d/\hat{M}_{\kappa'}} f\bigl(g_{\kappa'} n(x) g\bigr)\, dx = 0\,.$$

Taking for $\Lambda \subset \Lambda_0$ a non-zero ideal in $\mathcal{O}$ divisible by all primes that contribute denominators of the a_j, we can arrange that $a_j\Lambda \subset \hat{M}_{\kappa_j}$ for all j. Thus, we conclude that the integral in (45) vanishes. So $F_{\kappa,0}\bigl(f|T(\mathfrak{a}^2)\bigr) = 0$. □

4.2. Action on cusp forms. We turn to the action of the Hecke operators on the cuspidal space $L^{2\text{cusp}}_{\xi}(\Gamma\backslash G,\chi)$, and define the eigenvalues of Hecke operators that occur in Theorem 1.1.

Since all $T(\mathfrak{a}^2)$ act as self-adjoint bounded operators and they all commute with the Casimir operators C_j, $j=1,\dots,d$, we can arrange the orthogonal system $\{V_\varpi\}$ of irreducible cuspidal subspaces so that $V_\varpi|\mathcal{H}\subset V_\varpi$ for each ϖ. Hence we have for each ϖ a character χ_ϖ of $\mathcal{H}$ that gives the eigenvalue of the Hecke operators on the irreducible space V_ϖ. Since the convolution operator determined by $T(\mathfrak{a}^2)$ is symmetric, the value $\chi_\varpi\bigl(T(\mathfrak{a}^2)\bigr)$ is real for all $\mathfrak{a}$ prime to I. By (27), (28) and (33) we have

$$\bigl|\chi_\varpi\bigl(T(\mathfrak{a}^{2k})\bigr)\bigr|\le\prod_{\mathfrak{p}\in P}\#\bigl(\bar\Gamma_{\mathfrak{p}}\backslash\Delta(\mathfrak{p}^{2k_{\mathfrak{p}}})\bigr)\;=\;\prod_{\mathfrak{p}\in P}\sum_{j=0}^{2k}\mathrm{N}(\mathfrak{p})^j\,. \tag{46}$$

In (32) we have assigned a parameter $\nu_{\mathfrak{p}}$ to each character of a local Hecke algebra $\mathcal{H}_{\mathfrak{p}}$. To χ_ϖ corresponds, at the place $\mathfrak{p}$ outside I, a parameter $\nu_{\mathfrak{p}}\in\mathbb{C}$ such that $\lambda_{\varpi,\mathfrak{p}}=\mathrm{N}(\mathfrak{p})^{1/2}\left(\mathrm{N}(\mathfrak{p})^{\nu_{\mathfrak{p}}}+\mathrm{N}(\mathfrak{p})^{-\nu_{\mathfrak{p}}}\right)$. It satisfies

$$-\mathrm{N}(\mathfrak{p})-2-\mathrm{N}(\mathfrak{p})^{-1}\le\mathrm{N}(\mathfrak{p})^{2\nu_{\mathfrak{p}}}+\mathrm{N}(\mathfrak{p})^{-2\nu_{\mathfrak{p}}}\le\mathrm{N}(\mathfrak{p})+\mathrm{N}(\mathfrak{p})^{-1}\,.$$

By (31) we have

$$\chi_\varpi\bigl(T(\mathfrak{p}^{2k})\bigr)=\mathrm{N}(\mathfrak{p})^k\sum_{j=0}^{2k}\mathrm{N}(\mathfrak{p})^{2(k-j)\nu_{\varpi,\mathfrak{p}}}\,. \tag{47}$$

We note that if the operator $T(\mathfrak{p})$ can be defined, the system $\bigl\{V_\varpi\bigr\}$ can be rearranged so that $T(\mathfrak{p})$ acts by $\pm\lambda_{\varpi,\mathfrak{p}}\cdot\mathrm{Id}$ on V_ϖ. In that case $\lambda_{\varpi,\mathfrak{p}}$ is real.

If $\mathfrak{a}=\prod_{\mathfrak{p}\in P}\mathfrak{p}^{k_{\mathfrak{p}}}$ is prime to I then

$$\chi_\varpi\bigl(T(\mathfrak{a}^2)\bigr)\;=\;\prod_{\mathfrak{p}\in P}S_{\mathfrak{p},2k_{\mathfrak{p}}}(\lambda_{\varpi,\mathfrak{p}})\,, \tag{48}$$

where $S_{\mathfrak{p},2k}$ is the only even polynomial of degree $2k$ such that

$$S_{\mathfrak{p},2k}\bigl(\sqrt{\mathrm{N}(\mathfrak{p})}\,(X+X^{-1}\bigr)\;=\;\mathrm{N}(\mathfrak{p})^k\sum_{j=0}^{2k}X^{2(k-j)}\,. \tag{49}$$

Now we are in a position to give the relation between the eigenvalues $\chi_\varpi\bigl(T(\mathfrak{a}^2)\bigr)$ and the Fourier coefficients of the cuspidal automorphic representation ϖ:

Theorem 4.5. *Let $\mathfrak{a}$ be a non-zero integral ideal in $\mathcal{O}$ relatively prime to I, and let $r\in\mathcal{O}'\smallsetminus\{0\}$. With the notations of Proposition 4.3 we have for each irreducible cuspidal space V_ϖ invariant under the Casimir operators and the Hecke operators $T(\mathfrak{p}^2)$ with $\mathfrak{p}\nmid I$ the following relation for each non-zero ideal $\mathfrak{a}$ in $\mathcal{O}$ prime to I:*

$$\chi_\varpi\bigl(T(\mathfrak{a}^2)\bigr)c^{\infty,r}(\varpi)\;=\;\sum_{\mathfrak{b}|\mathfrak{a}^2\,,\;r\in\mathfrak{b}^{-1}\mathfrak{a}^2\mathcal{O}'}\mathrm{N}(\mathfrak{b})\;\hat\chi\bigl(g_{\kappa(\mathfrak{b})}p(\mathfrak{b})\bigr)^{-1}\,|\mathrm{N}(a(\mathfrak{b}))|\cdot\left(\prod_j\mathrm{Sign}\,\bigl(a_j(\mathfrak{b})\bigr)^{\xi_j}\right)\,e^{2\pi iS(rb(\mathfrak{b})/a(\mathfrak{b})^3)}\,c^{\kappa(\mathfrak{b}),r/a(\mathfrak{b})^2}(\varpi)\,. \tag{50}$$

Proof. We pick a weight q occurring in V_ϖ and use (22):

$$\begin{aligned}\chi_\varpi\big(T(\mathfrak{a}^2)\big)\, c^{\infty,r}(\varpi)\, d^r(q,\nu_\varpi)\, W_q(r,\nu_\varpi;g) \\ = \sum_{\mathfrak{b}|\mathfrak{a}^2\,,\; r\in\mathfrak{b}^{-1}\mathfrak{a}^2\mathcal{O}'} \mathrm{N}(\mathfrak{b})\; \hat\chi\big(g_{\kappa(\mathfrak{b})}p(\mathfrak{b})\big)^{-1} c^{\kappa(\mathfrak{b}),r/a(\mathfrak{b})^2}(\varpi)\; d^{r/a(\mathfrak{b})^2}(q,\nu_\varpi) \\ \cdot W_q(r\,a(\mathfrak{b})^{-2},\nu_\varpi;p(\mathfrak{b})g)\,.\end{aligned}$$

Formulas (1.12) and (2.28) in [8] imply that this is equal to

$$\begin{aligned}\sum_{\mathfrak{b}|\mathfrak{a}^2\,,\; r\in\mathfrak{b}^{-1}\mathfrak{a}^2\mathcal{O}'} \mathrm{N}(\mathfrak{b})\; \hat\chi\big(g_{\kappa(\mathfrak{b})}p(\mathfrak{b})\big)^{-1} c^{\kappa(\mathfrak{b}),r/a(\mathfrak{b})^2}(\varpi)\; \big|\mathrm{N}(a(\mathfrak{b}))\big| \\ \cdot d^r(q,\nu_\varpi)\, e^{2\pi i S(rb(\mathfrak{b})/a(\mathfrak{b})^3)}\, W_q(r,\nu_\varpi;g) \prod_j \mathrm{Sign}\,\big(a_j(\mathfrak{b})\big)^{\xi_j}\,.\end{aligned}$$

This yields statement (50) in the theorem. □

Remark 4.6. This theorem generalizes the classical relation between eigenvalues of Hecke operators T_p with p prime on a cuspidal eigenform and the Fourier coefficient of order p of that form. In the classical context one uses the normalization of the eigenform by taking its Fourier coefficient of order 1 equal to 1. This normalization does not extend to the present situation in a straightforward way, since there is in general no obvious Fourier term order r in $\mathcal{O}' \smallsetminus \{0\}$ to play the role of 1. Hence we give a formulation in which $r \in \mathcal{O}' \smallsetminus \{0\}$ can be chosen freely.

4.3. Proof of Theorem 1.1. In this subsection, we give a proof of Theorem 1.1 in two steps. First we prove Proposition 4.8, which is a version of Theorem 1.1 with the characteristic functions of the intervals $J_\mathfrak{p}$ replaced by polynomials. Next, §4.3.2 gives the extension of this result to characteristic functions of intervals.

4.3.1. Asymptotic formula for polynomials.

Lemma 4.7. *Let P be a finite set of primes of F. Then there are elements $r \in \mathcal{O}'$ such that $r \notin \mathfrak{p}\mathcal{O}'$ for all $\mathfrak{p} \in P$.*

Proof. For any fractional ideal $\mathfrak{a}$ with prime decomposition $\mathfrak{a} = \prod_\mathfrak{p} \mathfrak{p}^{a_\mathfrak{p}}$ the quotient $\mathfrak{a}/\big(\mathfrak{a}\prod_{\mathfrak{p}\in P}\mathfrak{p}\big) \cong \prod_{\mathfrak{p}\in P}\big(\mathfrak{p}^{a_\mathfrak{p}}/\mathfrak{p}^{a_\mathfrak{p}+1}\big)$ contains elements x such that $x_\mathfrak{p} + \mathfrak{p}^{a_\mathfrak{p}+1} \neq \mathfrak{p}^{a_\mathfrak{p}+1}$ for all $\mathfrak{p} \in P$. □

Proposition 4.8. *Let $t \mapsto \Omega_t$ be a family of sets in $\mathbb{R}^d$ as in* (3). *Let P be a finite set of primes not dividing I, and let $\lambda_{\varpi,\mathfrak{p}}$ and Φ be as in* (7) *and* (10) *respectively. Then, for any choice of even polynomials $q_\mathfrak{p}$, $\mathfrak{p} \in P$, and any $r \in \mathcal{O}'$ such that $r \notin \mathfrak{p}\mathcal{O}'$ for each $\mathfrak{p} \in P$, we have as $t \to \infty$:*

$$\begin{aligned}(51)\qquad \sum_{\varpi,\,\lambda_\varpi\in\Omega_t} \big|c^{\infty,r}(\varpi)\big|^2 \prod_{\mathfrak{p}\in P} q_\mathfrak{p}(\lambda_{\varpi,\mathfrak{p}}) \\ = \frac{2\sqrt{|D_F|}\,\mathrm{Vol}\,(\Gamma\backslash G)}{(2\pi)^d}\,\mathrm{Pl}(\Omega_t) \prod_{\mathfrak{p}\in P}\Phi_\mathfrak{p}(q_\mathfrak{p}) + o\big(V_1(\Omega_t)\big)\,.\end{aligned}$$

Proof. Let $\mathfrak{a} = \prod_{\mathfrak{p}\in P} \mathfrak{p}^{k_\mathfrak{p}}$ with $k_\mathfrak{p} \geq 0$. The sum in (50) is finite, so Theorems 4.5 and B.1 give for any $r \in \mathcal{O}' \smallsetminus \{0\}$:

$$\sum_{\varpi\,,\ \lambda_\varpi\in\Omega_t} \chi_\varpi\bigl(T(\mathfrak{a}^2)\bigr)\,\bigl|c^{\infty,r}(\varpi)\bigr|^2 \;=\; \sum_{\mathfrak{b}|\mathfrak{a}^2,\, r\in\mathfrak{b}^{-1}\mathfrak{a}^2\mathcal{O}'} \mathrm{N}(\mathfrak{b})\;\hat{\chi}\bigl(g_{\kappa(\mathfrak{b})}p(\mathfrak{b})\bigr)^{-1}\,|\mathrm{N}(a(\mathfrak{b}))|$$
$$\cdot\left(\prod_j \mathrm{Sign}\,\bigl(a_j(\mathfrak{b})\bigr)^{\xi_j}\right)\, e^{2\pi i S(r\,b(\mathfrak{b})/a(\mathfrak{b})^3)}\,\delta_{\infty,\kappa(\mathfrak{b})}\,\delta_\kappa(r, r/a(\mathfrak{b})^2)$$
$$\cdot\,\frac{2\sqrt{|D_F|}\,\mathrm{Vol}\,(\Gamma\backslash G)}{(2\pi)^d}\,\mathrm{Pl}(\Omega_t) + o\bigl(V_1(\Omega_t)\bigr)\,.$$

In the terms in the sum the factor $\delta_{\infty,\kappa(\mathfrak{b})}$ is nonzero (and equal to 1) only if $\kappa(\mathfrak{b}) = \infty$. Then either $\mathfrak{b} = \mathfrak{a}$ and $a(\mathfrak{b}) = 1$, or $a(\mathfrak{b}) \in F^* \smallsetminus \mathcal{O}^*$, by Lemma 4.2. Then we see in §A.2.3 that $\delta_\kappa\bigl(r, r/a(\mathfrak{b})^2\bigr) = 0$ if $a(\mathfrak{b}) \notin \mathcal{O}^*$, and $\delta_\kappa\bigl(r, r/a(\mathfrak{b})^2\bigr) = 1$ in the case $\mathfrak{b} = \mathfrak{a}$. Thus, we arrive at

$$\sum_{\varpi\,,\ \lambda_\varpi\in\Omega_t} \chi_\varpi\bigl(T(\mathfrak{a}^2)\bigr)\,\bigl|c^{\infty,r}(\varpi)\bigr|^2 = \begin{cases} \mathrm{N}(\mathfrak{a})\,\frac{2\sqrt{|D_F|}\,\mathrm{Vol}\,(\Gamma\backslash G)}{(2\pi)^d}\,\mathrm{Pl}(\Omega_t) & \text{if } r\in\mathfrak{a}\,\mathcal{O}' \\ 0 & \text{otherwise} \end{cases} + o\bigl(V_1(\Omega_t)\bigr)\,. \tag{52}$$

We use (48), and note that $S_{\mathfrak{p},0} = 1$, to find

$$\sum_{\varpi,\,\lambda_\varpi\in\Omega_t} \bigl|c^{\infty,r}(\varpi)\bigr|^2 \prod_{\mathfrak{p}\in P} S_{\mathfrak{p},2k_\mathfrak{p}}(\lambda_{\varpi,\mathfrak{p}}) = \begin{cases} \frac{2\sqrt{|D_F|}\,\mathrm{Vol}\,(\Gamma\backslash G)}{(2\pi)^d}\mathrm{Pl}(\Omega_t)\prod_{\mathfrak{p}\in P}\mathrm{N}(\mathfrak{p})^{k_\mathfrak{p}} & \text{if } r\in\mathcal{O}'\cdot\prod_{\mathfrak{p}\in P}\mathfrak{p}^{k_\mathfrak{p}} \\ 0 & \text{otherwise} \end{cases} + o\bigl(V_1(\Omega_t)\bigr)\,. \tag{53}$$

We have to consider this for $r \in \mathcal{O}'$ such that $r \notin \mathfrak{p}\mathcal{O}'$ for any $\mathfrak{p} \in P$. That means that we obtain only the term $o\bigl(V_1(\Omega_t)\bigr)$ except in the case that all $k_\mathfrak{p}$ vanish.

A computation shows that the measure $\Phi_\mathfrak{p}$ in (10) satisfies

$$\Phi_\mathfrak{p}(S_{\mathfrak{p},2k}) \;=\; \begin{cases} 1 & \text{if } k = 0\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{54}$$

So we can replace the right hand side in (53) by that in (51) with $q_\mathfrak{p} = S_{\mathfrak{p},2k}$. Since the $S_{\mathfrak{p},2k}$, $k \geq 0$, form a basis of the even polynomials in $X_\mathfrak{p}$, we have completed the proof. □

4.3.2. Asymptotic formula for characteristic functions. We complete the proof of Theorem 1.1 by extension steps also used in [6] and [9].

For the families under consideration we can have $\mathrm{Pl}(\Omega_t) = 0$ for all t. That occurs if $E \neq \emptyset$ in (3), and $\mathrm{Pl}\bigl([A_j, B_j]\bigr) = 0$ for at least one $j \in E$. In this case, equation (11) follows directly from (83).

For all other families under consideration we have $\mathrm{Pl}(\Omega_t) \to \infty$. Then we may view (51) as a limit formula

$$\lim_t \mu_t(p) \;=\; \mu(p)\,, \tag{55}$$

for positive measures on the compact space $\prod_{p\in P} \mathcal{J}_{\mathfrak{p}}$ with $\mathcal{J}_{\mathfrak{p}}$ as in (7):

$$\begin{aligned} \mu_t(f) \;&=\; \frac{1}{\mathrm{Pl}(\Omega_t)} \sum_{\varpi,\, \lambda_\varpi \in \Omega_t} \left|c^{\infty,r}(\varpi)\right|^2 f\big((\lambda_{\varpi,\mathfrak{p}^2})_{\mathfrak{p}\in P}\big)\,, \\ \mu(f) \;&=\; \frac{2\sqrt{|D_F|}\,\mathrm{Vol}\,(\Gamma\backslash G)}{(2\pi)^d} \Big(\bigotimes_{\mathfrak{p}\in P} \Phi_{\mathfrak{p}}\Big)(f)\,. \end{aligned} \tag{56}$$

Equation (51) gives (55) on tensor products of even polynomials. By the Stone-Weierstrass theorem we get (55) for all continuous functions.

For $\mathfrak{p} \in P$ let $J_{\mathfrak{p}}$ be an interval contained in $\mathcal{J}_{\mathfrak{p}}$, and denote by χ the characteristic function of $\prod_{\mathfrak{p}\in P} J_{\mathfrak{p}}$. For a given $\varepsilon > 0$, there exist continuous functions c and C on X_P such that $\mu(C - c) \le \varepsilon$, and $0 \le c \le \chi \le C$. From

$$\begin{array}{ccccc} \mu_t(c) & \le & \mu_t(\chi) & \le & \mu_t(C) \\ \downarrow & & & & \downarrow \\ \mu(c) & \le & \mu(\chi) & \le & \mu(C) \end{array}$$

and $\mu(C) - \mu(c) \le \varepsilon$ we conclude that

$$\mu(\chi) - 2\varepsilon \;\le\; \liminf_t \mu_t(\chi) \;\le\; \limsup_t \mu_t(\chi) \;\le\; \mu(\chi) + 2\varepsilon\,.$$

Since ε is arbitrary, then equation (55) holds for $p = \chi$ and the theorem now follows.

Appendix A. Kuznetsov sum formula. The proof of (4) in [9] is based on the sum formula in [8]. Similarly, the proof of Theorem B.1 in §B will be based on a generalization, Theorem A.2, of such a sum formula. In this section we discuss how to adapt and extend the sum formula in [8] to our present requirements, showing how the proof in *loc. cit.* can be modified to give Theorem A.2. We shall need an estimate of generalized Kloosterman sums that is discussed in §A.1.3.

A.1. Kloosterman sums. The sum formula relates Fourier coefficients of automorphic representations to Kloosterman sums, which we discuss now.

A.1.1. Bruhat decomposition. It is well known that

$$\begin{aligned} &\mathrm{SL}_2(F) \;=\; P_F \sqcup C_F \qquad \text{(Bruhat decomposition)}\,, \\ &P_F \;=\; \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}_2(F)\right\}, \qquad C_F \;=\; \left\{\begin{pmatrix} * & * \\ \neq 0 & * \end{pmatrix} \in \mathrm{SL}_2(F)\right\}. \end{aligned} \tag{57}$$

For $\kappa, \kappa' \in \mathcal{P}$ we put

$${}^{\kappa'}\!\mathcal{C}^{\kappa} \;=\; \left\{c \in F^* \;:\; \exists\gamma \in \Gamma \;:\; g_{\kappa'}^{-1}\gamma g_\kappa \;=\; \begin{pmatrix} * & * \\ c & * \end{pmatrix}\right\}, \tag{58}$$

and we let ${}^{\kappa'}\!\mathcal{S}^{\kappa}(c)$ denote a system of representatives of

$$\Gamma_{N^{\kappa'}} \backslash \left(\Gamma \cap g_{\kappa'} C(c) g_\kappa^{-1}\right) / \Gamma_{N^\kappa}\,, \qquad C(c) \;=\; \left\{\begin{pmatrix} * & * \\ c & * \end{pmatrix}\right\}.$$

Note that $-{}^{\kappa'}\!\mathcal{C}^{\kappa} = {}^{\kappa'}\!\mathcal{C}^{\kappa}$, and that we may use $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot {}^{\kappa'}\!\mathcal{S}^{\kappa}(c)$ as a system of representatives of $\Gamma_{N^{\kappa'}} \backslash \left(\Gamma \cap g_{\kappa'} C(-c) g_\kappa^{-1}\right) / \Gamma_{N^\kappa}$.

A.1.2. Kloosterman sums. For the present situation the Kloosterman sums are defined, for $\kappa, \kappa' \in \mathcal{P}$, $c \in {}^{\kappa'}\mathcal{C}^{\kappa}$, $r \in \tilde{M}'_{\kappa}$, $r' \in \tilde{M}'_{\kappa'}$, by:

$$S_\chi(\kappa, r; \kappa', r'; c) \;=\; \sum_{\gamma \in {}^{\kappa'}\mathcal{S}^{\kappa}(c)} \chi(\gamma)^{-1} e^{2\pi i S(\frac{r'a+rd}{c})}\,, \tag{59}$$

where $g_{\kappa'}\gamma g_{\kappa}^{-1} = \left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right)$. For the cusp ∞ this simplifies to a more familiar Kloosterman sum:

$$\begin{aligned} S_\chi(r, r'; c) &:= S_\chi(\infty, r; \infty, r'; c) \\ &= \sum_{a,d \bmod (c)\,,\; ad \equiv 1 \bmod c} \chi \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} e^{2\pi i S(\frac{r'a+rd}{c})}\,, \end{aligned}$$

with $c \in I$, $c \neq 0$, and $r, r' \in \mathcal{O}'$.

Since $\left({}^{\kappa'}\mathcal{S}^{\kappa}(c)\right)^{-1}$ is a system of representatives for the double quotient space $\Gamma_{N^{\kappa}} \backslash \left(\Gamma \cap g_\kappa C(-c) g_{\kappa'}^{-1}\right)/\Gamma_{N^{\kappa'}}$, we have

$$\overline{S_\chi(\kappa, r; \kappa', r'; c)} \;=\; S_\chi(\kappa', r'; \kappa, r; -c) \;=\; \chi(-1)\, S_\chi(\kappa', r'; \kappa, r; c)\,, \tag{60}$$

and we can use ${}^{\kappa'}\mathcal{S}^{\kappa}(c)$ as a system of representatives ${}^{\kappa}\mathcal{S}^{\kappa'}(c)$.

A.1.3. Weil type estimate. The proof of the sum formula in the form that we will need requires a Weil type estimate for the Kloosterman sums occurring in the formula. We will also need this estimate in the use of the sum formula, when we prove the asymptotic formula in Theorem B.1.

PROPOSITION A.1. *For $\kappa, \kappa' \in \mathcal{P}$, $r \in \tilde{M}'_{\kappa} \smallsetminus \{0\}$, $r' \in \tilde{M}'_{\kappa'} \smallsetminus \{0\}$, there is a finite set S of prime ideals in $\mathcal{O}_F$ such that, for each $\varepsilon > 0$ and for all $c \in {}^{\kappa'}\mathcal{C}^{\kappa}$:*

$$S_\chi(\kappa, r; \kappa', r'; c) \ll_{F,I,\kappa,r,\kappa',r',\varepsilon} \prod_{\mathfrak{p} \in S} \mathrm{N}(\mathfrak{p})^{v_\mathfrak{p}(c)} \left(\prod_{\mathfrak{p} \notin S} \mathrm{N}(\mathfrak{p})^{v_\mathfrak{p}(c)} \right)^{\frac{1}{2}+\varepsilon}\,, \tag{61}$$

where $v_\mathfrak{p}$ denotes the valuation at the prime $\mathfrak{p}$.

This estimate is weaker than what we may expect to be true. See for instance the estimate stated in (13) in [13].

The proof of (61) is relatively easy. First one establishes a product formula, reducing the task to estimating local Kloosterman sums. We put all places where something special happens (places dividing I, places where $\mathcal{O}_\mathfrak{p} \otimes_{\mathcal{O}} M_\kappa \neq \mathcal{O}_\mathfrak{p}$, places where $g_\kappa \notin \mathrm{SL}_2(\mathcal{O}_\mathfrak{p})$, and similarly for κ') into S, and estimate the corresponding local Kloosterman sum trivially. At the remaining places the local Kloosterman sum is a standard one and can be estimated in the usual way.

A.2. Sum formula. The sum formula in Theorem A.2 relates four terms, each depending on a test function. We discuss first the test functions and the four terms.

A.2.1. Test functions. The class of test functions is the same as in [9], §2.1.1: functions of product type $\varphi(\nu) = \prod_j \varphi_j(\nu_j)$, where the factor φ_j is defined on a strip $|\operatorname{Re}\nu_j| \le \tau$ with $\frac{1}{4} < \tau < \frac{1}{2}$, and on the discrete set $\frac{1+\xi_j}{2} + \mathbb{N}_0$. The factor φ_j satisfies on its domain $\varphi_j(\nu_j) \ll (1 + |\nu_j|)^{-a}$ for some $a > 2$, and is even and holomorphic on the strip $|\operatorname{Re}\nu_j| \le \tau$.

The ν_j occurring in this way are related to spectral data. The eigenvalues $\lambda_{\varpi,j}$ of the Casimir operators in V_ϖ can be written as $\lambda_{\varpi,j} = \frac{1}{4} - \nu_{\varpi,j}^2$, which we can choose in $(0,\infty) \cup i[0,\infty)$. So the test functions φ can be viewed as functions defined on a neighborhood of the set of possible values of the vectors $\nu_\varpi = (\nu_{\varpi,j})_j$.

A.2.2. Spectral terms. The sum formula is based on the choice of two pairs (κ, r) and (κ', r') of cuspidal representatives κ and κ' in $\mathcal{C}$, and non-zero Fourier term orders r and r' at those cusps. In this paper, we need only the case that r/r' is totally positive.

Closely related to the counting function in (82) is the cuspidal term

$$\tilde{\mathrm{N}}^{\kappa,r;\kappa',r'}(\varphi) \;=\; \sum_{\varpi} \overline{c^{\kappa,r}(\varpi)}\, c^{\kappa',r'}(\varpi)\, \varphi(\nu_\varpi)\,. \tag{62}$$

The Fourier coefficients of Eisenstein series enter into the Eisenstein term

$$\begin{aligned}\mathrm{Eis}^{\kappa,r;\kappa',r'}(\varphi) \;&=\; 2\sum_{\lambda\in\mathcal{P}} c_\lambda \sum_{\mu\in\Lambda_{\lambda,\chi}} \int_0^\infty \overline{D_\xi^{\kappa,r}(\lambda,\chi;iy,i\mu)} \\ &\quad \cdot D_\xi^{\kappa,r}(\lambda,\chi;iy+i\mu)\,\varphi(iy+i\mu)\,dy\,,\end{aligned} \tag{63}$$

with $c_\lambda > 0$, and $\Lambda_{\lambda,\chi}$ as in §2.1.2 of [9]. (The c_λ differ from those in [9] due to the difference in normalization discussed in §2.5. Their actual value is not important for the present purpose.)

A.2.3. Delta term. The delta term $\Delta^{\kappa,r;\kappa',r'}(\varphi)$ can be non-zero only if $\kappa = \kappa'$ and r/r' satisfies a strong condition.

We put

$$\begin{aligned}\delta_\kappa(r,r') \;&=\; \delta_\kappa(\chi,\xi;r,r') \\ &=\; \frac{1}{2} \sum_{\gamma\in\Gamma_{P^\kappa}/\Gamma_{N^\kappa}\,,\; r/r'=\varepsilon^2} \chi(\gamma)\, e^{-2\pi i S(r\beta\varepsilon)} \prod_j (\mathrm{Sign}\,\varepsilon_j)^{\xi_j}\,,\end{aligned} \tag{64}$$

where $\gamma = g_\kappa \begin{pmatrix} \varepsilon & \beta \\ 0 & \varepsilon^{-1}\end{pmatrix} g_\kappa^{-1}$. If $\kappa = \infty$ the ε occurring in $\begin{pmatrix} \varepsilon & \beta \\ 0 & \varepsilon^{-1}\end{pmatrix} \in \Gamma$ are the units of $\mathcal{O}$. For other $\xi \in \mathcal{C}$, the ε occurring in $g_\kappa \begin{pmatrix} \varepsilon & \beta \\ 0 & \varepsilon^{-1}\end{pmatrix} g_\kappa^{-1} \in \Gamma$ form also a subgroup of F^* isomorphic to $(\mathbb{Z}/2)\times\mathbb{Z}^{d-1}$. Only if r/r' is the square of an element of this subgroup the sum in (64) is non-empty, and then consists of two equal summands.

The delta term is

$$\Delta^{\kappa,r;\kappa',r'}(\varphi) \;=\; \frac{2\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)}{(2\pi)^d}\, \delta_{\kappa,\kappa'}\, \delta_\kappa(r,r')\, \widetilde{\mathrm{Pl}}(\varphi)\,, \tag{65}$$

where $\widetilde{\mathrm{Pl}} = \otimes_j \widetilde{\mathrm{Pl}}_{\xi_j}$ is the Plancherel measure in (2) written in terms of the spectral parameter ν:

$$\begin{aligned}\int f\, d\widetilde{\mathrm{Pl}}_0 \;&=\; 2i\int_0^{i\infty} f(\nu)\, \tan\pi\nu\; \nu\, d\nu \\ &\quad + \sum_{b\geq 2\,,\; b\equiv 0 \bmod 2} (b-1)\, f\left(\tfrac{b-1}{2}\right)\,, \\ \int f\, d\widetilde{\mathrm{Pl}}_1 \;&=\; -2i\int_0^{i\infty} f(\nu)\, \cot\pi\nu\; \nu\, d\nu \\ &\quad + \sum_{b\geq 3\,,\; b\equiv 1 \bmod 2} (b-1)\, f\left(\tfrac{b-1}{2}\right)\,.\end{aligned} \tag{66}$$

A.2.4. Sum of Kloosterman sums. The Bessel transform $\mathrm{B}^{\mathbf{s}}_{\xi}$ is the same as in (34) of [9], with $\mathbf{s} \in \{1,-1\}^d$, $\mathbf{s}_j = \mathrm{Sign}\,(r_j)$. For each test function φ it provides a function $\mathrm{B}^{\mathbf{s}}\varphi$ on $(\mathbb{R}^*)^d$. The Kloosterman term in the formula is

$$\mathrm{K}^{\kappa,r;\kappa',r'}(\mathrm{B}^{\mathbf{s}}\varphi) \;=\; \sum_{c \in {}^{\kappa}\mathcal{C}^{\kappa}} \frac{S_\chi(\kappa',r';\kappa,r;c)}{|\mathrm{N}(c)|}\,\mathrm{B}^{\mathbf{s}}\varphi\left(\tfrac{4\pi\sqrt{rr'}}{c}\right). \tag{67}$$

We restrict ourselves to stating the formula in the equal sign case $\mathrm{Sign}\,(r) = \mathrm{Sign}\,(r')$, since this is the case needed in this paper. With a different Bessel transform, the formula goes through if $\mathrm{Sign}\,(r) \neq \mathrm{Sign}\,(r')$.

THEOREM A.2. (Spectral sum formula) *Let $\kappa,\kappa' \in \mathcal{P}$ and $r \in \tilde{M}'_\kappa \smallsetminus \{0\}$, $r' \in \tilde{M}'_{\kappa'} \smallsetminus \{0\}$ such that $\mathbf{s} = \mathrm{Sign}\,(r) = \mathrm{Sign}\,(r')$. For any test function φ the sums and integrals $\tilde{\mathrm{N}}^{\kappa,r;\kappa',r'}(\varphi)$, $\mathrm{Eis}^{\kappa,r;\kappa',r'}(\varphi)$, $\widetilde{\mathrm{Pl}}(\varphi)$ and $\mathrm{K}^{\kappa,r;\kappa',r'}(\mathrm{B}^{\mathbf{s}}\varphi)$ converge absolutely, and*

$$\begin{aligned} &\frac{\tilde{\mathrm{N}}^{\kappa,r;\kappa',r'}(\varphi) + \mathrm{Eis}^{\kappa,r;\kappa',r'}(\varphi)}{\mathrm{Vol}\,(\Gamma\backslash G)} \\ &\quad= \frac{2\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)}{(2\pi)^d}\,\delta_{\kappa,\kappa'}\,\delta_\kappa(r,r')\,\widetilde{\mathrm{Pl}}(\varphi) + \mathrm{K}^{\kappa,r;\kappa',r'}(\mathrm{B}^{\mathbf{s}}\varphi). \end{aligned}$$

A.3. Proof of the sum formula. We shall go through the proof in §3 of [8], indicating where changes are needed to deal with Fourier coefficients at cusps $\kappa \neq \infty$.

A.3.1. Poincaré series. Let $\kappa \in \mathcal{P}$. If the function h^κ on G satisfies the transformation rule

$$h^\kappa(g_\kappa n(x) g) \;=\; e^{2\pi i S(rx)} h^\kappa(g_\kappa g)$$

with $r \in \tilde{M}'_\kappa$ and the estimate $h^\kappa(g_\kappa n a(y) k) \ll \prod_j \min\left(y_j^\alpha, y_j^{-\beta}\right)$ with $\alpha > 1$ and $\alpha + \beta > 0$, then

$$P^\kappa h^\kappa(g) = \sum_{\gamma \in \Gamma_{N^\kappa}\backslash\Gamma} \chi(\gamma)^{-1} h^\kappa(\gamma g) \tag{68}$$

converges absolutely and defines the function $P^\kappa h^\kappa$ on G that is a square integrable, (Γ,χ)-automorphic function. In the first step of the proof of convergence, the sum over the units in Lemma 2.3 of [8] is replaced by a sum over $\Gamma_{N^\kappa}\backslash\Gamma_{P^\kappa}$. The method in §8 of [5] works well for this sum. The second step is a reduction to the convergence of the Eisenstein series, which is a fact that we can assume. In Lemma 2.4 of [8], in place of (2.52) and (2.53), we have, as $N(y) := \prod_j y_j \to \infty$:

$$\begin{aligned} P^\kappa h^\kappa(g_\kappa n a(y) k) &\ll_{\alpha,\beta,\varepsilon} \max\bigl(N(y)^{1-\alpha+\varepsilon}, N(y)^{-\beta+\varepsilon}\bigr), \\ P^\kappa h^\kappa(g_\lambda n a(y) k) &\ll_{\alpha,\varepsilon} N(y)^{1-\alpha+\varepsilon}, \end{aligned} \tag{69}$$

for $\lambda \in \mathcal{P} \smallsetminus \{\kappa\}$.

For h^κ and f of weight q, equation (3.1) in [8] takes the form

$$\langle P^\kappa h^\kappa, f\rangle \;=\; \frac{\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)}{\mathrm{Vol}\,(\Gamma\backslash G)} \int_A h^\kappa(g_\kappa a)\overline{F_{\kappa,r} f(a)}\,|a|^{-1}\,da. \tag{70}$$

A.3.2. Fourier coefficient of Poincaré series. For (κ, r) and (κ', r') as in Theorem A.2 and for $h^{\kappa'}$ satisfying the conditions above:

$$F_{\kappa,r}P^{\kappa'}h^{\kappa'}(g) = \frac{1}{\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)} \sum_{\gamma\in\Gamma_{N^{\kappa'}}\backslash\Gamma} \chi(\gamma)^{-1} \cdot \int_{\mathbb{R}^d/M_\kappa} e^{-2\pi i S(rx)} h^{\kappa'}(\gamma g_\kappa n(x) g)\, dx\,. \tag{71}$$

We write $g_{\kappa'}^{-1}\gamma g_\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Only if $\kappa = \kappa'$ there can be a contribution with $c = 0$. With $\gamma = g_\kappa \begin{pmatrix} \varepsilon & \beta \\ 0 & 1/\varepsilon \end{pmatrix} g_\kappa^{-1}$ as in §A.2.3, we obtain:

$$\begin{aligned} &\frac{1}{\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)} \sum_{\gamma\in\Gamma_{N^\kappa}\backslash\Gamma_{P^\kappa}} \chi(\gamma)^{-1} \cdot \int_{\mathbb{R}^d/M_\kappa} e^{-2\pi i S(rx)} h^\kappa\left(g_\kappa n(\beta\varepsilon + \varepsilon^2 x) h(\varepsilon) g\right)\, dx \\ &= \frac{1}{\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)} \int_{\mathbb{R}^d/M_\kappa} e^{2\pi i S((r-\varepsilon^2 r')x)}\, dx\; h^\kappa\left(g_\kappa h(\varepsilon) g\right) \\ &= 2\overline{\delta_\kappa(r,r')}\, h^\kappa(g_\kappa a(\varepsilon^2) g)\,, \end{aligned} \tag{72}$$

where $\varepsilon \in \mathcal{O}_F^*$ satisfies $\varepsilon^2 = r/r'$ if there are elements of the form $\begin{pmatrix} \varepsilon & \beta \\ 0 & 1/\varepsilon \end{pmatrix}$ in $g_\kappa^{-1}\Gamma g_\kappa$.

For all combinations of κ and κ' the number of terms with $c \neq 0$ is large. We write $g_{\kappa'}^{-1}\gamma g_\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = n(a/c)w(c)n(d/c)$, where $w(y) = \begin{pmatrix} 0 & -y^{-1} \\ y & 0 \end{pmatrix}$, and obtain, in the notations in §2.1:

$$\begin{aligned} &\frac{1}{\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)} \sum_{c\in{}^{\kappa'}\mathcal{C}^\kappa} \sum_{\gamma\in{}^{\kappa'}\mathcal{S}^\kappa(c)} \sum_{\delta\in\Gamma_{N^\kappa}} \chi(\gamma\delta)^{-1} \cdot \int_{\mathbb{R}^d/M_\kappa} e^{-2\pi i S(rx)} h^{\kappa'}\left(\gamma g_\kappa n(x) g\right)\, dx \\ &= \frac{1}{\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)} \sum_{c\in{}^{\kappa'}\mathcal{C}^\kappa} \sum_{\gamma\in{}^{\kappa'}\mathcal{S}^\kappa(c)} \chi(\gamma)^{-1} \cdot \int_{\mathbb{R}^d} e^{-2\pi i S(r(x-d/c)) + 2\pi i S(r'a/c)} h^{\kappa'}\left(g_{\kappa'} w(c) n(x) g\right)\, dx \\ &= \sum_{c\in{}^{\kappa'}\mathcal{C}^\kappa} \frac{S_\chi(\kappa, r; \kappa', r'; c)}{\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)} \int_{\mathbb{R}^d} e^{-2\pi i S(rx)} h^{\kappa'}\left(g_{\kappa'} w(c) n(x) g\right)\, dx\,. \end{aligned} \tag{73}$$

A.3.3. Whittaker transform. *§3.2* goes through almost verbatim. (We again indicate references to [8] by italics.) In *Definition 3.4*, we put

$$w_q^{\kappa,r}\eta(g_\kappa g) = w_q^r \eta(g)\,. \tag{74}$$

We insert g_κ at appropriate places. The last formula in *Theorem 3.8*, implies

$$\int_{\mathbb{R}^d} e^{-2\pi i S(rx)} w_q^{\kappa',r'}\eta\left(g_{\kappa'} w(c) n(x) g\right)\, dx = w_q^{\kappa,r}\tilde{\eta}(g_\kappa g)\,. \tag{75}$$

A.3.4. Restricted version of the formula. As in §*3.3*, the scalar product of Poincaré series $P^\kappa w_q^{\kappa,r}\eta$ and $P^{\kappa'} w_q^{\kappa',r'}\eta'$ is computed in two ways.

In the spectral computation, we obtain the following

$$\langle P^\kappa w_q^{\kappa,r}\eta, \psi_{\varpi,q}\rangle \;=\; 8^{d/2}\,\pi^d \Big(\frac{\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)}{\mathrm{Vol}\,(\Gamma\backslash G)}\Big)^{1/2} |\mathrm{N}(r)|^{1/2}\,\overline{c^{\kappa,r}(\varpi)}$$
$$\cdot\,\eta(\nu_\varpi)\prod_j \frac{e^{-\pi i q_j}}{\Gamma(\frac12+\bar\nu_{\varpi,j}+\frac{q_j\mathbf{s}_j}{2})}\,,$$

together with a similar expression for the scalar product with an Eisenstein series. (Compare with *(3.39)*.) We have to modify *(3.40)* and *(3.41)* by inserting the cusps κ and κ' into the Fourier coefficients. We modify the definition of the measure by defining $d\sigma_{\chi,\xi}^{\kappa,r;\kappa',r'}$ by the expression in *(3.43)* with κ and κ' inserted into the Fourier coefficients. Then we obtain in place of *(3.46)*:

$$\text{(76)}\qquad \langle P^\kappa w_q^{\kappa,r}\eta, P^{\kappa'} w_q^{\kappa',r'}\eta'\rangle \;=\; \frac{(8\pi^2)^d\sqrt{\mathrm{N}(rr')}}{\mathrm{Vol}\,(\Gamma\backslash G)}\int_{Y_\xi}\vartheta_q^{r,r'}(\nu)\,d\sigma_{\chi,\xi}^{\kappa,r;\kappa',r'}(\nu)\,,$$

with

$$\vartheta_q^{r,r'}(\nu) = \prod_{j=1}^d \frac{\eta_j(\nu_j)\overline{\eta_j'(\bar\nu_j)}}{\Gamma(\frac12-\nu_j+\frac{q_j\,\mathrm{Sign}\,r_j}{2})\Gamma(\frac12-\nu_j+\frac{q_j\,\mathrm{Sign}\,r_j}{2})}\,,$$

and $d\sigma_{\chi,\xi}^{\kappa,r;\kappa',r'}$ adapted to the new normalization.

A comparison shows that the cuspidal subspace contributes

$$(8\pi^2)^d\sqrt{\mathrm{N}(rr')}\,\mathrm{Vol}\,(\Gamma\backslash G)^{-1}\,\tilde{\mathrm{N}}^{\kappa,r;\kappa',r'}(\vartheta_q^{r,r'})\,,$$

with a similarly modified expression for the scalar product with Eisenstein series.

The geometric computation of the scalar product is carried out as in §*3.3.4*. The contribution from (72) is equal to

$$\text{(77)}\qquad 2\delta_{\kappa,\kappa'}\,\delta_\kappa(r,r')\,\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)\,(4\pi)^d|\mathrm{N}(r)|\int_{Y_\xi}\vartheta_q^{r,r'}(\nu)\,d\widetilde{\mathrm{Pl}}(\nu)\,.$$

This agrees with *(3.50)*, with the substitutions

$$\text{(78)}\qquad \alpha(\chi,\xi;r,r')\mapsto 2\delta_{\kappa,\kappa'}\delta_\kappa(r,r')\,,\qquad \sqrt{|D_F|}\mapsto \mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)\,.$$

The remaining contribution to the scalar product is given by (73):

$$\text{(79)}\qquad \mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)\int_A w_q^{\kappa,r}\eta(g_\kappa a)\cdot\sum_{c\in{}^\kappa\mathcal{C}^\kappa}\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)^{-1}\overline{S_\chi(\kappa,r;\kappa',r')}$$
$$\cdot\int_{\mathbb{R}^d} e^{2\pi i S(rx)}\overline{w_q^{\kappa',r'}\eta'\,(g_{\kappa'}w(c)n(x)a)}\,dx\,|a|^{-1}\,da\,.$$

Using (60) and *Theorem 3.8*, we see that the expression in (79) is equal to

$$\sum_{c\in{}^{\kappa'}\mathcal{S}^\kappa} S_\chi(\kappa',r';\kappa,r)\,\chi(-1)\int_A w_q^{\kappa,r}\eta(g_\kappa a)\,\overline{w_q^{\kappa,r}\tilde\eta'(g_\kappa a)}\,|a|^{-1}\,da\,.$$

The transition $r \mapsto r'$ under the conjugation is present in *Theorem 3.8*. The transition $\kappa' \mapsto \kappa$ is a consequence of the definitions, and has to be checked. With *(3.52)* the integral over A is given by

$$\frac{(8\pi^2)^d}{|\mathrm{N}(c)|}\,\mathrm{N}(rr')^{1/2}\,\chi(-1)\,(\mathrm{B}^{\mathbf{s}}\vartheta_q^{r,r'})\big(\tfrac{4\pi\sqrt{rr'}}{c}\big)\,.$$

This gives the final result of this contribution:

$$(8\pi^2)^d\,\mathrm{N}(rr')^{1/2}\,\mathrm{K}^{\kappa,r;\kappa',r'}\big(\mathrm{B}^{\mathbf{s}}\vartheta_q^{r,r'}\big)\,. \tag{80}$$

Division by $(8\pi^2)^d\,\mathrm{N}(rr')^{1/2}$ gives the restricted sum formula

$$\frac{1}{\mathrm{Vol}\,(\Gamma\backslash G)}\int_{Y_\xi}\vartheta(\nu)\,d\sigma_{\chi,\xi}^{\kappa,r;\kappa',r'}(\nu) \;=\; \delta_{\kappa,\kappa'}\frac{2\mathrm{Vol}\,(\mathbb{R}^r/M_\kappa)}{(2\pi)^d}\int_{Y_\xi}\vartheta(\nu)\,d\widetilde{\mathrm{Pl}}(\nu) + \mathrm{K}^{\kappa,r;\kappa',r'}(\mathrm{B}^{\mathbf{s}}\vartheta)\,. \tag{81}$$

for all ϑ indicated in *Proposition 3.9*.

At this point we notice a minor error in [8]. With the sum of Kloosterman sums as defined in *(3.60)*, it should read $\mathrm{K}_\chi^{r',r}$ instead of $K_\chi^{r,r'}$ in *(3.61)*. The solution that we will adopt from now on, is to take the expression in *(3.60)* as the definition of $\mathrm{K}_\chi^{r,r'}$, like in (67) here.

The proof of the convergence of the sum of Kloosterman sums *(Proposition 3.14)* has to be revisited. In [8] the prime ideals are split according to whether they divide the ideal I or not. The finite set S in Proposition A.1 may tun out to be a larger set of prime ideals than those dividing I. Nevertheless, the method in *(3.72)*, goes through.

A.3.5. Extension. The rest of the proof of the formula in [8] is based on the restricted formula in *Proposition 3.9*, and consists of extending the space of test functions for which the formula holds. The dependence on the cusps κ and κ' in (81) here, is immaterial in these extension steps. In §*3.5* we use *Proposition 3.16*, which goes through unchanged. Thus we arrive at the sum formula as stated in Theorem A.2.

Appendix B. Asymptotic formula. In the proof of Proposition 4.8 we have used the generalization (83) of (4), where $|c^r(\varpi)|^2 = |c^{\infty,r}(\varpi)|^2$ is replaced by $\overline{c^{\kappa,r}(\varpi)}\,c^{\kappa',r'}(\varpi)$. The aim of this section is to show that the methods in [9] can be extended to give the asymptotic result (83) below.

We define for $\kappa,\kappa'\in\mathcal{P}$, $r,r'\in\tilde{M}'_\kappa\smallsetminus 0$ and compact sets $\Omega\subset\mathbb{R}^d$

$$\mathrm{N}^{\kappa,r;\kappa',r'}(\Omega) \;=\; \sum_{\varpi\,,\;\lambda_\varpi\in\Omega}\overline{c^{\kappa,r}(\varpi)}\,c^{\kappa',r'}(\varpi)\,, \tag{82}$$

where ϖ runs through a maximal orthogonal system of irreducible subspaces of $L_\xi^{2,\mathrm{cusp}}(\Gamma\backslash G;\chi)$. This generalizes the counting function on the left hand side of (1).

THEOREM B.1. *Let $t\mapsto\Omega_t$ be a family of bounded sets in $\mathbb{R}^d$ as in* (3), *or satisfying the conditions indicated in* §*B.1 below. Let $\kappa,\kappa'\in\mathcal{P}$, and let $r\in\tilde{M}'_\kappa\smallsetminus\{0\}$, $r'\in\tilde{M}'_{\kappa'}\smallsetminus\{0\}$, such that* $\mathrm{Sign}\,r=\mathrm{Sign}\,r'$. *Then, as $t\to\infty$*

$$\mathrm{N}^{\kappa,r;\kappa',r'}(\Omega_t) \;=\; \delta_{\kappa,\kappa'}\,\delta_\kappa(r,r')\,\frac{2\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)\,\mathrm{Vol}\,(\Gamma\backslash G)}{(2\pi)^d}\,\mathrm{Pl}(\Omega_t)+o\big(V_1(\Omega_t)\big)\,, \tag{83}$$

with the notations in (65)*,* (2) *and* (5)*.*

The restriction to $\operatorname{Sign} r = \operatorname{Sign} r'$ is necessary to us. We have not been able to estimate Bessel transforms suitably in the unequal sign case.

B.1. Conditions. In §B.3 we will show that Theorem B.1 is valid for families $t \mapsto \Omega_t$ as used in [9]. Such families have product form

$$\Omega_t \;=\; \hat{C}_t^+ \times \hat{C}_t^- \times \prod_{j\in E} [A_j, B_j]\,,$$

based on a partition $\{1, \ldots, d\} = Q^+ \sqcup Q^- \sqcup E$ of the archimedean places of F. The bounded intervals $[A_j, B_j]$ do not depend on t, and the endpoints should not be of the form $\frac{b}{2}\bigl(1 - \frac{b}{2}\bigr)$ with $b \equiv \xi_j \bmod 2$, $b > 1$. The sets $\hat{C}_t^+$ and $\hat{C}_t^-$ are compact sets contained in $\prod_{j\in Q^+} \bigl[\frac{5}{4}, \infty\bigr)$, respectively $\prod_{j\in Q^-} (\infty, 0]$, such that the corresponding sets $C_t^\pm = \prod_{j\in Q^\pm} \bigl(i[1,\infty) \cup [0,\infty)\bigr)$ in the variable ν with $\lambda = \frac{1}{4} - \nu^2$ satisfy the conditions in (97) or (102) in [9]. The proof of Theorem 1.1 is based on the asymptotic formula, so the statement of the theorem holds for all families $t \mapsto \Omega_t$ satisfying these conditions.

A family $t \mapsto \Omega_t$ as in (3) in the introduction does not satisfy these conditions directly. We write $\Omega_t = \bigsqcup_p \Omega_t^{(p)}$ with

$$\Omega_t^{(p)} \;=\; \bigl[\tfrac{5}{4}, t\bigr]^{Q^+} \times \bigl[-t, -\tfrac{1}{2}\bigr] \times \bigl[-\tfrac{1}{2}, \tfrac{5}{4}\bigr]\,,$$

and let $p = (Q^+, Q^-, Q^0)$ run over the partitions $Q = Q^+ \sqcup Q^- \sqcup Q^0$ such that $Q^0 \neq Q$. All sets $\Omega_t^{(p)}$ satisfy the conditions in [9]. Summing the asymptotic formulas (83) applied to each of the families $t \mapsto \Omega_t^{(p)}$ gives it for $t \mapsto \Omega_t$.

There are more families for which the asymptotic formula can be proved, by expressing them in families satisfying the conditions in (97) or (102) in [9]. Hence the statement of Theorem 1.1 holds for all these families, in particular for the families in §1.2.4–13 and §6 in [9].

B.2. Estimates of Fourier coefficients of Eisenstein series. In [9], (32), (33), we quoted from [6] and [8] estimates of $D_\xi^{\infty,r}(\lambda, \chi; iy, i\mu)$. These estimates have to be generalized to $D_\xi^{\kappa,r}(\lambda, \chi; iy, i\mu)$.

The estimations in §5.1–2 of [6] are based on the fact that Eisenstein series for $\Gamma_0(I)$ are linear combinations of Eisenstein series for the subgroup $\Gamma(I)$. In §4.2 of [8] it is shown that this result extends to the situation with weights in $\mathbb{Z}$ instead of $2\mathbb{Z}$. The character χ is trivial on $\Gamma(I)$, hence it only influences the coefficients in the linear combination.

These estimations concern the Fourier coefficients $D_\xi^{\infty,r}$. Let $\kappa \in \mathcal{P}$ be another cusp. The function $\tilde{E}_q : g \mapsto E_q(\lambda, \chi; \nu, i\mu; g_\kappa g)$ is an Eisenstein series on the group $\Gamma_1 = g_\kappa^{-1}\Gamma_0(I) g_\kappa$ for a character χ_1 determined by conjugation. Actually, depending on the normalizations, it might be a multiple of an Eisenstein series on Γ_1, with a factor in which the influence of ν and μ is of the form $t^\nu t_1^{i\mu}$ with $t > 0$, $t_1 > 0$. So this factor is unimportant for estimates. Since $g_\kappa \in \mathrm{SL}_2(F)$, the group Γ_1 is commensurable with $\Gamma_0(I)$, and contains a principal congruence subgroup $\Gamma(I_1)$, with $I_1 \subset I$. The character χ is trivial on $\Gamma(I)$, so we can arrange I_1 such that χ_1 is trivial on $\Gamma(I_1)$. Thus, for the Fourier coefficients of $\tilde{E}_q$ at ∞ with nonzero order, we have an estimate like in Proposition 4.2 in [8]. The Fourier coefficients of $E_q(\lambda, \chi; \nu, i\mu)$ at

κ of nonzero order can be expressed in those of $\tilde{E}_q$ at ∞. The consequence is that $D_\xi^{\kappa,r}(\lambda,\chi;\nu,i\mu)$ satisfies an estimate like that in Proposition 4.2 of [8]. As in (33) of [9], we have:

$$D_\xi^{\kappa,r}(\lambda,\chi;iy,i\mu) \ll_{F,I,\kappa,r,\kappa',r'} \Big(\log\big(2+\sum_j |y+\mu_j|\big)\Big)^7 . \tag{84}$$

B.3. Derivation of the asymptotic formula. The method of proof is the same as in §2-5 in [9], so we shall only indicate the points where the argument departs from the approach therein.

In the proofs in [9] we mostly use the spectral parameter ν instead of the eigenvalue $\lambda = \frac{1}{4} - \nu^2$. In terms of the spectral parameter the counting function in (82) has the form

$$\tilde{\mathrm{N}}^{\kappa,r;\kappa',r'}(\tilde{\Omega}) \;=\; \sum_{\varpi\,,\;\nu_\varpi\in\tilde{\Omega}} \overline{c^{\kappa,r}(\varpi)}\, c^{\kappa',r'}(\varpi)\,. \tag{85}$$

We choose the test functions in the same way as in Lemma 2.2 in [9]. The considerations in §2.2.1 and §2.2.4, *loc. cit.*, do not depend on the Fourier term order, and go through unchanged. We use the Weil type estimate of Kloosterman sums in Proposition A.1, and, in §2.2.2, *loc. cit.*, we replace the distinction between $\mathfrak{p} \mid I$ and $\mathfrak{p} \nmid I$, by the distinction, $\mathfrak{p} \in S$ and $\mathfrak{p} \notin S$, where S is a finite set of primes as in Proposition A.1.

In §2.2.3 of [9] we use the estimate (84) of Fourier coefficients of Eisenstein series. The statement of Proposition 2.4 in [9] goes through for the counting quantity $\tilde{N}^{\kappa,r;\kappa',r'}(\varphi(q,\cdot))$. The factor $\sqrt{|D_F|}$ is a specialization of $\mathrm{Vol}\,(\mathbb{R}^d/M_\kappa)$, and a factor $\delta_{\kappa,\kappa'}\,\delta_\kappa(r,r')$ is inserted. Hence the delta term is present only if the cusps κ and κ' are equal, and also r/r' is the square of a generalized unit.

The remaining proofs in [9] are based on the estimate in Proposition 2.4 of [9], and hence go through for Fourier coefficients at different cusps.

REFERENCES

[1] T. Barnet-Lamb, T. Gee, and D. Geraghty, *The Sato-Tate conjecture for Hilbert modular forms*, J. Amer. Math. Soc., 24 (2011), pp. 411–469.

[2] V. Blomer and F. Brumley, *On the Ramanujan conjecture over number fields*, Ann. Math., 174 (2011), pp. 581–604.

[3] A. Borel, *Density and maximality of arithmetic subgroups*, J. reine und angew. Mathematik, 224 (1966), pp. 78–89.

[4] R. W. Bruggeman, *Fourier coefficients of cusp forms*, Invent. math., 45 (1978), pp. 1–18.

[5] R. W. Bruggeman and R. J. Miatello, *Sum formula for* SL_2 *over a number field and Selberg type estimate for exceptional eigenvalues*, GaFA, 8 (1998), pp. 627–655.

[6] R. W. Bruggeman, R. J. Miatello, and I. Pacharoni, *Density results for automorphic forms on Hilbert modular groups*, Geometric and Functional Analysis, 13 (2003), pp. 681–719.

[7] R. W. Bruggeman and R. J. Miatello, *Distribution of square integrable automorphic forms on Hilbert modular groups*, preprint 2007.

[8] R. W. Bruggeman and R. J. Miatello, *Sum formula for* SL_2 *over a totally real number field*, Memoirs, 919, AMS, 2009.

[9] R. W. Bruggeman and R. J. Miatello, *Density results for automorphic forms on Hilbert modular groups II*, Transactions Amer. Math. Soc., 362 (2010), pp. 3841–3881.

[10] S. S. Gelbart, *Automorphic forms on adele groups*, Ann. Math. Studies, 83, Princeton Univerity Press, 1975.

[11] H. H. Kim and F. Shahidi, *Cuspidality of symmetric powers with applications*, Duke Math. J., 112 (2002), pp. 177–197.

[12] J. T. Tate, *Algebraic cycles and poles of zeta functions*, in "Arithmetic Algebraic Geometry", Proceedings Conf. at Purdue Univ., December 5–7, 1963; ed. O. F. G. schilling; Harper & Row, New York, 1965.

[13] A. Venkatesh, *Beyond endoscopy and special forms on* GL(2), J. reine angew. Math. 577 (2004), pp. 23–80.