ASIAN J. MATH.
Vol. 17, No. 3, pp. 561–568, September 2013

007

MANIFOLDS WITH NEF COTANGENT BUNDLE*

ANDREAS HÖRING†

Abstract. Generalising a classical theorem by Ueno, we prove structure results for manifolds with nef or semiample cotangent bundle.

Key words. Cotangent bundle, nef, Iitaka fibration, algebraic foliation.

AMS subject classifications. 14F10, 14D06, 14E30, 14J40, 32J27.

1. Introduction. If X is a submanifold of a complex torus, then by a classical result of Ueno [Uen75, Thm.10.9] the manifold X is an analytic fibre bundle with fibre a torus T over a manifold Y with ample canonical bundle. Moreover if X is projective, then it decomposes (after finite étale cover) as a product $Y \times T$. Since for a submanifold of a complex torus the cotangent bundle Ω_X is globally generated, it is natural to ask if there are analogues of Ueno's result under a weaker positivity assumption. Generalising a conjecture by Yau on compact Kähler manifolds with nonpositive bisectional curvature, Wu und Zheng [WZ02] proposed the following problem.

CONJECTURE 1.1. *Let X be a compact Kähler manifold with nef cotangent bundle. Then there exists a finite étale cover $X' \to X$ such that the Iitaka fibration $X' \to Y'$ is a smooth fibration onto a projective manifold Y with ample canonical bundle and all the fibres are complex tori.*

In this note we prove this conjecture for projective manifolds with semiample canonical bundle, i.e. some positive multiple mK_X is generated by its global sections.

THEOREM 1.2. *Let X be a projective manifold with nef cotangent bundle Ω_X and semiample canonical bundle K_X. Then Conjecture 1.1 holds for X.*

Since $K_X = \det \Omega_X$ is nef, the abundance conjecture [Rei87, Sec.2] claims that the semiampleness condition is redundant. So far this conjecture is known to hold if $\dim X \leq 3$; see [Kwc92]. Note however that a projective manifold with nef cotangent bundle does not contain any rational curves, so the abundance conjecture reduces to the weaker nonvanishing conjecture [HPR11, Thm.1.5]. In particular our statement holds for fourfolds with $\kappa(X) \geq 0$.

For manifolds with nonpositive bisectional curvature one expects the torus fibration to be locally trivial [WZ02, p.264]. This is no longer true if we assume only that Ω_X is nef: universal families over compact curves in the moduli space of abelian varieties (polarised and with level three structure) provide immediate counter-examples. However if we assume that the cotangent bundle Ω_X itself is semiample we obtain a precise analogue of Ueno's theorem:

THEOREM 1.3. *Let X be a projective manifold with semiample cotangent bundle, i.e. for some positive integer $m \in \mathbb{N}$, the symmetric product $S^m\Omega_X$ is globally generated. Then there exists a finite étale cover $X' \to X$ such that $X' \simeq Y \times A$ where Y has ample canonical bundle and A is an abelian variety.*

*Received November 20, 2011; accepted for publication June 13, 2012. Partially supported by the A.N.R. project "CLASS".

†Laboratoire de Mathématiques J. A. Dieudonné, UMR 7351 CNRS, Université de Nice Sophia Antipolis, 06108 Nice cedex 02, IMJ, France (hoering@unice.fr).

This generalises a theorem of Fujiwara [Fuj92, Thm.II].

While many of our arguments also work for compact Kähler manifolds, a crucial tool is a theorem of Kawamata [Kaw91, Thm.2] which allows us to exclude the existence of higher-dimensional fibres. In low dimension an elementary argument works also in the Kähler case (cf. Lemma 3.1), so we obtain:

THEOREM 1.4. *Let X be a compact Kähler manifold with nef cotangent bundle. If $\dim X \leq 3$, then Conjecture 1.1 holds for X.*

This improves a result of Kratz [Kra97, Thm.1].

On a technical level the key point is that in our situation the tangent bundle is numerically flat with respect to the Iitaka fibration. This allows to combine techniques used by Demailly, Peternell and Schneider in the study of manifolds with nef tangent bundles [DPS94] with those introduced by Kollár [Kol93] and Nakayama [Nak99] to understand torus fibrations.

Acknowledgements. I want to thank Simone Diverio, Noburo Nakayama, Thomas Peternell and Maxime Wolff for helpful discussions.

Notation. We work over the complex field $\mathbb{C}$. For positivity notions of vector bundles on compact Kähler and projective varieties we refer to [DPS94] and [Laz04].

A fibration is a proper surjective morphism $\varphi : X \to Y$ with connected fibres from a complex manifold onto a normal complex variety Y. We say that the fibration φ

- is almost smooth if for every $y \in Y$ the reduction F_{red} of the fibre $F := \varphi^{-1}(y)$ is smooth and has the expected dimension;
- is smooth in codimension one if there exists an analytic subset $Z \subset Y$ of codimension at least two such that $(X \setminus \varphi^{-1}(Z)) \to (Y \setminus Z)$ is a smooth fibration;
- has generically constant moduli if there exists a manifold F_0 such that every generic fibre F is isomorphic to F_0. By a theorem of Fischer and Grauert [FG65] this is equivalent to the property that φ is locally trivial over some Zariski open set.

If $\varphi : X \to Y$ is a fibration and $\mu : X' \to X$ a finite étale cover, there exists a fibration $\varphi' : X' \to Y'$ and a finite map $\mu' : Y' \to Y$ such that $\varphi \circ \mu = \mu' \circ \varphi'$. Since we never consider $\mu' : Y' \to Y$ we call the fibration $\varphi' : X' \to Y'$ the Stein factorisation (of φ and μ).

2. A structure result for fibrations. Recall that a vector bundle E on a compact Kähler variety is numerically flat [DPS94, Defn.1.17] if both E and E^* are nef. This is equivalent to the property that E is nef and $\det E$ is numerically trivial, i.e. $c_1(E) = 0$.

If $\varphi : X \to Y$ is a fibration from a Kähler manifold onto a normal variety and E a vector bundle on X, we say that E is φ-nef (resp. φ-numerically flat) if this property holds for any variety $Z \subset Y$ that is contracted by φ, i.e. such that $\varphi(Z) = pt$. We note that if the cotangent bundle Ω_X is φ-nef, then any subvariety $Z \subset X$ contracted by φ has nef cotangent sheaf: indeed Ω_Z is a quotient of $\Omega_X|_Z$, so it is nef. Moreover in this case φ does not contract any rational curves: if $f : \mathbb{P}^1 \to X$ is a non-constant morphism such that $\varphi \circ f$ is constant, the tangent map gives a non-zero map $f^*\Omega_X \to \Omega_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, which violates the nefness assumption.

LEMMA 2.1. *Let X be a Kähler manifold that admits an equidimensional fibration $\varphi : X \to Y$ onto a normal variety Y such that the tangent bundle T_X is φ-numerically flat. Then the following holds:*

1.) The fibration φ is almost smooth. Moreover every set-theoretical fibre F_{red} is a finite étale quotient $T \to F_{\mathrm{red}}$ of a torus T.

2.) There exists a finite étale cover $X' \to X$ such that the Stein factorisation $\varphi' : X' \to Y'$ is smooth in codimension one and the smooth fibres are tori.

3.) If moreover X is projective, there exists a finite étale cover $X' \to X$ such that the Stein factorisation $\varphi' : X' \to Y'$ is an abelian group scheme.

4.) If X is compact and φ has generically constant moduli, there exists a finite étale cover $X' \to X$ such that the Stein factorisation $\varphi' : X' \to Y'$ is smooth and locally trivial. If moreover X is projective, then (after finite étale cover) one has $X' \simeq Y' \times A$ with A an abelian variety.

REMARK. The statement does not generalise to non-Kähler manifolds. In fact there are examples of compact non-Kähler surfaces X admitting an elliptic fibration onto $\mathbb{P}^1$ that is almost smooth with a unique singular fibre. Arguing as in [BHPVdV04, V.13.2] one sees that one cannot remove the multiple fibre by an étale cover $X' \to X$.

Proof. **Step 1: φ almost smooth in codimension one, i.e. there exists a subvariety $Z \subset Y$ of codimension at least two such that $(X \backslash \varphi^{-1}(Z)) \to (Y \backslash Z)$ is almost smooth.**

We argue by contradiction. Choosing a generic disc that meets a codimension one component of the φ-singular locus in a generic point, we reduce the problem to the case where Y is a curve. Let F be a fibre such that the reduction F_{red} is not smooth. We decompose the divisor $F = \sum_{i=1}^{k} a_i F_i$ where the F_i are pairwise distinct prime divisors. Since F_1 is contained in a φ-fibre, the bundle $\Omega_X|_{F_1}$ is numerically flat. Thus its quotient Ω_{F_1} is nef, so on the one hand the dualising sheaf ω_{F_1} is nef. On the other hand by adjunction one has $\omega_{F_1} \simeq (\omega_X \otimes \mathcal{O}_X(F_1))|_{F_1}$. Since $\omega_X|_{F_1}$ and $\mathcal{O}_X(F)|_{F_1}$ are numerically trivial, we see that

$$\omega_{F_1} \sim_{\mathbb{Q}} \mathcal{O}_{F_1}(-\sum_{i=2}^{k} \frac{a_i}{a_1}(F_i \cap F_1)).$$

Thus ω_{F_1} is nef and anti-effective, hence trivial. By connectedness of the fibre, we have $k = 1$, i.e. F is irreducible. Since T_X is φ-nef, a result of Demailly-Peternell-Schneider [DPS94, Prop.5.1] (see also Remark 2.5) now shows that F_{red} is smooth, a contradiction.

Thus φ is almost smooth in codimension one, and if F is a fibre such that F_{red} is smooth, its normal bundle $N_{F_{\mathrm{red}}/X}$ is numerically flat [DPS94, Prop.5.1]. In particular by adjunction $K_{F_{\mathrm{red}}} \equiv 0$ and as we have seen above, the cotangent bundle $\Omega_{F_{\mathrm{red}}}$ is nef. The Chern class inequalities [DPS94, Thm.2.5.]

$$0 = c_1^2(\Omega_{F_{\mathrm{red}}}) \geq c_2(\Omega_{F_{\mathrm{red}}}) \geq 0$$

show that $c_2(F_{\mathrm{red}}) = 0$. Thus a classical result of Bieberbach [Kob87, Cor.4.15] shows that F_{red} is a finite étale quotient of a torus.

Step 2: Proof of Statement 2). Let $N \subset Y$ be a subvariety of codimension at least two. Since φ is equidimensional, $\varphi^{-1}(N)$ has codimension at least two. Hence we have an isomorphism of fundamental groups $\pi_1(X) \simeq \pi_1(X \setminus \varphi^{-1}(N))$ and any étale cover $(X \setminus \varphi^{-1}(N))' \to (X \setminus \varphi^{-1}(N))$ extends to an étale cover $X' \to X$. Thus by Step 1) we can suppose without loss of generality that we are in the situation of the following lemma.

Lemma 2.2. *Let $\varphi : X \to Y$ be an almost smooth fibration from a Kähler manifold X onto a manifold Y. Suppose that φ is smooth in the complement of a smooth divisor $D \subset Y$. Suppose moreover that for every fibre F, the set-theoretical fibre F_{red} is a finite étale quotient $T \to F_{\mathrm{red}}$ of a torus T. Then there exists a finite étale cover $X' \to X$ such that the Stein factorisation $\varphi' : X' \to Y'$ is smooth in codimension one and the smooth fibres are tori.*

Remark. This result is certainly well-known to experts. In fact the fibration being almost smooth, the local monodromies of the variation of Hodge structures around D are finite. The existence of the cover $X' \to X$ then follows analogously to the proof of [Kol93, Thm.6.3]. For the convenience of the reader we follow an argument indicated by Noburo Nakayama.

Proof of Lemma 2.2. We can cover Y by polydiscs Δ of dimension $m := \dim Y$ such that

$$\Delta \cap D = \{(w_1, \ldots, w_m) \in \Delta \mid w_m = 0\}$$

and for $y \in \Delta \cap D$ and $x \in \varphi^{-1}(\Delta)$ there exist local coordinates $z_1, \ldots, z_n$ around x such that φ is given by $(z_1, \ldots, z_n) \to (z_1, \ldots, z_{m-1}, z_m^k)$, where k is the multiplicity of the fibre F. Let $\Delta' \to \Delta$ be a finite map from some m-dimensional disc Δ' that ramifies exactly along $\Delta \cap D$ with multiplicity k. Let $X_{\Delta'}$ be the normalisation of the fibre product $\Delta' \times_\Delta X$, then a local computation shows that $X_{\Delta'} \to \varphi^{-1}(\Delta) \subset X$ is étale and the fibration $X_{\Delta'} \to \Delta'$ is smooth. Since Δ' retracts onto a point we have an isomorphism $\pi_1(F) \simeq \pi_1(X_{\Delta'})$, where F is any fibre. The cover $X_{\Delta'} \to \varphi^{-1}(\Delta)$ being étale and surjective this shows that we have an injection

$$\pi_1(F) \hookrightarrow \pi_1(\varphi^{-1}(\Delta)).$$

By [Nak99, Thm.7.8] this implies that φ is bimeromorphically equivalent to a fibration $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$ which becomes smooth after a finite étale cover. As we have just seen for such a fibration the natural morphism $\pi_1(\tilde{F}) \to \pi_1(\tilde{X})$ is injective. Since the fibrations φ and $\tilde{\varphi}$ are bimeromorphic, this shows that

$$\pi_1(F) \to \pi_1(X)$$

is injective. Thus by [Nak99, Thm.8.6] (which is the analogue of [Kol93, Thm.6.3] for the Kähler case) there exists a finite étale cover $X' \to X$ such that the Stein factorisation $\varphi' : X' \to Y'$ is bimeromorphically equivalent to a smooth torus fibration $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$. Up to blowing up $\tilde{Y}$ and excluding the image of the exceptional locus we can suppose without loss of generality that $\tilde{Y} = Y'$. Since in codimension one the φ'-fibres do not contain any rational curves, there exists a codimension two set $B \subset Y'$ such that the restriction of the bimeromorphic map $\mu : \tilde{X} \dashrightarrow X'$ to $\tilde{X} \setminus \tilde{\varphi}^{-1}(B)$ is a morphism and an isomorphism onto its image. Since $\tilde{\varphi}$ is smooth, this proves the statement. □

Step 3: φ is almost smooth. This property does not change under finite étale cover, so we can assume by Step 2) that φ is smooth in codimension one. Moreover φ is equidimensional, so the relative cotangent sheaf $\Omega_{X/Y}$ is locally free in codimension one and has determinant $\mathcal{O}_X(K_{X/Y})$. We consider the foliation $\mathcal{F} \subset T_X$ defined by the reduction of every φ-fibre F, i.e. on the non-singular locus $F_{\mathrm{red,nons}} \subset F_{\mathrm{red}}$ we have

$$(*) \qquad T_{F_{\mathrm{red,nons}}} = \mathcal{F}|_{F_{\mathrm{red,nons}}}.$$

Since φ is smooth in codimension one, the sheaves $T_{X/Y} := \Omega^*_{X/Y}$ and $\mathcal{F}$ coincide in codimension one, hence $\det \mathcal{F} \simeq \mathcal{O}_X(-K_{X/Y})$. We claim that the foliation $\mathcal{F}$ is regular which obviously implies that the reduction of every fibre is smooth.

Proof of the claim. The inclusion $\mathcal{F} \subset T_X$ induces a map $\alpha : \det \mathcal{F} \to \bigwedge^{\mathrm{rk}\mathcal{F}} T_X$ and by [DPS94, Lemma 1.20] it is sufficient to show that α has rank one in every point. By $(*)$ the restriction of α to $F_{\mathrm{red,nons}}$ identifies to the map induced by $T_{F_{\mathrm{red,nons}}} \subset T_X|_{F_{\mathrm{red,nons}}}$, hence $\alpha|_{F_{\mathrm{red}}}$ is not zero on any irreducible component of F_{red}. Since $\det \mathcal{F} \simeq \mathcal{O}_X(-K_{X/Y})$ and $\bigwedge^{\mathrm{rk}\mathcal{F}} T_X$ are φ-numerically flat, we know by [CP91, Prop.1.2(12)] that $\alpha|_{F_{\mathrm{red}}}$ does not vanish in any point of F_{red}. Thus α does not vanish in any point of X.

Step 4: Proof of Statement 3). By what precedes we know that φ is almost smooth and (after finite étale cover) smooth in codimension one. Since X is projective we know by [Kol93, Thm.6.3] that (after finite étale cover) the fibration φ is birational to an abelian group scheme $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$. Since $\tilde{\varphi}$ is a group scheme, there exists a section $s : \tilde{Y} \to \tilde{X}$. Let Z be the strict transform of $s(\tilde{Y})$ under the birational map $\tilde{X} \dashrightarrow X$. Then $\varphi|_Z : Z \to Y$ is birational, i.e. Z is generically a section of φ. In particular for a general fibre F we have $F \cdot Z = 1$. Since for any fibre F_0 we have $[F_0] = m[F]$ with m the multiplicity of the fibre F_0, we see that all the fibres are reduced. Thus the almost smooth fibration φ is smooth.

Step 5: Proof of Statement 4). By Statements 1) and 2) we know that (after finite étale cover) the almost smooth fibration φ has tori as general fibres. If φ has generically constant moduli, we have (after finite étale cover) that $q(X) = q(Y) + \dim F$ [CP00, Prop.6.7]. Since the reduction of every φ-fibre is an étale quotient of a torus, the Albanese map $\alpha_X : X \to \mathrm{Alb}(X)$ maps each φ-fibre isomorphically onto a fibre of the locally trivial fibration $\varphi_* : \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$. By the universal property of the fibre product we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Alb}(X) \times_{\mathrm{Alb}(Y)} Y \arrow[dr, "\psi"'] & X \arrow[l] \arrow[r, "\alpha_X"] \arrow[d, "\varphi"'] & \mathrm{Alb}(X) \arrow[d, "\varphi_*"] \\
 & Y \arrow[r, "\alpha_Y"] & \mathrm{Alb}(Y)
\end{tikzcd}$$

The map ψ is the pull-back of φ_* by the fibre product, so it is a locally trivial fibration. The base Y is normal, so the total space $\mathrm{Alb}(X) \times_{\mathrm{Alb}(Y)} Y$ is normal. By what precedes the morphism $X \to \mathrm{Alb}(X) \times_{\mathrm{Alb}(Y)} Y$ is bimeromorphic and finite, hence an isomorphism by Zariski's main theorem. In particular $\varphi = \psi$ is smooth and locally trivial.

If X is projective, the Albanese torus is an abelian variety. Thus we know by Poincaré's reducibility theorem [BL04, Thm.5.3.5] that (after finite étale cover) one has $\mathrm{Alb}(X) \simeq \mathrm{Alb}(Y) \times F$, hence the fibre product $\mathrm{Alb}(X) \times_{\mathrm{Alb}(Y)} Y$ is isomorphic to $Y \times F$. □

As a corollary of the proof we obtain the following statement.

Corollary 2.3. *Let X be a Kähler manifold that admits an equidimensional almost smooth fibration $\varphi : X \to Y$ onto a normal variety Y such that the general fibre F is a finite étale quotient $T \to F$ of a torus T. Then there exists a finite étale cover $X' \to X$ such that the Stein factorisation $\varphi' : X' \to Y'$ is smooth in codimension one and the smooth fibres are tori.*

Note that by [Cla10, Lemme 2.3] for a torus fibration that is smooth in codimension one the map $\pi_1(F) \to \pi_1(X)$ is injective. Thus in the situation above φ has generically large fundamental group along the general fibre [Kol93, Defn.6.1], i.e. the statement is a natural inverse to [Kol93, Thm.6.3].

We can also deduce a simplified version of [DPS94, Prop.5.1]:

Corollary 2.4. *Let X be a quasi-projective manifold that admits a fibration $\varphi : X \to Y$ onto a normal variety Y such that the tangent bundle T_X is φ-nef. Then φ is equidimensional and almost smooth. If X is projective, there exists a finite étale cover $X' \to X$ such that the Stein factorisation $\varphi' : X' \to Y'$ is smooth.*

Remark 2.5. *Solá Conde and Wiśniewski [SCW04, Ch.4.2] point out that the proof of the "First case" of [DPS94, Prop.5.1] has a gap and give a completely different proof under the additional condition that φ is a Mori contraction [SCW04, Thm.4.4]. Note that we used [DPS94, Prop.5.1] in the proof of Lemma 2.1, but only for a fibration over a curve which corresponds to the "Second case" of their proof.*

Proof. If K_X is not φ-nef, we know by the relative contraction theorem [KM98, Thm.3.25] that there exists an elementary Mori contraction $\mu : X \to Z$ that factors φ, i.e. there exists a fibration $\psi : Z \to Y$ such that $\varphi = \psi \circ \mu$. Applying [SCW04, Thm.4.4] to μ we see that μ and Z are smooth, in particular T_Z is ψ-nef. Since a composition of equidimensional and almost smooth fibrations is equidimensional and almost smooth, we can argue inductively and suppose without loss of generality that K_X is φ-nef. Since T_X is also φ-nef, it is φ-numerically flat. Hence Ω_X is also φ-nef, so the φ-fibres do not contain any rational curves. By a theorem of Kawamata [Kaw91, Thm.2] this shows that φ is equidimensional. Conclude by Lemma 2.1,1) and 3). □

3. Proofs of the main statements.

Proof of Theorem 1.2. Since K_X is semiample we can consider the Iitaka fibration $\varphi : X \to Y$. Note that the anticanonical divisor $-K_X$ is φ-numerically trivial. Since Ω_X is nef, hence φ-nef, the tangent bundle T_X is φ-numerically flat. By Corollary 2.4 the fibration φ is equidimensional. We conclude by Lemma 2.1,3) that there exists a finite étale cover such that the Iitaka fibration is an abelian group scheme. By [Kol93, 5.9.1] the projective manifold Y is of general type, so in order to see that K_Y is ample it is sufficient to show that Y does not contain any rational curves[1]. Yet the abelian group scheme $X \to Y$ has a section, so any rational curve $\mathbb{P}^1 \to Y$ would lift to X. This is excluded by the nefness of Ω_X. □

Proof of Theorem 1.3. By Theorem 1.2 we can suppose (after finite étale cover) that the Iitaka fibration $\varphi : X \to Y$ is smooth with abelian fibres. Let $F = \varphi^{-1}(y)$ be any smooth fibre, then we have an exact sequence

$$0 \to (\varphi^*\Omega_Y)|_F \to \Omega_X|_F \to \Omega_F \simeq \mathcal{O}_F^{\oplus \dim F} \to 0.$$

Since $\Omega_X|_F$ is semiample and $\det \Omega_F$ is trivial we know by [Fuj92, Cor.4] that the exact sequence splits. In particular the Kodaira spencer map is zero in y. Since this holds for all y we see that φ has constant moduli. Conclude by Lemma 2.1,4). □

Before we can prove Theorem 1.4 we need a technical lemma which is a first step towards a generalisation of [Kaw91, Thm.2] to the Kähler case.

[1]This is a well-known consequence of cone theorem, base-point free theorem and [Kaw91, Thm.2].

Lemma 3.1. *Let X be a compact Kähler threefold and let $\varphi : X \to S$ be a fibration onto a projective surface such that $-K_X$ is φ-nef. Let $D \subset X$ be a divisor that is contracted by φ. Then D is uniruled.*

Remark. The proof is based on the fact that a compact Kähler surface D with Gorenstein singularities is uniruled if the canonical sheaf ω_D is not pseudoeffective. This is well-known if D is smooth and standard arguments (cf. the proof of [HPR11, Lemma 4.2]) allow to generalise to singular D. Note that for projective manifolds the implication

$$K_D \text{ not pseudoeffective } \Rightarrow D \text{ uniruled}$$

is a famous theorem [BDPP04], but for Kähler manifolds this is only known up to dimension three [Bru06].

Proof. We fix a Kähler form α on X. Let H be an effective divisor passing through $\varphi(D)$, then we can write $\varphi^* H = H' + mD$ with $m \in \mathbb{N}$ and $D \not\subset \operatorname{supp} H'$ but $D \cap H' \neq 0$. Since $\varphi^* H \cdot D = 0$ we have

$$\alpha \cdot (\varphi^* H)^2 = \alpha \cdot \varphi^* H \cdot (H' + mD) = \alpha \cdot (H')^2 + \alpha \cdot H' \cdot mD,$$

and developing the left hand side implies $\alpha \cdot H' \cdot mD = -\alpha \cdot m^2 D^2$. Since $H' \cap D$ is an effective non-zero cycle, we see that $\alpha \cdot D^2 < 0$. By the adjunction formula we have $\omega_D \simeq \mathcal{O}_D(K_X + D)$, so our computation shows that

$$\omega_D \cdot \alpha|_D = (K_X + D) \cdot D \cdot \alpha < 0.$$

Therefore ω_D is not pseudoeffective, hence D is uniruled. $\square$

Proof of Theorem 1.4. Since K_X is nef and $\dim X \leq 3$, it is semiample [Pet01, Thm.1], [DP03]. Let $\varphi : X \to Y$ be the Iitaka fibration, then the anticanonical divisor $-K_X$ is φ-trivial. Since Ω_X is nef, hence φ-nef and $-K_X$ is φ-trivial, the tangent bundle T_X is φ-trivial. If $\dim Y = 1$ we conclude by Lemma 2.1,2). The cases $\dim Y = 0$ or 3 being trivial, we are left with case $\dim Y = 2$:

By Lemma 3.1, the fibration φ is equidimensional. Thus it is almost smooth and (after finite étale cover) smooth in codimension one by Lemma 2.1,2). Since every complete family of elliptic curves is isotrivial, φ has generically constant moduli. We conclude by Lemma 2.1,4). $\square$

REFERENCES

[BDPP04] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom., 22 (2013), pp. 201–248.

[BHPVdV04] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, volume 4 of "Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge", Springer-Verlag, Berlin, second edition, 2004.

[BL04] C. Birkenhake and H. Lange, *Complex abelian varieties*, volume 302 of "Grundlehren der Mathematischen Wissenschaften", Springer-Verlag, Berlin, second edition, 2004.

[Bru06] M. Brunella, *A positivity property for foliations on compact Kähler manifolds*, Internat. J. Math., 17:1 (2006), pp. 35–43.

[Cla10] B. Claudon, *Invariance de la Γ-dimension pour certaines familles kählériennes de dimension 3*, Math. Z., 266:2 (2010), pp. 265–284.

[CP91] F. Campana and T. Peternell, *Projective manifolds whose tangent bundles are numerically effective*, Math. Ann., 289:1 (1991), pp. 169–187.

[CP00] F. Campana and T. Peternell, *Complex threefolds with non-trivial holomorphic 2-forms*, J. Algebraic Geom., 9:2 (2000), pp. 223–264.

[DP03] J.-P. Demailly and T. Peternell, *A Kawamata-Viehweg vanishing theorem on compact Kähler manifolds*, J. Differential Geom., 63:2 (2003), pp. 231–277.

[DPS94] J.-P. Demailly, T. Peternell, and M. Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom., 3:2 (1994), pp. 295–345.

[FG65] W. Fischer and H. Grauert, *Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, 1965 (1965), pp. 89–94.

[Fuj92] T. Fujiwara, *Varieties of small Kodaira dimension whose cotangent bundles are semiample*, Compositio Math., 84:1 (1992), pp. 43–52.

[HPR11] A. Höring, T. Peternell, and I. Radloff, *Uniformisation in dimension four: towards a conjecture of Iitaka*, Mathematische Zeitschrift, 274:1-2 (2013), pp. 483–497.

[Kaw91] Y. Kawamata, *On the length of an extremal rational curve*, Invent. Math., 105:3 (1991), pp. 609–611.

[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, volume 134 of "Cambridge Tracts in Mathematics", Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti.

[Kob87] S. Kobayashi, *Differential geometry of complex vector bundles*, volume 15 of "Publications of the Mathematical Society of Japan", Princeton University Press, Princeton, NJ, 1987. Kanô Memorial Lectures, 5.

[Kol93] J. Kollár, *Shafarevich maps and plurigenera of algebraic varieties*, Invent. Math., 113:1 (1993), pp. 177–215.

[Kra97] H. Kratz, *Compact complex manifolds with numerically effective cotangent bundles*, Doc. Math., 2 (1997), pp. :183–193 (electronic).

[Kwc92] J. Kollár (with 14 coauthors), *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[Laz04] R. Lazarsfeld, *Positivity in algebraic geometry. II*, volume 49 of "Ergebnisse der Mathematik und ihrer Grenzgebiete", Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[Nak99] N. Nakayama, *Compact Kähler manifolds whose universal covering spaces are biholomorphic to* $\mathbf{C}^n$, RIMS preprint, 1230, 1999.

[Pet01] T. Peternell, *Towards a Mori theory on compact Kähler threefolds. III*, Bull. Soc. Math. France, 129:3 (2001), pp. 339–356.

[Rei87] M. Reid, *Tendencious survey of 3-folds*, in "Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), volume 46 of Proc. Sympos. Pure Math., pp. 333–344. Amer. Math. Soc., Providence, RI, 1987.

[SCW04] L. E. Solá Conde and J. A. Wiśniewski, *On manifolds whose tangent bundle is big and 1-ample*, Proc. London Math. Soc. (3), 89:2 (2004), pp. 273–290.

[Uen75] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Springer-Verlag, Berlin, 1975. Notes written in collaboration with P. Cherenack, Lecture Notes in Mathematics, Vol. 439.

[WZ02] H.-H. Wu and F. Zheng, *Compact Kähler manifolds with nonpositive bisectional curvature*, J. Differential Geom., 61:2 (2002), pp. 263–287.