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VOLUME GROWTH, EIGENVALUE AND COMPACTNESS FOR SELF-SHRINKERS*

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Abstract. In this paper, we show an optimal volume growth for self-shrinkers, and estimate a lower bound of the first eigenvalue of $\mathcal{L}$ operator on self-shrinkers, inspired by the first eigenvalue conjecture on minimal hypersurfaces in the unit sphere by Yau [14]. By the eigenvalue estimates, we can prove a compactness theorem on a class of compact self-shrinkers in $\mathbb{R}^3$ obtained by Colding-Minicozzi under weaker conditions.

Key words. Self-shrinkers, self similar solution, volume growth, eigenvalue estimates, compactness theorem.

AMS subject classifications. 53A07, 53A10, 53C21, 53C44.

1. Introduction. Let $X : M^n \to \mathbb{R}^{n+m}$ be an isometric immersion from an n-dimensional manifold M^n to Euclidean space $\mathbb{R}^{n+m}(m \geq 1)$ with the tangent bundle TM and the normal bundle NM along M. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connections on M and $\mathbb{R}^{n+m}$, respectively. Then we define the second fundamental form B by $B(V,W) = (\overline{\nabla}_V W)^N = \overline{\nabla}_V W - \nabla_V W$ for any $V, W \in \Gamma(TM)$, where $(\cdots)^N$ stands for the orthogonal projection into the normal bundle NM. The mean curvature vector H of M is given by $H = \mathrm{trace}(B) = \sum_{i=1}^n B(e_i, e_i)$, where $\{e_i\}$ is a local orthonormal frame field of M.

M^n is said to be a *self-shrinker* in $\mathbb{R}^{n+m}$ if it satisfies

$$H = -\frac{X^N}{2}. \tag{1.1}$$

Here, the factor $-\frac{1}{2}$ (when the codimension $m = 1$, the definition here is as the same as [4]) could be replaced by other negative number, while Ecker-Huisken defines $H = -X^N$ [8]. Self-shrinkers play an important role in the study of mean curvature flow. They are not only special solutions to the mean curvature flow equations (those where later time slices are rescalings of earlier), but they also describe all possible blow ups at a given type I singularity of a mean curvature flow (abbreviated by MCF in what follows).

After the pioneer work on self-shrinking hypersurfaces of G. Huisken [11][12], T. H. Colding and W. P. Minicozzi II gave a comprehensive study for self-shrinking hypersurfaces [4]. Their papers reveal the importance of the subject. For higher codimension, there is a few study, see [15] for example.

There are several other ways to characterize self-shrinkers (see [5] for hypersurfaces, and high codimensional situation is similar):
(1) The one-parameter family of submanifolds $\sqrt{-t}M \subset \mathbb{R}^{n+m}$ satisfies MCF equations.
(2) M is a minimal submanifold in $\mathbb{R}^{n+m}$ endowed with the conformally flat metric of the conformal factor $e^{-\frac{|X|^2}{2n}}$.

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(3) M is a critical point for the functional F defined on immersed submanifolds in $\mathbb{R}^{n+m}$ by

$$F(M) = (4\pi)^{-n/2}\int_M e^{-\frac{|X|^2}{4}}d\mu. \tag{1.2}$$

Self-shrinkers satisfy elliptic equations(systems) of the second order, see (1.1). It is an important class of submanifolds, which is closely related to minimal surface theory. We expect certain technique in minimal surface theory (see [17]) could be modified to study self-shrinkers.

For a complete non-compact manifold the volume growth is important. By easy arguments we can show that any complete non-compact self-shrinker properly immersed in Euclidean space with arbitrary codimension has Euclidean volume growth, just like the trivial self-shrinker: planes. It is in a sharp contrast to the complete minimal submanifolds in Euclidean space. Even for complete stable minimal hypersurfaces, it is still unclear whether they have Euclidean volume growth.

THEOREM 1.1. *Any complete non-compact properly immersed self-shrinker M^n in $\mathbb{R}^{n+m}$ has Euclidean volume growth at most.*

It is natural to raise a counterpart of the Calabi-Chern problem on minimal surfaces in $\mathbb{R}^3$. Is there a complete non-compact self-shrinker in Euclidean space, which is contained in a Euclidean ball?

REMARK 1.2. *It is worthy to compare the above Theorem with the interesting result of Cao-Zhou on the volume growth of complete gradient shrinking Ricci soliton [3].*

Let Σ^n be a compact embedded minimal hypersurface in $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$. It is well known that the coordinate functions are eigenfunctions of Laplacian operator on Σ with eigenvalue n. In [14], S. T. Yau conjectured that the first eigenvalue of Σ would be n. Choi-Wang, [2], proved that the first eigenvalue of Σ is bounded below by $n/2$.

In [1], H. I. Choi and R. Schoen gave the compactness theorem for minimal surfaces using the first eigenvalue estimates for Laplacian operator on Σ. Precisely, let N be a compact 3-dimensional manifold with positive Ricci curvature, then the space of compact embedded minimal surfaces of fixed topological type in N is compact in the C^k topology for any $k \geq 2$.

In [5], Colding-Minicozzi proved a compactness theorem for complete embedded self shrinkers in $\mathbb{R}^3$. Such compactness theorem acts a key role for proving a long-standing conjecture (of Huisken) classifying the singularities of mean curvature flow starting from a generic closed embedded surface (see [4]).

Let Δ, div and $d\mu$ be Laplacian, divergence and volume element on M, respectively. There is a linear operator

$$\mathcal{L} = \Delta - \frac{1}{2}\langle X, \nabla(\cdot)\rangle = e^{\frac{|X|^2}{4}}\operatorname{div}(e^{-\frac{|X|^2}{4}}\nabla\cdot).$$

On Euclidean space, this operator is so-called Ornstein-Uhlenbeck operator in stochastic analysis. So it can be seen as a generalization of Ornstein-Uhlenbeck operator. The $\mathcal{L}$ operator was introduced and studied firstly on self-shrinkers by Colding-Minicozzi in [4], where the authors also showed that $\mathcal{L}$ is self-adjoint respect to the measure $e^{-\frac{|X|^2}{4}}d\mu$. It is a weighted Laplacian and closely related to the self-shrinkers.

In Euclidean space the eigenvalues of the Ornstein-Uhlenbeck are well-known. On self-shrinkers it is interesting to study the eigenvalues of $\mathcal{L}$ operator. Now, we estimate its first eigenvalue (please see Chapter 3 for the definition) in a manner analogous to the arguments in [2]. For compact self-shrinkers the estimates are rather neat. It is also enough for the compactness applications.

THEOREM 1.3. *Let M^n be a compact embedded self-shrinker in $\mathbb{R}^{n+1}$, then the first eigenvalue λ_1 for the operator $\mathcal{L}$ on M satisfies $\lambda_1 \in [\frac{1}{4}, \frac{1}{2}]$.*

With the help of Theorem 1.3, uniform volume growth for compact embedded self-shrinkers could be estimated by genus and this will yield a compactness theorem. Give a non-negative integer g and a constant $D > 0$, and let $S_{g,D}$ denote the space of all compact embedded self-shrinkers in $\mathbb{R}^3$ with genus at most g, and diameter at most D. We have a compactness theorem as follows.

THEOREM 1.4. *For each fixed g and D, the space $S_{g,D}$ is compact. Namely, any sequence in $S_{g,D}$ has a subsequence that converges uniformly in the C^k topology (for any $k \geq 0$) to a surface in $S_{g,D}$.*

Colding-Minicozzi gave in [5] a compactness theorem for a class of self-shrinkers with bounded entropy in $\mathbb{R}^3$. The assumption of bounded entropy is natural since Colding-Minicozzi proved the conjecture of Huisken in [4] and these automatically satisfy such a bound. However, Theorem 1.4 shows the compactness theorem holds without the assumption of bounded entropy for compact case (please see Corollary 8.2 of [4]).

In this paper, we always suppose that M is an $n-$dimensional smooth submanifold in $\mathbb{R}^{n+m}$ with $n \geq 2$, the function $\rho = e^{-\frac{|X|^2}{4}}$ with $X = (x_1, \cdots, x_{n+m}) \in \mathbb{R}^{n+m}$. Let $\langle \cdot, \cdot \rangle$ be standard inner product of $\mathbb{R}^{n+m}$, and B_r be a standard ball in $\mathbb{R}^{n+m}$ with radius r and centered at the origin, and $D_r = M \cap B_r$ for $r > 0$. When $m = 1$ (codimension is 1), let ν be unit outward normal field of M, and $\langle H, \nu \rangle$ be mean curvature of M. We also write $H = \langle H, \nu \rangle$ if there is no ambiguity in the context. We agree with the following range of indices

$$1 \leq i, j, k, \cdots \leq n + m, \quad 1 \leq \alpha, \beta, \gamma, \cdots \leq n.$$

2. Volume growth of self shrinkers. Let M be an n-dimensional complete self shrinkers in $\mathbb{R}^{n+m}$. By (1.1) we have $\Delta X = H = -\frac{1}{2}X^N$ for any $X = (x_1, \cdots, x_{n+m}) \in \mathbb{R}^{n+m}$, then(see also [4])

$$\mathcal{L}x_i = \Delta x_i - \frac{1}{2}\langle X, \nabla x_i \rangle = -\frac{1}{2}\langle X^N, E_i \rangle - \frac{1}{2}\langle X, (E_i)^T \rangle = -\frac{1}{2}x_i, \tag{2.1}$$

where $\{E_i\}_{i=1}^{n+m}$ is a standard basis of $\mathbb{R}^{n+m}$ and $(\cdots)^T$ denotes the orthogonal projection into the tangent bundle TM. Moreover,

$$\mathcal{L}|X|^2 = 2x_i\mathcal{L}x_i + 2|\nabla X|^2 = 2n - |X|^2, \tag{2.2}$$

and

$$\Delta|X|^2 = 2\langle X, \Delta X \rangle + 2|\nabla X|^2 = 2\langle X, H \rangle + 2n = 2n - 4|H|^2. \tag{2.3}$$

Now, we give an analytic lemma which will be used in proving volume growth.

LEMMA 2.1. *If $f(r)$ is a monotonic increasing nonnegative function on $[0, +\infty)$ with $f(r) \leq C_1 r^n f(\frac{r}{2})$ on $[C_2, +\infty)$ for some positive constant n, C_1, C_2, here $C_2 > 1$,* *then* $f(r) \le C_3 e^{2n(\log r)^2}$ *on* $[C_2, +\infty)$ *for some positive constant* C_3 *depending only on* $n, C_1, C_2, f(C_2)$.

Proof. If $f(\frac{C_2}{2}) = 0$, then $f(r) = 0$ for $r \ge \frac{C_2}{2}$ and this Lemma holds obviously. Hence we could assume $f(\frac{C_2}{2}) > 0$, then the function $g(r) = \log f(r)$ on $[C_2, \infty)$ is well defined. By the assumption, one has

$$g(r) \le g(\frac{r}{2}) + \log C_1 + n \log r, \qquad \text{on} \quad [C_2, +\infty).$$

Let $k = [\frac{\log \frac{r}{C_2}}{\log 2}] + 1$, then $\frac{C_2}{2} \le \frac{r}{2^k} < C_2$, which implies $\frac{r}{2^{k-1}} \ge C_2 > 1$. By iteration,

$$\begin{aligned} g(r) \le & g(\frac{r}{2^2}) + 2\log C_1 + n(\log r + \log \frac{r}{2}) \le \cdots \\ \le & g(\frac{r}{2^k}) + k \log C_1 + n \sum_{j=0}^{k-1} \log \frac{r}{2^j}. \end{aligned}$$

Then we have

$$\begin{aligned} g(r) \le & g(C_2) + k(\log C_1 + n \log r) \\ \le & g(C_2) + (\frac{\log \frac{r}{C_2}}{\log 2} + 1)(\log C_1 + n \log r) \\ \le & \log C_3 + 2n(\log r)^2, \end{aligned} \tag{2.4}$$

where C_3 is a positive constant depending only on $n, C_1, C_2, f(C_2)$. By the definition of g, (2.4) implies

$$f(r) \le C_3 e^{2n(\log r)^2}$$

on $[C_2, +\infty)$. □

For a complete non-compact $n-$submanifold M in $\mathbb{R}^{n+m}$, we say that M has *Euclidean volume growth at most* if there is a constant C so that for all $r \ge 1$,

$$\int_{D_r} 1 d\mu \le C r^n.$$

For a complete self-shrinker M^n in $\mathbb{R}^{n+m}$, we define a functional F_t on any set $\Omega \subset M$ (see also [4] for the definition of F_t) by

$$F_t(\Omega) = \frac{1}{(4\pi t)^{n/2}} \int_\Omega e^{-\frac{|X|^2}{4t}} d\mu, \quad \text{for} \quad t > 0.$$

THEOREM 2.2. *Any complete non-compact properly immersed self-shrinker* M^n *in* $\mathbb{R}^{n+m}$ *has Euclidean volume growth at most. Precisely,* $\int_{D_r} 1 d\mu \le C r^n$ *for* $r \ge 1$, *where* C *is a constant depending only on* n *and the volume of* D_{8n}.

Proof. We differential $F_t(D_r)$ with respect to t,

$$F_t'(D_r) = (4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} \int_{D_r} (-\frac{n}{2} + \frac{|X|^2}{4t}) e^{-\frac{|X|^2}{4t}} d\mu.$$

A straightforward calculation shows

$$
\begin{aligned}
-e^{\frac{|X|^2}{4t}}\operatorname{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2) &= -\Delta|X|^2 + \frac{1}{4t}\nabla|X|^2\cdot\nabla|X|^2 \\
&= -2\langle H, X\rangle - 2n + \frac{1}{4t}4|X^T|^2 \\
&= |X^N|^2 + \frac{|X^T|^2}{t} - 2n \\
&\geq \frac{|X|^2}{t} - 2n \quad (\ when\ t\geq 1\ ),
\end{aligned} \tag{2.5}
$$

where the third equality above uses the self-shrinker's equation (1.1). Since

$$\nabla|X|^2 = 2X^T$$

and the unit normal vector to ∂D_r is $\frac{X^T}{|X^T|}$, then for $t\geq 1$ one gets

$$
\begin{aligned}
F_t'(D_r) \leq& \pi^{-\frac{n}{2}}(4t)^{-(\frac{n}{2}+1)}\int_{D_r} -\operatorname{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2)d\mu \\
=& \pi^{-\frac{n}{2}}(4t)^{-(\frac{n}{2}+1)}\int_{\partial D_r} -2|X^T|e^{-\frac{|X|^2}{4t}} \leq 0.
\end{aligned} \tag{2.6}
$$

We now integrate $F_t'(D_r)$ over t from 1 to $r^2\geq 1$ and get $F_{r^2}(D_r)\leq F_1(D_r)$, namely,

$$(4\pi r^2)^{-\frac{n}{2}}\int_{D_r} e^{-\frac{|X|^2}{4r^2}}d\mu \leq (4\pi)^{-\frac{n}{2}}\int_{D_r} e^{-\frac{|X|^2}{4}}d\mu.$$

Then

$$
\begin{aligned}
\frac{1}{r^n}e^{-\frac{1}{4}}\int_{D_r} 1d\mu \leq& \frac{1}{r^n}\int_{D_r} e^{-\frac{|X|^2}{4r^2}}d\mu \leq \int_{D_r} e^{-\frac{|X|^2}{4}}d\mu = \int_{(D_r\setminus D_{\frac{r}{2}})\cup D_{\frac{r}{2}}} e^{-\frac{|X|^2}{4}}d\mu \\
\leq& e^{-\frac{r^2}{16}}\int_{D_r} 1d\mu + \int_{D_{\frac{r}{2}}} 1d\mu.
\end{aligned} \tag{2.7}
$$

Let $f(r) = \int_{D_r} 1d\mu$, then (2.7) implies

$$f(r) \leq 2e^{\frac{1}{4}}r^n f(\frac{r}{2}) \quad for \quad r\geq 8n.$$

With the help of Lemma 2.1, we obtain

$$f(r) \leq C_4 e^{2n(\log r)^2} \quad for \quad r\geq 8n,$$

where C_4 is a constant depending only on $n, f(8n)$. Hence

$$
\begin{aligned}
\int_M e^{-\frac{|X|^2}{4}}d\mu =& \sum_{j=0}^{\infty}\int_{D_{8n(j+1)}\setminus D_{8nj}} e^{-\frac{|X|^2}{4}}d\mu \leq \sum_{j=0}^{\infty} e^{-\frac{(8nj)^2}{4}} f(8n(j+1)) \\
\leq& C_4\sum_{j=0}^{\infty} e^{-\frac{(8nj)^2}{4}} e^{2n(\log(8n)+\log(j+1))^2} \leq C_5,
\end{aligned}
$$

where C_5 is a constant depending only on $n, f(8n)$. Observe (2.7), we finish the proof. □

If M is an entire graphic self shrinking hypersurface in $\mathbb{R}^{n+1}$, then M has Euclidean volume growth. Then by [8], we know that M is a hyperplane, which is also obtained in [16].

Remark 2.3. *Let M^n be a complete properly immersed self-shrinker in $\mathbb{R}^{n+m}$, then* (2.2) *yields(see also [4])*

$$\int_M |X|^2\rho = 2n\int_M \rho. \tag{2.8}$$

For any $0<\epsilon<\sqrt{2n}$ we obtain

$$2\sqrt{2n}\epsilon\int_{M\setminus D_{\sqrt{2n}+\epsilon}}\rho \le \int_{M\setminus D_{\sqrt{2n}+\epsilon}}(|X|^2-2n)\rho = \int_{D_{\sqrt{2n}+\epsilon}}(2n-|X|^2)\rho \le 2n\int_{D_{\sqrt{2n}}}\rho.$$

Let η be a cut-off function satisfying $\eta\mid_{B_{\sqrt{2n}-\epsilon}}\equiv 1$, $\eta\mid_{\mathbb{R}^{n+m}\setminus B_{\sqrt{2n}}}\equiv 0$ and $|\bar\nabla\eta|\le\frac{1}{\epsilon}$, then

$$\sqrt{2n}\epsilon\int_{D_{\sqrt{2n}-\epsilon}}\rho \le \int_{D_{\sqrt{2n}}}(2n-|X|^2)\eta\rho = 2\int_{D_{\sqrt{2n}}}\nabla\eta\cdot\frac{X^T}{|X|}\rho \quad \le \frac{2}{\epsilon}\int_{D_{\sqrt{2n}}\setminus D_{\sqrt{2n}-\epsilon}}\rho.$$

Hence, we conclude that there is a constant C_6, C_7 depending only on n and ϵ, such that

$$\int_M \rho d\mu \le C_6\int_{M\cap(B_{\sqrt{2n}+\epsilon}\setminus B_{\sqrt{2n}-\epsilon})}\rho d\mu. \tag{2.9}$$

Combining (2.7) *and* (2.9)*, we have*

$$\int_{D_r} 1 d\mu \le C_7 r^n\int_{M\cap(B_{\sqrt{2n}+\epsilon}\setminus B_{\sqrt{2n}-\epsilon})} 1 d\mu.$$

Corollary 2.4. *Let M^n be a complete non-compact properly immersed self-shrinker in $\mathbb{R}^{n+m}$, then $F_t(M)\le F_1(M)$ for any $t>0$.*

Proof. Let ζ is a cut-off function such that

$$\zeta(|X|) = \begin{cases} 1 & \text{if} \quad X\in B_r \\ linear & \text{if} \quad X\in B_{2r}\setminus B_r \\ 0 & \text{if} \quad X\in\mathbb{R}^{n+m}\setminus B_{2r}, \end{cases}$$

Combining (2.5), for any $t>0$ one gets

$$\begin{aligned} &\left|\int_{D_r} -\mathrm{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2)d\mu\right| \\ \le&\left|\int_M -\mathrm{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2)\zeta d\mu\right| + \left|\int_{M\setminus D_r} -\mathrm{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2)\zeta d\mu\right| \\ \le&\left|\int_M (\nabla|X|^2\cdot\nabla\zeta)e^{-\frac{|X|^2}{4t}}d\mu\right| + \int_{M\setminus D_r}\left||X^N|^2+\frac{|X^T|^2}{t}-2n\right|e^{-\frac{|X|^2}{4t}}d\mu \\ \le&\frac{2}{r}\int_M |X|e^{-\frac{|X|^2}{4t}}d\mu + \int_{M\setminus D_r}\left((1+\frac{1}{t})|X|^2+2n\right)e^{-\frac{|X|^2}{4t}}d\mu, \end{aligned}$$

which implies

$$\lim_{r\to\infty}\int_{D_r} -\operatorname{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2)d\mu = 0. \tag{2.10}$$

When $0 < t \leq 1$, from (2.5), we have

$$-e^{\frac{|X|^2}{4t}}\operatorname{div}(e^{-\frac{|X|^2}{4t}}\nabla_M|X|^2) \leq \frac{|X|^2}{t} - 2n,$$

then

$$F_t'(D_r) \geq \pi^{-\frac{n}{2}}(4t)^{-(\frac{n}{2}+1)}\int_{D_r} -\operatorname{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2)d\mu. \tag{2.11}$$

Combining (2.10) and (2.11), we know

$$F_t'(M) = \lim_{r\to\infty} F_t'(D_r) \geq 0,$$

which implies

$$F_t(M) \leq F_1(M), \quad for\ 0 < t \leq 1.$$

On the other hand, by (2.6), we have

$$(4\pi R^2)^{-n/2}\int_{D_r} e^{-\frac{|X|^2}{4R^2}}d\mu \leq (4\pi)^{-n/2}\int_{D_r} e^{-\frac{|X|^2}{4}}d\mu \qquad for\ any \quad R \geq 1.$$

Let r go to infinity, thus we finish the proof. □

In hypersurface case, Colding and Minicozzi II (Lemma 7.10 in [4]) has proved a more general result than Corollary 2.4.

3. The first eigenvalue of self-shrinkers and compactness theorem. Let $\mathbb{R}^{n+1}$ be Euclidean space with the canonical metric, with Levi-Civita connection $\overline{\nabla}$, Laplacian operator $\overline{\Delta}$, and divergence $\overline{\operatorname{div}}$. Let $\overline{\mathcal{L}} = \overline{\Delta} - \frac{1}{2}\langle X, \overline{\nabla}\cdot\rangle$. Reilly derived a useful integral formula for Laplacian operator [13](see also [2]). Now, we derive a Reilly type formula for the operator $\mathcal{L}$.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^{n+1}$ with smooth boundary. Suppose that f satisfies*

$$\begin{cases} \overline{\mathcal{L}}f = g & \text{in} \quad \Omega \\ f = u & \text{on} \quad \partial\Omega, \end{cases}$$

where g is a smooth function on Ω and u is a smooth function on $\partial\Omega$, then

$$\int_\Omega g^2\rho = \int_\Omega |\overline{\nabla}^2 f|^2\rho + \frac{1}{2}\int_\Omega |\overline{\nabla} f|^2\rho + 2\int_{\partial\Omega} f_\nu\mathcal{L}u\rho - \int_{\partial\Omega} h(\nabla u, \nabla u)\rho - \int_{\partial\Omega} f_\nu^2\left(\frac{\langle X,\nu\rangle}{2} + H\right)\rho,$$

where $h(\cdot,\cdot) = \langle B(\cdot,\cdot),\nu\rangle$, B is the second fundamental form on $\partial\Omega$, ν is the outward unit normal vector field on $\partial\Omega$ and mean curvature $H = \operatorname{trace}(h)$.

Proof. Let $\{\frac{\partial}{\partial x_i}\}_{i=1}^{n+1}$ be a canonical basis of $\mathbb{R}^{n+1}$, $f_i = \frac{\partial f}{\partial x_i}$, and so on. Since $\overline{\mathcal{L}}f = g$, then we have

$$\overline{\mathcal{L}}f_i = \sum_j (f_{ijj} - \frac{1}{2}x_j f_{ij}) = \frac{\partial}{\partial x_i}(g + \frac{1}{2}\sum_j x_j f_j) - \frac{1}{2}\sum_j x_j f_{ij} = g_i + \frac{f_i}{2},$$

and

$$\frac{1}{2}\overline{\mathcal{L}}|\overline{\nabla} f|^2 = \sum_{i,j} f_{ij}^2 + f_i\overline{\mathcal{L}} f_i = |\overline{\nabla}^2 f|^2 + \langle\overline{\nabla} f, \overline{\nabla} g\rangle + \frac{1}{2}|\overline{\nabla} f|^2. \tag{3.1}$$

Integrating the equality (3.1) by parts we get

$$\begin{aligned}
\frac{1}{2}\int_\Omega \overline{\mathcal{L}}|\overline{\nabla} f|^2\rho &= \int_\Omega |\overline{\nabla}^2 f|^2\rho + \int_\Omega \langle\overline{\nabla} f, \overline{\nabla} g\rangle\rho + \frac{1}{2}\int_\Omega |\overline{\nabla} f|^2\rho \\
&= \int_\Omega |\overline{\nabla}^2 f|^2\rho + \frac{1}{2}\int_\Omega |\overline{\nabla} f|^2\rho + \int_\Omega (\overline{\mathrm{div}}(\rho g\overline{\nabla} f) - g\overline{\mathrm{div}}(\rho\overline{\nabla} f)) \\
&= \int_\Omega |\overline{\nabla}^2 f|^2\rho + \frac{1}{2}\int_\Omega |\overline{\nabla} f|^2\rho + \int_{\partial\Omega} f_\nu g\rho - \int_\Omega g^2\rho.
\end{aligned} \tag{3.2}$$

On the other hand, we select an orthonormal frame field $\{e_1, \cdots, e_{n+1}\}$ near the boundary of Ω such that $\{e_1, \cdots, e_n\}$ are tangential to $\partial\Omega$, and $\nabla_{e_\alpha} e_\beta = \overline{\nabla}_{e_{n+1}} e_i = 0$ at a considered point in $\partial\Omega$ and $\nu = e_{n+1}$ is the outward unit normal vector. Let $h_{\alpha\beta} = \langle\overline{\nabla}_{e_\alpha} e_\beta, \nu\rangle = \langle B(e_\alpha, e_\beta), \nu\rangle$, then integrating by parts gives

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \overline{\mathcal{L}}|\overline{\nabla} f|^2\rho = \frac{1}{2}\int_\Omega \overline{\mathrm{div}}(\rho\overline{\nabla}|\overline{\nabla} f|^2) = \int_{\partial\Omega}\sum_{i=1}^{n+1}(e_i f)(e_{n+1}e_i f)\rho \\
&= \int_{\partial\Omega} f_\nu(e_{n+1}e_{n+1}f)\rho + \sum_{\alpha=1}^n\int_{\partial\Omega}(e_\alpha f)(e_\alpha e_{n+1} f)\rho + \sum_{\alpha=1}^n\int_{\partial\Omega}[e_{n+1}, e_\alpha](f)(e_\alpha f)\rho \\
&= \int_{\partial\Omega} f_\nu(e_{n+1}e_{n+1}f)\rho - \int_{\partial\Omega} f_\nu(\mathcal{L}u)\rho + \sum_{\alpha,\beta=1}^n\int_{\partial\Omega} h_{\alpha\beta}e_\beta(f)e_\alpha(f)\rho.
\end{aligned} \tag{3.3}$$

Moreover,

$$\begin{aligned}
e_{n+1}e_{n+1}f &= \sum_{i=1}^{n+1}\left(e_ie_if - \left(\overline{\nabla}_{e_i}e_i\right)f\right) - \sum_{\alpha=1}^n(e_\alpha e_\alpha f) + \sum_{\alpha=1}^n\left(\overline{\nabla}_{e_\alpha}e_\alpha\right)f \\
&= \overline{\Delta} f - \Delta f + \sum_{\alpha=1}^n h_{\alpha\alpha}f_\nu = \overline{\mathcal{L}} f - \mathcal{L} u + \frac{\langle X, \nu\rangle}{2}f_\nu + Hf_\nu.
\end{aligned} \tag{3.4}$$

Combining (3.2)-(3.4), we complete the proof. □

We would use the above Reilly type formula to estimate the first eigenvalue of $\mathcal{L}$ operator on a self-shrinker in Euclidean space. Now the ambient space is not compact. We need the following boundary gradient estimate for $\overline{\mathcal{L}}$.

LEMMA 3.2. *Let Σ be a compact embedded hypersurface in $\mathbb{R}^{n+1}$, Ω be a bounded domain in $\mathbb{R}^{n+1}$ with $\partial\Omega = \Sigma \cup S_R$. Here S_R is an n-sphere with radius R and centered at the origin for any $R \geq \sqrt{2(n+1)} + diam(\Sigma)$. We consider Dirichlet problem*

$$\begin{cases} \overline{\mathcal{L}} f = 0 & \text{in} \quad \Omega \\ f\,|_\Sigma = u\ ,\ f\,|_{S_R} = 0\ , \end{cases}$$

where u is a smooth function on Σ, then $|\overline{\nabla} f(X_0)| \leq 3\max_{X\in\Sigma}|u(X)|R$ for any $X_0 \in S_R$.

Proof. For any $X_0 \in S_R$, there is a unique $Y_0 \in \mathbb{R}^{n+1}$ such that $\overline{B_R(0)} \cap \overline{B_R(Y_0)} = X_0$. Let $u_0 = \max_{X\in\Sigma} |u(X)|$ and define two barrier functions $w^{\pm}(d) = \pm 3u_0 \left(1 - \exp\left(-\frac{d^2-R^2}{2}\right)\right)$, $d(X) = |X - Y_0|$ on the ball $B_{\sqrt{R^2+1}}(Y_0)$. Now, we prove that the two functions $w^{\pm}$ satisfy

(i) $\pm\overline{\mathcal{L}}w^{\pm} < 0$ in $B_{\sqrt{R^2+1}}(Y_0) \cap \Omega$,

(ii) $w^{\pm}(X_0) = f(X_0) = 0$,

(iii) $w^-(X) \le f(X) \le w^+(X)$, $X \in \partial B_{\sqrt{R^2+1}}(Y_0) \cap \Omega$.

Let $Y = X - Y_0$, then $d = |Y|$, $\overline{\nabla} d = \frac{Y}{|Y|}$ and $\overline{\Delta} d = \frac{n}{|Y|}$, hence

$$\overline{\mathcal{L}}w^+ = (w^+)'\overline{\mathcal{L}}d + (w^+)''|\overline{\nabla}d|^2 = (w^+)'' + (w^+)'\left(\frac{n}{|Y|} - \frac{1}{2}\frac{X\cdot Y}{|Y|}\right).$$

Since $(w^+)' = 3u_0 d e^{-\frac{d^2-R^2}{2}}$ and $(w^+)'' = 3u_0(1-d^2)e^{-\frac{d^2-R^2}{2}}$, then for any $X \in B_{\sqrt{R^2+1}}(Y_0) \cap \Omega$, we have $|X| \le R$, $d = |Y| \ge R$ and

$$\begin{aligned}\overline{\mathcal{L}}w^+ &\le 3u_0(1-d^2)e^{-\frac{d^2-R^2}{2}} + 3u_0 d e^{-\frac{d^2-R^2}{2}}\left(\frac{n}{d} + \frac{R}{2}\right)\\ &= 3u_0 e^{-\frac{d^2-R^2}{2}}\left(1 - d^2 + n + \frac{R}{2}d\right) \le 0 \quad (\textit{since } R \ge \sqrt{2(n+1)}\ ).\end{aligned}$$

Thus, (i) is proved. (ii) is obvious. By maximum principle, one can obtain

$$|f(X)| \le u_0, \quad \text{for any } X \in \Omega.$$

When $X \in \partial B_{\sqrt{R^2+1}}(Y_0) \cap \Omega$, $w^+(X) = 3u_0(1 - e^{-1/2}) \ge f(X)$ and $w^-(X) = -3u_0(1 - e^{-1/2}) \le f(X)$. Thus, (iii) is proved.

Comparison principle of elliptic equations gives

$$w^-(X) \le f(X) \le w^+(X), \quad X \in B_{\sqrt{R^2+1}}(Y_0) \cap \Omega.$$

Therefore, the normal derivatives of $w^{\pm}$ and f satisfy

$$\frac{\partial w^-}{\partial \nu}(X_0) \le \frac{\partial f}{\partial \nu}(X_0) \le \frac{\partial w^+}{\partial \nu}(X_0),$$

which completes the proof. □

We define the first (Neumann) eigenvalue λ_1 of the self-adjoint operator $\mathcal{L}$ in complete self-shrinkers M^n in $\mathbb{R}^{n+1}$ by

$$\lambda_1 = \inf_{f\in C^\infty(M)}\left\{\int_M |\nabla f|^2\rho; \quad \int_M f^2\rho = 1, \int_M f\rho = 0\right\}.$$

By (2.1), one has $\lambda_1 \le \frac{1}{2}$. From the following lemma, λ_1 can be arrived by the first eigenfunction u and $\lambda_1 > 0$ for any complete properly immersed self-shrinker.

LEMMA 3.3. *Let M^n be a complete properly immersed self-shrinker in $\mathbb{R}^{n+1}$, then there exists a smooth function u with $\int_M u^2\rho = 1, \int_M u\rho = 0$ such that $\mathcal{L}u + \lambda_1 u = 0$ and $\int_M |\nabla u|^2\rho = \lambda_1$.*

Proof. By the definition of λ_1, there exists a sequence $\{f_i\}$ satisfying

$$\int_M f_i^2\rho = 1, \int_M f_i\rho = 0 \quad \text{and} \quad \lim_{i\to\infty}\int_M |\nabla f_i|^2\rho = \lambda_1. \tag{3.5}$$

Since $\lambda_1 \le 1/2$, then there exists a N_0 such that, for any $i \ge N_0$, $\int_M |\nabla f_i|^2\rho \le 1$.

Define two Sobolev spaces $L^2(\Omega,\rho)$, $H^1(\Omega,\rho)$ for any set $\Omega \subset M$ by

$$\begin{aligned} L^2(\Omega,\rho) &= \{f\ ; \int_\Omega f^2\rho < \infty\ \},\\ H^1(\Omega,\rho) &= \{f\ ; \int_\Omega f^2\rho < \infty \text{ and } \int_\Omega |\nabla f|^2\rho < \infty\}, \end{aligned}$$

respectively. Since $H^1(D_r,\rho)$ is a Hilbert space, then there is a subsequence $\{f_{n_i}\}$ of $\{f_i\}$ converging to some $u_r \in H^1(D_r,\rho)$ weakly, and there is a subsequence $\{f_{n_{k_i}}\}$ of $\{f_{n_i}\}$ converging to some $u_{r+1} \in H^1(D_{r+1},\rho)$ weakly and so on. Hence we could choose a diagonal sequence, denoted by f_k for simplicity, such that f_k converges to some $u_K \in H^1(K,\rho)$ weakly for any compact set $K \subset M$, i.e., we can define u on M such that $u\,|_K = u_K$. By compact embedding theorem, sequence f_k converges to u in the space $L^2(K,\rho)$ strongly for any compact set K, then

$$\int_K u^2\rho = \lim_{k\to\infty}\int_K f_k^2\rho \le 1.$$

By weak convergence in $H^1(D_r,\rho)$, one has

$$\begin{aligned} &\int_K (\nabla u\cdot\nabla f_k)\rho = \lim_{j\to\infty}\int_K (\nabla f_j\cdot\nabla f_k)\rho\\ \le&\frac{1}{2}\lim_{j\to\infty}\int_K (|\nabla f_j|^2 + |\nabla f_k|^2)\rho \le \frac{\lambda_1}{2} + \frac{1}{2}\int_K |\nabla f_k|^2\rho. \end{aligned}$$

Hence

$$\int_M u^2\rho \le 1, \qquad \int_M |\nabla u|^2\rho \le \lambda_1. \tag{3.6}$$

For any sufficiently small $\epsilon > 0$ and compact set $K \subset M$, there exists a k such that $\int_K |u - f_k|^2\rho \le \epsilon$. Combining (3.5) and (3.6) and Cauchy inequality, we get

$$\begin{aligned} \left|\int_K u\rho\right| &\le \int_K |u-f_k|\rho + \left|\int_K f_k\rho\right| \le \int_K |u-f_k|\rho + \int_{M\setminus K}|f_k|\rho\\ &\le\sqrt{\int_K\rho\int_K|u-f_k|^2\rho} + \sqrt{\int_{M\setminus K}\rho\int_{M\setminus K}|f_k|^2\rho}\\ &\le\sqrt{\epsilon\int_K\rho} + \sqrt{\int_{M\setminus K}\rho}. \end{aligned} \tag{3.7}$$

Since M has Euclidean volume growth, then (3.7) implies

$$\int_M u\rho = 0.$$

Now let us prove $\int_M u^2\rho = 1$. By Logarithmic type Sobolev inequalities on self-shrinkers one has [7]

$$\int_M |X|^2 f_k^2\rho \le 16\int_M |\nabla f_k|^2\rho + 4n\int_M f_k^2\rho. \tag{3.8}$$

In fact, multiplying a smooth function g with compact support on the both sides of (2.2), then integrating by parts yield

$$\int_M (|X|^2 - 2n)g^2\rho = \int_M (\nabla|X|^2 \cdot \nabla g^2)\rho \leq \frac{1}{2}\int_M |X|^2 g^2\rho + 8\int_M |\nabla g|^2\rho.$$

Using such function g to approach f_k, we can also get (3.8).

For any $r > 0$ (3.8) implies

$$r^2 \int_{M\setminus D_r} f_k^2\rho \leq 16\int_M |\nabla f_k|^2\rho + 4n\int_M f_k^2\rho \leq 16 + 4n,$$

then

$$\int_{D_r} u^2\rho = \lim_{k\to\infty}\int_{D_r} f_k^2\rho = 1 - \lim_{k\to\infty}\int_{M\setminus D_r} f_k^2\rho \geq 1 - \frac{16+4n}{r^2}. \tag{3.9}$$

Combining (3.6) one arrives at $\int_M u^2\rho = 1$. By the definition of λ_1, we get $\int_M |\nabla u|^2\rho = \lambda_1$. Let us define a functional

$$I(f) = \int_M |\nabla f|^2\rho - 2\lambda_1\int_M fu\rho$$

and

$$\overline{f} = \frac{\int_M f\rho}{\int_M \rho},$$

then

$$\begin{aligned} I(f) &= \int_M |\nabla f|^2\rho - 2\lambda_1\int_M (f-\overline{f})u\rho \geq \int_M |\nabla f|^2\rho - \lambda_1\int_M \left((f-\overline{f})^2 + u^2\right)\rho \\ &= -\lambda_1 + \int_M |\nabla f|^2\rho - \lambda_1\int_M (f-\overline{f})^2\rho \geq -\lambda_1. \end{aligned}$$

Since $I(u) = -\lambda_1$, then the function u arrives at the minimum of the functional $I(\cdot)$. Hence $\frac{\partial}{\partial\epsilon}\mid_{\epsilon=0} I(u+\epsilon\varphi) = 0$ for any $\varphi \in C_c^\infty(M)$. By a simple calculation, we have

$$\int_M (\mathcal{L}u + \lambda_1 u)\varphi\rho = 0.$$

By the regularity theory of elliptic equations u is a smooth function (see [9] for example). We finish the proof. □

Now, we give a uniformly positive lower bound of λ_1 for compact embedded self shrinkers in $\mathbb{R}^{n+1}$.

Proof of Theorem 1.3. We have known $0 < \lambda_1 \leq \frac{1}{2}$ in the previous discussion. Let B_R be an n-ball with radius R and centered at the origin, then there is a R_0 such that $M \subset\subset B_{R_0}$. Set Ω_R be a bounded domain in $\mathbb{R}^{n+1}$ with $\partial\Omega_R = M \cup \partial B_R$ for $R \geq R_0$. We consider the following Dirichlet problem

$$\begin{cases} \overline{\mathcal{L}}f = 0 & \text{in} \quad \Omega_R \\ f\mid_M = u\ ,\ f\mid_{\partial B_R} = 0\ , \end{cases}$$

where u is the first eigenfunction of the self-adjoint operator $\mathcal{L}$ in M, i.e., $\mathcal{L}u+\lambda_1 u=0$ and $\int_M u^2\rho = 1$. By Lemma 3.2, we get $|\overline{\nabla} f(Y)| \leq 3\max_{X\in M}|u(X)|R$ for any $Y\in\partial B_R$. Integrating by parts gives

$$\int_M f_\nu \mathcal{L}u\rho = -\lambda_1\int_M f_\nu u\rho = -\lambda_1\int_{\Omega_R}\overline{\mathrm{div}}(\rho f\overline{\nabla} f) = -\lambda_1\int_{\Omega_R}|\overline{\nabla} f|^2\rho, \tag{3.10}$$

Combining (1.1), (3.10) and Theorem 3.1, we have

$$\begin{aligned}0\geq&\int_{\Omega_R}|\overline{\nabla}^2 f|^2\rho+\frac{1}{2}\int_{\Omega_R}|\overline{\nabla} f|^2\rho-2\lambda_1\int_{\Omega_R}|\overline{\nabla} f|^2\rho-\int_M h(\nabla u,\nabla u)\rho-\int_{\partial B_R}f_\nu^2\frac{R}{2}\rho\\ \geq&\int_{\Omega_R}|\overline{\nabla}^2 f|^2\rho+(\frac{1}{2}-2\lambda_1)\int_{\Omega_R}|\overline{\nabla} f|^2\rho-\int_M h(\nabla u,\nabla u)\rho-\frac{9}{2}\max_{X\in M}|u(X)|^2R^3\int_{\partial B_R}\rho.\end{aligned}\tag{3.11}$$

We may assume $\int_M h(\nabla u,\nabla u)\rho\leq 0$, or else we consider the bounded domain U with $\partial U = M$ instead of Ω_R. By trace theorems in Sobolev spaces (see [10] for example), there is a positive constant C depending only on n, R_0 and M such that

$$\int_{\Omega_{R_0}}|\overline{\nabla}^2 f|^2+\int_{\Omega_{R_0}}|\overline{\nabla} f|^2\geq C\int_M|\overline{\nabla} f|^2.$$

Then for $R > R_0$ one has

$$\begin{aligned}&\int_{\Omega_R}|\overline{\nabla}^2 f|^2\rho+\int_{\Omega_R}|\overline{\nabla} f|^2\rho\\ \geq&\int_{\Omega_{R_0}}|\overline{\nabla}^2 f|^2\rho+\int_{\Omega_{R_0}}|\overline{\nabla} f|^2\rho\\ \geq& e^{-\frac{R_0^2}{4}}\left(\int_{\Omega_{R_0}}|\overline{\nabla}^2 f|^2+\int_{\Omega_{R_0}}|\overline{\nabla} f|^2\right)\geq e^{-\frac{R_0^2}{4}}C\int_M|\overline{\nabla} f|^2\\ \geq& e^{-\frac{R_0^2}{4}}C\int_M|\nabla u|^2\rho = e^{-\frac{R_0^2}{4}}C\lambda_1>0.\end{aligned}\tag{3.12}$$

If $\lambda_1 < \frac{1}{4}$, then let R go to infinite in (3.11), which deduces a contradiction by (3.12). Hence we get $\lambda_1\geq\frac{1}{4}$. □

COROLLARY 3.4. *Let M^n be a compact embedded self-shrinker in $\mathbb{R}^{n+1}$, then for any $f\in C^1(M)$ there is a Poincaré inequality*

$$\int_M (f-\overline{f})^2\rho\leq 4\int_M|\nabla f|^2\rho,$$

where $\overline{f}=\frac{\int_M f\rho}{\int_M\rho}$.

Proof. If f is not a constant, let $g = (\int_M f^2\rho-\overline{f}^2\cdot\int_M\rho)^{-1/2}(f-\overline{f})$, then $\int_M g\rho = 0$ and $\int_M g^2\rho = 1$. Since λ_1 is the first eigenvalue of self-adjoint operator $\mathcal{L}$, then combining Theorem 1.3 we complete the proof. □

Now, let us recall a classical result of P. Yang and S. T. Yau [18].

THEOREM 3.5. *(Yang-Yau) Let (Σ_g^2, ds^2) be an orientable Riemann surface of genus g with area $Area(\Sigma_g)$. Then we have*

$$\lambda_1(\Sigma_g)\leq\frac{8\pi(1+g)}{Area(\Sigma_g)},$$

where $\lambda_1(\Sigma_g)$ is the first eigenvalue of Δ on Σ_g.

Let $N^3 = (\mathbb{R}^3, (\rho\delta_{ij}))$ and M^2 be a compact embedded self-shrinker in $\mathbb{R}^3$. Let $\widetilde{g} = \widetilde{g}_{ij} d\theta_i d\theta_j$, $\widetilde{\nabla}$ and $\widetilde{\Delta}$ be the metric, the Levi-Civita connection and the Laplacian operator of M induced from N^3, respectively. Denote the self-shrinker M with metric $\widetilde{g}$ by $\widetilde{M}$. Let $g_{ij} d\theta_i d\theta_j$ and $d\mu$ be the metric and the volume element of M induced from $\mathbb{R}^3$, then $\widetilde{g}_{ij} = \rho g_{ij}$. Denote the first eigenvalue of $\widetilde{M}$ by $\lambda_1(\widetilde{M})$, then

$$\begin{aligned} \lambda_1(\widetilde{M}) &= \inf_{\int_M f\rho = 0} \frac{\int_M |\widetilde{\nabla} f|^2 \rho d\mu}{\int_M f^2 \rho d\mu} = \inf_{\int_M f\rho = 0} \frac{\int_M \widetilde{g}^{ij} f_i f_j \rho d\mu}{\int_M f^2 \rho d\mu} \\ &\geq \inf_{\int_M f\rho = 0} \frac{\int_M g^{ij} f_i f_j \rho d\mu}{\int_M f^2 \rho d\mu} = \lambda_1 \geq \frac{1}{4}, \end{aligned} \tag{3.13}$$

where we have used Theorem 1.3 in the last inequality in (3.13).

COROLLARY 3.6. *Let M be a compact embedded self-shrinker in $\mathbb{R}^3$ with genus g, then*

$$\int_M \rho \leq 32\pi(1+g),$$

moreover,

$$\int_{D_r} 1 d\mu \leq 32 e^{\frac{1}{4}} \pi (1+g) r^2 \quad \text{for} \quad r \geq 1.$$

Proof. Combining (3.13) and Theorem 3.5, we get

$$\int_M \rho \leq 32\pi(1+g). \tag{3.14}$$

Combining (2.7) and (3.14) for $r \geq 1$ gives

$$\frac{1}{r^2} e^{-\frac{1}{4}} \int_{D_r} 1 d\mu \leq \frac{1}{r^2} \int_{D_r} e^{-\frac{|X|^2}{4r^2}} d\mu \leq \int_{D_r} e^{-\frac{|X|^2}{4}} d\mu \leq 32\pi(1+g), \tag{3.15}$$

which yields

$$\int_{D_r} 1 d\mu \leq 32 e^{\frac{1}{4}} \pi (1+g) r^2, \quad \text{for} \quad r \geq 1. \tag{3.16}$$

∎

For a non-negative integer g and a constant $D > 0$, let $S_{g,D}$ denote the space of all compact embedded self-shrinkers in $\mathbb{R}^3$ with genus at most g, and diameter at most D. Now, we are in a position to prove a compactness theorem.

Proof of Theorem 1.4. For any compact surface Σ in $\mathbb{R}^3$, using Gauss-Bonnet formula one has

$$\int_\Sigma |B|^2 = \int_\Sigma H^2 - 2\int_\Sigma K = \int_\Sigma H^2 - 4\pi\chi(\Sigma), \tag{3.17}$$

where $\chi(\Sigma)$ is the Euler number of surface Σ. If $\Sigma \in S_{g,D}$, then (2.8) implies there is a $X \in \Sigma$ with $|X| = \sqrt{2n}$ and $\Sigma \subset B_{D+\sqrt{2n}}$. Combining (2.3) and (3.16) gives

$$\int_{\Sigma} H^2 = \int_{\Sigma} 1 d\mu \leq 32 e^{\frac{1}{4}} \pi (1+g)(D+\sqrt{2n})^2. \tag{3.18}$$

Then (3.17) and (3.18) implies

$$\int_{\Sigma} |B|^2 = \int_{\Sigma} H^2 - 4\pi(2-2g) \leq 32 e^{\frac{1}{4}} \pi (1+g)(D+\sqrt{2n})^2 + 8\pi(g-1). \tag{3.19}$$

According to Proposition 5.10 of [6](see also [1] and [5]), we complete the proof. □

REFERENCES

[1] H. I. Choi and R. Schoen, *The space of minimal embeddings of a surface into a three dimensional manifold of positive Ricci curvature*, Invent. Math., 81 (1985), pp. 387–394.
[2] H. I. Choi and A. N. Wang, *A first eigenvalue estimate for minimal hypersurfaces*, J. Differential Geom., 18 (1983), pp. 559–562.
[3] H.-D. Cao and D. Zhou, *On complete gradient shrinking Ricci soliton*, J. Differential Geom., 85 (2010), pp. 175–185.
[4] T. H. Colding and W. P. Minicozzi II, *Generic mean curvature flow I; generic singularities*, Ann. of Math. (2), 175:2 (2012), pp. 755–833.
[5] T. H. Colding and W. P. Minicozzi II, *Smooth Compactness of Self-shrinkers*, Comment. Math. Helv., 87:2 (2012), pp. 463–475.
[6] T. H. Colding and W. P. Minicozzi II, *Compactness and convergence of minimal surfaces in 3-manifolds.*
[7] K. Ecker, *Logarithmic Sobolev inequalities on submanifolds of Euclidean space*, J. reine angew. Math., 522 (2000), pp. 105–118.
[8] K. Ecker and G. Huisken, *Mean curvature evolution of entire graphs*, Annals of Mathematics, 130 (1989), pp. 453–471.
[9] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin-New York, (1983).
[10] D. D. Haroske and H. Triebel, *Distributions, Sobolev spaces, Elliptic equations*, European Mathematical Society, (2008).
[11] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom., 31 (1990), pp. 285–299.
[12] G. Huisken, *Local and global behaviour of hypersurfaces moving by mean curvature*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., 54, Part 1, Amer. Math. Soc., Providence, RI, (1993), pp. 175–191.
[13] R. C. Reilly, *Extrinsic rigidity theorems for compact submanifolds of the sphere*, J. Differential geom., 4 (1970), pp. 487–497.
[14] R. Schoen and S. T. Yau, *Lectures on differential geometry*, International Press, (1994).
[15] K. Smoczyk, *Self-shrinkers of the mean curvature flow in arbitrary codimension*, Int. Math. Res. Not., 2005:48 (2005), pp. 2983–3004.
[16] L. Wang, *A Bernstein type theorem for self-similar shrinkers*, Geom. Dedicata, 151 (2011), pp. 297–303.
[17] Y. L. Xin, *Minimal submanifolds and related topics*, World Scientific Publ., (2003).
[18] P. Yang, and S. T. Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa, 7 (1980), pp. 55–63.