

WEIGHTED THERMODYNAMIC FORMALISM ON SUBSHIFTS AND APPLICATIONS*

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Abstract. We examine the interplay between the thermodynamic formalism and the multifractal formalism on the so-called self-affine symbolic spaces, under the specification property assumption. We investigate the properties of a weighted variational principle to derive a new result concerning the approximation of any invariant probability measure μ by sequences of weighted equilibrium states whose weighted entropies converge to the weighted entropy of μ . This is a key property in the estimation of the Hausdorff dimension of sets of generic points, and then in the multifractal analysis of *non homogeneous* Birkhoff averages.

Key words. Thermodynamic formalism, equilibrium states, symbolic dynamics, affine invariant sets, multifractal analysis, Hausdorff dimension.

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1. Introduction. The interplay between the thermodynamic formalism and the multifractal formalism has been rigorously examined in the literature for expanding conformal dynamical systems (see, e.g. [15, 40, 38, 6]). In this paper, we study this relationship on the so-called *self-affine symbolic spaces* defined as follows.

Let $k \geq 2$. Assume that (X_i, T_i) ($i = 1, \dots, k$) are subshifts over finite alphabets \mathcal{A}_i such that X_{i+1} is a factor of X_i with a one-block factor map $\pi_i : X_i \rightarrow X_{i+1}$ for $i = 1, \dots, k - 1$ (see Section 2 for the definitions). For convenience, we use π_0 to denote the identity map on X_1 . Define $\tau_i : X_1 \rightarrow X_{i+1}$ by $\tau_i = \pi_i \circ \pi_{i-1} \circ \dots \circ \pi_0$ for $i = 0, 1, \dots, k - 1$.

Let $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ so that $a_1 > 0$ and $a_i \geq 0$ for $i > 1$. Define an ultrametric distance $d_{\mathbf{a}}$ on X_1 by

$$(1.1) \quad d_{\mathbf{a}}(x, y) = \max \left(e^{-\frac{|\tau_{i-1}(x) \wedge \tau_{i-1}(y)|}{a_1 + \dots + a_i}} : 1 \leq i \leq k \right),$$

where

$$|u \wedge v| = \begin{cases} 0, & \text{if } u_1 \neq v_1, \\ \max\{n : u_j = v_j \text{ for } 1 \leq j \leq n\} & \text{if } u_1 = v_1 \end{cases}$$

for $u = (u_j)_{j=1}^{\infty}, v = (v_j)_{j=1}^{\infty} \in X_i$. The metric space $(X_1, d_{\mathbf{a}})$ is called a self-affine symbolic space. It is a natural model used to characterize the geometry of compact invariant sets on the k -torus under a diagonal endomorphism [7, 35, 28].

For $1 \leq i \leq k$, let $\mathcal{M}(X_i, T_i)$ denote the set of all T_i -invariant Borel probability measures on X_i , endowed with the weak-star topology. Let $\mathcal{E}(X_i, T_i)$ denote the set of ergodic measures in $\mathcal{M}(X_i, T_i)$. For $\mu \in \mathcal{M}(X_1, T_1)$, define

$$h_{\mu}^{\mathbf{a}}(T_1) = \sum_{i=1}^k a_i h_{\mu \circ \tau_{i-1}^{-1}}(T_i),$$

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which is called the **a-weighted entropy** of μ . It was proved by Kenyon and Peres [28] that

$$(1.2) \quad \dim_H \mu = h_\mu^{\mathbf{a}}(T_1)$$

for each $\mu \in \mathcal{E}(X_1, T_1)$, here $\dim_H \mu$ denotes the Hausdorff dimension of μ in the metric space $(X_1, d_{\mathbf{a}})$, that is, $\dim_H \mu = \inf\{\dim_H E : \text{Borel } E \subset X_1 \text{ with } \mu(X_1 \setminus E) = 0\}$.

For $\mu \in \mathcal{M}(X_1, T_1)$, the set of *generic points* of μ is defined by

$$(1.3) \quad G_\mu(X_1, T_1) = \left\{ x \in X_1 : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(T_1^j x) = \int \phi d\mu \text{ for all } \phi \in C(X_1) \right\},$$

where $C(X_1)$ denotes the set of all real continuous functions on X_1 .

We are going to establish a connection between the thermodynamic formalism and the multifractal formalism on $(X_1, d_{\mathbf{a}})$ by considering the Hausdorff dimension of $G_\mu(X_1, T_1)$ and the multifractal analysis of (non-homogeneous) Birkhoff averages (which will be defined a little bit later) of continuous functions in $(X_1, d_{\mathbf{a}})$.

A key notion introduced for the above study is the weighted topological pressure, defined for each $\phi \in C(X_1)$ by

$$P^{\mathbf{a}}(T_1, \phi) = \sup \left\{ \int \phi d\mu + h_\mu^{\mathbf{a}}(T_1) : \mu \in \mathcal{M}(X_1, T_1) \right\}.$$

Clearly the supremum is attainable, since the weighted entropy $h_{(\cdot)}^{\mathbf{a}}(T_1)$ is upper semi-continuous on $\mathcal{M}(X_1, T_1)$. Each measure μ which attains the supremum is called an **a-weighted equilibrium state** of ϕ . When $\mathbf{a} = (1, 0, \dots, 0)$, the **a-weighted** topological pressure and **a-weighted** equilibrium states are reduced back to the classical topological pressure and equilibrium states (cf. [43, 44, 38]).

We say that the subshift X_1 satisfies *specification* if there exists $s \in \mathbb{N}$ such that, for any two words I and J that are legal in X_1 , there is a word K of length s such that the word IKJ is legal in X_1 . For more details about this definition, see Section 2.

For $n \in \mathbb{N}$, let $\mathcal{L}_n(X_1)$ denote the n -th language of X_1 (see Section 2.1). Furthermore, for $I \in \mathcal{L}_n(X_1)$, let $[I]$ denote the n -th cylinder in X_1 associated with I (see Section 2.1). Define

$$\Omega_n = \{(\eta([I]))_{I \in \mathcal{L}_n(X_1)} : \eta \in \mathcal{M}(X_1, T_1)\}.$$

Clearly Ω_n is a convex set. Let $\text{ri}(\Omega_n)$ denote the relative interior of Ω_n . Say that $f \in C(X_1)$ is an *n-symbol function* if $f(x)$ only depends on the first n -coordinates of x .

One of the main results of this paper is the following.

THEOREM 1.1. *Assume that (X_1, T_1) is a subshift satisfying specification.*

- (i) *Let $n \in \mathbb{N}$ and $\eta \in \mathcal{M}(X_1, T_1)$. If $(\eta([I]))_{I \in \mathcal{L}_n(X_1)} \in \text{ri}(\Omega_n)$, then there is a unique measure $\mu = \mu(\mathbf{a}, \eta, n)$ in $\mathcal{M}(X_1, T_1)$ attaining the following supremum*

$$\sup \{h_\mu^{\mathbf{a}}(T_1) : \mu([I]) = \eta([I]) \text{ for all } I \in \mathcal{L}_n(X_1)\}.$$

*Furthermore, $\mu(\mathbf{a}, \eta, n)$ is the **a-weighted equilibrium state** of some n -symbol function.*

(ii) For any $\eta \in \mathcal{M}(X_1, T_1)$, there exists $(\mu_n)_{n=1}^\infty \subset \mathcal{M}(X_1, T_1)$ converging to η in the weak-star topology such that for each n , μ_n is the \mathbf{a} -weighted equilibrium state of some n -symbol function and

$$h_{\mu_n}^{\mathbf{a}}(T_1) \geq h_{\mu}^{\mathbf{a}}(T_1), \quad \lim_{n \rightarrow \infty} h_{\mu_n}^{\mathbf{a}}(T_1) = h_{\mu}^{\mathbf{a}}(T_1).$$

The above result is well known in the special case that $\mathbf{a} = (1, 0, \dots, 0)$ and X_1 is an irreducible subshift of finite type, for which $\mu(\mathbf{a}, \eta, n)$ is a $(n - 1)$ -step Markov measure (see, e.g. [19]). We remark that in our general setting, $\mu(\mathbf{a}, \eta, n)$ is typically not a Markov measure of any order even in the full shift case. In the following, we give a simple example (see Section 4 for a proof).

EXAMPLE 1.2. Let $X_1 = \{a, b, c\}^{\mathbb{N}}$ and $X_2 = \{1, 2\}^{\mathbb{N}}$. Let $\pi : X_1 \rightarrow X_2$ be a one-block factor map induced by $a, b \mapsto 1$ and $c \mapsto 2$. Let $\mathbf{a} = (a_1, a_2)$ with $a_1, a_2 > 0$. Let η be a fully supported measure in $\mathcal{M}(X_1, T_1)$. If

$$(1.4) \quad \frac{\eta([ac])}{\eta([a])} = \frac{\eta([bc])}{\eta([b])} \quad \text{or} \quad \frac{\eta([ca])}{\eta([a])} = \frac{\eta([cb])}{\eta([b])},$$

then $\mu(\mathbf{a}, \eta, 2)$ is a one-step Markov measure; otherwise $\mu(\mathbf{a}, \eta, 2)$ is not a Markov measure of any order.

We point out that when $X_1 = \mathcal{A}_1^{\mathbb{N}}$ is a full shift and f is a continuous function on X_1 with sufficiently regularity (Hölder continuity, for instance), the \mathbf{a} -weighted equilibrium state of f is quasi-Bernoulli (see Theorem 3.5), furthermore it is the classical equilibrium state of some continuous function g on X_1 (see Remark 3.7). Recall that a probability measure μ on $\mathcal{A}_1^{\mathbb{N}}$ is called *quasi-Bernoulli* if there exists $C > 1$ such that

$$(1.5) \quad \frac{1}{C} \mu([I])\mu([J]) \leq \mu([IJ]) \leq C \mu([I])\mu([J]), \quad \forall I, J \in \mathcal{A}_1^* = \bigcup_{n=0}^{\infty} \mathcal{A}_1^n.$$

Theorem 1.1 might have its own interest in ergodic theory. It is crucial in our study of the Hausdorff dimension of generic points of invariant measures. Indeed, we have

THEOREM 1.3. Assume that (X_1, T_1) is a subshift satisfying specification. Then for any $\mu \in \mathcal{M}(X_1, T_1)$,

$$(1.6) \quad \dim_H G_{\mu}(X_1, T_1) = h_{\mu}^{\mathbf{a}}(T_1).$$

We remark that $G_{\mu}(X_1, T_1) \neq \emptyset$ for each $\mu \in \mathcal{M}(X_1, T_1)$ ([39, 20]). Theorem 1.3 is only known in the literature for the case that $\mathbf{a} = (1, 0, \dots, 0)$ (cf. [9, 11, 39, 20]), which corresponds to the conformal case. Our proof of the lower bound $\dim_H G_{\mu}(X_1, T_1) \geq h_{\mu}^{\mathbf{a}}(T_1)$ in the general case is based on a delicate concatenation of quasi-Bernoulli measures provided by Theorem 1.1. The upper bound is not a simple adaptation of McMullen argument; we need to overcome a difficulty coming from the fact that, in the specification case, weighted equilibrium states of Hölder continuous functions might not be quasi-Bernoulli. It is worth to point out that (1.6) still holds for any $\mu \in \mathcal{E}(X_1, T_1)$ without assuming the specification property of X_1 (see Remark 1.5); this fact was proved by Bowen [9] in the case that $\mathbf{a} = (1, 0, \dots, 0)$.

To formulate our result on the multifractal analysis, let $d \in \mathbb{N}$ and let $\Phi_j, j = 1, \dots, r$, be \mathbb{R}^d -valued continuous functions on X_1 . Let $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{R}^r$ with $c_i > 0$ for all i . For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$, define the level set

$$E_{\{\Phi_j\}_{j=1}^r, \mathbf{c}}(\alpha) = \left\{ x \in X_1 : \lim_{n \rightarrow \infty} \sum_{j=1}^r \frac{S_{\lfloor c_j n \rfloor} \Phi_j(x)}{\lfloor c_j n \rfloor} = \alpha \right\},$$

where $S_n \Phi_j(x) := \sum_{\ell=0}^{n-1} \Phi_j(T_1^\ell x)$, and $\lfloor x \rfloor$ denotes the largest integer not greater than x . The limit $\lim_{n \rightarrow \infty} \sum_{j=1}^r \frac{S_{\lfloor c_j n \rfloor} \Phi_j(x)}{\lfloor c_j n \rfloor}$, provided it exists, is called the *c-Birkhoff average* of (Φ_1, \dots, Φ_r) at x . When \mathbf{c} is a multiple of $(1, \dots, 1)$, this average reduces back to the Birkhoff average of $\sum_{j=1}^r \Phi_j$.

THEOREM 1.4. *Assume that the subshift X_1 satisfies specification. Let $\Phi = \sum_{j=1}^r \Phi_j$. Set $L_\Phi = \{ \int \Phi d\mu : \mu \in \mathcal{M}(X_1, T_1) \}$.*

1. *For $\alpha \in \mathbb{R}^d$, $E_{\{\Phi_j\}_{j=1}^r, \mathbf{c}}(\alpha) \neq \emptyset$ if and only if $\alpha \in L_\Phi$.*
2. *For $\alpha \in L_\Phi$, we have*

$$\begin{aligned} \dim_H E_{\{\Phi_j\}_{j=1}^r, \mathbf{c}}(\alpha) &= \max \{ h_\mu^{\mathbf{a}}(T_1) : \mu \in \mathcal{M}(X_1, T_1), \Phi_*(\mu) = \alpha \} \\ &= \inf \{ P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \Phi) - \alpha \cdot \mathbf{q} : \mathbf{q} \in \mathbb{R}^d \}, \end{aligned}$$

where $\mathbf{a} \cdot \mathbf{b}$ denotes the standard dot product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$.

3. *Suppose that L_Φ is not a singleton. Then the set $X_1 \setminus \bigcup_{\alpha \in L_\Phi} E_{\{\Phi^{(j)}\}, \mathbf{c}}(\alpha)$ has the same Hausdorff dimension as X_1 .*

Theorem 1.4 is concerned with the multifractal analysis of the level set of non-homogeneous Birkhoff averages, which was initially motivated by the study of the multifractal analysis of \mathbf{a} -weighted Gibbs measures (see Section 6). It also provides a unified way to study the multifractal analysis of Birkhoff averages and Gibbs measures. As far as we know, this result is new when $r \geq 2$, even in the case $\mathbf{a} = (1, 0, \dots, 0)$. The level sets $E_{\{\Phi_j\}_{j=1}^r, \mathbf{c}}(\alpha)$ do depend on \mathbf{c} (see Example 6.2). In the literature, there are some works considering the multifractal analysis of Birkhoff averages of Hölder continuous functions, and Gibbs measures on self-affine sponges [31, 37, 2, 3]. However no relation between the Hausdorff spectra and any dynamical quantity like entropy was found in these papers. Moreover, the methods employed in these papers do not provide the whole Hausdorff spectrum, and they can not be used to study the cases of general continuous functions and weak Gibbs measures.

REMARK 1.5. Our proof of the upper bound for the Hausdorff dimension of level sets of non homogeneous Birkhoff averages does not use the specification property; it follows that the upper bound $\dim_H G_\mu(X_1, T_1) \leq h_\mu^{\mathbf{a}}(T_1)$ holds for any $\mu \in \mathcal{M}(X_1, T_1)$ without this property. Consequently, due to (1.2), (1.6) holds for any $\mu \in \mathcal{E}(X_1, T_1)$ without assuming the specification property of X_1 . Furthermore, the upper bound

$$\dim_P G_\mu(X_1, T_1) \leq h_\mu^{\mathbf{a}}(T_1)$$

also holds for any $\mu \in \mathcal{M}(X_1, T_1)$ in the general subshift setting, where \dim_P denotes the packing dimension (see Remark 5.10 for details). Hence we have $\dim_P G_\mu(X_1, T_1) = h_\mu^{\mathbf{a}}(T_1)$ in the setting of Theorem 1.3.

The paper is organized as follows. Some definitions and known results on sub-additive thermodynamic formalism on subshifts are given in Section 2. In Section 3,

we study the weighted topological pressure and equilibrium states for sub-additive potentials. In Section 4, we prove Theorem 1.1 and the statement given in Example 1.2. Then, in Section 5 we prove Theorems 1.3-1.4. In Section 6, we give some extensions of the multifractal analysis to asymptotically additive potentials and weighted equilibrium states. In Appendix A, we summarize the main notation and conventions used in this paper.

2. Sub-additive thermodynamic formalism on subshifts. In this section, we present some definitions and known results about the sub-additive thermodynamic formalism on subshifts, which plays an important role in our study of weighted topological pressures and weighted equilibrium states.

2.1. Subshifts over finite alphabets. Let $p \geq 2$ be an integer and $\mathcal{A} = \{1, \dots, p\}$. Denote

$$\mathcal{A}^{\mathbb{N}} = \{(x_i)_{i=1}^{\infty} : x_i \in \mathcal{A} \text{ for } i \geq 1\}.$$

Then $\mathcal{A}^{\mathbb{N}}$ is compact endowed with the product discrete topology (cf. [33]). We say that (X, T) is a *subshift over \mathcal{A}* , if X is a compact subset of $\mathcal{A}^{\mathbb{N}}$ and $T(X) \subseteq X$, where T is the left shift map on $\mathcal{A}^{\mathbb{N}}$ defined as

$$T((x_i)_{i=1}^{\infty}) = (x_{i+1})_{i=1}^{\infty}, \quad \forall (x_i)_{i=1}^{\infty} \in \mathcal{A}^{\mathbb{N}}.$$

In particular, (X, T) is called the *full shift over \mathcal{A}* if $X = \mathcal{A}^{\mathbb{N}}$. For any $n \in \mathbb{N}$ and $I \in \mathcal{A}^n$, we write

$$[I] = \{(x_i)_{i=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} : x_1 \dots x_n = I\}$$

and call it an *n -th cylinder* in $\mathcal{A}^{\mathbb{N}}$.

The *language $\mathcal{L}(X)$* of a subshift X is the set of all finite words (including the empty word ε) that occur as consecutive strings $x_1 \dots x_n$ in the sequences $x = (x_i)_{i=1}^{\infty}$ which comprise X . That is,

$$\mathcal{L}(X) = \{I \in \mathcal{A}^* : I = x_1 \dots x_n \text{ for some } x = (x_i)_{i=1}^{\infty} \in X \text{ and } n \geq 1\} \cup \{\varepsilon\}.$$

Denote by $|I|$ the length of a word I . For $n \geq 0$, denote

$$\mathcal{L}_n(X) = \{I \in \mathcal{L}(X) : |I| = n\},$$

and we call $\mathcal{L}_n(X)$ the *n -th language of X* .

Let $s \in \mathbb{N}$. A subshift X is said to satisfy *s -specification* if for any $I, J \in \mathcal{L}(X)$, there exists $K \in \mathcal{L}_s(X)$ such that $IKJ \in \mathcal{L}(X)$. We say that X satisfies *specification* if it satisfies s -specification for some $s \in \mathbb{N}$.

Let (X, T) and (Y, S) be two subshifts over finite alphabets \mathcal{A} and \mathcal{D} , respectively. We say that Y is a *factor* of X , if there is a continuous surjective map $\pi : X \rightarrow Y$ such that $\pi T = S\pi$. Here π is called a *factor map*. Furthermore π is called a *one-block factor map* if there exists a map $\pi : \mathcal{A} \rightarrow \mathcal{D}$ such that

$$\pi((x_i)_{i=1}^{\infty}) = (\pi(x_i))_{i=1}^{\infty}, \quad \forall (x_i)_{i=1}^{\infty} \in X.$$

It is well known (see, e.g. [33, Proposition 1.5.12]) that each factor map $\pi : X \rightarrow Y$ between two subshifts X and Y , will become a one-block factor map if we enlarge the alphabet for X and recode X appropriately.

2.2. Sub-additive thermodynamic formalism. Let (X, T) be a subshift over a finite alphabet \mathcal{A} . Let $\Phi = (\log \phi_n)_{n=1}^\infty$ be a sequence of real functions on X . We say that Φ is a *sub-additive potential* and write $\Phi \in \mathcal{C}_s(X, T)$ if ϕ_n is non-negative and upper semi-continuous¹ on X for each n and there exists a constant $c > 0$ such that

$$\phi_{n+m}(x) \leq c\phi_n(x)\phi_m(T^n x), \quad \forall x \in X, n, m \in \mathbb{N}.$$

(we admit that ϕ_n takes the value zero). More generally, $\Phi = (\log \phi_n)_{n=1}^\infty$ is said to be an *asymptotically sub-additive potential* and write $\Phi \in \mathcal{C}_{ass}(X, T)$ if for any $\varepsilon > 0$, there exists a sub-additive potential $\Psi = (\log \psi_n)_{n=1}^\infty$ on X such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in X} |\log \phi_n(x) - \log \psi_n(x)| \leq \varepsilon,$$

where we take the convention $\log 0 - \log 0 = 0$. Furthermore Φ is called an *asymptotically additive potential* and write $\Phi \in \mathcal{C}_{asa}(X, T)$ if both Φ and $-\Phi$ are asymptotically sub-additive, where $-\Phi$ denotes $(\log(1/\phi_n))_{n=1}^\infty$. We say that Φ is *almost additive* if ϕ_n is positive and continuous on X for each n and there is a constant $c > 0$ such that

$$\frac{1}{c}\phi_n(x)\phi_m(T^n x) \leq \phi_{n+m}(x) \leq c\phi_n(x)\phi_m(T^n x), \quad \forall x \in X, n, m \in \mathbb{N}.$$

For convenience, we denote by $\mathcal{C}_{aa}(X, T)$ the collection of almost additive potentials on X . In particular, Φ is called *additive* if each ϕ_n is a continuous positive-valued function so that $\phi_{n+m}(x) = \phi_n(x)\phi_m(T^n x)$ for all $x \in X$ and $m, n \in \mathbb{N}$; in this case, there is a continuous real function g on X such that $\phi_n(x) = \exp(\sum_{i=0}^{n-1} g(T^i x))$ for each n .

For $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{ass}(X, T)$, and a compact set $K \subseteq X$, define

$$(2.1) \quad P_n(T, \Phi, K) = \sum_{I \in \mathcal{A}^n, [I] \cap K \neq \emptyset} \sup_{x \in [I] \cap K} \phi_n(x).$$

and

$$(2.2) \quad P(T, \Phi, K) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(T, \Phi, K).$$

For $\mu \in \mathcal{M}(X, T)$, let $h_\mu(T)$ denote the measure-theoretic entropy of μ with respect to T , and write

$$(2.3) \quad \Phi_*(\mu) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \log \phi_n(x) d\mu(x).$$

The existence of the limit (which may take value $-\infty$) in (2.3) follows from the sub-additivity of Φ . We list below some basic properties of asymptotically sub-additive potentials.

LEMMA 2.1 ([23], Appendix A). *Let $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{ass}(X, T)$. Then we have the following properties.*

- (i) *Let $\mu \in \mathcal{M}(X, T)$. The limit $\lambda_\Phi(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \phi_n(x)$ exists (which may take value $-\infty$) for μ -a.e. $x \in X$, and $\int \lambda_\Phi(x) d\mu(x) = \Phi_*(\mu)$. When μ is ergodic, $\lambda_\Phi(x) = \Phi_*(\mu)$ for μ -a.e. $x \in X$.*

¹In the previous definition in [12, 23], ϕ_n is assumed to be non-negative and continuous.

- (ii) The map $\Phi_* : \mathcal{M}(X, T) \rightarrow \mathbb{R} \cup \{-\infty\}$ is upper semi-continuous, and there is $C \in \mathbb{R}$ such that for all $\mu \in \mathcal{M}(X, T)$, $\lambda_\Phi(x) \leq C$ μ -a.e and $\Phi_*(\mu) \leq C$. If $\Phi \in \mathcal{C}_{asa}(X, T)$, Φ_* is continuous on $\mathcal{M}(X, T)$.
- (iii) $\Phi \in \mathcal{C}_{asa}(X, T)$ if and only if for any $\varepsilon > 0$, there exists a continuous function g on X such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in X} |\log \phi_n(x) - S_n g(x)| \leq \varepsilon,$$

where $S_n g(x) := \sum_{j=0}^{n-1} g(T^j x)$.

The following variational principle plays a key role in our analysis.

PROPOSITION 2.2 ([12]). *Let $P(T, \Phi, X)$ be defined as in (2.2). Then for any $\Phi \in \mathcal{C}_{ass}(X, T)$, we have the following variational principle:*

$$(2.4) \quad P(T, \Phi, X) = \sup\{\Phi_*(\mu) + h_\mu(T) : \mu \in \mathcal{M}(X, T)\}.$$

We call $P(T, \Phi) := P(T, \Phi, X)$ the *topological pressure of Φ* . We remark that the variational principle for sub-additive potentials has been studied in [17, 4, 21, 27, 30, 5, 36] under additional assumptions on the corresponding sub-additive potential and TDS. Say that $\mu \in \mathcal{M}(X, T)$ is an *equilibrium state* of Φ if the supremum in (2.4) is attained at μ . Note that $\Phi_*(\cdot)$ is upper semi-continuous on $\mathcal{M}(X, T)$ (cf. Lemma 2.1(ii)), and so is $h_{(\cdot)}(T)$ for subshifts. Hence any $\Phi \in \mathcal{C}_{ass}(X, T)$ has at least one equilibrium state.

PROPOSITION 2.3 ([5, 36]). *Let (X, T) be a full shift or mixing subshift of finite type. Let $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{aa}(X, T)$. Assume that Φ has the bounded distortion property. Then Φ has a unique equilibrium state μ . Furthermore, there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$ and $x = (x_i)_{i=1}^\infty \in X$,*

$$c^{-1} \leq \frac{\mu([x_1 \dots x_n])}{\exp(-nP(T, \Phi)) \phi_n(x)} \leq c.$$

REMARK 2.4.

- (1) Lemma 2.1 was proved in [23] under a slightly stronger assumption that ϕ_n is continuous on X . However it is easy to extend the result to the case that ϕ_n is upper semi-continuous, by using the property that, if f is an upper semi-continuous function on X , then so is the map $\mu \mapsto \int f d\mu$ on $\mathcal{M}(X)$ (cf. [16, (A8)]). Similarly, Proposition 2.2 was only stated in [12] for sub-additive potentials under a slight stronger assumption that ϕ_n is continuous. However, the proof given there works well for this new setting. Indeed, using the property we mentioned above, one sees that Lemma 2.3 in [12] can be extended to the case $\Phi \in \mathcal{C}_{ass}(X, T)$.
- (2) A special case of Proposition 2.3 was first proved in [24, 21] for the almost additive potentials given by

$$\phi_n(x) = \|M(x)M(Tx) \dots M(T^{n-1}x)\|, \quad n \in \mathbb{N},$$

where M is a Hölder continuous function taking values in the set of $d \times d$ positive matrices.

(3) According to Lemma 2.1(iii), for $\mu \in \mathcal{M}(X, T)$, the set $G_\mu(X, T_1)$ of generic points of μ defined as in (1.3) is just equal to

$$\left\{ x \in X : \lim_{n \rightarrow \infty} \frac{\log \phi_n(x)}{n} = \Phi_*(\mu), \quad \forall \Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{asa}(X, T) \right\}.$$

2.3. Relativized sub-additive thermodynamic formalism. Let $\pi : X \rightarrow Y$ be a one-block factor map between two subshifts (X, T) and (Y, S) . The following relativized variational principle was proved in [45] for sub-additive potentials $\Phi = (\log \phi_n)_{n=1}^\infty$ with ϕ_n being continuous, under a general random setting. It does hold for $\Phi \in \mathcal{C}_{ass}(X, T)$ by modifying the proof in [45] slightly. This extends the relativized variational principle of Ledrappier and Walters [32] for additive potentials.

PROPOSITION 2.5. *Let $\Phi \in \mathcal{C}_{ass}(X, T)$ and $\nu \in \mathcal{M}(Y, S)$. Then*

$$(2.5) \quad \sup\{\Phi_*(\mu) + h_\mu(T) - h_\nu(S)\} = \int_Y P(T, \Phi, \pi^{-1}(y)) \, d\nu(y),$$

where the supremum is taken over the set of $\mu \in \mathcal{M}(X, T)$ such that $\mu \circ \pi^{-1} = \nu$, $P(T, \Phi, \pi^{-1}(y))$ is defined as in (2.2).

By the upper semi-continuity of $\Phi_*(\cdot)$ and $h_{(\cdot)}(T)$ on $\mathcal{M}(X, T)$, the supremum in (2.5) is attainable. Any measure $\mu \in \mathcal{M}(X, T)$ for which the supremum in (2.5) is attained at μ is called a *conditional equilibrium state of Φ with respect to ν* .

3. Weighted thermodynamic formalism. In this section, we define the weighted topological pressure for general asymptotically sub-additive potentials, and we discuss the uniqueness and Gibbs properties of weighted equilibrium states. These properties are needed in the proofs of our main results listed in Section 1.

First we recall our basic settings. Let $k \geq 2$. Assume that (X_i, T_i) ($i = 1, \dots, k$) are subshifts over finite alphabets \mathcal{A}_i such that X_{i+1} is a factor of X_i with a one-block factor map $\pi_i : X_i \rightarrow X_{i+1}$ for $i = 1, \dots, k - 1$. For convenience, we use π_0 to denote the identity map on X_1 . Define $\tau_i : X_1 \rightarrow X_{i+1}$ by $\tau_i = \pi_i \circ \pi_{i-1} \circ \dots \circ \pi_0$ for $i = 0, 1, \dots, k - 1$.

Let $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ so that $a_1 > 0$ and $a_i \geq 0$ for $i > 1$. For $\Phi \in \mathcal{C}_{ass}(X_1, T_1)$. We define the *\mathbf{a} -weighted topological pressure of Φ* as

$$P^{\mathbf{a}}(T_1, \Phi) = \sup \{ \Phi_*(\mu) + h_\mu^{\mathbf{a}}(T_1) : \mu \in \mathcal{M}(X_1, T_1) \}.$$

By Lemma 2.1(ii), $\Phi_*(\cdot)$ is upper semi-continuous on $\mathcal{M}(X_1, T_1)$, and so is $h_{(\cdot)}^{\mathbf{a}}(T_1)$, whence the above supremum is attainable. Each measure μ which attains the supremum is called an *\mathbf{a} -weighted equilibrium state of Φ* .

For $i = 1, \dots, k - 1$, we define $\theta_i : \mathcal{C}_{ass}(X_i, T_i) \rightarrow \mathcal{C}_{ass}(X_{i+1}, T_{i+1})$ by $(\log \phi_n)_{n=1}^\infty \mapsto (\log \psi_n)_{n=1}^\infty$, where

$$\psi_n(y) = \left(\sum_{I \in \mathcal{A}_i^n : [I] \cap \pi_i^{-1}(y) \neq \emptyset} \sup_{x \in [I] \cap \pi_i^{-1}(y)} \phi_n(x)^{1/A_i} \right)^{A_i}$$

for $y \in X_{i+1}$, with $A_i = a_1 + \dots + a_i$. In particular, let \mathcal{S}_{ass} denote the collection of asymptotically sub-additive additive (scalar) sequences $(\log c_n)_{n=1}^\infty$ (a sequence $(\log c_n)_{n=1}^\infty$, where $c_n \geq 0$, is called *asymptotically sub-additive* if, for

any $\varepsilon > 0$, there exists a sequence $(d_n)_{n=1}^\infty$, so that $0 \leq d_{n+m} \leq d_n d_m$ and $\limsup_{n \rightarrow \infty} \frac{1}{n} |\log c_n - \log d_n| < \varepsilon$. Let $\theta_k : \mathcal{C}_{ass}(X_k, T_k) \rightarrow \mathcal{S}_{ass}$ be defined as $(\log \phi_n)_{n=1}^\infty \mapsto (\log c_n)_{n=1}^\infty$, where

$$c_n = \left(\sum_{I \in \mathcal{A}_k^n} \sup_{x \in [I]} \phi_n(x)^{1/A_k} \right)^{A_k}.$$

Our first result in this section is the following.

THEOREM 3.1. *Let $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{ass}(X_1, T_1)$. Then*

- (i) *For any $1 \leq i \leq k-1$, $P^{\mathbf{a}}(T_1, \Phi) = P^{(\sum_{j=1}^{i+1} a_j, a_{i+2}, \dots, a_k)}(T_{i+1}, \theta_i \circ \dots \circ \theta_1(\Phi))$.*
- (ii) *$P^{\mathbf{a}}(T_1, \Phi) = \lim_{n \rightarrow \infty} (1/n) \log c_n$, where $(c_n)_{n=1}^\infty = \theta_k \circ \dots \circ \theta_1(\Phi)$.*
- (iii) *$\mu \in \mathcal{M}(X_1, T_1)$ is an \mathbf{a} -weighted equilibrium state of Φ if and only if $\mu \circ \tau_{k-1}^{-1}$ is an equilibrium state of $\frac{\theta_{k-1} \circ \dots \circ \theta_1(\Phi)}{a_1 + \dots + a_k}$ and, for $i = k-2, k-3, \dots, 0$, $\mu \circ \tau_i^{-1}$ is a conditional equilibrium state of $\frac{\theta_i \circ \dots \circ \theta_1(\Phi)}{a_1 + \dots + a_{i+1}}$ with respect to $\mu \circ \tau_{i+1}^{-1}$.*

The above result establishes the relation between weighted topological pressures and non-weighted ones, as well as the relation between weighted equilibrium states, non-weighted equilibrium states and conditional equilibrium states. We remark that Theorem 3.1 was proved in [22] for a special class of sub-additive potentials $(\log \phi_n)_{n=1}^\infty$, where ϕ_n are assumed to be n -symbol functions. Yayama [47, 46] also considered the case $\Phi = 0$ independently.

The following simple lemma plays a key role in the proof of Theorem 3.1.

LEMMA 3.2. *Let $\pi : X \rightarrow Y$ be a one-block factor map between two subshifts (X, T) and (Y, S) . Let $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{ass}(X, T)$ and $\nu \in \mathcal{M}(Y, S)$. Then we have*

$$(3.1) \quad \sup\{\Phi_*(\mu) + h_\mu(T) - h_\nu(S) : \mu \in \mathcal{M}(X, T), \mu \circ \pi^{-1} = \nu\} = \Psi_*(\nu),$$

where $\Psi = (\log \psi_n)_{n=1}^\infty \in \mathcal{C}_{ass}(Y, S)$ is defined by

$$\psi_n(y) = \sum_{I \in \mathcal{A}^n : [I] \cap \pi^{-1}(y) \neq \emptyset} \sup_{x \in [I] \cap \pi^{-1}(y)} \phi_n(x).$$

Proof. By Proposition 2.5, the left-hand side of (3.1) equals $\int P(T, \Phi, \pi^{-1}(y)) d\nu(y)$. However by (2.2)-(2.1),

$$P(T, \Phi, \pi^{-1}(y)) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(T, \Phi, \pi^{-1}(y))$$

and

$$P_n(T, \Phi, \pi^{-1}(y)) = \sum_{I \in \mathcal{A}^n : [I] \cap \pi^{-1}(y)} \sup_{x \in [I] \cap \pi^{-1}(y)} \phi_n(x).$$

Clearly $\psi_n(y) = P_n(T, \Phi, \pi^{-1}(y))$. It is direct to check that $\Psi = (\log \psi_n)_{n=1}^\infty \in \mathcal{C}_{ass}(Y, S)$. Hence by Lemma 2.1,

$$\Psi_*(\nu) = \int \limsup_{n \rightarrow \infty} \frac{1}{n} \log \psi_n(y) d\nu(y) = \int P(T, \Phi, \pi^{-1}(y)) d\nu(y).$$

This finishes the proof of the lemma. \square

REMARK 3.3. In Lemma 3.2, it seems that ψ_n might be discontinuous when ϕ_n is continuous. However when both X and Y are full shifts, if ϕ_n is continuous, then so is ψ_n .

Proof of Theorem 3.1. To show (i), we only prove the case when $i = 1$. The general case then just follows by a recursive argument. Write $\mathbf{b} = (a_2, a_3, \dots, a_k)$ and $\mathbf{c} = (a_1 + a_2, a_3, \dots, a_k)$. Then we have

$$(3.2) \quad \begin{aligned} & \sup\{\Phi_*(\mu) + h_\mu^{\mathbf{a}}(T_1), \mu \in \mathcal{M}(X_1, T_1)\} \\ &= \sup\{\Phi_*(\mu) + a_1 h_\mu(T_1) + h_\nu^{\mathbf{b}}(T_2) : \nu \in \mathcal{M}(X_2, T_2), \mu \in \mathcal{M}(X_1, T_1), \mu \circ \pi_1^{-1} = \nu\} \\ &= \sup\{A(\nu) + h_\nu^{\mathbf{c}}(T_2) : \nu \in \mathcal{M}(X_2, T_2)\}, \end{aligned}$$

where

$$A(\nu) := a_1 \sup\left\{ \frac{1}{a_1} \Phi_*(\mu) + h_\mu(T_1) - h_\nu(T_2) : \mu \in \mathcal{M}(X_1, T_1), \mu \circ \pi_1^{-1} = \nu \right\}.$$

By Lemma 3.2, we have $A(\nu) = \Psi_*(\nu)$, where $\Psi = (\log \psi_n)_{n=1}^\infty \in \mathcal{C}_{ass}(Y, S)$ is defined as

$$\psi_n(y) = \left(\sum_{I \in \mathcal{A}^n: [I] \cap \pi^{-1}(y) \neq \emptyset} \sup_{x \in [I] \cap \pi^{-1}(y)} \phi_n(x)^{1/a_1} \right)^{a_1}.$$

That is, $\Psi = \theta_1(\Phi)$. Hence by (3.2), we have $P^{\mathbf{a}}(T_1, \Phi) = P^{\mathbf{c}}(T_2, \theta_1(\Phi))$, as desired. To see (ii), note that by (i) we have

$$P^{\mathbf{a}}(T_1, \Phi) = P^{(a_1 + \dots + a_k)}(T_k, \Gamma) = \sup\{\Gamma_*(\eta) + (a_1 + \dots + a_k)h_\eta(T_k), \eta \in \mathcal{M}(X_k, T_k)\},$$

where $\Gamma := \theta_{k-1} \circ \dots \circ \theta_1(\Phi)$. Then (ii) follows from Proposition 2.2.

By (3.2), μ is an \mathbf{a} -weighted equilibrium state of Φ if and only if $\mu \circ \pi_1^{-1}$ is a \mathbf{c} -weighted equilibrium state of $\theta_1(\Phi)$ and μ is a conditional equilibrium state of $\frac{1}{a_1}\theta_1(\Phi)$ with respect to $\mu \circ \pi_1^{-1}$. A recursive argument then yields (iii). \square

In the remaining part of this section, we assume that X_1 satisfies specification. Let $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{asa}(X_1, T_1)$. Define $\phi^{(0)} : \mathcal{L}(X_1) \rightarrow (0, \infty)$ by $\phi^{(0)}(I) = \sup_{x \in [I]} \phi_n(x)$ for $I \in \mathcal{L}_n(X_1)$. Furthermore, define $\phi^{(i)} : \mathcal{L}(X_{i+1}) \rightarrow [0, \infty)$ ($i = 1, \dots, k-1$) recursively by

$$(3.3) \quad \phi^{(i)}(J) = \left(\sum_{I \in \mathcal{L}_n(X_i): \pi_i I = J} \phi^{(i-1)}(I)^{\frac{1}{a_1 + \dots + a_i}} \right)^{a_1 + \dots + a_i}$$

for $n \in \mathbb{N}$, $J \in \mathcal{L}_n(X_{i+1})$. Furthermore, define $\phi^{(k)} : \mathbb{N} \rightarrow [0, \infty)$ by

$$\phi^{(k)}(n) = \sum_{I \in \mathcal{L}_n(X_k)} \phi^{(k-1)}(I)^{\frac{1}{a_1 + \dots + a_k}}.$$

Then we define the \mathbf{a} -weighted potential associated with Φ by

$$(3.4) \quad \Phi^{\mathbf{a}} = (\log \phi_n^{\mathbf{a}})_{n=1}^\infty, \text{ where } \phi_n^{\mathbf{a}}(x) = \phi^{(0)}(x|_n)^{1/A_1} \prod_{i=1}^{k-1} \phi^{(i)}(\tau_i(x|_n))^{1/A_{i+1} - 1/A_i},$$

where $A_i = a_1 + \dots + a_i$, and $x_{|n} := x_1 \dots x_n$ for $x = (x_i)_{i=1}^\infty \in X_1$. Since there exists a sequence $(g^{(p)})_{p \geq 1}$ of Hölder potentials such that $\lim_{p \rightarrow 0} \limsup_{n \rightarrow \infty} \|\Phi_n - S_n g^{(p)}\|_\infty / n = 0$ (see Lemma 2.1(iii)), it is easily seen that all the potentials $(\log \phi^{(i)}(\tau_{i-1}(\cdot|_n)))_{n=1}^\infty$ and $(\log \phi_n^{\mathbf{a}})_{n=1}^\infty$ belong to $\mathcal{C}_{asa}(X, T)$.

We say that $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{aa}(X_1, T_1)$ has the *bounded distortion property* if there exists a constant $c > 0$ such that

$$\frac{1}{c} \phi_n(y) \leq \phi_n(x) \leq c \phi_n(y) \quad \text{whenever } x, y \in X \text{ are in the same } n\text{-th cylinder.}$$

For two families of real numbers $\{a_i\}_{i \in \mathcal{I}}$ and $\{b_i\}_{i \in \mathcal{I}}$, we write

$$\begin{aligned} a_i \approx b_i & \quad \text{if there is } c > 0 \text{ such that } \frac{1}{c} b_i \leq a_i \leq c b_i \text{ for } i \in \mathcal{I}; \\ a_i \succ b_i & \quad \text{if there is } c > 0 \text{ such that } a_i \geq c b_i \text{ for } i \in \mathcal{I}. \end{aligned}$$

The following result was proved in [22].

THEOREM 3.4 (Theorem 7.3 in [22]). *Assume that X_1 satisfies specification. Suppose that $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{aa}(X_1, \sigma_{X_1})$ has the bounded distortion property. Then Φ has a unique \mathbf{a} -weighted equilibrium state μ . Furthermore, μ is ergodic and has the following properties: $\mu([I]) \approx \tilde{\phi}^*(I) \succ \tilde{\phi}(I)$ for $I \in \mathcal{L}(X_1)$, where $\tilde{\phi}, \tilde{\phi}^* : \mathcal{L}(X_1) \rightarrow [0, \infty)$ are defined respectively by*

$$\tilde{\phi}(I) = \left(\prod_{i=0}^{k-2} \frac{\phi^{(i)}(\tau_i I)^{\frac{1}{a_1 + \dots + a_{i+1}}}}{\phi^{(i+1)}(\tau_{i+1} I)^{\frac{1}{a_1 + \dots + a_{i+1}}}} \right) \cdot \frac{\phi^{(k-1)}(\tau_{k-1} I)^{\frac{1}{a_1 + \dots + a_k}}}{\phi^{(k)}(n)}$$

for $I \in \mathcal{L}_n(X_1)$, $n \in \mathbb{N}$, and

$$\tilde{\phi}^*(I) = \sup_{m, n \geq 0} \sum_{I_1 \in \mathcal{L}_m(X_1), I_2 \in \mathcal{L}_n(X_1): I_1 I_2 \in \mathcal{L}(X_1)} \tilde{\phi}(I_1 I_2), \quad I \in \mathcal{L}(X_1).$$

Applying Theorem 3.1(ii) and Theorem 3.4 to the full shift case, we obtain

THEOREM 3.5. *Assume that X_1 is a full shift.*

(i) *Let $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{asa}(X_1, T_1)$. Then*

$$P^{\mathbf{a}}(T_1, \Phi) = A_k \lim_{n \rightarrow \infty} (1/n) \log \phi^{(k)}(n).$$

(ii) *Suppose that $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{aa}(X_1, T_1)$ has the bounded distortion property. Then Φ has a unique \mathbf{a} -weighted equilibrium state μ . Furthermore, μ is fully supported and quasi-Bernoulli, and it satisfies the following Gibbs property*

$$(3.5) \quad \mu([I]) \approx \exp\left(\frac{-nP}{A_k}\right) \phi_n^{\mathbf{a}}(I), \quad I \in \mathcal{A}_1^n,$$

where $P = P^{\mathbf{a}}(T_1, \Phi)$, and $\phi_n^{\mathbf{a}}$ is defined as in (3.4). Consequently, for $i = 2, \dots, k$,

$$(3.6) \quad \mu_i([\tau_{i-1} I]) \approx \exp\left(\frac{-nP}{A_k}\right) \phi^{(i-1)}(\tau_{i-1} I)^{1/A_i} \prod_{j=i}^{k-1} \phi^{(j)}(\tau_j I)^{1/A_{j+1} - 1/A_j}, \quad I \in \mathcal{A}_1^n,$$

where $\mu_i := \mu \circ \tau_{i-1}^{-1}$. Furthermore,

$$\phi_n(x) \exp(-nP) \approx \prod_{i=1}^k \mu_i([\tau_{i-1} x_{|n}])^{a_i} \quad \text{for } x \in X_1, n \geq 1.$$

DEFINITION 3.6. A Borel probability measure μ (not necessarily to be invariant) on X_1 satisfying (3.5) is called an \mathbf{a} -weighted Gibbs measure for Φ .

REMARK 3.7. In the setting of Theorem 3.4, if $\Phi = (S_n f)_{n=1}^\infty$ for $f \in C(X_1)$ with sufficiently regularity (for instance, the Hölder continuity), then the \mathbf{a} -weighted Gibbs measure of Φ is the classical equilibrium state of a continuous function $g \in C(X_1)$ with weaker regularity. To see this, we need to use an invariance of the recent result of Chazottes and Ugalde [14, Theorem 4.1], which claims that if $\phi^{(i-1)}$ is a function on \mathcal{A}_i^* such that

$$\phi^{(i-1)}(x_{|n}) \approx \exp(S_n u(x)), \quad x \in X_i$$

for some function $u(x) \in C(X_i)$ satisfying $\text{var}_n(u) \leq D_1 \exp(-c_1 n^{\gamma_1})$, where $\text{var}_n(u) = \max_{x_{|n}=y_{|n}} |u(x) - u(y)|$, and let $\phi^{(i)}$ be defined as in (3.3), then there exist $v \in C(X_{i+1})$ and $D_2 > D_1$, $0 < \gamma_2 < \gamma_1$, $0 < c_2 < c_1$ so that

$$\phi^{(i)}(x_{|n}) \approx \exp(S_n v(x)), \quad x \in X_{i+1}$$

and $\text{var}_n(v) \leq D_2 \exp(-c_2 n^{\gamma_2})$. Using this claim repeatedly, we see that if $f \in C(X_1)$ satisfies $\text{var}_n(f) \leq \tilde{D} \exp(-\tilde{c} n^{\tilde{\gamma}})$ for some constants $\tilde{D}, \tilde{c}, \tilde{\gamma} > 0$, then the potential $\phi^{\mathbf{a}}$ defined as in (3.4) will satisfy

$$\phi^{\mathbf{a}}(x_{|n}) \approx \exp(S_n g(x)), \quad x \in X_1$$

for some $g \in C(X_1)$ satisfying $\text{var}_n(g) \leq D \exp(-cn^\gamma)$ with some constants $D, c, \gamma > 0$. Then according to (3.5), the \mathbf{a} -weighted equilibrium state μ of Φ is just the equilibrium state of g .

In the reminder of this section, we consider the question when $\Phi, \Psi \in \mathcal{C}_{aa}(X_1, T_1)$ have the same \mathbf{a} -weighted equilibrium state.

DEFINITION 3.8. We say that two almost additive potentials $\Phi = (\log \phi_n)_{n=1}^\infty$ and $\Psi = (\log \psi_n)_{n=1}^\infty$ are *cohomologous* if $\sup_n \|\log \phi_n - \log \psi_n\|_\infty < \infty$, where $\|f\|_\infty = \sup_{x \in X_1} |f(x)|$ for $f \in C(X_1)$. If there exists $C \in \mathbb{R}$ such that $\log \psi_n = Cn$, we say that Φ is *cohomologous to a constant*.

The following proposition is a direct consequence of Theorem 3.5.

PROPOSITION 3.9. Suppose $\Phi, \Psi \in \mathcal{C}_{aa}(X_1, T_1)$ satisfy the bounded distortion property. Then, Φ and Ψ share the same \mathbf{a} -weighted equilibrium state if and only if $\Phi - \Psi$ is cohomologous to a constant.

4. The proof of Theorem 1.1. In this section, we always assume that (X_1, T_1) is a subshift satisfying specification. To prove Theorem 1.1, we need the following result.

PROPOSITION 4.1. Let $\Phi_1, \dots, \Phi_d \in \mathcal{C}_{aa}(X_1, T_1)$ satisfy the bounded distortion property. Then the map $Q : \mathbb{R}^d \rightarrow \mathbb{R}$ defined as

$$(q_1, \dots, q_d) \mapsto P^{\mathbf{a}} \left(T_1, \sum_{i=1}^d q_i \Phi_i \right),$$

is C^1 over \mathbb{R}^d with

$$\nabla Q(q_1, \dots, q_d) = ((\Phi_1)_*(\mu_{\mathbf{q}}), \dots, (\Phi_d)_*(\mu_{\mathbf{q}})),$$

where ∇Q denotes the gradient of Q , $\mu_{\mathbf{q}}$ is the unique \mathbf{a} -weighted equilibrium state of $\sum_{i=1}^d q_i \Phi_i$.

To prove Proposition 4.1, we need the following result coming from convex analysis.

PROPOSITION 4.2 ([23], Proposition 2.3). *Let Z be a compact convex subset of a topological vector space which satisfies the first axiom of countability (i.e., there is a countable base at each point) and $U \subseteq \mathbb{R}^d$ a non-empty open set. Suppose $f : U \times Z \rightarrow \mathbb{R} \cup \{-\infty\}$ is a map satisfying the following conditions:*

- (i) $f(\mathbf{q}, z)$ is convex in \mathbf{q} ;
- (ii) $f(\mathbf{q}, z)$ is affine in z ;
- (iii) f is upper semi-continuous over $U \times Z$;
- (iv) $g(\mathbf{q}) := \sup_{z \in Z} f(\mathbf{q}, z) > -\infty$ for any $\mathbf{q} \in U$.

For each $\mathbf{q} \in U$, denote $\mathcal{I}(\mathbf{q}) := \{z \in Z : f(\mathbf{q}, z) = g(\mathbf{q})\}$. Then

$$\partial g(\mathbf{q}) = \bigcup_{z \in \mathcal{I}(\mathbf{q})} \partial f(\mathbf{q}, z),$$

where $\partial f(\mathbf{q}, z)$ denotes the subdifferential of $f(\cdot, z)$ at \mathbf{q} .

Proof of Proposition 4.1. In Proposition 4.2, we let $U = \mathbb{R}^d$, $Z = \mathcal{M}(X_1, T_1)$, and define $f : U \times Z \rightarrow \mathbb{R}$ by

$$f(\mathbf{q}, \mu) = \sum_{i=1}^d q_i (\Phi_i)_*(\mu) + h_{\mu}^{\mathbf{a}}(T_1), \quad \mathbf{q} = (q_1, \dots, q_d), \mu \in \mathcal{M}(X_1, T_1).$$

Set $g(\mathbf{q}) = \sup_{z \in Z} f(\mathbf{q}, z) = P^{\mathbf{a}}(T_1, \sum_{i=1}^d q_i \Phi_i)$. Since $\Phi_i \in \mathcal{C}_{aa}(X_1, T_1)$, $\mu \mapsto (\Phi_i)_*(\mu)$ is continuous on $\mathcal{M}(X_1, T_1)$ (see Lemma 2.1(ii)). Thus, f and g satisfy the assumptions (i)-(iv) in Proposition 4.2. Note that $\mathcal{I}(\mathbf{q})$ just corresponds to the set of all \mathbf{a} -weighted equilibrium states of $\sum_{i=1}^d q_i \Phi_i$. By Theorem 3.5, $\mathcal{I}(\mathbf{q}) = \{\mu_{\mathbf{q}}\}$ is a singleton for each $\mathbf{q} \in \mathbb{R}^d$. By Proposition 4.2, $\nabla g(\mathbf{q}) = ((\Phi_1)_*(\mu_{\mathbf{q}}), \dots, (\Phi_d)_*(\mu_{\mathbf{q}}))$. Since g is convex and differentiable on \mathbb{R}^d , it is C^1 on \mathbb{R}^d (see, e.g. [42, Corollary 25.5.1]). This finishes the proof of Proposition 4.1. \square

Before proving Theorem 1.1, we still need some notation and basic facts in convex analysis. Let $g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex and not identically equal to $+\infty$. Then the function $g^* : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathbf{s} \mapsto g^*(\mathbf{s}) := \sup\{\mathbf{s} \cdot \mathbf{x} - g(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\}$$

is called the *conjugate function* of g , where $\mathbf{s} \cdot \mathbf{x}$ denotes the standard dot product of \mathbf{s} and \mathbf{x} in \mathbb{R}^d . It is known that g^* is also convex and not identically equal to $+\infty$ (cf. [26, p. 211]). Let g^{**} denote the conjugate of g^* . The following result is well known in convex analysis (cf. [42, Theorem 12.2, Corollary 26.4.1]).

PROPOSITION 4.3.

- (i) Let $g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex and not identically equal to $+\infty$. Let $\mathbf{x} \in \mathbb{R}^d$. Assume that g is lower semi-continuous at \mathbf{x} , i.e., $\liminf_{\mathbf{y} \rightarrow \mathbf{x}} g(\mathbf{y}) \geq g(\mathbf{x})$. Then $g^{**}(\mathbf{x}) = g(\mathbf{x})$.
- (ii) Let g be real convex and differentiable on \mathbb{R}^d . Let $D = \{\nabla g(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\}$. Then $ri(\text{dom } g^*) \subseteq D$, where $\text{dom } g^* := \{\mathbf{x} \in \mathbb{R}^d : g^*(\mathbf{x}) \neq +\infty\}$, and $ri(A)$ denotes the relative interior of a convex set A .

Proof of Theorem 1.1. We first prove part (i) of the theorem. Fix $n \in \mathbb{N}$ and define a function $f : \Omega_n \rightarrow \mathbb{R}$ by

$$f(\mathbf{p}) = \sup \left\{ h_{\mu}^{\mathbf{a}}(T_1) : \mu \in \mathcal{M}(X_1, T_1) : (\mu([I]))_{I \in \mathcal{L}_n(X_1)} = \mathbf{p} \right\}.$$

Define $g : \mathbb{R}^{\mathcal{L}_n(X_1)} \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$g(\mathbf{p}) = \begin{cases} -f(\mathbf{p}) & \text{if } \mathbf{p} \in \Omega_n, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easily checked that g is convex and lower semi-continuous on $\mathbb{R}^{\mathcal{L}_n(X_1)}$. Let g^* denote the conjugate of g , and g^{**} the conjugate of g^* . By Proposition 4.3(i), $g^{**} = g$ on $\mathbb{R}^{\mathcal{L}_n(X_1)}$.

However, by the definition of f , we have for $\mathbf{q} = (q(I))_{I \in \mathcal{L}_n(X_1)}$,

$$\begin{aligned} g^*(\mathbf{q}) &= \sup \left\{ f(\mathbf{p}) + \mathbf{p} \cdot \mathbf{q} : \mathbf{p} \in \mathbb{R}^{\mathcal{L}_n(X_1)} \right\} = \sup \left\{ f(\mathbf{p}) + \mathbf{p} \cdot \mathbf{q} : \mathbf{p} \in \Omega_n \right\} \\ &= \sup \left\{ \left(\sum_{I \in \mathcal{L}_n(X_1)} q(I) \int \chi_{[I]} d\mu \right) + h_{\mu}^{\mathbf{a}}(T_1) : \mu \in \mathcal{M}(X_1, T_1) \right\} \\ &= P^{\mathbf{a}} \left(T_1, \sum_{I \in \mathcal{L}_n(X_1)} q(I) \Phi_I \right), \end{aligned}$$

where $\chi_{[I]}$ denotes the indicator function of $[I]$, and Φ_I denotes the additive potential $\left(\sum_{i=0}^{m-1} \chi_{[I]}(T_1^i x) \right)_{m=1}^{\infty}$. By Proposition 4.1, g^* is differentiable over $\mathbb{R}^{\mathcal{L}_n(X_1)}$ and

$$(4.1) \quad \nabla g^*(\mathbf{q}) = ((\Phi_I)_*(\mu_{\mathbf{q}}))_{I \in \mathcal{A}^n} = (\mu_{\mathbf{q}}([I]))_{I \in \mathcal{L}_n(X_1)},$$

where $\mu_{\mathbf{q}}$ denotes the unique \mathbf{a} -weighted equilibrium state of $\sum_{I \in \mathcal{L}_n(X_1)} q(I) \Phi_I$. Applying Proposition 4.3(ii) to g^* , we have

$$(4.2) \quad \left\{ \nabla g^*(\mathbf{q}) : \mathbf{q} \in \mathbb{R}^{\mathcal{L}_n(X_1)} \right\} \supseteq \text{ri}(\text{dom } g^{**}) = \text{ri}(\text{dom } g) = \text{ri}(\Omega_n).$$

Now let $\eta \in \mathcal{M}(X_1, T_1)$ so that $(\eta([I]))_{I \in \mathcal{L}_n(X_1)} \in \text{ri}(\Omega_n)$. Therefore by (4.2), there exists $\mathbf{q} \in \mathbb{R}^{\mathcal{L}_n(X_1)}$ so that $\nabla g^*(\mathbf{q}) = (\eta([I]))_{I \in \mathcal{L}_n(X_1)}$; and thus by (4.1), $(\mu_{\mathbf{q}}([I]))_{I \in \mathcal{L}_n(X_1)} = (\eta([I]))_{I \in \mathcal{L}_n(X_1)}$. Now assume that $\tilde{\mu}$ is a T_1 -invariant measure different from $\mu_{\mathbf{q}}$ such that $(\tilde{\mu}([I]))_{I \in \mathcal{L}_n(X_1)} = (\eta([I]))_{I \in \mathcal{L}_n(X_1)}$. Since $\mu_{\mathbf{q}}$ is the unique \mathbf{a} -weighted equilibrium state of $\sum_{I \in \mathcal{L}_n(X_1)} q(I) \Phi_I$, we have

$$\begin{aligned} P^{\mathbf{a}} \left(T_1, \sum_{I \in \mathcal{L}_n(X_1)} q(I) \Phi_I \right) &= \sum_{I \in \mathcal{L}_n(X_1)} q(I) \int \chi_{[I]} d\mu_{\mathbf{q}} + h_{\mu_{\mathbf{q}}}^{\mathbf{a}}(T_1) \\ &> \sum_{I \in \mathcal{L}_n(X_1)} q(I) \int \chi_{[I]} d\tilde{\mu} + h_{\tilde{\mu}}^{\mathbf{a}}(T_1). \end{aligned}$$

It follows that $h_{\mu_{\mathbf{q}}}^{\mathbf{a}}(T_1) > h_{\tilde{\mu}}^{\mathbf{a}}(T_1)$. This proves (i).

To prove (ii), let $\eta_0 \in \mathcal{M}(X_1, T_1)$ be the measure having the maximal \mathbf{a} -weighted entropy. Clearly, η_0 is the \mathbf{a} -weighted equilibrium state of $f \equiv 0$. If $\eta = \eta_0$, then

we just take $\mu_n = \eta_0$ for $n \geq 1$ and we are done. In the following we assume that $\eta \neq \eta_0$. For each $n \in \mathbb{N}$, we pick $\xi_n \in \mathcal{M}(X_1, T_1)$ such that $(\xi_n([I]))_{I \in \mathcal{L}_n(X_1)} \in \text{ri}(\Omega_n)$. Then choose a small $p_n > 0$ such that $p_n h_{\xi_n}^{\mathbf{a}}(T_1) + (1 - p_n) h_{\eta_0}^{\mathbf{a}}(T_1) > h_{\eta}^{\mathbf{a}}(T_1)$. Define $\tilde{\xi}_n = p_n \xi_n + (1 - p_n) \eta_0$. Let $\eta_n = (1 - 1/n) \eta + (1/n) \tilde{\xi}_n$. Then it is readily checked that $\eta_n \in \text{ri}(\Omega_n)$. Let $\mu_n = \mu(\mathbf{a}, \eta_n, n)$. Then the sequence (μ_n) has the desired properties. \square

REMARK 4.4. Let X_1 be a full shift and $\mu \in \mathcal{M}(X_1, T_1)$. Taking $\mathbf{a} = (1, \dots, 1)$ in Theorem 1.1 and using the upper semi-continuity of the entropy, we see that there is a sequence of quasi-Bernoulli measures $(\mu_n)_{n=1}^\infty$ which converges to μ in the weak-star topology, such that we have $\lim_{n \rightarrow \infty} h_{\mu_n \circ \tau_{i-1}^{-1}}(T_i) = h_{\mu \circ \tau_{i-1}^{-1}}(T_i)$ for all $1 \leq i \leq k$. Moreover, one can deduce from Theorem 3.5 that for any $\mathbf{a} = (a_1, \dots, a_k)$ with $a_1 > 0$ and $a_i \geq 0$ for $i \geq 2$, each invariant quasi-Bernoulli measure is the \mathbf{a} -weighted equilibrium state of some almost additive potential satisfying the bounded distortion property.

In the remainder of this section, we prove our statement in Example 1.2. Let $X_1 = \{a, b, c\}^\mathbb{N}$ and $X_2 = \{1, 2\}^\mathbb{N}$. Let $\pi : X_1 \rightarrow X_2$ be a one-block factor map induced by $a, b \mapsto 1$ and $c \mapsto 2$. We first prove the following lemma.

LEMMA 4.5. *Let η be a fully supported one-step Markov measure on X_1 . If*

$$(4.3) \quad \frac{\eta([ac])}{\eta([a])} = \frac{\eta([bc])}{\eta([b])} \quad \text{or} \quad \frac{\eta([ca])}{\eta([a])} = \frac{\eta([cb])}{\eta([b])},$$

then $\eta \circ \pi^{-1}$ is a one-step Markov measure on X_2 ; otherwise $\eta \circ \pi^{-1}$ is not a Markov measure of any order.

Proof. Our argument is inspired by the work of Chazottes and Ugalde [13]. Denote $p_{i,j} = \eta([ij])/\eta([i])$ for $i, j \in \{a, b, c\}$. Clearly

$$(4.4) \quad \eta([x_1 x_2 \dots x_n]) = \eta([x_1]) p_{x_1, x_2} \dots p_{x_{n-1}, x_n}, \quad \forall x_1 \dots x_n \in \{a, b, c\}^n.$$

According to (4.4), it is direct to check that

$$\eta \circ \pi^{-1}([y_1 y_2 \dots y_n]) = \mathbf{v}_{y_1} M_{y_1 y_2} \dots M_{y_{n-1} y_n} \mathbf{u}_{y_n}^T, \quad \forall y_1 \dots y_n \in \{1, 2\}^n,$$

where

$$\mathbf{v}_1 = (\eta([a]), \eta([b])), \quad \mathbf{v}_2 = \eta([c]), \quad \mathbf{u}_1 = (1, 1), \quad \mathbf{u}_2 = 1.$$

$$M_{11} = \begin{pmatrix} p_{a,a} & p_{a,b} \\ p_{b,a} & p_{b,b} \end{pmatrix}, \quad M_{12} = \begin{pmatrix} p_{ac} \\ p_{bc} \end{pmatrix}, \quad M_{21} = (p_{ca}, p_{cb}), \quad M_{22} = p_{cc}.$$

Note that $\eta \circ \pi^{-1}$ is a $(k - 1)$ -step Markov measure if and only if

$$(4.5) \quad \frac{\eta \circ \pi^{-1}([y_1 y_2 \dots y_n])}{\eta \circ \pi^{-1}([y_2 \dots y_n])} \text{ depends only on } y_1 \dots y_k \text{ when } n \geq k.$$

However, it is easy to check

$$\frac{\eta \circ \pi^{-1}([y_1 y_2 \dots y_n])}{\eta \circ \pi^{-1}([y_2 \dots y_n])} = \frac{\eta \circ \pi^{-1}([y_1 y_2 \dots y_i])}{\eta \circ \pi^{-1}([y_2 \dots y_i])},$$

provided that $y_i = 2$ for some $2 \leq i \leq n$. Thus to check whether (4.5) holds for all $y_1 y_2 \dots y_n$, it suffices to check the property for the words $1^n, 1^n 2, 21^n, 21^n 2$. However,

it is direct to check that (4.5) holds for the words $1^n, 1^n2, 21^n, 21^n2$ for some $k \geq 2$ if and only if \mathbf{v}_1 and M_{21} are the left eigenvectors of M_{11} , or, M_{21} and \mathbf{u}_1 are the right eigenvectors of M_{11} ; and this holds if and only if (4.3) holds. In the end, if (4.3) holds then, (4.5) holds for $k = 2$. This finishes the proof of the lemma. \square

Proof of Example 1.2. Let $\eta \in \mathcal{M}(X_1, T_1)$ be fully supported. Write $\mu = \mu(\mathbf{a}, \eta, 2)$. By Theorem 1.1, μ is the \mathbf{a} -weighted equilibrium state of some 2-symbol function f on X_1 . Define $\phi^{(0)} : \{a, b, c\}^* \rightarrow (0, \infty)$ and $\phi^{(1)} : \{1, 2\}^* \rightarrow (0, \infty)$ respectively by

$$\phi^{(0)}(I) = \sup_{x \in [I]} \exp(S_n f(x)) \text{ for } I \in \{a, b, c\}^n \text{ and}$$

$$\phi^{(1)}(J) = \left(\sum_{I \in \pi^{-1}(J)} \phi^{(0)}(I)^{1/a_1} \right)^{a_1} \text{ for } J \in \{1, 2\}^n,$$

here and afterwards, $S_n f(x)$ denotes the sum $\sum_{i=0}^{n-1} f(T_1^i x)$. Then by Theorem 3.5,

$$(4.6) \quad \mu([I]) \approx \exp\left(-\frac{nP}{a_1 + a_2}\right) \phi^{(0)}(I)^{\frac{1}{a_1}} \phi^{(1)}(\pi I)^{\frac{1}{a_1+a_2} - \frac{1}{a_1}}, \quad I \in \{a, b, c\}^*,$$

and

$$(4.7) \quad \mu \circ \pi^{-1}([J]) \approx \exp\left(-\frac{nP}{a_1 + a_2}\right) \phi^{(1)}(J)^{\frac{1}{a_1+a_2}}, \quad J \in \{1, 2\}^*,$$

where $P = P^{\mathbf{a}}(T_1, f)$. Let m be the equilibrium state of $\frac{1}{a_1}f$. Since $\frac{1}{a_1}f$ is a 2-symbol function, m is a one-step Markov measure on X_1 . Clearly

$$m([I]) \approx \exp(-nQ) \phi^{(0)}(I)^{1/a_1}, \quad m \circ \pi^{-1}([I]) \approx \exp(-nQ) \phi^{(1)}(\pi I)^{1/a_1}$$

for $I \in \{a, b, c\}^*$, where $Q := P(T_1, (1/a_1)f)$. By Lemma 4.5, $m \circ \pi^{-1}$ is either a one-step Markov measure, or not a Markov measure of any order. Hence, either there exists a 2-symbol function h on X_2 such that

$$(4.8) \quad \phi^{(1)}(J) \approx \sup_{y \in [J]} \exp(S_n h(y)), \quad J \in \{1, 2\}^n,$$

or (4.8) does not hold for any finite-symbol function h on X_2 .

Assume that μ is a Markov measure of finite order. Then there exists a finite-symbol function g on X_1 such that $\mu([I]) \approx \sup_{x \in [I]} \exp(S_n g(x))$ for $I \in \{a, b, c\}^n$. Combining this with (4.6) yields that there exists $k \geq 2$ such that (4.8) holds for some k -symbol function h on X_2 . Therefore (4.8) holds for some 2-symbol function \tilde{h} on X_2 . Applying this to (4.6) and (4.7), we see that both μ and $\mu \circ \pi^{-1}$ are one-step Markov measures. By Lemma 4.5, (1.4) must hold (noting that $\mu([I])$ and $\eta([I])$ coincide for all 2-nd cylinders $[I]$).

Conversely, assume that (1.4) holds. Define a measure $\tilde{\mu}$ on X_1 by

$$\tilde{\mu}([x_1 x_2 \dots x_n]) = \eta([x_1]) p_{x_1, x_2} \dots p_{x_{n-1}, x_n}, \quad \forall x_1 \dots x_n \in \{a, b, c\}^n,$$

where $p_{i,j} = \eta([ij])/\eta([i])$ for $i, j \in \{a, b, c\}$. Then $\tilde{\mu}$ is a one-step Markov measure so that $\tilde{\mu}([I])$ and $\eta([I])$ coincide for all 2-nd cylinders $[I]$. By Lemma 4.5, $\tilde{\mu} \circ \pi^{-1}$ is also a one-step Markov measure. Hence $h_{\tilde{\mu}}(T_1) \geq h_{\tilde{\eta}}(T_1)$ and $h_{\tilde{\mu} \circ \pi^{-1}}(T_2) \geq h_{\tilde{\eta} \circ \pi^{-1}}(T_2)$ for any $\tilde{\eta} \in \mathcal{M}(X_1, T_1)$ so that $\tilde{\eta}([I])$ and $\eta([I])$ coincide for all 2-nd cylinders $[I]$ (cf. [19, Propositions 1-2]). Therefore $\tilde{\mu} = \mu(\mathbf{a}, \eta, 2)$. \square

5. Proofs of Theorems 1.3 and 1.4.

5.1. Preliminary lemmas. Let us introduce some more notation and give useful preliminary facts.

For $1 \leq i \leq k$ and $n \in \mathbb{N}$, let

$$\ell_i(n) = \min\{p \in \mathbb{N} : p \geq (a_1 + \dots + a_i)n/a_1\},$$

and by convention set $\ell_0(n) = 0$. It is easy to check that

LEMMA 5.1. *In (X_1, d_a) , the closed ball centered at x of radius e^{-n/a_1} is given by*

$$B(x, e^{-n/a_1}) = \{y \in X_1 : \tau_{i-1}(y|_{\ell_i(n)}) = \tau_{i-1}(x|_{\ell_i(n)}) \text{ for all } 1 \leq i \leq k\}.$$

For convenience, sometimes for a measure μ on X_i ($i = 1, \dots, k$), we write $\mu(I) = \mu([I])$ for $I \in \mathcal{L}(X_i)$. The following result estimates the value of an \mathbf{a} -weighted Gibbs measure on a ball in (X_1, d_a) .

LEMMA 5.2. *Suppose that (X_1, T_1) is a full shift. Let $\Phi = (\log \phi_n)_{n=1}^\infty \in C_{aa}(X_1, T_1)$ satisfy the bounded distortion property. Let μ denote the \mathbf{a} -weighted Gibbs measure of Φ . Then, uniformly in $x = (x_i)_{i=1}^\infty \in X_1$ and $n \geq 1$, we have*

$$(5.1) \quad \mu(B(x, e^{-n/a_1})) \approx \prod_{i=1}^k \mu_i(\tau_{i-1}(x_{\ell_{i-1}(n)+1} \dots x_{\ell_i(n)}))$$

$$(5.2) \quad \approx \exp\left(\frac{-nP^{\mathbf{a}}(T, \Phi)}{a_1}\right) \phi_n(x)^{1/a_1} \prod_{j=1}^{k-1} \frac{\phi^{(j)}(\tau_j(x|_{\ell_{j+1}(n)}))^{1/A_{j+1}}}{\phi^{(j)}(\tau_j(x|_{\ell_j(n)}))^{1/A_j}},$$

where $\mu_i = \mu \circ \tau_{i-1}^{-1}$ for $1 \leq i \leq k$, and $\phi^{(j)}$, $j = 0, \dots, k-1$, are defined as in (3.3), $A_j = a_1 + \dots + a_j$ for $1 \leq j \leq k$.

Proof. We first prove (5.1). Let $x = (x_i)_{i=1}^\infty \in X_1$ and $n \geq 1$. For $i = 1, \dots, k$, write $U_i = x_{\ell_{i-1}(n)+1} \dots x_{\ell_i(n)}$. Let B denote $B(x, e^{-n/a_1})$. By Lemma 5.1, $B = \{y : \forall 1 \leq i \leq k, \tau_{i-1}(y) \in \tau_{i-1}([U_1 \dots U_i])\}$, so

$$\mu(B) = \sum_{(J_1, \dots, J_k) \in \Theta} \mu(J_1 \dots J_k),$$

where

$$\Theta := \left\{ (J_1, \dots, J_k) \in \prod_{i=1}^k \mathcal{A}_1^{\ell_i(n) - \ell_{i-1}(n)} : \forall 1 \leq i \leq k, \tau_{i-1}(J_i) = \tau_{i-1}(U_i) \right\}.$$

Applying $k-1$ times the quasi Bernoulli property of μ to each term of the above sum, we get

$$\mu(B) \approx \sum_{(J_1, \dots, J_k) \in \Theta} \mu(J_1) \dots \mu(J_k),$$

and (5.1) follows by summing over J_k, \dots, J_2 successively.

Now we prove (5.2). Let us transform (5.1) by using (3.6). Since each word U_i is of length $\ell_i(n) - \ell_{i-1}(n)$ and by construction $\ell_k(n)/A_k - n/a_1 = O(1/n)$, (3.6) yields

$$\begin{aligned} \mu(B) &\approx \exp\left(\frac{-\ell_k(n)P^{\mathbf{a}}(T_1, \Phi)}{A_k}\right) \prod_{i=1}^k \phi^{(i-1)}(\tau_{i-1}U_i)^{1/A_i} \prod_{j=i}^{k-1} \phi^{(j)}(\tau_j U_i)^{1/A_{j+1}-1/A_j} \\ &\approx \exp\left(\frac{-nP^{\mathbf{a}}(T_1, \Phi)}{a_1}\right) \left(\prod_{i=0}^{k-1} \phi^{(i)}(\tau_i U_{i+1})^{1/A_i}\right) \prod_{j=1}^{k-1} \prod_{i=1}^j \phi^{(j)}(\tau_j U_i)^{1/A_{j+1}-1/A_j} \\ &\approx \exp\left(\frac{-nP^{\mathbf{a}}(T_1, \Phi)}{a_1}\right) \phi^{(0)}(U_1)^{1/a_1} \prod_{j=1}^{k-1} \frac{\phi^{(j)}(\tau_j(U_1 \cdots U_{j+1}))^{1/A_{j+1}}}{\phi^{(j)}(\tau_j(U_1 \cdots U_j))^{1/A_j}} \\ &\approx \exp\left(\frac{-nP^{\mathbf{a}}(T_1, \Phi)}{a_1}\right) \phi_n(x)^{1/a_1} \prod_{j=1}^{k-1} \frac{\phi^{(j)}(\tau_j x_{|\ell_{j+1}(n)})^{1/A_{j+1}}}{\phi^{(j)}(\tau_j x_{|\ell_j(n)})^{1/A_j}}. \end{aligned}$$

This finishes the proof of the lemma. \square

5.2. Proof of Theorem 1.3. The main result of this section is the following.

THEOREM 5.3. *Suppose that X_1 satisfies the specification property. Let $\mu \in \mathcal{M}(X_1, T_1)$. Then $\dim_H G_\mu(X_1, T_1) \geq h_\mu^{\mathbf{a}}(T_1)$.*

Proof of Theorem 1.3. By Theorem 5.3, to complete the proof of Theorem 1.3 it remains to control $\dim_H G_\mu(X_1, T_1)$ from above.

For any $\varphi \in C(X_1)$, denote $\mu(\varphi) = \int \varphi d\mu$. Furthermore for $\alpha \in \mathbb{R}$, denote

$$E_\varphi(\alpha) = \left\{ x \in X_1 : \lim_{n \rightarrow \infty} \frac{1}{n} S_n \varphi(x) = \alpha \right\}.$$

Then by (1.3), $G_\mu(X_1, T_1) = \bigcap_{\varphi \in C(X_1)} E_\varphi(\mu(\varphi))$. Thus, by using Lemma 5.6 whose proof is independent of the present one, we obtain

$$\begin{aligned} \dim_H G_\mu(X_1, T_1) &\leq \inf_{\varphi \in C(X_1)} \dim_H E_\varphi(\mu(\varphi)) \\ &\leq \inf_{\varphi \in C(X_1)} \inf_{q \in \mathbb{R}} (P^{\mathbf{a}}(T_1, q\varphi) - q\mu(\varphi)) \\ &= \inf_{q \in \mathbb{R}} \inf_{\varphi \in C(X_1)} (P^{\mathbf{a}}(T_1, q\varphi) - q\mu(\varphi)) \\ &= \inf_{\varphi \in C(X_1)} (P^{\mathbf{a}}(T_1, \varphi) - \mu(\varphi)). \end{aligned}$$

Now we note that, on the one hand, the \mathbf{a} -weighted topological pressure is the Legendre-Fenchel transform of the \mathbf{a} -weighted entropy defined on the compact convex set $\mathcal{M}(X_1, T_1)$ of $C(X)^*$ endowed with the weak-star topology, and on the other hand, the \mathbf{a} -weighted entropy is upper semi-continuous. Hence we have $\inf_{\varphi \in C(X_1)} (P^{\mathbf{a}}(T_1, \varphi) - \mu(\varphi)) = h_\mu^{\mathbf{a}}(T_1)$ by mimicking the proof of Theorem 3.12 in [43]. This yields the conclusion. \square

Let $(\mathcal{A}_1^{\mathbb{N}}, T_1)$ denote the full shift over the alphabet \mathcal{A}_1 . Then X_1 can be viewed as an T_1 -invariant subset of $\mathcal{A}_1^{\mathbb{N}}$. For $p \in \mathbb{N}$, write $\mathcal{L}_p := \mathcal{L}_p(X_1)$ for short, where $\mathcal{L}_p(X_1)$ denotes the p -th language of X_1 (cf. Section 2.1). We use $(\mathcal{L}_p^{\mathbb{N}}, T_1^p)$ to denote the full shift over the alphabet \mathcal{L}_p . Again $\mathcal{L}_p^{\mathbb{N}}$ can be viewed as a subset $\mathcal{A}_1^{\mathbb{N}}$, and $X_1 \subset \mathcal{L}_p^{\mathbb{N}}$.

Proof of Theorem 5.3. Let us outline the main steps in our approach. Fix $\mu \in \mathcal{M}(X_1, T_1)$. We first construct a set $G_\mu \subset \mathcal{A}_1^{\mathbb{N}}$ such that

$$(5.3) \quad G_\mu \subset G_\mu(\mathcal{A}_1^{\mathbb{N}}, T_1) \text{ and } \dim_H G_\mu \geq h_\mu^{\mathbf{a}}(T_1),$$

where μ is considered as an element of $\mathcal{M}(\mathcal{A}_1^{\mathbb{N}}, T_1)$. Once this is done, we will define an injective map $\text{conc} : G \rightarrow X_1$ such that $\text{conc}(G_\mu) \subset G_\mu(X_1, T_1)$ and $\dim_H \text{conc}(G_\mu) = \dim_H G_\mu$. Hence $G_\mu(X_1, T_1) \geq \dim_H G_\mu \geq h_\mu^{\mathbf{a}}(T_1)$.

Now we start the rigorous construction of G_μ . Pick a countable family of Hölder continuous functions

$$\tilde{\mathcal{C}} = \{\varphi_m : m = 1, 2, \dots\} \subset C(\mathcal{A}_1^{\mathbb{N}})$$

so that it is dense in $C(\mathcal{A}_1^{\mathbb{N}})$. Next we construct a sequence (μ_p) of measures on $\mathcal{A}_1^{\mathbb{N}}$ as below.

Let $p \in \mathbb{N}$. Since $X_1 \subset \mathcal{L}_p^{\mathbb{N}}$, the measure μ has a trivial extension to the Borel subsets of $\mathcal{L}_p^{\mathbb{N}}$ defined by $A \mapsto \mu(A \cap X_1)$. We still denote this extension by μ ; it is T_1^p invariant. As $\mathcal{L}_p^{\mathbb{N}}$ is a full shift, by Remark 4.4, μ is the limit in the weak-star topology of a sequence of quasi-Bernoulli measures $(\eta_j)_{j \geq 1}$ on $(\mathcal{L}_p^{\mathbb{N}}, T_1^p)$ such that $h_{\eta_j \circ \tau_{i-1}^{-1}}(T_i^p)$ converges to $h_{\mu \circ \tau_{i-1}^{-1}}(T_i^p) = ph_{\mu \circ \tau_{i-1}^{-1}}(T_i)$ for all $1 \leq i \leq k$, and $\eta_j(S_p \varphi_m)$ converges to $\mu(S_p \varphi_m) = p\mu(\varphi_m)$ for all $1 \leq m \leq p$, as $j \rightarrow \infty$. Hence we may pick a large enough j , by setting $\mu_p = \eta_j$, we have

$$(5.4) \quad \begin{aligned} \left| h_{\mu_p \circ \tau_{i-1}^{-1}}(T_i^p) - ph_{\mu \circ \tau_{i-1}^{-1}}(T_i) \right| &\leq \frac{1}{2p}, \quad \forall 1 \leq i \leq k, \\ \left| \mu_p(S_p \varphi_m) - p\mu(\varphi_m) \right| &\leq \frac{1}{2p}, \quad 1 \leq m \leq p. \end{aligned}$$

As above, we can obtain a sequence $(\mu_p)_{p=1}^\infty$ so that (5.4) holds for each $p \in \mathbb{N}$; moreover μ_p is quasi-Bernoulli and ergodic on $(\mathcal{L}_p^{\mathbb{N}}, T_1^p)$, and it can be viewed as a T_1^p -invariant and ergodic measure on $\mathcal{A}_1^{\mathbb{N}}$. Let (κ_p) be an increasing sequence of positive numbers such that

$$(5.5) \quad \kappa_p^{-1} \mu_p([I]) \mu_p([J]) \leq \mu_p([IJ]) \leq \kappa_p \mu_p([I]) \mu_p([J]), \quad \forall I, J \in \bigcup_{i=0}^{\infty} \mathcal{L}_p^i.$$

For $p, N \in \mathbb{N}$, let $G(p, N)$ denote the set of points $x \in \mathcal{L}_p^{\mathbb{N}} \subset \mathcal{A}_1^{\mathbb{N}}$ such that

$$(5.6) \quad \begin{aligned} \left| \frac{\log \mu_p \circ \tau_{i-1}^{-1}(\tau_{i-1} x |_{pn})}{-n} - ph_{\mu \circ \tau_{i-1}^{-1}}(T_i) \right| &\leq \frac{1}{p} \quad \text{for } n \geq N \text{ and } 1 \leq i \leq k, \text{ and} \\ \left| \frac{S_{np} \varphi_m(x)}{n} - p\mu(\varphi_m) \right| &\leq \frac{1}{p} \quad \text{for } n \geq N \text{ and } 1 \leq m \leq p. \end{aligned}$$

By using the Shannon-McMillian-Brieman theorem (cf. [44, p. 93]) and the Birkhoff ergodic theorem to the ergodic measure dynamic systems $(\mathcal{A}_i^{\mathbb{N}}, T_i^p, \mu_p \circ \tau_{i-1}^{-1})$ ($1 \leq i \leq k$), we have for μ_p -a.e. x ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mu_p \circ \tau_{i-1}^{-1}(\tau_{i-1} x |_{pn}) = -h_{\mu_p}(T_i^p), \quad i = 1, \dots, k, \text{ and}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} S_{np} \varphi_m(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} (S_p \varphi_m)(T_1^{pi} x) = \mu_p(S_p \varphi_m), \quad 1 \leq m \leq p.$$

Combining them with (5.4) yields

$$\lim_{N \rightarrow \infty} \mu_p(G(p, N)) = 1.$$

Hence we can choose a sequence (N_p) of increasing positive integers such that

$$\mu_p(G(p, N_p)) \geq 1 - 2^{-p}, \quad p = 1, 2, \dots$$

Next construct a sequence (M_p) of positive integers such that

$$(5.7) \quad M_p \geq \max \{2^{M_{p-1}}, \kappa_p, 2^{N_{p+1}}\}, \quad p = 2, 3, \dots$$

and set

$$N'_p = N_p \times M_p, \quad p = 1, 2, \dots$$

Then, for $p \geq 1$ let $\tilde{\mu}_p$ be the discrete measure on $\mathcal{A}_1^{pN'_p}$ defined by

$$\tilde{\mu}_p(I) = \mu_p([I]).$$

Clearly, $\tilde{\mu}_p$ is supported on $\mathcal{L}_p^{N'_p}$. Now we define

$$(5.8) \quad G = \prod_{p=1}^{\infty} \mathcal{L}_p^{N'_p}, \quad \nu = \otimes_{p=1}^{\infty} \tilde{\mu}_p,$$

$$G_p = \left\{ I \in \mathcal{L}_p^{N'_p} : [I] \cap G(p, N_p) \neq \emptyset \right\}, \quad p = 1, 2, \dots$$

Finally, we define

$$G_\mu = \otimes_{p \geq 1} G_p = \{I_1 I_2 \cdots I_p \cdots \in \mathcal{A}_1^{\mathbb{N}} : \forall p \geq 1, I_p \in G_p\}.$$

By construction, the measure ν is supported on G and we have

$$\nu(G_\mu) = \prod_{p \geq 1} \tilde{\mu}_p(G_p) \geq \prod_{p \geq 1} \mu_p(G(p, N_p)) \geq \prod_{p \geq 1} (1 - 2^{-p}) > 0.$$

Now we prove that (5.3) holds for G_μ . It is enough to show that

$$(5.9) \quad G_\mu \subset G_\mu(\mathcal{A}_1^{\mathbb{N}}, T_1) \text{ and}$$

$$(5.10) \quad \liminf_{n \rightarrow \infty} \frac{\log \nu(B(x, e^{-n/a_1}))}{-n/a_1} \geq h_\mu^{\mathbf{a}}(T_1) \quad \text{for all } x \in G_\mu$$

since (5.10) implies $\dim_H G_\mu \geq h_\mu^{\mathbf{a}}(T_1)$ (see [18] for instance).

Proof of (5.9). Note that the sequence (φ_m) is dense in $C(\mathcal{A}_1^{\mathbb{N}})$, it suffices to show that for each $x \in G_\mu$ and $m \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} S_n \varphi_m(x) = \mu(\varphi_m).$$

Now we fix such a pair x, m , and write $\varphi = \varphi_m$. Write x in the concatenated form:

$$x = I_1 I_2 \dots I_p \dots, \quad I_j \in G_j \subset \mathcal{L}_j^{N'_j} \text{ for } j \geq 1.$$

For a large number n , let p be the unique integer so that $\sum_{j=1}^{p-1} jN'_j < n \leq \sum_{j=1}^p jN'_j$. That is, p is the integer such that the n -th digit of x is located in the word I_p . Furthermore let q be the unique integer $q \in [1, M_p]$ such that

$$\sum_{j=1}^{p-1} jN'_j + (q-1)pN_p < n \leq \sum_{j=1}^{p-1} jN'_j + qpN_p.$$

That is, if we write $I_p = W_1W_2 \dots W_{M_p}$ with $W_i \in \mathcal{L}_p^{N_p}$, then the n -th digit of x is located in the word W_q . The condition (5.7) guarantees that

$$\sum_{j=1}^{p-2} jN'_j = o((p-1)N'_{p-1}) = o(n), \quad pN_p = o(n).$$

Hence

$$(5.11) \quad S_n\varphi(x) = S_{(p-1)N'_{p-1}}\varphi(T_1^u x) + S_{qpN_p}\varphi\left(T_1^{u+(p-1)N'_{p-1}}x\right) + o(n),$$

where $u := \sum_{j=1}^{p-2} jN'_j$. Keep in mind that $T_1^u x \in [I_{p-1}]$ and $T_1^{u+(p-1)N'_{p-1}}x \in [I_p]$. Since $I_j \in G_j$, by the construction (5.8), $[I_j] \cap G(j, N_j) \neq \emptyset$ for $j = p-1, p$. Hence, by (5.6), there exist $y \in [I_{p-1}]$ and $z \in [I_p]$ such that

$$(5.12) \quad S_{(p-1)N'_{p-1}}\varphi(y) = (p-1)N'_{p-1}\mu(\varphi) + o(n), \quad S_{qpN_p}\varphi(z) = qpN_p\mu(\varphi) + o(n).$$

Since φ is Hölder continuous, it has the following bounded distortion property

$$(5.13) \quad \begin{aligned} S_{(p-1)N'_{p-1}}\varphi(T_1^u x) &= S_{(p-1)N'_{p-1}}\varphi(y) + O(1), \\ S_{qpN_p}\varphi\left(T_1^{u+(p-1)N'_{p-1}}x\right) &= S_{qpN_p}\varphi(z) + O(1). \end{aligned}$$

Combining (5.11)-(5.13) yields

$$S_n\varphi(x) = ((p-1)N'_{p-1} + qpN_p)\mu(\varphi) + o(n) = n\mu(\varphi) + o(n).$$

That is, $\lim_{n \rightarrow \infty} \frac{1}{n} S_n\varphi(x) = \mu(\varphi)$. This finishes the proof of (5.9). \square

Proof of (5.10). Fix $x \in G_\mu$. Write x in the concatenated form:

$$x = I_1 I_2 \dots I_p \dots, \quad I_j \in G_j \subset \mathcal{L}_j^{N'_j} \text{ for } j \geq 1.$$

Let n be a large integer. To estimate $\nu(B(x, e^{-n/a_1}))$, recall that (cf. Lemma 5.1)

$$(5.14) \quad B(x, e^{-n/a_1}) = \{y \in X_1 : \tau_{i-1}(y|_{\ell_i(n)}) = \tau_{i-1}(x|_{\ell_i(n)}) \text{ for all } 1 \leq i \leq k\}.$$

where

$$\ell_i(n) = \min\{p \in \mathbb{N} : p \geq (a_1 + \dots + a_i)n/a_1\},$$

in particular, $\ell_1(n) = n$. For convention write $\ell_0(n) = 0$.

For $i = 1, \dots, k$, let p_i denote the unique integer such that the $\ell_i(n)$ -th digit of x is located in the word I_{p_i} . Note that p_i depends only on n and i , and is independent of x . Due to (5.7), $|I_1 I_2 \dots I_{p-1}| = o(|I_p|)$, where $|I|$ denotes the length of the word I . Hence there are only two possibilities when n is large enough:

- (C1) $p_1 = p_2 = \dots = p_k (=: p)$;
 - (C2) there exists $2 \leq \iota \leq k$ so that $p_1 = \dots = p_{\iota-1} (=: p)$, $p_\iota = \dots = p_k = p + 1$;
- Without loss of generality we assume that $a_i > 0$ for all $1 \leq i \leq k$.

First we consider the case (C1). Write the word I_p in the following form

$$I_p = U_1 W_1 U_2 W_2 \dots U_k W_k U_{k+1}$$

in a way such that $|W_i| = pN_p$ for $i = 1, \dots, k$ and $|U_j|$ are multiples of pN_p (maybe 0) for $j = 1, \dots, k + 1$, and in particular, the $\ell_i(n)$ -th digit of x is located in the word W_i for $i = 1, \dots, k$. Clearly, the above decomposition of I_p is unique (i.e., $|U_j|$'s are uniquely determined). Now we consider the following concatenation

$$x = I_1 I_2 \dots I_{p-1} U_1 W_1 U_2 W_2 \dots U_k W_k U_{k+1} I_{p+1} \dots$$

Due to (5.7), we have

$$\begin{aligned} |I_1 I_2 \dots I_{p-2}| &= o(|I_{p-1}|) = o(n), & |W_i| &= o(n), \\ |I_{p-1}| + |U_1| &= \ell_1(n) + o(n), & |U_j| &= \ell_j(n) - \ell_{j-1}(n) + o(n) \end{aligned}$$

for $i = 1, \dots, k$ and $2 \leq j \leq k$.

By the construction of ν (cf. (5.8)), and the quasi-Bernoulli property of μ_p (cf. (5.5)), we have the following estimation

$$\nu([x]_{\ell_k(n)}) \leq \kappa_p^k \nu([I_1 \dots I_{p-2}]) \mu_{p-1}([I_{p-1}]) \prod_{i=1}^k \mu_p([U_i]).$$

Similarly for any $y \in G = \prod_{j=1}^\infty \mathcal{L}_j^{N'_j}$, if we write y in the same form

$$y = \tilde{I}_1 \tilde{I}_2 \dots \tilde{I}_{p-1} \tilde{U}_1 \tilde{W}_1 \tilde{U}_2 \tilde{W}_2 \dots \tilde{U}_k \tilde{W}_k \tilde{U}_{k+1} \tilde{I}_{p+1} \dots,$$

we have

$$\nu([y]_{\ell_k(n)}) \leq \kappa_p^k \nu([\tilde{I}_1 \dots \tilde{I}_{p-2}]) \mu_{p-1}([\tilde{I}_{p-1}]) \prod_{i=1}^k \mu_p([\tilde{U}_i]).$$

Then by (5.14), we obtain

(5.15)

$$\begin{aligned} \nu(B(x, e^{-n/a_1})) &\leq \kappa_p^k (\#\mathcal{L}_p^{N_p})^k \nu([I_1 \dots I_{p-2}]) \mu_{p-1}([I_{p-1}]) \prod_{i=1}^k \mu_p \circ \tau_{i-1}^{-1}([\tau_{i-1} U_i]) \\ &\leq \kappa_p^k (\#\mathcal{A}_1)^{pN_p k} \mu_{p-1}([I_{p-1}]) \prod_{i=1}^k \mu_p \circ \tau_{i-1}^{-1}([\tau_{i-1} U_i]). \end{aligned}$$

Since $[I_j] \cap G(j, N_j) \neq \emptyset$ for $j = p - 1, p$, and $|W_i| = o(n)$, by (5.6), we have

$$(5.16) \quad \log \mu_{p-1}([I_{p-1}]) = -(p - 1)N'_{p-1} h_\mu(T_1) + o(n),$$

and for $i = 1, \dots, k$,

$$\log \mu_p \circ \tau_{i-1}^{-1}([\tau_{i-1}(U_1 W_1 \dots U_i W_i)]) = -(|U_1| + \dots + |U_i|) h_{\mu \circ \tau_{i-1}^{-1}}(T_i) + o(n).$$

$$\log \mu_p \circ \tau_{i-1}^{-1}([\tau_{i-1}(U_1 W_1 \dots U_{i-1} W_{i-1})]) = -(|U_1| + \dots + |U_{i-1}|) h_{\mu \circ \tau_{i-1}^{-1}}(T_i) + o(n).$$

which implies that

$$(5.17) \quad \begin{aligned} \log \mu_p \circ \tau_{i-1}^{-1}([\tau_{i-1}U_i]) &= \log \mu_p \circ \tau_{i-1}^{-1}([\tau_{i-1}(U_iW_i)]) + o(n) \\ &= -|U_i| h_{\mu \circ \tau_{i-1}^{-1}}(T_i) + o(n), \end{aligned}$$

by using the quasi-Bernoulli property of μ_p and $\log \kappa_p = o(n)$. Combining (5.15)-(5.17) yields

$$\log \nu(B(x, e^{-n/a_1})) \leq -(|I_{p-1}| + |U_1|)h_\mu(T_1) + \sum_{i=2}^k |U_i| h_{\mu \circ \tau_{i-1}^{-1}}(T_i) + o(n).$$

Hence

$$(5.18) \quad \log \nu(B(x, e^{-n/a_1})) \leq (-n/a_1)h_\mu^{\mathbf{a}}(T_1) + o(n),$$

because $|I_{p-1}| + |U_1| = \ell_1(n) + o(n) = n + o(n)$ and $|U_i| = \ell_i(n) - \ell_{i-1}(n) + o(n) = a_i n/a_1 + o(n)$ for $2 \leq i \leq k$.

Next we consider the case (C2). Write the words I_p, I_{p+1} as

$$I_p = U_1W_1U_2W_2 \dots U_{\iota-1}W_{\iota-1}V_\iota, \quad I_{p+1} = U_\iota W_\iota \dots U_k W_k U_{k+1},$$

so that $|W_i| = pN_p$ for $i = 1, \dots, \iota - 1$, and $(p+1)N_{p+1}$ for $i = \iota, \dots, k$; furthermore, $|U_1|, \dots, |U_{\iota-1}|, |V_{\iota-1}|$ are multiples of pN_p , whilst $|U_j|$'s are multiples of $(p+1)N_{p+1}$ for $j = \iota, \dots, k+1$; in particular, the $\ell_i(n)$ -th digit of x is located in W_i for $i = 1, \dots, k$. In this way, we have the following concatenation

$$x = I_1 I_2 \dots I_{p-1} U_1 W_1 U_2 W_2 \dots U_{\iota-1} W_{\iota-1} V_\iota U_\iota W_\iota \dots U_k W_k U_{k+1} I_{p+2} \dots,$$

for which we have

$$\begin{aligned} |I_1 I_2 \dots I_{p-1}| &= o(|I_p|) = o(n), \quad |U_i| = \ell_i(n) - \ell_{i-1}(n) + o(n) \text{ for } 1 \leq i \leq \iota - 1 \text{ and} \\ |V_\iota| + |U_\iota| &= \ell_\iota(n) - \ell_{\iota-1}(n), \quad |U_i| = \ell_i(n) - \ell_{i-1}(n) + o(n) \text{ for } \iota + 1 \leq i \leq k, \\ |W_i| &= o(n) \text{ for } 1 \leq i \leq k. \end{aligned}$$

Similar to the case (C1), we can show that

$$\begin{aligned} \nu(B(x, e^{-n/a_1})) &\leq \kappa_p^k (\#\mathcal{A}_1)^{pN_p k} \left(\prod_{i=1}^{\iota-1} \mu_p \circ \tau_{i-1}^{-1}([\tau_{i-1}U_i]) \right) \mu_p \circ \tau_{\iota-1}^{-1}([\tau_{\iota-1}V_\iota]) \\ &\quad \cdot \prod_{j=\iota}^k \mu_{p+1} \circ \tau_{j-1}^{-1}([\tau_{j-1}U_j]). \end{aligned}$$

Then by a similar discussion as in case (C1), we obtain the estimation (5.18). This finishes the proof of (5.10). \square

To end the proof of Theorem 5.3, we need to define an injective map $\text{conc} : G \rightarrow X_1$ such that $\text{conc}(G_\mu) \subset G_\mu(X_1, T_1)$ and $\dim_H \text{conc}(G_\mu) = \dim_H G_\mu$. For this purpose, assume that X_1 satisfies s -specification for some integer $s \geq 1$. Then there exists a map $\theta : \mathcal{L}(X_1) \times \mathcal{L}(X_1) \rightarrow \mathcal{L}_s$ such that

$$I\theta(I, J)J \in \mathcal{L}_{|I|+|J|+s}, \quad \forall I, J \in \mathcal{L}(X_1).$$

For $x \in G = \prod_{j=1}^{\infty} \mathcal{L}_j^{N'_j}$, write x in the following concatenated form

$$x = W_{1,1}W_{1,2} \cdots W_{1,N'_1}W_{2,1}W_{2,2} \cdots W_{2,N'_2} \cdots W_{p,1}W_{p,2} \cdots W_{p,N'_p} \cdots,$$

where $W_{i,j} \in \mathcal{L}_i$ for $i \geq 1$ and $1 \leq j \leq N'_i$. Relabel the words $W_{i,j}$ to get

$$x = W_1W_2W_3 \cdots.$$

Then we define

$$\text{conc}(x) = W_1\theta_1W_2\theta_2W_3\theta_3 \cdots,$$

where the sequence (θ_n) is defined inductively by $\theta_1 = \theta(W_1, W_2)$, $\theta_2 = \theta(W_1\theta_1W_2, W_3)$, and once $\theta_1, \dots, \theta_{p-1}$ have been defined, then define $\theta_p = \theta(W_1\theta_1W_2 \dots \theta_{p-1}W_p, W_{p+1})$.

Clearly, $\text{conc}(x) \in X_1$, and the map $\text{conc} : G \rightarrow X_1$ is injective. Since $|W_j| \rightarrow \infty$ as $j \rightarrow \infty$, conc is almost Lipschitz, thus it preserves the Hausdorff dimension (i.e., $\dim_H \text{conc}(E) = \dim_H E$ for any $E \subseteq G$). Furthermore, $\text{conc}(G_\mu) \subset G_\mu(X_1, T_1)$. To see this, it is enough to show that if $\lim \frac{1}{n} S_n \varphi(x) = \alpha$, then $\lim \frac{1}{n} S_n \varphi(\text{conc}(x)) = \alpha$. This can be done in a way similar to the proof of Proposition 6 in [19]. Thus we finish the proof of Theorem 5.3.

REMARK 5.4. The specification property used in this paper can be weakened to cover a wider class of systems. For instance, one can ask that there exists $s \in \mathbb{N}$ such that for all $I, J \in \mathcal{L} := \mathcal{L}(X_1)$, one can find a word K in $\bigcup_{p=0}^s \mathcal{L}_p$ such that $IKJ \in \mathcal{L}$ (this covers transitive subshifts of finite type). A more general situation is that for all $I, J \in \mathcal{L}$, one can find a word $K \in \mathcal{L}$ such that $IKJ \in \mathcal{L}$ and $|K| = o(\min(|I|, |J|))$ as $\min(|I|, |J|)$ tends to ∞ [39]. Alternatively, one can ask that for all $I, J \in \mathcal{L}$, one can write $I = \tilde{I}\tilde{I}$ and $J = \tilde{J}\tilde{J}$ so that $\tilde{I}\tilde{J} \in \mathcal{L}$ and $|\tilde{I}| + |\tilde{J}| = o(\min(|I|, |J|))$ as $\min(|I|, |J|)$ tends to ∞ [39]. In all these cases our approach can be used to obtain the same conclusion as in Theorem 5.3. The only difference is that in these cases the mapping conc introduced in the end of the proof of Theorem 5.3 may take a slightly different form, and it has no reason to be one to one. However, this mapping naturally satisfies the property $\text{conc}(G_\mu) \subset G_\mu(X_1, T_1)$, and a (tedious, that we omit here) combinatoric argument shows that on the one hand, conc is α -Hölder continuous for all $\alpha \in (0, 1)$, and on the other hand there exists a \mathbb{N} -valued function $\mathcal{N}(r)$ ($r > 0$) such that $\lim_{r \rightarrow 0^+} \log \mathcal{N}(r) / |\log r| = 0$ and for any open ball B of radius $r > 0$ in $\text{conc}(G_\mu)$, $\text{conc}^{-1}(B)$ can be covered by at most $\mathcal{N}(r)$ balls of radius r . Thus conc preserves the Hausdorff dimension.

5.3. Proof of Theorem 1.4. (1) We prove that the following assertions are equivalent:

- (i) $\alpha \in L_\Phi$;
- (ii) $E_{\{\Phi^{(j)}\}, c}(\alpha) \neq \emptyset$;
- (iii) $\inf \{P^a(T_1, \mathbf{q} \cdot \Phi) - \alpha \cdot \mathbf{q} : \mathbf{q} \in \mathbb{R}^d\} \geq 0$;
- (iv) $\inf \{P^a(T_1, \mathbf{q} \cdot \Phi) - \alpha \cdot \mathbf{q} : \mathbf{q} \in \mathbb{R}^d\} > -\infty$;

For $\alpha \in L_\Phi$ let $f_\Phi(\alpha) = \max\{h_\mu^a(T_1) : \mu \in \mathcal{M}(X_1, T_1), \Phi_*(\mu) = \alpha\}$. Since the mapping $\mu \in \mathcal{M}(X_1, T_1) \mapsto h_\mu^a(T_1)$ is upper semi-continuous and affine, the equality $f_\Phi(\alpha) = \inf \{P^a(T_1, \mathbf{q} \cdot \Phi) - \alpha \cdot \mathbf{q} : \mathbf{q} \in \mathbb{R}^d\}$ for $\alpha \in L_\Phi$ is obtained by exactly the same arguments as those used to prove Theorem 5.2(iii) in [23]; one just replaces the usual entropy by the \mathbf{a} -weighted one. Similarly, the proof of the equivalence

between (i), (iii) and (iv) follow the same lines as that of Theorem 5.2 (ii) in [23]. The equivalence of (ii) and the other assertions will follow from the proof of Theorem 1.4(2) below.

(2) We only need to show

$$(5.19) \quad E_{\{\Phi_j\},c}(\alpha) \neq \emptyset \text{ and } \dim_H E_{\{\Phi_j\},c}(\alpha) \geq f_{\Phi}(\alpha) \text{ if } \alpha \in L_{\Phi};$$

$$(5.20) \quad \dim_H E_{\{\Phi_j\},c}(\alpha) \leq \inf \{P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \Phi) - \alpha \cdot \mathbf{q} : \mathbf{q} \in \mathbb{R}^d\} \text{ if } E_{\{\Phi_j\},c}(\alpha) \neq \emptyset,$$

This yields the equivalence of (i) and (ii) above, as well as the value of $\dim_H E_{\{\Phi_j\},c}(\alpha)$.

Assertion (5.19) is an immediate consequence of Theorem 1.3 and the following lemma.

LEMMA 5.5. *Let $\alpha = (\alpha_1, \dots, \alpha_d) \in L_{\Phi}$ and $\mu \in \mathcal{M}(X_1, T_1)$ such that $\int \Phi d\mu = \alpha$. We have $G_{\mu}(X_1, T_1) \subset E_{\{\Phi_j\},c}(\alpha)$.*

Proof of Lemma 5.5. Write each Φ_j as $(\varphi_{j,1}, \dots, \varphi_{j,d})$. By definition of Φ , we have $\alpha_i = \sum_{j=1}^r \int \varphi_{j,i} d\mu$ for each $1 \leq i \leq d$. Moreover, by the definition of $G_{\mu}(X_1, T_1)$, we have $G_{\mu}(X_1, T_1) \subset E_{\varphi_{j,i}}(\int \varphi_{j,i} d\mu)$ for each $1 \leq j \leq r$ and $1 \leq i \leq d$, hence for each $x \in G_{\mu}(X_1, T_1)$ we have $\lim_{n \rightarrow \infty} \sum_{j=1}^r \frac{S_{[c_j n]} \varphi_{j,i}(x)}{[c_j n]} = \alpha_i$ for each $1 \leq i \leq d$. This yields $G_{\mu}(X_1, T_1) \subset E_{\{\Phi_j\},c}(\alpha)$. \square

Now we establish (5.20). We define the following sequence of functions

$$\Phi_{\mathbf{c},n} = n \sum_{j=1}^r \frac{S_{[c_j n]} \Phi_j}{[c_j n]}.$$

We first treat the case where (X_1, T_1) is a full shift.

The upper bound in the full shift case. In this case we have the following lemma, which yields (5.20).

LEMMA 5.6. *Assume that (X_1, T_1) is a full shift. Fix $\alpha \in \mathbb{R}^d$ and suppose that $E_{\{\Phi_j\},c}(\alpha) \neq \emptyset$. For every $\varepsilon > 0$ and $\mathbf{q} \in \mathbb{R}^d$, we have*

$$\dim_H E_{\{\Phi_j\},c}(\alpha, \varepsilon) \leq P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \Phi) - \alpha \cdot \mathbf{q} + (4|\mathbf{q}| + a_1)\varepsilon,$$

where $E_{\{\Phi_j\},c}(\alpha, \varepsilon) = \{x \in X_1 : \limsup_{n \rightarrow \infty} |\Phi_{\mathbf{c},n}(x)/n - \alpha| \leq \varepsilon\}$. Consequently, if $E_{\{\Phi_j\},c}(\alpha) \neq \emptyset$, then $\dim_H E_{\{\Phi_j\},c}(\alpha) \leq \inf_{\mathbf{q} \in \mathbb{R}^d} P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \Phi) - \alpha \cdot \mathbf{q}$, i.e., (5.20) holds.

Proof of Lemma 5.6. Fix $\varepsilon > 0$ and $\mathbf{q} \in \mathbb{R}^d$. For each $1 \leq j \leq r$, choose a continuous function $\tilde{\Phi}_j = (\tilde{\varphi}_{j,1}, \dots, \tilde{\varphi}_{j,d})$ in $C(X_1)^d$ such that each $\tilde{\varphi}_{j,d}$ is Hölder continuous and

$$\sup_{1 \leq i \leq d} \|\tilde{\varphi}_{j,d} - \varphi_{j,d}\|_{\infty} \leq \varepsilon/r.$$

Then we define $\tilde{\Phi} = \sum_{j=1}^r \tilde{\Phi}_j$, and the sequence of functions

$$(5.21) \quad \tilde{\Phi}_{\mathbf{c},n} = n \sum_{j=1}^r \frac{S_{[c_j n]} \tilde{\Phi}_j}{[c_j n]} \quad (n \geq 1).$$

Endow the space \mathbb{R}^d with the norm $|(z_1, \dots, z_d)| = \max_{1 \leq i \leq d} |z_i|$. By construction we have $\limsup_{n \rightarrow \infty} \|\tilde{\Phi}_{\mathbf{c},n} - \Phi_{\mathbf{c},n}\|_\infty/n \leq \varepsilon$ so

$$E_{\{\Phi_j\},\mathbf{c}}(\alpha, \varepsilon) \subset E_{\{\tilde{\Phi}_j\},\mathbf{c}}(\alpha, 2\varepsilon) = \left\{ x \in X_1 : \limsup_{n \rightarrow \infty} |\tilde{\Phi}_{\mathbf{c},n}(x)/n - \alpha| \leq 2\varepsilon \right\}.$$

The definition of the \mathbf{a} -weighted topological pressure implies

$$(5.22) \quad |P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \tilde{\Phi}) - P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \Phi)| \leq |\mathbf{q}|\varepsilon.$$

Let us denote by $\mu_{\mathbf{q}}$ the unique \mathbf{a} -weighted equilibrium state of $\mathbf{q} \cdot \tilde{\Phi}$ (see Theorem 3.5). The following key property holds.

LEMMA 5.7. *Let $\mathbf{q} \in \mathbb{R}^d$. For all $x \in X_1$, we have $\limsup_{n \rightarrow \infty} f_n(x)^{1/n} \geq 1$, where*

$$f_n(x) = \frac{\mu_{\mathbf{q}}(B(x, e^{-n/a_1}))}{\exp((\mathbf{q} \cdot \tilde{\Phi}_{\mathbf{c},n}(x) - nP^{\mathbf{a}}(T_1, \mathbf{q} \cdot \tilde{\Phi}))/a_1)}.$$

It is worth mentioning that the idea of considering the asymptotic behavior of such a function f_n at each point of X_1 goes back to [35] for the upper bound estimate of $\dim_H X_1$ when $k = 2$. The proof of Lemma 5.7 will be given later. To finish the proof of Lemma 5.6, we need the following classical lemma.

LEMMA 5.8 ([8], Ch. 14). *Let E be a non-empty subset of a compact metric space (Y, d) endowed with an ultrametric distance. Let ν be a positive Borel measure on Y . Then $\dim_H E \leq \sup_{x \in E} \liminf_{r \rightarrow 0^+} \frac{\log \nu(B(x, r))}{\log r}$.*

Now, if $x \in E_{\{\tilde{\Phi}_j\},\mathbf{c}}(\alpha, 2\varepsilon)$ then, due to Lemma 5.7, for infinitely many n we have simultaneously $f_n(x) \geq \exp(-n\varepsilon)$, and $\exp(\mathbf{q} \cdot \tilde{\Phi}_{\mathbf{c},n}(x)) \geq \exp(n\alpha \cdot \mathbf{q}) - 3|\mathbf{q}|\varepsilon n$. Consequently,

$$\liminf_{n \rightarrow \infty} \frac{\log \mu_{\mathbf{q}}(B(x, e^{-n/a_1}))}{-n/a_1} \leq P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \tilde{\Phi}) - \alpha \cdot \mathbf{q} + (3|\mathbf{q}| + a_1)\varepsilon.$$

Now, Lemma 5.8 and (5.22) yield

$$\begin{aligned} \dim_H E_{\{\Phi_j\},\mathbf{c}}(\alpha, \varepsilon) &\leq \dim_H E_{\{\tilde{\Phi}_j\},\mathbf{c}}(\alpha, 2\varepsilon) \leq P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \tilde{\Phi}) - \alpha \cdot \mathbf{q} + (3|\mathbf{q}| + a_1)\varepsilon \\ &\leq P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \Phi) - \alpha \cdot \mathbf{q} + (4|\mathbf{q}| + a_1)\varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we obtain $\dim_H E_{\{\Phi_j\},\mathbf{c}}(\alpha) \leq P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \Phi) - \alpha \cdot \mathbf{q}$. Since $\mathbf{q} \in \mathbb{R}^d$ is arbitrarily given, we have

$$\dim_H E_{\{\Phi_j\},\mathbf{c}}(\alpha) \leq \inf_{\mathbf{q} \in \mathbb{R}^d} P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \Phi) - \alpha \cdot \mathbf{q}.$$

This finishes the proof of Lemma 5.6. \square

Before we prove Lemma 5.7, we give some auxiliary lemmas.

LEMMA 5.9 ([28], Lemma 4.1). *Let $m \geq 1$ be an integer. For $1 \leq j \leq m$ let $f_j : \mathbb{N} \rightarrow \mathbb{R}$, $\beta_j > 0$ and $\lambda_j > 0$. If $\sup_{n \geq 1} |f_j(n+1) - f_j(n)| < \infty$ for each j , then*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{j=1}^m \left(\beta_j f_j \left(\left\lfloor \frac{t}{\lambda_j} \right\rfloor \right) - f_j \left(\left\lfloor \frac{\beta_j t}{\lambda_j} \right\rfloor \right) \right) \geq 0.$$

Proof of Lemma 5.7. Fix $\mathbf{q} \in \mathbb{R}^d$ and $x \in X_1$. By Lemma 5.2, we have

$$\mu_{\mathbf{q}}(B(x, e^{-n/a_1})) \approx \exp\left(\frac{-nP^{\mathbf{a}}(T_1, \mathbf{q} \cdot \tilde{\Phi}) + S_n(\mathbf{q} \cdot \tilde{\Phi})(x)}{a_1}\right) \prod_{i=1}^{k-1} \frac{\tilde{\phi}^{(i)}(\tau_i(x|_{\ell_{i+1}(n)}))^{1/A_{i+1}}}{\tilde{\phi}^{(i)}(\tau_i(x|_{\ell_i(n)}))^{1/A_i}}$$

for certain functions $\tilde{\phi}^{(i)}$ on \mathcal{A}_i^* ($1 \leq i \leq k$). Combining this with the definition of $f_n(x)$ yields

$$(5.23) \quad f_n(x) \approx \exp\left(\frac{S_n(\mathbf{q} \cdot \tilde{\Phi})(x) - \tilde{\Phi}_{\mathbf{c},n}(x)}{a_1}\right) \prod_{i=1}^{k-1} \frac{\tilde{\phi}^{(i)}(\tau_i(x|_{\ell_{i+1}(n)}))^{1/A_{i+1}}}{\tilde{\phi}^{(i)}(\tau_i(x|_{\ell_i(n)}))^{1/A_i}}.$$

Notice that $\tilde{\Phi}_{\mathbf{c},n}$ has the following form (cf. (5.21)):

$$\tilde{\Phi}_{\mathbf{c},n} = S_n(\mathbf{q} \cdot \tilde{\Phi}) + \sum_{j=1}^r \left(\frac{nS_{\lfloor c_j n \rfloor}(\mathbf{q} \cdot \tilde{\Phi}_j)}{\lfloor c_j n \rfloor} - S_n(\mathbf{q} \cdot \tilde{\Phi}_j) \right).$$

Now, for $n \geq 1$, let us define

$$\begin{cases} u^{(j)}(n) = a_1^{-1} S_n \tilde{\Phi}_j(x) & \text{for } 1 \leq j \leq r, \\ \tilde{u}^{(i)}(n) = a_1^{-1} \log \tilde{\phi}^{(i)}(\tau_i(x|_n)) & \text{for } 1 \leq i \leq k-1. \end{cases}$$

Since the potentials $\tilde{\Phi}$ and $\tilde{\Phi}_j$ are Hölder continuous, for any $v \in \{u^{(j)}, \tilde{u}^{(i)} : 1 \leq j \leq r, 1 \leq i \leq k-1\}$ the sequence $(v(n))_{n \geq 1}$ satisfies $v(n+1) - v(n) = O(1)$. Then, by using (5.23) we can get

$$\frac{\log f_n(x)}{n} = \frac{1}{n} \left(\sum_{j=1}^r \left(u^{(j)}(n) - \frac{1}{c_j} u^{(j)}(\lfloor c_j n \rfloor) \right) + \sum_{i=1}^{k-1} \left(\frac{\tilde{u}^{(i)}(\lfloor \tilde{c}_{i+1} n \rfloor)}{\tilde{c}_{i+1}} - \frac{\tilde{u}^{(i)}(\lfloor \tilde{c}_i n \rfloor)}{\tilde{c}_i} \right) + O(1) \right),$$

where $\tilde{c}_i := A_i/a_1$ for $i = 1, \dots, k$. Then, the fact that $\limsup_{n \rightarrow \infty} \frac{\log f_n(x)}{n} \geq 0$ comes from Lemma 5.9. This finishes the proof of Lemma 5.7. \square

The upper bound in the general case. We show that the upper bound for $\dim_H E_{\{\Phi_j\}, \mathbf{c}}(\alpha)$ is valid without any assumption like specification on (X_1, T_1) .

For each $p \geq 1$, let Γ_p be the natural injection of $(X_1, d_{\mathbf{a}})$ into $(\mathcal{L}_p^{\mathbb{N}}, d_{\mathbf{a}/p})$. The map Γ_p is easily seen to be a bi-Lipschitz map from X_1 onto its image, so it preserves Hausdorff dimension.

For each $1 \leq j \leq r$, the \mathbb{R}^d valued continuous potentials Φ_j can be extended to an \mathbb{R}^d valued continuous potentials $\hat{\Phi}_j$ defined on $(\mathcal{A}_1^{\mathbb{N}}, d_{\mathbf{a}})$. By construction, for each $\alpha \in L_{\Phi}$, we have $\Gamma_p(E_{\{\Phi_j\}, \mathbf{c}}(\alpha)) \subset E_{\{S_p \hat{\Phi}_j/p\}, \mathbf{c}}(\alpha)$ so the above study applied to the full shift $(\mathcal{L}_p^{\mathbb{N}}, T_1^p, d_{\mathbf{a}/p})$ yields

$$\dim_H E_{\{\Phi_j\}, \mathbf{c}}(\alpha) \leq \max \left\{ h_{\mu}^{\mathbf{a}/p}(T_1^p) : \mu \in \mathcal{M}(\mathcal{L}_p^{\mathbb{N}}, T_1^p), \int S_p \hat{\Phi}/p d\mu = \alpha \right\}.$$

Let ν_p be one measure realizing the above maximum and extend it as an invariant measure on $(\mathcal{A}_1^{\mathbb{N}}, T_1^p)$ (notice that ν_p is supported by $\mathcal{L}_p^{\mathbb{N}}$). Then define on $\mathcal{A}_1^{\mathbb{N}}$ the T_1 -invariant measure $\mu_p = \frac{1}{p} \sum_{j=0}^{p-1} \nu_p \circ T_1^{-j}$. By the affinity of $h_{(\cdot)}^{\mathbf{a}/p}(T_1^p)$ we have $h_{\mu_p}^{\mathbf{a}/p}(T_1^p) = h_{\nu_p}^{\mathbf{a}/p}(T_1^p)$ since all the measures $\nu_p \circ T_1^{-j}$ have the same \mathbf{a}/p -weighted

entropy with respect to T_1^p . Consequently, $h_{\nu_p}^{\mathbf{a}/p}(T_1^p) = h_{\mu_p}^{\mathbf{a}/p}(T_1^p) = ph_{\mu_p}^{\mathbf{a}/p}(T_1) = h_{\mu_p}^{\mathbf{a}}(T_1)$. Finally, by construction,

$$(5.24) \quad \dim_H E_{\{\Phi_j\},\mathbf{c}}(\alpha) \leq h_{\mu_p}^{\mathbf{a}}(T_1) \text{ and } \int \widehat{\Phi} d\mu_p = \alpha.$$

Now, without loss of generality we can suppose that μ_p converges to an element $\mu \in \mathcal{M}(\mathcal{A}_1^{\mathbb{N}}, T_1)$ in the weak-star topology. We have both $h_{\mu}^{\mathbf{a}}(T_1) \geq \limsup_{p \rightarrow \infty} h_{\mu_p}^{\mathbf{a}}(T_1)$ and $\int \widehat{\Phi} d\mu = \alpha$. Assume that μ is supported on X_1 . Then by (5.24) we have $\dim_H E_{\{\Phi_j\},\mathbf{c}}(\alpha) \leq \max \{h_{\mu}^{\mathbf{a}}(T_1) : \mu \in \mathcal{M}(X_1, T_1), \int \widehat{\Phi} d\mu = \alpha\}$ as desired. Now, the fact that μ is supported on X_1 follows from the same argument as that used in the proof of [28, Theorem 1.1] to build an invariant measure of full Hausdorff dimension on X_1 .

(3) We will use a modification of the Moran construction achieved in the proof of Theorem 5.3. To do so we need some preparation.

Let ν_1 be an invariant measure on X_1 such that $h_{\nu_1}^{\mathbf{a}}(T_1) = \dim_H X_1$ (the existence of such a measure was first proved in [28]; this fact is also a consequence of Theorem 1.4 applied to the null potential). Fix $\varepsilon > 0$, and for each $1 \leq i \leq k$ let $h_i = h_{\nu_1 \circ \tau_{i-1}^{-1}}(T_i) - \varepsilon/(a_1 + \dots + a_k)$. By our assumption, L_{Φ} is not a singleton. Let $\nu \in \mathcal{M}(X_1, T_1)$ such that $\nu(\Phi) \neq \nu_1(\Phi)$. Then take a large positive integer n so that

$$h_{\nu_2 \circ \tau_{i-1}^{-1}}(T_i) \geq h_{\nu_1 \circ \tau_{i-1}^{-1}}(T_i) - \varepsilon/(a_1 + \dots + a_k), \quad (1 \leq i \leq k)$$

where $\nu_2 = (1 - 1/n)\nu_1 + (1/n)\nu$. Note that $\delta = |\nu_2(\Phi) - \nu_1(\Phi)| > 0$.

By Remark 4.4, for each $l \in \{1, 2\}$ there exists a sequence $(\mu_{l,p})_{p=1}^{\infty}$ of quasi-Bernoulli measures such that $\mu_p \in \mathcal{M}(\mathcal{L}_p^{\mathbb{N}}, T_1^p)$ and

$$h_{\mu_{l,p} \circ \tau_{i-1}^{-1}}(T_i^p) \geq p(h_{\nu_l \circ \tau_{i-1}^{-1}}(T_i) - \varepsilon/(a_1 + \dots + a_k)) - 1/(2p) = ph_i - 1/(2p)$$

for each $1 \leq i \leq k$ and for $p \geq 1$.

For each $1 \leq j \leq r$ and $1 \leq i \leq d$, take $\tilde{\varphi}_{j,i} \in C(\mathcal{A}_1^{\mathbb{N}})$ such that $\tilde{\varphi}_{j,i}$ is Hölder continuous and $\|\tilde{\varphi}_{j,i} - \varphi_{j,i}\|_{\infty} \leq \delta/8r$. Moreover, for each $l \in \{1, 2\}$ the sequence $(\mu_{l,p})_{p=1}^{\infty}$ can be chosen so that $|\mu_{l,p}(\tilde{\varphi}_{j,i}) - \nu_l(\tilde{\varphi}_{j,i})| \leq \delta/8r$.

We are going to construct a sequence $(N'_p)_{p=1}^{\infty}$ such that the set

$$D = \bigcap_{l=1}^2 \left\{ x \in G = \prod_{p=1}^{\infty} \mathcal{L}_p^{N'_p} : \liminf_{n \rightarrow \infty} |\Phi_{\mathbf{c},n}(x) - n\nu_l(\Phi)|/n \leq \delta/4 \right\},$$

which is a subset of $\mathcal{A}_1^{\mathbb{N}} \setminus \bigcup_{\alpha \in L_{\Phi}} E_{\{\Phi_j\},\mathbf{c}}(\alpha)$, contains a Moran subset G_{μ} with $\dim_H G_{\mu} \geq \dim_H X_1 - 2\varepsilon$. Then, by using the same imbedding *conc* as in the proof of Theorem 5.3, we will get $\text{conc}(G_{\mu}) \subset X_1 \setminus \bigcup_{\alpha \in L_{\Phi}} E_{\{\Phi_j\},\mathbf{c}}(\alpha)$ and $\dim_H \text{conc}(G_{\mu}) \geq \dim_H X_1 - \varepsilon$.

Now we briefly explain how to modify the Moran construction done in the proof of Theorem 5.3 to build G and G_{μ} . At first, without loss of generality, we suppose that the c_j 's are greater than 1. Also, we include the potentials $\tilde{\varphi}_{j,i}$ in the family $\tilde{\mathcal{C}}$. Then, the only changes are that for each $p \geq 1$, one takes $\mu_{2p-1} = \nu_{1,p}$ and $\mu_{2p} = \nu_{2,p}$. Then, for $p \geq 1$, let $n_p = \sum_{i=1}^{p-1} iN'_i + \sqrt{pN'_p}$. For p large enough, for each $1 \leq j \leq r$ we have

$$[c_j n_p] \in \left[\sum_{i=1}^{p-1} iN'_i + \sqrt{pN'_p}, \sum_{i=1}^p iN'_i \right],$$

so that for each $x \in G_\mu$, $1 \leq j \leq r$ and $1 \leq i \leq d$ we have

$$\limsup_{p \rightarrow \infty} |S_{[c_j n_{2p-1}]} \tilde{\varphi}_{j,i}(x) / [c_j n_{2p-1}] - \nu_1(\tilde{\varphi}_{j,i})| \leq \delta/8r$$

and

$$\limsup_{p \rightarrow \infty} |S_{[c_j n_{2p}]} \tilde{\varphi}_{j,i}(x) / [c_j n_{2p}] - \nu_2(\tilde{\varphi}_{j,i})| \leq \delta/8r.$$

Consequently, by construction for each $x \in G_\mu$, we have

$$\limsup_{p \rightarrow \infty} |\Phi_{\mathbf{c}, n_{2p-1}}(x) / n_{2p-1} - \nu_1(\Phi)| \leq \delta/4 \text{ and } \lim_{p \rightarrow \infty} |\Phi_{\mathbf{c}, n_{2p}}(x) / n_{2p} - \nu_2(\Phi)| \leq \delta/4,$$

so $G_\mu \subset D$. Moreover, the simultaneous controls from below of the entropies $h_{\nu_i, p \circ \tau_i^{-1}}(T_i^p)$ by the $ph_i - 1/(2p)$ yield, for every $x \in G_\mu$,

$$\liminf_{n \rightarrow \infty} \frac{\log \nu(B(x, e^{-n/a_1}))}{-n/a_1} \geq \sum_{i=1}^k a_i h_i \geq \dim_H X_1 - \varepsilon.$$

□

REMARK 5.10. Without assuming the specification property, for any $\mu \in \mathcal{M}(X_1, T_1)$, we have $\dim_P G_\mu(X_1, T_1) \leq h_\mu^{\mathbf{a}}(T_1)$, where \dim_P denotes the packing dimension (cf. [34]). This is done by using a recent result of Reeves [41]. To see it, let $(\phi_n)_{n=1}^\infty$ be a family of continuous functions on X_1 which is dense in $C(X_1)$. For $m \geq 1$, define $\Phi_m = (\phi_1, \dots, \phi_m)$ and

$$E_m = \left\{ x \in X_1 : \lim_{n \rightarrow \infty} \frac{1}{n} S_n \Phi_m(x) = \left(\int \phi_1 d\mu, \dots, \int \phi_m d\mu \right) \right\}.$$

By Reeves' result [41],

$$\dim_P E_m \leq \sum_{i=1}^k \sup \left\{ a_i h_{\eta \circ \pi^{-1}}(T_i) : \int \phi_j d\eta = \int \phi_j d\mu \text{ for } 1 \leq j \leq m \right\}.$$

(Indeed, Reeves only proved the above result in the case $k = 2$ and $m = 1$ in the full shift case with the equality rather than \leq ; however, it can be extended to the above general form without additional difficulty.) Since $(\phi_n)_{n=1}^\infty$ is dense in $C(X_1)$, by using the upper semi-continuity of the entropy function, we obtain

$$\limsup_{n \rightarrow \infty} \dim_P E_m \leq \sum_{i=1}^k a_i h_{\mu \circ \pi^{-1}}(T_i) = h_\mu^{\mathbf{a}}(T_1).$$

Since $G_\mu(X_1, T_1) \subseteq E_m$ for each m , we obtain the upper bound $\dim_P G_\mu(X_1, T_1) \leq h_\mu^{\mathbf{a}}(T_1)$.

6. Multifractal analysis of asymptotically additive potentials and application to weighted equilibrium states. For $\Phi = (\Phi_1, \dots, \Phi_d) \in \mathcal{C}_{asa}(X_1, T_1)^d$, where $\Phi_i = (\log \phi_{n,i})_{n=1}^\infty =: (\Phi_{n,i})_{n=1}^\infty$, and $\mu \in \mathcal{M}(X_1, T_1)$, write $\Phi_*(\mu) = ((\Phi_1)_*(\mu), \dots, (\Phi_d)_*(\mu))$ and define $L_\Phi = \{\Phi_*(\mu) : \mu \in \mathcal{M}(X_1, T_1)\}$.

Let $\{\Phi^{(j)}\}_{1 \leq j \leq r}$ be a family of elements of $\mathcal{C}_{asa}(X_1, T_1)^d$. Let $\mathbf{c} = (c_1, \dots, c_r)$ be a real vector with positive entries. For $\alpha \in \mathbb{R}^d$, define

$$E_{\{\Phi^{(j)}\}, \mathbf{c}}(\alpha) = \left\{ x \in X_1 : \lim_{n \rightarrow \infty} \sum_{j=1}^r \frac{\Phi^{(j)}_{[c_j n]}(x)}{[c_j n]} = \alpha \right\}.$$

Theorem 1.4 has the following easy extension, which is obtained thanks to the density of continuous additive potentials in $\mathcal{C}_{asa}(X_1, T_1)^d$. This extension is useful to get results on the multifractal analysis of weighted equilibrium states.

THEOREM 6.1. *Let $\Phi = \sum_{j=1}^r \Phi^{(j)}$.*

1. *For $\alpha \in \mathbb{R}^d$, $E_{\{\Phi^{(j)}\}, \mathbf{c}}(\alpha) \neq \emptyset$ if and only if $\alpha \in L_\Phi$.*
2. *For $\alpha \in L_\Phi$, we have*

$$\begin{aligned} \dim_H E_{\{\Phi^{(j)}\}, \mathbf{c}}(\alpha) &= \max \{h_\mu^{\mathbf{a}}(T_1) : \mu \in \mathcal{M}(X_1, T_1), \Phi_*(\mu) = \alpha\} \\ &= \inf \{P^{\mathbf{a}}(T_1, \mathbf{q} \cdot \Phi) - \alpha \cdot \mathbf{q} : \mathbf{q} \in \mathbb{R}^d\}. \end{aligned}$$

3. *Suppose that L_Φ is not a singleton. Then the set $X \setminus \bigcup_{\alpha \in L_\Phi} E_{\{\Phi^{(j)}\}, \mathbf{c}}(\alpha)$ is of full Hausdorff dimension.*

EXAMPLE 6.2. Generally, the level sets $E_{\{\Phi^{(j)}\}, \mathbf{c}}(\alpha)$ depend on \mathbf{c} . For example, let $X = \{0, 1\}^{\mathbb{N}}$, and let $g \in C(X)$ be given by $g(x) = x_1$ for $x = (x_i)_{i=1}^\infty \in X$. Set $\Phi^{(1)} = (S_n g)_{n=1}^\infty$ and $\Phi^{(2)} = (-S_n g)_{n=1}^\infty$. Then $E_{\{\Phi^{(j)}\}_{j=1, (1,1)}^2}(0) = X$, however $E_{\{\Phi^{(j)}\}_{j=1, (1,2)}^2}(0) \neq X$ (it is easy to check that $x = 0^1 1^2 0^4 1^8 \dots 0^{2^{2n}} 1^{2^{2n+1}} \dots \notin E_{\{\Phi^{(j)}\}_{j=1, (1,2)}^2}(0)$).

Application to the multifractal analysis of \mathbf{a} -weighted Gibbs measures.

In this section we suppose that (X_1, T_1) is the full shift $(\mathcal{A}_1^{\mathbb{N}}, T_1)$.

Let $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{aa}(X_1, T_1)$ and suppose that Φ satisfies the bounded distortion property. Let μ be the \mathbf{a} -weighted Gibbs measure μ associated with Φ .

Due to the quasi-Bernoulli property of μ , for each $1 \leq i \leq k$, the potential $\Psi_\mu^{(i)} := (\log \mu \circ \tau_{i-1}^{-1}(\tau_{i-1}(x|_n)))_{n=1}^\infty$ belongs to $\mathcal{C}_{aa}(X_1, T_1)$.

We have the following result about the multifractal analysis of μ .

THEOREM 6.3. *For $\alpha \in \mathbb{R}_+$ let*

$$E_\mu(\alpha) = \left\{ x \in X_1 : \lim_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}.$$

Let $\Psi_\mu = \sum_{i=1}^k a_i \Psi_\mu^{(i)}$. Let $L_\mu = L_{-\Psi_\mu} = \{-(\Psi_\mu)_(\lambda) : \lambda \in \mathcal{M}(X_1, T_1)\}$. Then, for all $\alpha \geq 0$, $E_\mu(\alpha) \neq \emptyset$ if and only if $\alpha \in L_\mu$. Moreover, for $\alpha \in L_\mu$ we have*

$$\begin{aligned} \dim_H E_\mu(\alpha) &= \max \{h_\lambda^{\mathbf{a}}(T_1) : \lambda \in \mathcal{M}(X_1, T_1), (\Psi_\mu)_*(\lambda) = -\alpha\} \\ &= \inf \{P^{\mathbf{a}}(T_1, q \Psi_\mu) + \alpha q : q \in \mathbb{R}\}. \end{aligned}$$

Proof. This result is just a corollary of Theorem 6.1. Indeed, thanks to (5.1) we can write

$$\begin{aligned} \frac{\log \mu(B(x, e^{-n/a_1}))}{-n/a_1} &= -a_1 \frac{\Psi_{\mu,n}^{(1)}(x)}{n} - a_1 \sum_{i=2}^k \frac{\Psi_{\mu,\ell_i(n)}^{(i)}(x)}{n} - \frac{\Psi_{\mu,\ell_{i-1}(n)}^{(i)}(x)}{n} + O(1/n) \\ &= -a_1 \frac{\Psi_{\mu,n}^{(1)}(x)}{n} - a_1 \sum_{i=2}^k \frac{b_i \Psi_{\mu,[b_i n]}^{(i)}(x)}{[b_i n]} - \frac{b_{i-1} \Psi_{\mu,[b_{i-1} n]}^{(i)}(x)}{[b_{i-1} n]} + O(1/n), \end{aligned}$$

with $b_i = (a_1 + \dots + a_i)/a_1$. Thus, any set $E_\mu(\alpha)$ takes the form $E_{\{\Phi^{(j)}\},c}(\alpha)$, with $\sum_{j=1}^r \Phi^{(j)} = -\Psi_\mu$. \square

More geometric applications. A parallelepiped is a subset of X_1 of the form

$$R(I_1, \dots, I_k) = \bigcap_{i=1}^k \tau_{i-1}^{-1}(I_i), \text{ with } I_i \in \bigcup_{n \geq 0} \mathcal{A}_i^n.$$

If we fix $0 \leq \lambda_1 \leq \dots \leq \lambda_k$ and set

$$R_n(\lambda_1, \dots, \lambda_k, x) = R\left(x_{\lfloor \lambda_1 n \rfloor}, \dots, \tau_{i-1}(x_{\lfloor \lambda_i n \rfloor}), \dots, \tau_{k-1}(x_{\lfloor \lambda_k n \rfloor})\right),$$

then

$$\log \mu(R_n(\lambda_1, \dots, \lambda_k, x)) = \sum_{i=1}^k \Psi_{\mu, \lfloor \lambda_i n \rfloor}^{(i)}(x) - \Psi_{\mu, \lfloor \lambda_{i-1} n \rfloor}^{(i)}(x) + O(1),$$

with the convention $\lambda_0 = 0$. Consequently, Theorem 6.1 makes it also possible to compute the Hausdorff dimension of the sets

$$\bigcap_{m=1}^M \left\{ x \in X_1 : \lim_{n \rightarrow \infty} \frac{\log \mu(R_n(\lambda_1^{(m)}, \dots, \lambda_k^{(m)}, x))}{-n} = \beta_m \right\},$$

where $\beta \in \mathbb{R}_+^M$ and each $(\lambda_i^{(m)})_{1 \leq i \leq m}$ satisfies $0 \leq \lambda_1^{(m)} \leq \dots \leq \lambda_k^{(m)}$.

REMARK 6.4. Let $\Phi \in \mathcal{C}_{asa}(X_1, T_1)$. We say that a fully supported Borel probability measure μ (not necessarily to be shift invariant) on X_1 is an **a-weighted weak Gibbs measure associated with Φ** if

$$\mu(I) \approx_n \exp\left(\frac{-nP}{A_k}\right) \phi_n^{\mathbf{a}}(I), \quad I \in \mathcal{A}^n,$$

where $P = P^{\mathbf{a}}(T_1, \Phi)$, $A_k = a_1 + \dots + a_k$, $\Phi^{\mathbf{a}} = (\log \phi_n^{\mathbf{a}}) \in \mathcal{C}_{asa}(X_1, T_1)$ is defined as in (3.4), and \approx_n means that there exists a sequence of positive numbers $(\kappa_n)_{n=1}^\infty$ with $\lim_{n \rightarrow \infty} (1/n) \log \kappa_n = 0$, such that the ratio between the left and right hand sides of \approx_n lies in $(\kappa_n^{-1}, \kappa_n)$.

This notion reduces back to classical weak Gibbs measures when $\mathbf{a} = (1, 0, \dots, 0)$ and Φ is the sequence of Birkhoff sums associated with a continuous potential over X_1 [48, 29]. It turns out that such a measure always exists for each $\Phi \in \mathcal{C}_{asa}(X_1, T_1)$, and Theorem 6.3 can be extended to weak **a-weighted equilibrium states** (for details, see an earlier version [1] of this paper).

Appendix A. Main notation and conventions. For the reader's convenience, we summarize in Table A.1 the main notation and typographical conventions used in this paper.

TABLE A.1
Main notation and conventions

$\Phi = (\log \phi_n)_{n=1}^{\infty}$	(Asymptotically sub-additive) potential (Section 1)
$\mathcal{M}(X, T)$	Set of T -invariant Borel probability measures on X
$h_{\mu}(T)$	Measure-theoretic entropy of T with respect to μ (Section 1)
$\Phi_*(\mu)$	$\lim_{n \rightarrow \infty} \frac{1}{n} \int \log \phi_n(x) d\mu(x)$.
$P(T, \Phi, K)$	(cf. (2.2))
$P(T, \Phi)$	Topological pressure of Φ (Section 2.2)
$h_{\mu}^{\mathbf{a}}(T_1)$	\mathbf{a} -weighted measure-theoretic entropy of T_1 with respect to μ (Section 1)
$P^{\mathbf{a}}(T_1, \Phi)$	\mathbf{a} -weighted topological pressure of Φ (Section 1)
$\mathcal{C}_s(X, T)$	Collection of sub-additive potentials on X (Section 1)
$\mathcal{C}_{ass}(X, T)$	Collection of asymptotically sub-additive potentials on X (Section 1)
$\mathcal{C}_{asa}(X, T)$	Collection of asymptotically additive potentials on X (Section 1)
$\mathcal{C}_{aa}(X, T)$	Collection of almost additive potentials on X (Section 1)
$\mathcal{C}(X)$	Collection of real continuous functions on X
$G_{\mu}(X, T)$	Set of generic points of $\mu \in \mathcal{M}(X, T)$ (cf. (1.3))
$\Phi = (\Phi_1, \dots, \Phi_d)$	A family of asymptotically sub-additive potentials
$\Phi_*(\mu)$	$((\Phi_1)_*(\mu), \dots, (\Phi_d)_*(\mu))$
L_{Φ}	$\{\Phi_*(\mu) : \mu \in \mathcal{M}(X_1, T_1)\}$
$\lambda_{\Phi}(x)$	Lyapunov exponent of Φ at x (Section 2.2)
$E_{\{\Phi_j\}_{j=1}^r, c}(\alpha)$	non-homogeneous α -level set of $\{\Phi_j\}_{j=1}^r$ (cf. Section 1)
$a_n \approx b_n$	$(1/c)b_n \leq a_n \leq cb_n$ for a constant $c > 0$
$a_n \approx_n b_n$	$(1/\kappa_n)b_n \leq a_n \leq \kappa_n b_n$ for a sequence of positive numbers (κ_n) with $\lim_{n \rightarrow \infty} (1/n) \log \kappa_n = 0$
$\Phi^{\mathbf{a}}$	(cf. (3.4))
$d_{\mathbf{a}}$	“self-affine” metric (cf. (1.1)).

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