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ON TWO-VARIABLE PRIMITIVE p-ADIC L-FUNCTIONS*

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Abstract. We construct a two variable p-adic L-function which lead to the p-adic interpolation of values of ***primitive*** Hecke L-functions, and use it to give a modification of Yager's theorem which relate the p-adic L-function to a certain Iwasawa module.

Key words. Elliptic curve, complex multiplication, Iwasawa theory, p-adic L-function.

AMS subject classifications. 11R23, 11G40.

0. Introduction. Let E be an elliptic curve defined over an imaginary quadratic field K, with complex multiplication by the ring of integers $\mathcal{O}_K$. Let $-d_K$ be the discriminant of K. Since K has class number 1, we choose a global minimal Weierstrass equation for E

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{1}$$

whose coefficient a_i belong to $\mathcal{O}$. We also fix an embedding of K into the field of complex numbers $\mathbb{C}$, and write $\mathcal{L}$ for the period lattice of the Néron differential

$$\omega = \frac{dx}{2y + a_1x + a_3}$$

on E. Since $\mathcal{O}$ is a principal ideal domain, $\mathcal{L}$ is a free $\mathcal{O}$-module of rank 1, and we fix Ω_∞ in $\mathcal{L}$ such that $\mathcal{L}$ is equal to $\Omega_\infty\mathcal{O}$. For each integral ideal $\mathfrak{a} = (\alpha)$ of $\mathcal{O}_K$, let E_α(or $E_{\mathfrak{a}}$) be the kernel of the endomorphism α of E and set E_{α^∞}(or $E_{\mathfrak{a}^\infty}$) to be $\bigcup_{n\geq 0} E_{\alpha^n}$. We shall denote by ψ the Grossencharacter of E over K, and write $\mathfrak{f} = (f)$ for the conductor of ψ. We fix for the rest of this paper a rational prime p different from 2 and 3, which splits as $(p) = \mathfrak{p}\mathfrak{p}^*$ in K, and where E has good reduction at both $\mathfrak{p}$ and $\mathfrak{p}^*$. Put $\pi = \psi(\mathfrak{p})$, $\pi^* = \psi(\mathfrak{p}^*)$, and $\mathcal{G} = Gal(K(E_{p^\infty})/K)$. Let $\chi_{\mathfrak{p}} : \mathcal{G} \to \mathbb{Z}_p^\times$ and $\chi_{\mathfrak{p}^*} : \mathcal{G} \to \mathbb{Z}_p^\times$ be respectively the characters giving the action of $\mathcal{G}$ on E_{π^∞} and $E_{\pi^{*\infty}}$. We also use the same notations for their restrictions to subgroups or some quotient of $\mathcal{G}$ (it is well defined modulo certain subgroup of $\mathbb{Z}_p$) .

For any positive integer k and any ideal $\mathfrak{g}$ of $\mathcal{O}_K$ which is divisible by the conductor of $\bar{\psi}^k$, we define the Hecke L-function $L_{\mathfrak{g}}(\bar{\psi}^k, s)$ to be the analytic continuation of

$$L_{\mathfrak{g}}(\bar{\psi}^k, s) = \sum \bar{\psi}^k(\mathfrak{a})N\mathfrak{a}^{-s}, \qquad Re(s) > 1 + k/2,$$

where the sum is taken over all integral ideals $\mathfrak{a}$ which is prime to $\mathfrak{g}$. In general it is imprimitive, and we will omit the subscript $\mathfrak{g}$ if it's ***primitive*** (i.e. $\mathfrak{g}$ equals to the conductor of $\bar{\psi}^k$).

We denote by $\mathscr{I}$ the ring of integers of the completion of the maximal unramified extension of $K_{\mathfrak{p}}$, where $K_{\mathfrak{p}}$ is the completion of K at $\mathfrak{p}$. Let $\mathscr{I}(\mathcal{G})$ be the Iwasawa algebra of $\mathcal{G}$:

$$\mathscr{I}(\mathcal{G}) = \varprojlim \mathscr{I}[\mathcal{G}/\mathcal{H}]$$

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where $\mathcal{H}$ runs over all open subgroups of $\mathcal{G}$.

Katz[9] and Mannin-Vishik[13] first showed the existence of a two variable p-adic L-function in $\mathscr{I}(\mathcal{G})$, which lead to the p-adic interpolation of values of (in general imprimitive) Hecke L-functions. Yager[19] gave an elementary construction of a p-adic measure on certain ray class group $Gal(\mathscr{R}(\mathfrak{g}p^{\infty})/K)$. He exploited an identity known to Eisenstein, and sought a natural choice of the p-adic period $\Omega_{\mathfrak{p}}$. Moreover, in the spirit of Iwasawa and of Coates and Wiles, Yager[18] proved an important theorem which related the p-adic L-function to the structure of a certain Iwasawa module attached to the elliptic curve E. The theorem provided a key step to the main conjecture, which was finally proved by Rubin[15].

In this paper, we use an important congruence from Yager[19] to construct a two variable p-adic L-function on the Galois group $Gal(K(E_{p^{\infty}})/K)$, which lead to the p-adic interpolation of values of ***primitive*** Hecke L-functions:

THEOREM 0.1. *There exists a p-adic period $\Omega_{\mathfrak{p}}$ in $\mathscr{I}^{\times}$ and a unique p-adic measure $\mu \in \mathscr{I}(\mathcal{G})$ such that, for all integers k, j with $k > j \geq 0$, we have*

$$\Omega_{\mathfrak{p}}^{-(k+j)} \int_{\mathcal{G}} \chi_{\mathfrak{p}}^{k} \chi_{\mathfrak{p}^*}^{-j} d\mu = \Omega_{\infty}^{-(k+j)} (k-1)! (\frac{2\pi}{\sqrt{d_K}})^j L_{\infty}(\bar{\psi}^{k+j}, k)$$

where

$$L_{\infty}(\bar{\psi}^{k+j}, k) = (1 - \frac{\psi^{k+j}(\mathfrak{p})}{(N\mathfrak{p})^{j+1}})(1 - \frac{\bar{\psi}^{k+j}(\mathfrak{p}^*)}{(N\mathfrak{p}^*)^k}) L(\bar{\psi}^{k+j}, k)$$

Note that the π appears in the factor $\frac{2\pi}{\sqrt{d_K}}$ denote the real number 3.14..., and for all other cases $\pi = \psi(\mathfrak{p})$. We also remark that the one variable p-adic L-function was constructed by Katz[10], Manin-Vishik[13], Coates-Wiles[6], Lichtenbaum[11] and others, but Bernadi-Goldstein-Stephens[1] first dealt with ***primitive*** Hecke L-functions and the trivial eigenspace. Moreover, we will use the two variable ***primitive*** p-adic L-function to give a slight modification of Yager's theorem, which we now describe:

We put $K_n = K(E_{p^{n+1}})$. Let $U_{n,v}$ be the local units of the completion of K_n at a prime v which are congruent to 1 module v, and put $\mathbf{U}_n = \prod_{v|\mathfrak{p}} U_{n,v}$, where the product is taken over all primes of K_n lying above $\mathfrak{p}$. We will define a group of local units $\bar{C}_n$ for the field K_n (in the end of Section 3) which is a slight modification of Robert's elliptic units in Yager[18], and define

$$Y_{\infty} = \varprojlim (\mathbf{U}_n / \bar{C}_n)$$

where the projective limit is taken relative to the norm maps, has a natural structure as a module over the Iwasawa algebra

$$\mathbb{Z}_p(\mathcal{G}) = \varprojlim \mathbb{Z}_p[\mathcal{G}/\mathcal{H}]$$

where $\mathcal{H}$ runs over all open subgroups of $\mathcal{G}$. We will prove that Y_{∞} is a finite generated torsion $\mathbb{Z}_p(\mathcal{G})$-module, so it follows from the structure theory that there are non-zero divisors f_1, ..., f_r of $\mathbb{Z}_p(\mathcal{G})$ such that $\oplus_{n=1}^{r} \mathbb{Z}_p(\mathcal{G})/(f_n)$ is pseudo-isomorphic to Y_{∞}. We may define the characteristic ideal $ch_{\mathcal{G}}(Y_{\infty})$ of Y_{∞} by

$$ch_{\mathcal{G}}(Y_{\infty}) = f_1 ... f_r \mathbb{Z}_p(\mathcal{G}),$$

see the Appendix of Coates-Sujatha[3] for details. Recall that μ is the p-adic measure in theorem 0.1. Now we can state the modification of Yager's theorem:

THEOREM 0.2. $ch_{\mathcal{G}}(Y_\infty)\mathscr{I}(\mathcal{G}) = \mu\mathscr{I}(\mathcal{G})$.

Our method of proof is strongly motivated by the original work of Yager, who established the analogue of Theorem 0.2 with an imprimitive p-adic L-function instead of our μ, but used a different group of elliptic units to us. It is important for the arithmetic applications of Theorem 0.2, in particular for the proof of the two variable main conjecture (proved in Rubin[15]) to have a result with the primitive L-function.

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1. Notation. We begin by introducing further notation, which will be used throughout the paper. Let F_m denote the the field $K(E_{\pi^{*m+1}})$ and $K_{n,m}$ the field $F_m(E_{\pi^{n+1}})$. It is well known that the extension $K_{n,m}$ over F_m is totally ramified at the primes above $\mathfrak{p}$, and that $\mathfrak{p}$ is unramified in F_m. In fact, From the relation between the Grossencharacter and the action of the Galois group on points of finite order, we see that the number of primes of F_m lying above $\mathfrak{p}$, which we denote by r_m is given by the index of the subgroup generated by π in $(\mathcal{O}_K/\mathfrak{p}^{*m+1})^\times$. Hence, there exists an integer M such that $r_m = r_0 p^m$ for $m < M$ and $r_m = r_0 p^M$ for $m \geq M$.

We choose and fix a prime $\mathfrak{p}_M$ of F_M lying above $\mathfrak{p}$, and let $\mathfrak{p}_m$ be the unique prime of F_M lying above or below $\mathfrak{p}_M$.

We write $\mathfrak{p}_{n,m}$ for the unique prime of $K_{n,m}$ lying above $\mathfrak{p}_m$. If ω is any prime of F_m above $\mathfrak{p}$, we let $H_{n,m,\omega}$ be the completion of $K_{n,m}$ at the unique prime above ω, and we let $\Phi_{m,\omega}$ denote the completion of F_m at ω. We shall write $\mathcal{I}_{m,\omega}$ for the ring of integers of $\Phi_{m,\omega}$, and we shall also write ω for the maximal ideal of $\mathcal{I}_{m,\omega}$. Put $F_\infty = \bigcup_{m\geq 0} F_m$, and let φ denote the Artin symbol $(\mathfrak{p}, F_\infty/K)$ for the extension F_∞ over K. Note that we always view our global fields as lying inside the complex numbers, and equipped with embeddings into their completions.

The rings $H_{n,m} = \prod H_{n,m,\omega}$ and $\Phi_m = \prod \Phi_{m,\omega}$, where the product is taken over the set of primes ω of F_m lying above $\mathfrak{p}$, have a natural action of the Galois group $\mathcal{G}$ as follows. Let $\alpha_{k,\omega}(k = 0, 1, 2, ...)$ be a Cauchy sequence of elements of $K_{n,m}$ (or F_m) which converge to α_ω in $H_{n,m,\omega}$ (or $\Phi_{m,\omega}$). Then the ω^σ component of $(\alpha_\omega)^\sigma$ is the limit of the Cauchy sequence $\alpha^\sigma_{k,\omega}(k = 0, 1, 2, ...)$. We embed $K_{n,m}$ and F_m in these rings via the diagonal map, and it is easy to verify that the usual norm and trace maps on $H_{n,m}$, Φ_m, $K_{n,m}$ and F_m, as well as the Galois action, all commute with these embeddings.

It's well known that $\mathcal{G} = \Gamma \times \Delta$, where Γ is the Galois group of $K(E_{p^\infty})$ over $K_{0,0}$, and Δ is the product of two cyclic groups of order $p-1$ which can be identified with the Galois group of $K_{0,0}$ over K. If A is any $\mathbb{Z}_p[\Delta]$-module, we define $A^{(i_1,i_2)}$ to be the submodule of A on which Δ acts via $\chi_{\mathfrak{p}}^{i_1}\chi_{\mathfrak{p}^*}^{i_2}$. Thus, we have the canonical decomposition

$$A = \bigoplus_{i_1,i_2 \bmod (p-1)} A^{(i_1,i_2)}.$$

Let Λ be the ring of formal power series in the commuting indeterminates T_1 and T_2 with coefficients in $\mathbb{Z}_p$. Choose a topological generator u of $(1+p\mathbb{Z}_p)^\times$ and let γ_1 and γ_2 be the elements of Γ for which $\chi_{\mathfrak{p}}(\gamma_1) = \chi_{\mathfrak{p}^*}(\gamma_2) = u$ and $\chi_{\mathfrak{p}}(\gamma_2) = \chi_{\mathfrak{p}^*}(\gamma_1) = 1$. Any compact $\mathbb{Z}_p$-module B on which Γ acts continuously can be endowed with a unique Λ-module structure such that $\gamma_1 x = (1+T_1)x$ and $\gamma_2 x = (1+T_2)x$ for all x in B.

Let $\widehat{E}$ be the formal group giving the kernel of reduction mod $\mathfrak{p}$ on E with parameter $t = -x/y$. Since $\widehat{E}$ is defined over $\mathcal{O}_{\mathfrak{p}}$, we have the power series expansions

$$x = t^{-2}a(t),\ y = -t^{-3}a(t)$$

where $a(t)$ has coefficients in $\mathcal{O}_{\mathfrak{p}}$ and constant term equal to 1. Denote by $\widehat{E}_{\pi^{n+1}}$ the kernel of the endomorphism $[\pi^{n+1}]$ on $\widehat{E}$, which we identify with $E_{\pi^{n+1}}$.

Finally, we denote by $U'_{n,m,\omega}$ the units of $H_{n,m,\omega}$ and by $U_{n,m,\omega}$ the subgroup consisting of those units which are congruent to 1 modulo the maximal ideal. Put $\mathbf{U}'_{n,m} = \prod_\omega U'_{n,m,\omega}$ and $\mathbf{U}_{n,m} = \prod_\omega U_{n,m,\omega}$, where again the product is taken over the primes ω of F_m lying above $\mathfrak{p}$. Let $\mathbf{U}'_\infty$ and $\mathbf{U}_\infty$ denote the projective limits of the $\mathbf{U}'_{n,m}$ and $\mathbf{U}_{n,m}$ respectively relative to the norm maps. We endow $\mathbf{U}_\infty$ with its natural structure as a $\mathbb{Z}_p[\mathcal{G}]$-module. In particular $\mathbf{U}_\infty$ is a compact Γ-module, and thus also a Λ-module and $\mathbb{Z}_p(\mathcal{G})$-module.

2. Coleman power series. In this section, we will first recall some basic facts about Coleman power series, and associate a two-variable power series with each $\beta \in \mathbf{U}'_\infty$, then we will produce some important maps using these power series.

Let T_π denote the Tate module $\varprojlim \widehat{E}_{\pi^{n+1}}$, where the limit is taken relative to the usual projection maps given by multiplication by powers of π. We fix a basis (u_n) of T_π, and let $\beta = (\beta_{n,m,\omega})$ be an element of $\mathbf{U}'_\infty$. Coleman[7] has shown that for each integer $m \geq 0$ and each prime ω of F_m lying above $\mathfrak{p}$, there is a unique power series $c_{m,\omega,\beta}(T) \in \mathcal{I}_{m,\omega}[[T]]$ such that

$$\beta_{n,m,\omega} = c^{\varphi^{-n}}_{m,\omega,\beta}(u_n) \text{ for all } n \geq 0.$$

Moreover, these power series satisfy the functional equation

$$c^{\varphi}_{m,\omega,\beta}([\pi]T) = \prod_{\eta \in \widehat{E}_\pi} c_{m,\omega,\beta}(T \oplus \eta) \tag{2}$$

where $[\pi]T$ is the endomorphism of $\widehat{E}$ induced by π, and $T \oplus \eta$ denotes the sum of T and η under the addition on $\widehat{E}$.

We will denote by $c_{m,\beta}(T)$ the element $(c_{m,\omega,\beta}(T)) \in \prod_\omega \mathcal{I}_{m,\omega}[[T]]$, which we shall write as $\mathcal{I}_m[[T]]$, with the obvious Galois structure inherited from the structure on Φ_m. It is plain that, for each $m \geq 0$ and each prime ω of F_m lying above $\mathfrak{p}$, the Coleman series $c_{m,\omega,\beta}(T)$ attached to an element β of $\mathbf{U}'_\infty$ is a unit in $\mathcal{I}_{m,\omega}[[T]]$. We denote by $g_{m,\beta}(T)$ the vector whose ω-component $(g_{m,\beta}(T))_\omega$ is given by $\frac{1}{p}\log(c_{m,\omega,\beta}(T)^p / c^{\varphi}_{m,\omega,\beta}([\pi]T))$. Observe that φ induces the Frobenius automorphism for the extension Φ_m over $K_{\mathfrak{p}}$, and that $[\pi]T \equiv T^p \bmod \omega$, we get

$$c_{m,\omega,\beta}(T)^p / c^{\varphi}_{m,\omega,\beta}([\pi]T) \equiv 1 \bmod \omega,$$

which shows that $g_{m,\beta}(T) \in \mathcal{I}_m[[T]]$.

Lemma 2.1. *Let $m' \geq m \geq 0$ and let $Tr_{m',m}$ denote the trace map from $\mathcal{I}_{m'}[[T]]$ to $\mathcal{I}_m[[T]]$. Then, for each β in $\mathbf{U}'_\infty$,*

$$g_{m,\beta}(T) = Tr_{m',m}(g_{m',\beta}(T)).$$

Proof. This is clear from Lemma 2 of Yager[18], which asserts the norm compatibility of Coleman power series. The only point is the Galois action commutes with the logarithmic operator. ◻

The following theorem provides the key to the rest of this paper.

THEOREM 2.2. *For each β in $\mathbf{U}'_\infty$, there is a unique power series $g_\beta(T_1,T_2) \in \mathscr{I}[[T_1,T_2]]$ such that*

$$g_\beta(T_1,T_2) \equiv \sum_{\sigma\in Gal(F_m/K)} (g^\sigma_{m,\beta}(T_1))_{\mathfrak{p}_m}(1+T_2)^{\chi_{\mathfrak{p}^*}(\sigma)} \bmod ((1+T_2)^{p^{m+1}}-1)$$

Proof. Observe firstly that $(1+T_2)^{\chi_{\mathfrak{p}^*}(\sigma)}$ is well defined modulo $((1+T_2)^{p^{m+1}}-1)$ for all $\sigma \in Gal(F_m/K)$. All that we need check is the appropriate compatibilities are satisfied. Let $m' \geq m$. Then by Lemma 2.1 we have

$$(g^\sigma_{m,\beta}(T_1))_{\mathfrak{p}_m} = \sum_{\substack{\theta\in Gal(F_{m'}/K)\\ \theta|_{F_m}=\sigma}} (g^\theta_{m',\beta}(T_1))_{\mathfrak{p}_{m'}}$$

Consequently

$$\sum_{\substack{\theta\in Gal(F_{m'}/K)\\ \theta|_{F_m}=\sigma}} (g^\theta_{m',\beta}(T_1))_{\mathfrak{p}_{m'}}(1+T_2)^{\chi_{\mathfrak{p}^*}(\theta)}$$

$$\equiv (g^\sigma_{m,\beta}(T_1))_{\mathfrak{p}_m}(1+T_2)^{\chi_{\mathfrak{p}^*}(\sigma)} \bmod ((1+T_2)^{p^{m+1}}-1)$$

which is sufficient to prove the theorem. ◻

We write $\lambda : \widehat{E} \simeq G_a$ for the logarithm map of $\widehat{E}$, where G_a is the additive formal group. Let $k \geq 1$ and $j \geq 0$. We define for each $\beta \in \mathbf{U}_\infty$:

$$\delta_{k,j}(\beta) = (\lambda'(T_1)^{-1}\frac{\partial}{\partial T_1})^k((1+T_2)\frac{\partial}{\partial T_2})^j g_\beta(T_1,T_2)|_{(0,0)}. \tag{3}$$

The following lemma summarizes the basic properties of these maps.

LEMMA 2.3. *Let $k \geq 1$ and $j \geq 0$. Then $\delta_{k,j}$ is a homomorphism of $\mathbb{Z}_p$-modules from $\mathbf{U}_\infty$ to $\mathscr{I}$, and for all $\beta \in \mathbf{U}_\infty$ and all $\sigma \in \mathcal{G}$,*

$$\delta_{k,j}(\beta^\sigma) = \chi_{\mathfrak{p}}(\sigma)^k\chi_{\mathfrak{p}^*}(\sigma)^{-j}\delta_{k,j}(\beta). \tag{4}$$

In particular, if $\beta \in \mathbf{U}_\infty^{(i_1,i_2)}$, then $\delta_{k,j}(\beta) = 0$ unless $(k,-j) \equiv (i_1,i_2) \bmod (p-1)$, and if $h(T_1,T_2) \in \Lambda$,

$$\delta_{k,j}(h(T_1,T_2)\beta) = h(u^k-1,u^{-j}-1)\delta_{k,j}(\beta). \tag{5}$$

Proof. It is clear that $\delta_{k,j}$ is a $\mathbb{Z}_p$-homomorphism. Note that for each $\sigma \in \mathcal{G}$ and $n \geq 0$, $u_n^\sigma = [\chi_{\mathfrak{p}}(\sigma)](u_n)$, so by definition we have

$$g_{m,\beta^\sigma}(T) = g^\sigma_{m,\beta}([\chi_{\mathfrak{p}}(\sigma)]T),$$

and from this it is easy to see that

$$g_{\beta^\sigma}(T_1,T_2) = g_\beta([\chi_{\mathfrak{p}}(\sigma)]T_1,(1+T_2)^{\chi_{\mathfrak{p}^*}(\sigma)^{-1}}-1).$$

Then equation(4) is evident from the definition of $\delta_{k,j}$. The next assertion follows from the first two if we take $\sigma \in \Delta$. For equation(5), it is just a restatement of equation(3) if we take $h(T_1,T_2)$ to be either $1+T_1$ (corresponding to γ_1) or $1+T_2$ (corresponding to γ_2), and follows in general by linearity and continuity. ◻

3. Elliptic units. We will define and establish a number of results about elliptic units in this section. Let $\mathfrak{g}=(g)$ be an integral ideal of K which divides $\mathfrak{f}$. For each integer $m \geq 0$, we write $\mathfrak{g}_m$(resp. g_m) for $\mathfrak{g}\mathfrak{p}^{*m+1}$(resp. $g\pi^{*m+1}$), where we recall that $\pi^* = \psi(\mathfrak{p}^*)$. Let Δ_m be the Galois group of $K(E_{g_m})/K(E_g)$. Plainly Δ_m is isomorphic to $Gal(F_m/K)$, and $Gal(K(E_{g_m})/F_m)$ is isomorphic to $Gal(K(E_g)/K)$ under the natural restriction maps, since $(\mathfrak{p}^*, \mathfrak{g}) = 1$. We use $\sigma_{\mathfrak{a}}$ denote the Artin symbol of $\mathfrak{a}$ for the extension $K(E_{g_m})/K$ for every integral ideal which is prime to $\mathfrak{f}\mathfrak{p}^*$. Let us first define some rational functions which are crucial for our definition of elliptic units.

Lemma 3.1. *Let λ be an element in $\mathcal{O}_K \backslash \mathcal{O}_K^\times$ which is prime to 6, then there exists a unique $c_E(\lambda)$ in $K^\times$ such that the rational function*

$$\mathscr{R}_\lambda(P) = c_E(\lambda) \prod_{R \in (E_\lambda \backslash \{0\})/\pm 1} (x(P) - x(R))^{-1}$$

satisfies:

$$\mathscr{R}_\lambda(\alpha(P)) = \prod_{R \in E_\alpha} \mathscr{R}_\lambda(P \oplus R) \tag{6}$$

for all $\alpha \in \mathcal{O}_K$ with $(\alpha, \lambda) = 1$. Moreover $c_E(\lambda)$ can only be divisible by primes where E has bad reduction, in particular it is a unit at the prime $\mathfrak{p}$.

Proof. See the Appendix of Coates[2]. ☐

We choose a primitive $\mathfrak{g}_m$-division point $V_{\mathfrak{g}_m}$ on E for each non-negative integer m such that $\pi^* V_{\mathfrak{g}_{m+1}} = V_{\mathfrak{g}_m}$. Let $B_{\mathfrak{g}}$ be a minimal set of integral ideals of K, prime to $\mathfrak{f}\mathfrak{p}^*$, such that $Gal(K(E_{g_m})/F_m) = \{\sigma_{\mathfrak{v}} : \mathfrak{v} \in B_{\mathfrak{g}}\}$, and similarly let B_m be a minimal set of integral ideals of K, prime to $\mathfrak{f}\mathfrak{p}^*$, such that $\Delta_m = \{\sigma_{\mathfrak{u}} : \mathfrak{u} \in B_m\}$. If $\mathfrak{g} = (1)$ we set $B_{\mathfrak{g}} = \{(1)\}$. For each $\lambda \in \mathcal{O}_K \backslash \mathcal{O}_K^\times$ which satisfies $(\lambda, 6p\mathfrak{f}) = 1$, we put

$$\mathscr{R}_{\lambda, V_{\mathfrak{g}_m}}(P) = \mathscr{R}_\lambda(P \oplus V_{\mathfrak{g}_m}) \tag{7}$$

and

$$\Lambda_{\lambda, V_{\mathfrak{g}_m}}(P) = \prod_{\mathfrak{v} \in B_{\mathfrak{g}}} \mathscr{R}_{\lambda, V_{\mathfrak{g}_m}^{\sigma_{\mathfrak{v}}}}(P). \tag{8}$$

Obviously $\mathscr{R}_\lambda(P)$ and $\Lambda_{\lambda, V_{\mathfrak{g}_m}}(P)$ are rational functions on E with coefficients in K and F_m respectively.

The following proposition was proved in Coates-Sujatha[4]:

Proposition 3.2. *Let $\Lambda_{\lambda, V_{\mathfrak{g}_m}}(t)$ and $\mathscr{R}_{\lambda, V_{\mathfrak{g}_m}}(t)$ be the t-expansions of $\Lambda_{\lambda, V_{\mathfrak{g}_m}}(P)$ and $\mathscr{R}_{\lambda, V_{\mathfrak{g}_m}}(P)$ respectively, then we have:*
(i) *$\mathscr{R}_{\lambda, V_{\mathfrak{g}_m}}(t)$ and $\Lambda_{\lambda, V_{\mathfrak{g}_m}}(t)$ are units in $\mathscr{I}[[t]]$.*
(ii) $\Lambda_{\lambda, V_{\mathfrak{g}_m}^{\sigma_{\mathfrak{p}}}}(\psi(\mathfrak{p})P) = \prod_{R \in E_{\mathfrak{p}}} \Lambda_{\lambda, V_{\mathfrak{g}_m}}(P \oplus R)$.
(iii) $\Lambda_{\lambda, V_{\mathfrak{g}_m}}(\psi(\mathfrak{p}^*)P) = \prod_{R \in E_{\mathfrak{p}^*}} \Lambda_{\lambda, V_{\mathfrak{g}_{m+1}}}(P \oplus R)$.

Proof. We will use V_m to instead $V_{\mathfrak{g}_m}$ for simplicity. To proof the first statement, we see it suffices to show that for each primitive $\mathfrak{g}_m$-division point V_m and each $R \in E_\lambda \backslash \{0\}$, the t-expansion of

$$x(P \oplus V_m) - x(R)$$

is a unit in $\mathscr{I}[[t]]$. Note that neither V_m nor R can lie in the kernel of the formal group of E at $\mathfrak{p}$ since $(\mathfrak{g}_m, \mathfrak{p}) = (\lambda, \mathfrak{p}) = 1$. Thus all of the coordinates $x(V_m)$, $y(V_m)$, $x(R)$, $y(R)$ must lie in $\mathscr{I}$. In particular, it follows from the expansion of $x(P)$ as a power series in t that

$$x(P) - x(V_m) = t^{-2}(1 + \sum_{n=1}^{\infty} \alpha_n t^n),$$

where all of the coefficients α_n belong to $\mathscr{I}$. As $y = -x/t$, we conclude easily that

$$\frac{y(P) - y(V_m)}{x(P) - x(V_m)} = -t^{-1}(1 + \sum_{n=1}^{\infty} \beta_n t^n),$$

where all of the coefficients β_n belong to $\mathscr{I}$. Now, by the addition law on E, we have

$$x(P \oplus V_m) - x(R) = (\frac{y(P) - y(V_m)}{x(P) - x(V_m)})^2 + a_1(\frac{y(P) - y(V_m)}{x(P) - x(V_m)}) - a_2 - x(P) - x(V_m) - x(R).$$

which is obviously a unit in $\mathscr{I}[[t]]$ from the explicit t-expansion above.

For the second statement, we first note that $V_m^{\sigma_\mathfrak{p}} = \pi V_m$. Then by the definitions (7) and (8), and the equation (6), we have

$$\begin{aligned}
\Lambda_{\lambda, V_m^{\sigma_\mathfrak{p}}}(\pi P) &= \prod_{\mathfrak{v} \in B_\mathfrak{g}} \mathscr{R}_{\lambda, (\pi V_m)^{\sigma_\mathfrak{v}}}(\pi P) \\
&= \prod_{\mathfrak{v} \in B_\mathfrak{g}} \mathscr{R}_\lambda(\pi(P \oplus V_m^{\sigma_\mathfrak{v}})) \\
&= \prod_{\mathfrak{v} \in B_\mathfrak{g}} \prod_{R \in E_\mathfrak{p}} \mathscr{R}_\lambda(P \oplus V_m^{\sigma_\mathfrak{v}} \oplus R) \\
&= \prod_{R \in E_\mathfrak{p}} \Lambda_{\lambda, V_m}(P \oplus R)
\end{aligned}$$

The last statement is similar:

$$\begin{aligned}
\Lambda_{\lambda, V_m}(\pi^* P) &= \prod_{\mathfrak{v} \in B_\mathfrak{g}} \mathscr{R}_{\lambda, V_m^{\sigma_\mathfrak{v}}}(\pi^* P) \\
&= \prod_{\mathfrak{v} \in B_\mathfrak{g}} \mathscr{R}_\lambda(\pi^* P \oplus V_m^{\sigma_\mathfrak{v}}) \\
&= \prod_{\mathfrak{v} \in B_\mathfrak{g}} \mathscr{R}_\lambda(\pi^*(P \oplus V_{m+1}^{\sigma_\mathfrak{v}})) \\
&= \prod_{\mathfrak{v} \in B_\mathfrak{g}} \prod_{R \in E_{\mathfrak{p}^*}} \mathscr{R}_\lambda(P \oplus V_{m+1}^{\sigma_\mathfrak{v}} \oplus R) \\
&= \prod_{R \in E_{\mathfrak{p}^*}} \Lambda_{\lambda, V_{m+1}}(P \oplus R)
\end{aligned}$$

□

Let I denote the set of elements of $\mathcal{O}_K \backslash \mathcal{O}_K^\times$ which are prime to $6p\mathfrak{f}$, and let

$$\mathcal{S} = \{\nu : I \to \mathbb{Z} |\ \nu(\lambda) = 0 \text{ for almost all } \lambda \in I \text{ and } \sum_{\lambda \in I} (N\lambda - 1)\nu(\lambda) = 0\},$$

where $N\lambda$ denote the norm of λ. If $\nu \in \mathcal{S}$, we set

$$\Lambda_{\nu,V_{\mathfrak{g}_m}}(P) = \prod_{\lambda \in I} \Lambda_{\lambda,V_{\mathfrak{g}_m}}(P)^{\nu(\lambda)}$$

Let P_n be the π^{n+1}-division point of E corresponding to $u_n \in \widehat{E}_{\pi^{n+1}}$, and choose $\varepsilon_n \in \mathcal{O}$ such that $\varepsilon_n \pi^* \equiv 1 \bmod \mathfrak{p}^{n+1}$. Observe that π^* is a unit in $\mathbb{Z}_p$, and that we have $[\pi^{*-(m+1)}]u_n$ corresponding to $\varepsilon_n^{m+1} P_n$.

Robert[12] has shown that $\Lambda_{\nu,V_{\mathfrak{g}_m}}(\varepsilon_n^{m+1} P_n)$ is a unit of $K_{n,m}$. The reader could also see De Shalit[8] for more details. Let $\mathscr{P}(z,\mathcal{L})$ be the Weierstrass $\mathscr{P}$-function of $\mathcal{L}$, then we have the Weierstrass isomorphism $\xi(z,\mathcal{L})$ between $\mathbb{C}/\mathcal{L}$ and $E(\mathbb{C})$ defined by

$$\xi(z,\mathcal{L}) = (\mathscr{P}(z,\mathcal{L}) - \frac{a_1^2 + 4a_2}{12},\ \frac{1}{2}(\mathscr{P}'(z,\mathcal{L}) - a_1(\mathscr{P}(z,\mathcal{L}) - \frac{a_1^2 + 4a_2}{12}) - a_3)).$$

From now on, we will fix the primitive $\mathfrak{g}_m$-division point $V_{\mathfrak{g}_m}$ to be $\xi(\rho_m, \mathcal{L})$ for each $\mathfrak{g}$ divides $\mathfrak{f}$ and non-negative integer m, where $\rho_m = \frac{\Omega_\infty}{g_m}$.

We set:

$$C'_{n,m}(\mathfrak{g}) = \{\Lambda_{\nu,V_{\mathfrak{g}_m}}(\varepsilon_n^{m+1} P_n) | \nu \in \mathcal{S}\}.$$

Obviously $C'_{n,m}(\mathfrak{g})$ is a group of units of $K_{n,m}$, and it is stable under the action of $\mathcal{G}$ by Lemma 20 of Coates-Wiles[5].

Lemma 3.3. *Let $\mathfrak{g}$ be an integral ideal divides $\mathfrak{f}$, $\nu \in \mathcal{S}$, and put*

$$e_{n,m}(\mathfrak{g},\nu) = \Lambda_{\nu,V_{\mathfrak{g}_m}}^{\varphi^{-n}}(\varepsilon_n^{m+1} P_n), \quad e(\mathfrak{g},\nu) = (e_{n,m}(\mathfrak{g},\nu)).$$

Then $e(\mathfrak{g},\nu) \in \mathbf{U}'_\infty$.

Proof. We only need to show $e_{n,m}(\mathfrak{g},\nu)$ is just the image of $e_{n+1,m+1}(\mathfrak{g},\nu)$ under the norm map $N_{n+1,n}^{m+1,m}$ from $K_{n+1,m+1}$ to $K_{n,m}$. Take $P = \varepsilon_{n+1}^{m+2} P_{n+1}$ and $P = \varepsilon_n^{m+2} P_n$ in Proposition 3.2 (ii) and (iii) respectively, we see that

$$N_{n+1,n}^{m+1}(\Lambda_{\nu,V_{\mathfrak{g}_{m+1}}}^{\varphi^{-(n+1)}}(\varepsilon_{n+1}^{m+2} P_{n+1})) = \Lambda_{\nu,V_{\mathfrak{g}_{m+1}}}^{\varphi^{-n}}(\varepsilon_n^{m+2} P_n)$$

and

$$N_n^{m+1,m}(\Lambda_{\nu,V_{\mathfrak{g}_{m+1}}}^{\varphi^{-n}}(\varepsilon_n^{m+2} P_n)) = \Lambda_{\nu,V_{\mathfrak{g}_m}}^{\varphi^{-n}}(\varepsilon_n^{m+1} P_n)$$

where $N_{n+1,n}^{m+1}$ and $N_n^{m+1,m}$ are the norm maps from $K_{n+1,m+1}$ to $K_{n,m+1}$ and from $K_{n,m+1}$ to $K_{n,m}$ respectively. Now this lemma follows from the compatibility of the norm maps. ◻

Theorem 3.4. *Let $\nu \in \mathcal{S}$. Then the Coleman power series $c_{m,e(\mathfrak{g},\nu)}(T) \in \mathcal{I}_m[[T]]$ attached to $e(\mathfrak{g},\nu)$ are given by*

$$c_{m,e(\mathfrak{g},\nu)}(T) = \Lambda_{\nu,V_{\mathfrak{g}_m}}(\pi^{*-(m+1)} T).$$

Proof. Proposition 3.2 (i) tells us that $\Lambda_{\nu,V_{\mathfrak{g}_m}}(\pi^{*-(m+1)} T) \in \mathcal{I}_m[[T]]$, then the theorem is an direct consequence from the definition of Coleman power series and $e(\mathfrak{g},\nu)$. ◻

Finally, we define a group for each pair of non-negative integers n and m as follows:

$$\bar{C}'_{n,m} = \prod_{i_1,i_2 \bmod (p-1)} (\bar{C}'_{n,m}(\mathfrak{g}_{i_1,i_2}))^{(i_1,i_2)}$$

where $\mathfrak{g}_{i_1,i_2} = (g_{i_1,i_2})$ is the conductor of the Grossencharacter $\psi^{i_1-i_2}$ and $\bar{C}'_{n,m}(\mathfrak{g}_{i_1,i_2})$ is the closure of $C'_{n,m}(\mathfrak{g}_{i_1,i_2})$ in $\mathbf{U}'_{n,m}$. By definition it is also stable under the action of $\mathcal{G}$. We shall denote $e(\nu)$ the unit whose (i_1,i_2) -th branch is just $e(\mathfrak{g}_{i_1,i_2},\nu)^{(i_1,i_2)}$. We write $\bar{C}'_\infty$ for the projective limits of the $\bar{C}'_{n,m}$ with respect to the norm maps, clearly $e(\nu) \in \bar{C}'_\infty$ for all $\nu \in \mathcal{S}$. We will use $<\beta>$ to denote the projection from $\mathbf{U}'_\infty$ to $\mathbf{U}_\infty$ for each $\beta \in \mathbf{U}'_\infty$.

4. Eisenstein series and Yager's p-adic period. Let us first recall some basic definitions of certain Eisenstein-Kronecker series, see Weil[14], Coates-Wiles[5], and Yager[19] for details. For each positive integer k, we write $K_k(z,s)$ for the analytic continuation to the whole complex s-plane of

$$K_k(z,s) = \sum_{w\in L, w\neq -z} \frac{\overline{z+w}}{|z+w|^{2s}}, \qquad Re(s) > 1 + k/2,$$

and for integers $k > j \geq 0$ we set

$$E_{j,k}(z) = (k-1)!(2\pi/\sqrt{d_K})^j |\Omega_\infty|^{-2j} K_{k+j}(z,k),$$

$$E_k(z) = E_{0,k}(z).$$

Clearly both $K_k(z,s)$ and $E_{j,k}(z)$ are periodic in z with period lattice L. The values of the Eisenstein-Kronecker series at division points of L are closely related to the values of Hecke L-functions, see Coates-Wiles[5] and Yager[19]. We now give a slightly different form.

We define $\delta_\mathfrak{g} = [K(E_g) : \mathscr{R}(\mathfrak{g})] \times m_\mathfrak{g}$, where $m_\mathfrak{g}$ denotes the number of roots of unity which are congruent to 1 mod $\mathfrak{g}$ and $\mathscr{R}(\mathfrak{g})$ is the ray class field of K modulo $\mathfrak{g}$. It's clear that $\delta_\mathfrak{g}$ divides w_K, which is the number of roots of unity in K. Recall that B_m, B_g, and ρ_m are defined in the section above.

LEMMA 4.1. *Let $k > j \geq 0$ be such that the conductor of ψ^{k+j} divides $\mathfrak{g}$, then we have*

$$\delta_\mathfrak{g}(k-1)!(\frac{2\pi}{\sqrt{d_K}})^j \Omega_\infty^{-(k+j)} (1 - \frac{\bar{\psi}^{k+j}(\mathfrak{p}^*)}{(N\mathfrak{p}^*)^k}) L_\mathfrak{g}(\bar{\psi}^{k+j}, k)$$

$$= \frac{N\mathfrak{g}_m^j}{g_m^{k+j}} \sum_{\mathfrak{u}\in B_m} \sum_{\mathfrak{v}\in B_\mathfrak{g}} E_{j,k}(\psi(\mathfrak{u}\mathfrak{v})\rho_m).$$

Proof. We divide the proof into 4 steps.
1. As $\mathfrak{v}$ runs over $B_\mathfrak{g}$, $\mathfrak{u}$ runs over B_m, and c runs over $\mathfrak{g}_m$, the ideal $(\psi(\mathfrak{u}\mathfrak{v}) + c)$ runs over all integral ideals of K which prime to $\mathfrak{g}\mathfrak{p}^*$, precisely $\delta_\mathfrak{g}$ times. This is easily deduced from the fact that $Gal(\mathscr{R}(\mathfrak{g})/K) = (O_K/\mathfrak{g})^\times/\tilde{\mu}_K$, where $\tilde{\mu}_K$ is the image of

the group of roots of unity μ_K in $(O_K/\mathfrak{g})^\times$ and $\mathfrak{g} \neq (1)$.
2. From the infinite sum expansions of the Hecke L-functions we have

$$(1 - \frac{\bar{\psi}^{k+j}(\mathfrak{p}^*)}{(N\mathfrak{p}^*)^k}) L_{\mathfrak{g}}(\bar{\psi}^{k+j}, k) = L_{\mathfrak{g}\mathfrak{p}^*}(\bar{\psi}^{k+j}, k).$$

3. For $\mathfrak{v} \in B_{\mathfrak{g}}$, $\mathfrak{u} \in B_m$, and $c \in \mathfrak{g}_m$, we have

$$\begin{aligned} \bar{\psi}^{k+j}((\psi(\mathfrak{u}\mathfrak{v}) + c)) &= \bar{\psi}^{k+j}(\mathfrak{u}\mathfrak{v}) \bar{\psi}^{k+j}((1 + \tfrac{c}{\psi(\mathfrak{u}\mathfrak{v})})) \\ &= \bar{\psi}^{k+j}(\mathfrak{u}\mathfrak{v}) \overline{(1 + \tfrac{c}{\psi(\mathfrak{u}\mathfrak{v})})}^{k+j} \\ &= \overline{(\psi(\mathfrak{u}\mathfrak{v}) + c)}^{k+j} \end{aligned}$$

The second equality is true since the conductor of ψ^{k+j} divides $\mathfrak{g}$.
4. We compare the infinite sum expansions of the two sides of the equation in this lemma:

$$\begin{aligned} \delta_{\mathfrak{g}} L_{\mathfrak{g}\mathfrak{p}^*}(\bar{\psi}^{k+j}, k) &= \sum_{\mathfrak{u} \in B_m} \sum_{\mathfrak{v} \in B_{\mathfrak{g}}} \sum_{c \in \mathfrak{g}_m} \tfrac{\bar{\psi}^{k+j}((\psi(\mathfrak{u}\mathfrak{v})+c))}{|\psi(\mathfrak{u}\mathfrak{v})+c|^{2k}} \\ &= \sum_{\mathfrak{u} \in B_m} \sum_{\mathfrak{v} \in B_{\mathfrak{g}}} \sum_{c \in \mathfrak{g}_m} \tfrac{\overline{(\psi(\mathfrak{u}\mathfrak{v})+c)}^{k+j}}{|\psi(\mathfrak{u}\mathfrak{v})+c|^{2k}} \\ &= \tfrac{N\mathfrak{g}_m^j}{g_m^{k+j}} \Omega_\infty^{k+j} |\Omega_\infty|^{-2j} \sum_{\mathfrak{u} \in B_m} \sum_{\mathfrak{v} \in B_{\mathfrak{g}}} \sum_{w \in L} \tfrac{\overline{(\psi(\mathfrak{u}\mathfrak{v})\rho_m + w)}^{k+j}}{|\psi(\mathfrak{u}\mathfrak{v})\rho_m + w|^{2k}} \end{aligned}$$

Now the lemma follows from the definition of $E_{j,k}(z)$. □

The following lemma relates the Kronecker-Eisenstein series to certain logarithmic derivatives of those rational functions defined in last section. We will use $\mathscr{R}_{\lambda, V_{\mathfrak{g}_m}}(z)$ and $\Lambda_{\lambda, V_{\mathfrak{g}_m}}(z)$ for their expansions at z respectively.

Lemma 4.2. *Suppose $\lambda \in I$, then for any $\mathfrak{u} \in B_m$, $\mathfrak{v} \in B_{\mathfrak{g}}$ and $k \geq 1$, we have*

$$(\frac{d}{dz})^k \log \mathscr{R}_{\lambda, V_{\mathfrak{g}_m}^{\sigma_{\mathfrak{u}\mathfrak{v}}}}(z) |_{z=0} =$$

$$(-1)^{k-1}((N\lambda) E_k(\psi(\mathfrak{u}\mathfrak{v})\rho_m) - \psi^k((\lambda)) E_k(\psi(\lambda\mathfrak{u}\mathfrak{v})\rho_m)) \tag{9}$$

and consequently for any $\mathfrak{u} \in B_m$, $k \geq 1$ and $\nu \in \mathcal{S}$, we have

$$(\frac{d}{dz})^k \log \Lambda_{\nu, V_{\mathfrak{g}_m}^{\sigma_{\mathfrak{u}}}}(\pi^{*-(m+1)} z) |_{z=0} = (-1)^{k-1} \pi^{*-k(m+1)}$$

$$\times \sum_{\mathfrak{v} \in B_{\mathfrak{g}}} \sum_{\lambda \in I} \nu(\lambda)((N\lambda) E_k(\psi(\mathfrak{u}\mathfrak{v})\rho_m) - \psi^k((\lambda)) E_k(\psi(\lambda\mathfrak{u}\mathfrak{v})\rho_m)) \tag{10}$$

Proof. The lemma is almost a re-writing version of Corollary 12 of Yager[18]. □

Yager[19] used these Eisenstein series to construct a canonical p-adic period:

YAGER'S THEOREM. *There exists a non-negative integer r such that*

$$\Omega_{\mathfrak{p}} = -p^r(\lim_{m\to\infty} \bar{g}_m E_1(\rho_m))^{-1}$$

is a p-adic period, i.e. there is an isomorphism : $t = \delta(w) = \Omega_{\mathfrak{p}} w + \ldots$ between the multiplicative formal group G_m and $\widehat{E}$ with the leading coefficient $\Omega_{\mathfrak{p}} \in \mathscr{I}^\times$. Moreover, $\Omega_{\mathfrak{p}}$ is independent of the choice of the ideal $\mathfrak{g}$, and r is zero except finite good ordinary primes. □

Furthermore it was shown in De Shalit[8] that r is always zero, so we can define $\Omega_{\mathfrak{p}} = -(\lim_{m\to\infty} \bar{\pi}^{*m} E_1(\frac{\Omega_\infty}{\pi^{*m}}))^{-1}$ to be the canonical p-adic period appeared in Theorem 0.1. The following lemma shows the usefulness of Yager's p-adic period.

LEMMA 4.3. *For all integers $k > j \geq 0$ and $m > 0$ and for all $\mathfrak{u} \in B_m$, $\mathfrak{v} \in B_{\mathfrak{g}}$ we have*

$$(N\mathfrak{g}_m^j / g_m^{k+j}) E_{j,k}(\psi(\mathfrak{u}\mathfrak{v})\rho_m)$$

$$\equiv \Omega_{\mathfrak{p}}^{-j} \chi_{\mathfrak{p}^*}(\sigma_{\mathfrak{u}}) g_m^{-k} E_k(\psi(\mathfrak{u}\mathfrak{v})\rho_m) \qquad \mathrm{mod}\ \mathfrak{p}^m \mathscr{I}.$$

Proof. See Yager[19], Lemma 5.2, only notice the fact that $\chi_{\mathfrak{p}^*}(\sigma_{\mathfrak{u}\mathfrak{v}}) = \chi_{\mathfrak{p}^*}(\sigma_{\mathfrak{u}})$ since $\sigma_{\mathfrak{v}} \in Gal(K(E_g)/K)$. □

5. Two Λ-homomorphisms. Recall that $\delta(T_1) = \Omega_{\mathfrak{p}} T_1 + \ldots$ is a power series in $\mathscr{I}[[T_1]]$, where $\delta(\cdot)$ is the isomorphism between G_m and $\widehat{E}$ which defined in Yager's Theorem. Let $\beta \in \mathbf{U}_\infty$ and put $h_\beta(T_1, T_2) = g_\beta(\delta(T_1), T_2)$, where $g_\beta(T_1, T_2)$ is defined in Theorem 2.1.

LEMMA 5.1. *The $\mathscr{I}$-valued measure on $\mathbb{Z}_p^2$ corresponding to $h_\beta(T_1, T_2)$ for each $\beta \in \mathbf{U}_\infty$ is supported on $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. Moreover let D_i be the operator $(1 + T_i)\partial/\partial T_i$ on $\mathscr{I}[[T_1, T_2]]$ for $i = 1, 2$, and let $k > j \geq 0$ be two integers, we have that*

$$D_1^k D_2^j h_\beta(T_1, T_2)|_{(0,0)} = \Omega_{\mathfrak{p}}^k \delta_{k,j}(\beta). \tag{11}$$

Proof. Firstly, from equation (2) we have

$$(g_{m,\beta}(T))_\omega = \log c_{m,\omega,\beta}(T) - \frac{1}{p} \sum_{\eta \in \widehat{E}_\pi} \log c_{m,\omega,\beta}(T \oplus \eta)$$

for each prime ω of F_m over $\mathfrak{p}$. By compositing with $\delta(T_1)$ we get

$$(g_{m,\beta}(\delta(T_1)))_\omega = \log c_{m,\omega,\beta}(\delta(T_1)) - \frac{1}{p} \sum_{\zeta^p = 1} \log c_{m,\omega,\beta}(\zeta(1 + \delta(T_1)) - 1),$$

which shows that the one-variable p-adic measure corresponding to $(g_{m,\beta}(\delta(T_1)))_\omega$ is supported on $\mathbb{Z}_p^\times$ by Lemma 19 of Yager[18]. Moreover from Theorem 2.2 we have

$$h_\beta(T_1, T_2) \equiv \sum_{\sigma \in Gal(F_m/K)} (g_{m,\beta}^\sigma(\delta(T_1)))_{\mathfrak{p}_m} (1 + T_2)^{\chi_{\mathfrak{p}^*}(\sigma)} \ \mathrm{mod}\ ((1 + T_2)^{p^{m+1}} - 1).$$

Since $\chi_{\mathfrak{p}^*}$ takes values in $\mathbb{Z}_p^\times$, it follows from equation (32) of Yager[18] that the two-variable measure corresponding to $h_\beta(T_1, T_2)$ is supported on $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$.

Secondly, let $\iota(T) \in \mathscr{I}[[T]]$ be the inverse of $\delta(T)$, then by the uniqueness of the exponential map of $\widehat{E}$ we see that $\iota(T) = \exp(\Omega_{\mathfrak{p}}^{-1}\lambda(T)) - 1$, which shows

$$(\Omega_{\mathfrak{p}}\lambda'(T)^{-1}\frac{d}{dT}f(T))|_{T=\delta(T_1)} = (1+T_1)\frac{d}{dT_1}f(T_1).$$

Now equation (11) follows from the definition (3) of $\delta_{k,j}$. □

Now we get a two variable p-adic measure $\mu_\beta \in \mathscr{I}(\mathcal{G})$ which corresponds to $h_\beta(T_1,T_2)$ for each $\beta \in \mathbf{U}_\infty$, since $\mathcal{G}$ is isomorphic to $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. Using this measure we can define a Λ-homomorphism as the following theorem. Here we need some basic results of the two-variable Γ-transform, which could be found in Yager[18].

THEOREM 5.2. *Let i_1 and i_2 be integers modulo $(p-1)$, and let $\beta \in \mathbf{U}_\infty$. Then there is a unique power series $\mathscr{G}_\beta^{(i_1,i_2)}(T_1,T_2) \in \mathscr{I}[[T_1,T_2]]$ such that for all $k > j \geq 0$ satisfying $(k,-j) \equiv (i_1,i_2) \bmod (p-1)$,*

$$\mathscr{G}_\beta^{(i_1,i_2)}(u^k-1,u^{-j}-1) = \Omega_{\mathfrak{p}}^k\delta_{k,j}(\beta). \tag{12}$$

Moreover, if $h \in \Lambda$,

$$\mathscr{G}_{h\beta}^{(i_1,i_2)}(T_1,T_2) = h(T_1,T_2)\mathscr{G}_\beta^{(i_1,i_2)}(T_1,T_2). \tag{13}$$

In particular, the map $\beta \to \mathscr{G}_\beta^{(i_1,i_2)}$ is a Λ-homomorphism from $\mathbf{U}_\infty$ to $\mathscr{I}[[T_1,T_2]]$.

Proof. By Lemma 16 of Yager[18], we know that there exists a power series $h_\beta^{(i_1,-i_2)}(T_1,T_2) \in \mathscr{I}[[T_1,T_2]]$ such that

$$h_\beta^{(i_1,-i_2)}(u^k-1,u^j-1) = \Gamma_{h_\beta}^{(i_1,-i_2)}(k,j),$$

where $\Gamma_{h_\beta}^{(i_1,-i_2)}(s_1,s_2)$ is the $(i_1,-i_2)$-th Γ-transform of h_β or μ_β. Furthermore, we could connect this value with some derivative of the original power series from Lemma 18 of Yager[18]:

$$\Gamma_{h_\beta}^{(i_1,-i_2)}(k,j) = D_1^kD_2^jh_\beta(T_1,T_2)|_{(0,0)}.$$

Set $\mathscr{G}_\beta^{(i_1,i_2)}(T_1,T_2) = h_\beta^{(i_1,-i_2)}(T_1,(1+T_2)^{-1}-1)$, then it's followed from equation (11) of Lemma 5.1 that equation (12) will be satisfied. Such a power series is clearly unique, and so equation (13) follows immediately from equation (5) of Lemma 2.3. □

Wintenberger[17] has studied the structure of $\mathbf{U}_\infty$, and Yager[18] used his result to give another Λ-homomorphism which will be stated as follows:

LEMMA 5.3. *Let i_1 and i_2 be integers* mod $(p-1)$. *Then there is an injection $W^{(i_1,i_2)} : \mathbf{U}_\infty^{(i_1,i_2)} \to \Lambda$ which is a homomorphism of Λ-modules. Moreover, if $(i_1,i_2) \not\equiv (1,1)$ mod $(p-1,(p-1)/r_0)$, $W^{(i_1,i_2)}$ is an isomorphism; and if $(i_1,i_2) \equiv (1,1)$ mod $(p-1,(p-1)/r_0)$, the image of $W^{(i_1,i_2)}$ is the ideal of Λ generated by $1+T_1-u$ and $(1+T_2)^{p^M}-u^{p^M}$.*

Proof. See Lemma 24 of Yager[18]. □

In future, we shall denote the image of $W^{(i_1,i_2)}$ by $\mathcal{H}^{(i_1,i_2)}$. The following theorem proved in Yager[18] establish a connection between the two Λ-homomorphisms $\mathscr{G}^{(i_1,i_2)}$ and $W^{(i_1,i_2)}$.

THEOREM 5.4. *Let i_1 and i_2 be integers modulo $p-1$. Then there is a power series $\Phi^{(i_1,i_2)}(T_1,T_2) \in \mathscr{I}[[T_1,T_2]]$ such that, for all $\beta \in \mathbf{U}_\infty^{(i_1,i_2)}$,*

$$\mathscr{G}_\beta^{(i_1,i_2)}(T_1,T_2) = \Phi^{(i_1,i_2)}(T_1,T_2)W^{(i_1,i_2)}(\beta) \tag{14}$$

Proof. See Theorem 27 of Yager[18]. □

6. p-adic interpolation of special values of L-functions. In this section we shall produce power series giving p-adic interpolations of the numbers $L_\infty(\bar{\psi}^{k+j},k)$, and prove Theorem 0.1. In the process we shall determine the image under $W^{(i_1,i_2)}$ of the Λ-submodule D of $\mathbf{U}_\infty$ generated by $\{< e(\nu) >: \nu \in \mathcal{S}\}$.

Before doing that, we shall make some remarks about the relationship between this submodule D and the group $\bar{C}'_{n,m}$ which is defined in the end of Section 3. We denote $\bar{C}'_{n,m} \bigcap \mathbf{U}_{n,m}$ and $\bar{C}'_\infty \bigcap \mathbf{U}_\infty$ by $\bar{C}_{n,m}$ and $\bar{C}_\infty$ respectively. It is plain that $\bar{C}_\infty$ is a Λ-submodule of $\mathbf{U}_\infty$ containing D. Moreover, the image of D under the projection map from $\mathbf{U}_\infty$ to $\mathbf{U}_{n,m}$ is precisely $\bar{C}_{n,m}$.

We now begin to consider the values of the homomorphisms $\delta_{k,j}$ at $< e(\nu) >$ for any two integers $k > j \geq 0$. Recall that $g_{k,-j}$ is the generator of $\mathfrak{g}_{k,-j}$, where $\mathfrak{g}_{k,-j}$ is the conductor of ψ^{k+j}, and $\delta_{\mathfrak{g}_{k,-j}}$ is the integer defined in Section 4. We will denote $g_{k,-j}$ and $\delta_{\mathfrak{g}_{k,-j}}$ by g and c respectively for simplicity. The following theorem could be seen as a primitive version of Theorem 15 of Yager[18], which was proved using the formulae in Katz[9]. Here we will calculate the values directly using the formulae stated in the previous sections.

THEOREM 6.1. *Let $\nu \in \mathcal{S}$ and let k and j be integers such that $k > j \geq 0$. g and c are defined as above. Then we have*

$$\delta_{k,j}(< e(\nu) >) = (-1)^{k-1}(k-1)!cg^k \sum_{\lambda \in I} \nu(\lambda)(N\lambda - \psi^k((\lambda))\bar{\psi}^{-j}((\lambda)))$$

$$\times(\frac{2\pi}{\sqrt{d_K}})^j \Omega_{\mathfrak{p}}^j \Omega_\infty^{-(k+j)} L_\infty(\bar{\psi}^{k+j},k). \tag{15}$$

Proof. By Lemma 2.3, we have $\delta_{k,j}(< e(\nu) >) = \delta_{k,j}(< e(\nu) >^{(k,-j)})$, and it's easy to see that $< e(\nu) >^{(k,-j)}$ coincides with $< e(\mathfrak{g},\nu) >^{(k,-j)}$ where we simply write $\mathfrak{g} = (g)$ to be the conductor of ψ^{k+j}, so we get $\delta_{k,j}(< e(\nu) >) = \delta_{k,j}(< e(\mathfrak{g},\nu) >)$. Let $\beta =< e(\mathfrak{g},\nu) >$, now we begin to calculate the logarithmic derivative explicitly using

Definition (3), Theorem 2.2 and lemmas in Section 4:

$$\begin{aligned}
\delta_{k,j}(\beta) &= (\lambda'(T_1)^{-1}\tfrac{\partial}{\partial T_1})^k((1+T_2)\tfrac{\partial}{\partial T_2})^j g_\beta(T_1,T_2)|_{(0,0)} \\
&\equiv \sum_{\sigma\in Gal(F_m/K)} (\lambda'(T_1)^{-1}\tfrac{\partial}{\partial T_1})^k((g^\sigma_{m,\beta}(T_1))_{\mathfrak{p}_m})\,|_{T_1=0} \\
&\qquad \times((1+T_2)\tfrac{\partial}{\partial T_2})^j((1+T_2)^{\chi_{\mathfrak{p}^*}(\sigma)})\,|_{T_2=0} \qquad \bmod p^m\mathscr{I} \\
&= \sum_{\mathfrak{u}\in B_m} \chi^j_{\mathfrak{p}^*}(\sigma_{\mathfrak{u}})(1-\pi^{k+j}/p^{j+1})(\tfrac{d}{dz})^k \log\Lambda_{\nu,V^{\sigma_{\mathfrak{u}}}_{\mathfrak{g}m}}(\pi^{*-(m+1)}z)\,|_{z=0} \\
&= (-1)^{k-1}\pi^{*-k(m+1)}(1-\pi^{k+j}/p^{j+1})\sum_{\mathfrak{u}\in B_m}\sum_{\mathfrak{v}\in B_{\mathfrak{g}}}\sum_{\lambda\in I}\nu(\lambda)((N\lambda)\chi^j_{\mathfrak{p}^*}(\sigma_{\mathfrak{u}}) \\
&\qquad \times E_k(\psi(\mathfrak{uv})\rho_m) - \psi^k((\lambda))\chi^j_{\mathfrak{p}^*}(\sigma_{\mathfrak{u}})E_k(\psi(\lambda\mathfrak{uv})\rho_m)) \\
&= (-1)^{k-1}\pi^{*-k(m+1)}\sum_{\mathfrak{u}\in B_m}\sum_{\mathfrak{v}\in B_{\mathfrak{g}}}\sum_{\lambda\in I}\nu(\lambda)((N\lambda)-\psi^k((\lambda))\bar{\psi}^{-j}((\lambda))) \\
&\qquad \times\chi^j_{\mathfrak{p}^*}(\sigma_{\mathfrak{u}})(1-\pi^{k+j}/p^{j+1})E_k(\psi(\mathfrak{uv})\rho_m) \\
&\equiv (-1)^{k-1}g^k\Omega^j_{\mathfrak{p}}\sum_{\mathfrak{u}\in B_m}\sum_{\mathfrak{v}\in B_{\mathfrak{g}}}\sum_{\lambda\in I}\nu(\lambda)((N\lambda)-\psi^k((\lambda))\bar{\psi}^{-j}((\lambda))) \\
&\qquad \times(N\mathfrak{g}^j_m/g^{k+j}_m)(1-\pi^{k+j}/p^{j+1})E_{j,k}(\psi(\mathfrak{uv})\rho_m) \qquad \bmod p^m\mathscr{I} \\
&= (-1)^{k-1}(k-1)!cg^k\sum_{\lambda\in I}\nu(\lambda)(N\lambda-\psi^k((\lambda))\bar{\psi}^{-j}((\lambda))) \\
&\qquad \times(\tfrac{2\pi}{\sqrt{d_K}})^j\Omega^j_{\mathfrak{p}}\Omega_\infty^{-(k+j)}L_\infty(\bar{\psi}^{k+j},k)
\end{aligned}$$

Since m can be arbitrary, the proof is complete. ∎

In order to get rid of some extra terms, we give the following lemma proved in Yager[18]. If $\nu\in\mathcal{S}$ and i_1 and i_2 are integers modulo $p-1$, we define

$$h^{(i_1,i_2)}_\nu(T_1,T_2) = \sum_{\lambda\in I}\nu(\lambda)(N\lambda-\omega^{i_1}(\psi((\lambda)))\omega^{i_2}(\bar{\psi}((\lambda)))$$

$$\times(1+T_1)^{l(\psi((\lambda)))}(1+T_2)^{l(\bar{\psi}((\lambda)))}),$$

where ω is the Teichmüller character on $\mathbb{Z}_p$, so we can write $x=\omega(x)<x>$ for $x\in\mathbb{Z}_p$, and l is the homomorphism from $\mathbb{Z}_p^\times$ to $\mathbb{Z}_p$ such that $<x>=u^{l(x)}$ for all $x\in\mathbb{Z}_p^\times$. Observe that for all $(k,-j)\equiv(i_1,i_2)\bmod (p-1)$,

$$h^{(i_1,i_2)}_\nu(u^k-1,u^{-j}-1) = \sum_{\lambda\in I}\nu(\lambda)(N\lambda-\psi^k((\lambda))\bar{\psi}^{-j}((\lambda))).$$

Lemma 6.2. *Let $H^{(i_1,i_2)}$ be the Λ-module generated by $\{h^{(i_1,i_2)}_\nu(T_1,T_2):\nu\in\mathcal{S}\}$. Then $H^{(i_1,i_2)}=\Lambda$ unless $(i_1,i_2)\equiv(0,0)$ or $(1,1)\bmod (p-1)$, $H^{(0,0)}$ is the Λ-module generated by T_1 and T_2 and $H^{(1,1)}$ is the module generated by T_1+1-u and T_2+1-u.*

Proof. See Lemma 28 of Yager[18]. □

THEOREM 6.3. *Let i_1 and i_2 be integers modulo $p-1$. Then there is a power series $\mathscr{G}^{(i_1,i_2)}(T_1,T_2) \in \mathscr{I}[[T_1,T_2]]$ such that for all $k > j \geq 0$ satisfying $(k,-j) \equiv (i_1,i_2) \bmod (p-1)$,*

$$\mathscr{G}^{(i_1,i_2)}(u^k-1,u^{-j}-1) = (k-1)!\Omega_{\mathfrak{p}}^{k+j}\Omega_{\infty}^{-(k+j)}(\frac{2\pi}{\sqrt{d_K}})^j L_\infty(\bar{\psi}^{k+j},k). \tag{16}$$

Moreover

$$W^{(i_1,i_2)}(D^{(i_1,i_2)}) = \Phi^{(i_1,i_2)}(T_1,T_2)^{-1}\mathscr{G}^{(i_1,i_2)}(T_1,T_2)H^{(i_1,i_2)}. \tag{17}$$

Proof. The proof is similar to Theorem 29 of Yager[18], only notice the complex L-function in Theorem 6.1 is primitive. Equations (12) and (15) together show that if $\nu \in \mathcal{S}$, the value of $\mathscr{G}^{(i_1,i_2)}_{<e(\nu)>}(T_1,T_2)$ at $(u^k-1,u^{-j}-1)$ is

$$(-1)^{k-1}(k-1)!cg^k\sum_{\lambda\in I}\nu(\lambda)(N\lambda-\psi^k((\lambda))\bar{\psi}^{-j}((\lambda)))(\frac{2\pi}{\sqrt{d_K}})^j\Omega_{\mathfrak{p}}^{k+j}\Omega_{\infty}^{-(k+j)}L_\infty(\bar{\psi}^{k+j},k).$$

Observe that $(-1)^{i_1}c\omega^{i_1}(g)(1+T_1)^{l(g)}$ is a unit power series in Λ whose value at $(u^k-1,u^{-j}-1)$ is $(-1)^{k-1}cg^k$ whenever $(k,-j)\equiv(i_1,i_2) \bmod (p-1)$, here c and g have the same meaning with in Theorem 6.1. It follows by linearity of equation (13) in Theorem 5.2 that for each element $h \in H^{(i_1,i_2)}$ there is a corresponding element e_h of D such that:

$$\mathscr{G}^{(i_1,i_2)}_{e_h}(u^k-1,u^{-j}-1) = h(u^k-1,u^{-j}-1)(k-1)!(\frac{2\pi}{\sqrt{d_K}})^j\Omega_{\mathfrak{p}}^{k+j}\Omega_{\infty}^{-(k+j)}L_\infty(\bar{\psi}^{k+j},k). \tag{18}$$

And conversely for each e in D, there is an $h \in H^{(i_1,i_2)}$ such that (18) holds.

The theorem is now clear from Lemma 6.2 and Theorem 5.4 unless $(i_1,i_2)\equiv(0,0)$ or $(1,1) \bmod (p-1)$.

Suppose $(i_1,i_2)\equiv(0,0) \bmod (p-1)$, and let e_0 be the element of D corresponding to the power series T_2 in $H^{(0,0)}$ as in equation (18). Observe that $\mathscr{G}^{(0,0)}_{e_0}(u^k-1,0)=0$ for all $k\geq 1$ such that $k\equiv 0 \bmod (p-1)$, and so $\mathscr{G}^{(0,0)}_{e_0}(T_1,T_2)=T_2\mathscr{G}^{(0,0)}(T_1,T_2)$ for some power series $\mathscr{G}^{(0,0)}(T_1,T_2)\in\mathscr{I}[[T_1,T_2]]$. It is clear from equation (18) that $\mathscr{G}^{(0,0)}(T_1,T_2)$ has the desired properties.

Suppose $(i_1,i_2)\equiv(1,1) \bmod (p-1)$, and let e_1 be the element of D corresponding to the power series T_1+1-u in $H^{(1,1)}$. Observe that $\mathscr{G}^{(1,1)}_{e_1}(u-1,u^{-j}-1)=0$ for all $j\geq 0$ such that $j\equiv -1 \bmod (p-1)$(note that since we could use Eisenstein series to instead the L-values, the restriction that $k>j$ would not cause any problem), and so $\mathscr{G}^{(1,1)}_{e_1}(T_1,T_2)=(T_1+1-u)\mathscr{G}^{(1,1)}(T_1,T_2)$ for some power series $\mathscr{G}^{(1,1)}(T_1,T_2)\in\mathscr{I}[[T_1,T_2]]$. It is clear from equation (18) that $\mathscr{G}^{(0,0)}(T_1,T_2)$ has the desired properties. This completes the proof. □

Proof of Theorem 0.1. Firstly by Lemma A.1.1 of Coates-Sujatha[3], we know there exists a unique element $\mu \in \Lambda(\mathcal{G})$ such that

$$\mu^{(i_1,i_2)} = \mathscr{G}^{(i_1,i_2)}(T_1,T_2)$$

for i_1 and i_2 modulo $p-1$. Secondly, by Lemma 3.6.2 of Coates-Sujatha[3] we get:

$$\int_{\mathcal{G}}\chi_{\mathfrak{p}}^k\chi_{\mathfrak{p}^*}^{-j}d\mu = \mathscr{G}^{(i_1,i_2)}(u^k-1,u^{-j}-1),$$

for $(k,-j)\equiv(i_1,i_2) \bmod (p-1)$, which complete the proof of Theorem 0.1. □

7. The structure of Y_∞.

THEOREM 7.1. *Let i_1 and i_2 be integers modulo $p-1$. Then there is an element $G^{(i_1,i_2)}(T_1,T_2)$ of Λ which generates the same ideal in $\mathscr{I}[[T_1,T_2]]$ as $\mathscr{G}^{(i_1,i_2)}(T_1,T_2)$. Moreover, $Y_\infty^{(i_1,i_2)}$ is isomorphic to $\mathcal{H}^{(i_1,i_2)}/G^{(i_1,i_2)}(T_1,T_2)H^{(i_1,i_2)}$.*

Proof. The proof is similar to Theorem 30 of Yager[18], only notice that the power series $\mathscr{G}^{(i_1,i_2)}(T_1,T_2)$ here interpolates the primitive complex L-functions. We recall that in Section 5 we defined $\mathcal{H}^{(i_1,i_2)}$ to be the image of $W^{(i_1,i_2)}$ and that this is Λ unless $(i_1,i_2)\equiv(1,1) \bmod (p-1,(p-1)/r_0)$, in which case $\mathcal{H}^{(i_1,i_2)}$ is generated by T_1+1-u and $(T_2+1)^{p^M}-u^{p^M}$.

The projection map $p_{n,m}:\mathbf{U}_\infty^{(i_1,i_2)}\to\mathbf{U}_{n,m}^{(i_1,i_2)}$ has as its image those elements of $\mathbf{U}_{n,m}^{(i_1,i_2)}$ for which the local norm to $K_{\mathfrak{p}}$ of each component is 1. It is clear that $\bigcap_{n,m\geq 0}\ker p_{n,m}=1$. As we have already observed

$$p_{n,m}(D^{(i_1,i_2)})=\bar{C}_{n,m}^{(i_1,i_2)}. \tag{19}$$

Let $j_{n,m}$ be the composition of $p_{n,m}$ with the canonical surjection of $\mathbf{U}_{n,m}^{(i_1,i_2)}$ onto $\mathbf{U}_{n,m}^{(i_1,i_2)}/\bar{C}_{n,m}^{(i_1,i_2)}$. The image of $j_{n,m}$ is precisely the image of $Y_\infty^{(i_1,i_2)}$ under the projection onto $\mathbf{U}_{n,m}^{(i_1,i_2)}/\bar{C}_{n,m}^{(i_1,i_2)}$. In view of equation (19), it is plain that the kernel of $j_{n,m}$ is $D^{(i_1,i_2)}\ker p_{n,m}$, and that $j_{n,m}$ is a Λ-homomorphism. Thus

$$Y_\infty^{(i_1,i_2)}\cong\varprojlim \mathbf{U}_\infty^{(i_1,i_2)}/D^{(i_1,i_2)}\ker p_{n,m}.$$

But $\bigcap_{n,m\geq 0}\ker p_{n,m}=1$ and so it follows that $Y_\infty^{(i_1,i_2)}\cong\mathbf{U}_\infty^{(i_1,i_2)}/D^{(i_1,i_2)}$. The theorem is now clear from Theorems 5.4 and 6.3. □

Proof of Theorem 0.2. We see from the the above theorem that we have the following exact sequence of Λ-modules:

$$0\to A\to\frac{\mathcal{H}^{(i_1,i_2)}}{G^{(i_1,i_2)}(T_1,T_2)H^{(i_1,i_2)}}\to\frac{\Lambda}{G^{(i_1,i_2)}(T_1,T_2)}\to\frac{\Lambda}{\mathcal{H}^{(i_1,i_2)}+G^{(i_1,i_2)}(T_1,T_2)\Lambda}\to 0,$$

where

$$A=\frac{\mathcal{H}^{(i_1,i_2)}\bigcap G^{(i_1,i_2)}(T_1,T_2)\Lambda}{G^{(i_1,i_2)}(T_1,T_2)H^{(i_1,i_2)}}.$$

Clearly A injects into $\Lambda/H^{(i_1,i_2)}$, and $H^{(i_1,i_2)}$ and $\mathcal{H}^{(i_1,i_2)}+G^{(i_1,i_2)}(T_1,T_2)\Lambda$ are clearly contained in no proper principal ideal of Λ, and so $Y_\infty^{(i_1,i_2)}$ is pseudo-isomorphic to $\Lambda/G^{(i_1,i_2)}(T_1,T_2)\Lambda$, which means $G^{(i_1,i_2)}(T_1,T_2)$ is a characteristic power series of $Y_\infty^{(i_1,i_2)}$. Theorem 7.1 also tells us that $G^{(i_1,i_2)}(T_1,T_2)$ generates the same ideal with $\mu^{(i_1,i_2)}=\mathscr{G}^{(i_1,i_2)}(T_1,T_2)$. This proves Theorem 0.2. □

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