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EXTEND MEAN CURVATURE FLOW WITH FINITE INTEGRAL CURVATURE*

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Abstract. In this note, we prove that the solution of certain mean curvature flow on a finite time interval $[0, T)$ can be extended over time T if the space-time integration of the mean curvature is finite. Moreover, we show that the condition is optimal in some sense.

Key words. Mean curvature flow, maximal existence time, second fundamental form, integral curvature.

AMS subject classifications. 53C44; 53C21.

1. Introduction. Let M^n be a complete n-dimensional manifold without boundary, and let $F_t : M^n \to \mathbb{R}^{n+1}$ be a one-parameter family of smooth hypersurfaces immersed in Euclidean space. We say that $M_t = F_t(M)$ is a solution of the mean curvature flow if F_t satisfies

$$\begin{cases} \frac{\partial}{\partial t}F(x,t) &= -H(x,t)\nu(x,t) \\ F(x,0) &= F_0(x), \end{cases}$$

where $F(x,t) = F_t(x)$, $H(x,t)$ is the mean curvature, $\nu(x,t)$ is the unit outward normal vector, and F_0 is some given initial hypersurface.

K. Brakke [1] studied the mean curvature flow from the view point of geometric measure theory firstly. For the classical solution of the mean curvature flow, G. Huisken (see [5, 6]) showed that for a smooth complete initial hypersurface with bounded second fundamental form the solution exists on a maximal time interval $[0, T)$, $0 < T \leq \infty$. If the initial hypersurface is closed and convex, he showed in [6] that the mean curvature flow will converge to a round point in finite time. He also proved that if the second fundamental form is uniformly bounded, then the mean curvature flow can be extended.

By a blow up argument, N. Šešum [9] proved that if the Ricci curvature is uniformly bounded on $M \times [0, T)$, then the Ricci flow can be extended over T. In [10], B. Wang obtained some integral conditions to extend the Ricci flow. A natural question is that, what is the optimal condition for the mean curvature flow to be extended? By a different method, we investigate the integral conditions to extend the mean curvature flow. We will prove that the mean curvature flow can be extended if the space-time integration of the mean curvature is finite and the second fundamental tensor is bounded from below.

THEOREM 1.1. *Let $F_t : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ be a solution of the mean curvature flow of closed hypersurfaces on a finite time interval $[0, T)$. If*
(1) *there is a positive constant C such that $h_{ij} \geq -C$ for $(x,t) \in M \times [0, T)$,*
(2) $||H||_{\alpha, M\times[0,T)} = \left(\int_0^T \int_{M_t} |H|^\alpha d\mu_t dt\right)^{\frac{1}{\alpha}} < +\infty$ *for some $\alpha \geq n+2$,*

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then this flow can be extended over time T.

When the initial hypersurface is mean convex, we have the following

THEOREM 1.2. *Let $F_t : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ be a solution of the mean curvature flow of closed hypersurfaces on a finite time interval $[0,T)$. If*
(1) $H > 0$ *at* $t = 0$,
(2) $||H||_{\alpha, M\times[0,T)} = \left(\int_0^T \int_{M_t} |H|^\alpha d\mu_t dt\right)^{\frac{1}{\alpha}} < +\infty$ *for some* $\alpha \geq n+2$,
then this flow can be extended over time T.

The following example shows that the condition $\alpha \geq n+2$ in Theorems 1.1 and 1.2 are optimal.

EXAMPLE. Set $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i^2 = 1\}$. Let F be the standard isometric embedding of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$. It is clear that $F(t) = \sqrt{1-2nt}F$ is the solution to the mean curvature flow, where $T = \frac{1}{2n}$ is the maximal existence time. By a simple computation, we have $g_{ij}(t) = (1-2nt)g_{ij}$, $H(t) = \frac{n}{\sqrt{1-2nt}}$ and $h_{ij}(t) \geq 0$. Hence

$$\begin{aligned} ||H||_{\alpha, M\times[0,T)} &= \left(\int_0^T \int_{M_t} |H|^\alpha d\mu_t dt\right)^{\frac{1}{\alpha}} \\ &= C_1 \left(\int_0^T (T-t)^{\frac{n-\alpha}{2}} dt\right)^{\frac{1}{\alpha}}, \end{aligned}$$

where C_1 is a positive constant. It follows that

$$||H||_{\alpha, M\times[0,T)} \begin{cases} = \infty, & for \quad \alpha \geq n+2, \\ < \infty, & for \quad \alpha < n+2. \end{cases}$$

This implies that the condition $\alpha \geq n+2$ in Theorems 1.1 and 1.2 are optimal.

2. An upper bound of the mean curvature by its L^{n+2}-norm. Let $F : M^n \to \mathbb{R}^{n+1}$ be a compact immersed hypersurface. Denote by $g = \{g_{ij}\}$ the induced metric, $A = \{h_{ij}\}$ the second fundamental form, ∇ the induced Levi-Civita connection and $\triangle$ the induced Laplacian. The volume form on M is $d\mu = \sqrt{\det(g_{ij})}dx$, and the mean curvature H is the trace of the second fundamental form.

In this section we obtain an inequality relating the mean curvature and its L^{n+2}-norm in the space-time. We first recall some evolution equations (see [3, 14]).

LEMMA 2.1. *Along the mean curvature flow we have the following evolution equations.*

$$\begin{aligned} &\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}, \\ &\frac{\partial}{\partial t} d\mu_t = -H^2 d\mu_t, \\ &\frac{\partial}{\partial t} H = \triangle H + |A|^2 H, \\ &\frac{\partial}{\partial t} |A|^2 = \triangle |A|^2 - 2|\nabla A|^2 + 2|A|^4. \end{aligned}$$

The following Sobolev inequality can be found in [8] and [12].

LEMMA 2.2. *Let M^n be an n-dimensional $(n \geq 3)$ closed submanifold of a Riemannian manifold N^{n+p} with codimension $p \geq 1$. Suppose that the sectional curvature of N^{n+p} is non-positive. Then for any $s \in (0, +\infty)$ and $f \in C^1(M)$ such that $f \geq 0$,*

$$\int_M |\nabla f|^2 d\mu \geq \frac{(n-2)^2}{4(n-1)(1+s)} \left[\frac{1}{C^2(n)} \left(\int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} - H_0^2 \left(1 + \frac{1}{s}\right) \int_M f^2 d\mu \right],$$

where $H_0 = \max_{x \in M} |H|$, $C(n) = \frac{2^n (1+n)^{(1+\frac{1}{n})}}{(n-1)\sigma_n}$, and σ_n is the volume of the unit ball in $\mathbb{R}^{n+1}$.

The following estimate is very useful in the proofs of our theorems.

THEOREM 2.3. *Suppose that $F_t : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ is a mean curvature flow solution for $t \in [0, T_0]$, and the second fundamental form is uniformly bounded on the time interval $[0, T_0]$. Then*

$$\max_{(x,t) \in M \times [\frac{T_0}{2}, T_0]} H^2(x,t) \leq C_2 \left(\int_0^{T_0} \int_{M_t} |H|^{n+2} d\mu_t dt \right)^{\frac{2}{n+2}},$$

where C_2 is a constant depending on n, T_0 and $\sup_{(x,t) \in M \times [0,T_0]} |A|$.

Proof. The evolution equation of H^2 is

$$\frac{\partial}{\partial t} H^2 = \triangle H^2 - 2|\nabla H|^2 + 2|A|^2 H^2. \tag{1}$$

Since A is bounded, we obtain the following estimate from (1) that

$$\frac{\partial}{\partial t} H^2 \leq \triangle H^2 + \beta H^2, \tag{2}$$

where β is a constant depending only on $\sup_{(x,t) \in M \times [0,T_0]} |A|$.

Denoting $f = H^2$, from the inequality in (2) and the evolution equation of the volume form in Lemma 2.1, we obtain that for any $p \geq 2$,

$$\begin{aligned} \frac{\partial}{\partial t} \int_{M_t} f^p d\mu_t &= \int_{M_t} p f^{p-1} \frac{\partial}{\partial t} f d\mu_t - \int_{M_t} f^{p+1} d\mu_t \\ &\leq \int_{M_t} p f^{p-1} \left(\triangle f + \beta f \right) d\mu_t \\ &= -\frac{4(p-1)}{p} \int_{M_t} |\nabla f^{\frac{p}{2}}|^2 d\mu_t + \beta p \int_{M_t} f^p d\mu_t. \end{aligned}$$

Thus

$$\frac{\partial}{\partial t} \int_{M_t} f^p d\mu_t + \frac{4(p-1)}{p} \int_{M_t} |\nabla f^{\frac{p}{2}}|^2 d\mu_t \leq \beta p \int_{M_t} f^p d\mu_t. \tag{3}$$

For any $0 < \tau < \tau' < T_0$, define a function ψ on $[0, T_0]$:

$$\psi(t) = \begin{cases} 0 & 0 \le t \le \tau, \\ \frac{t-\tau}{\tau'-\tau} & \tau \le t \le \tau', \\ 1 & \tau' \le t \le T_0. \end{cases}$$

Then by (3) we have

$$\begin{aligned} \frac{\partial}{\partial t}\left(\psi \int_{M_t} f^p d\mu_t\right) &= \psi' \int_{M_t} f^p d\mu_t + \psi \frac{\partial}{\partial t}\left(\int_{M_t} f^p d\mu_t\right) \\ &\le \psi' \int_{M_t} f^p d\mu_t + \psi\bigg(-\frac{4(p-1)}{p} \int_{M_t} |\nabla f^{\frac{p}{2}}|^2 d\mu_t \\ &\quad + \beta p \int_{M_t} f^p d\mu_t \bigg). \end{aligned} \tag{4}$$

For any $t \in [\tau', T_0]$, integrating both sides of the inequality in (4) on $[\tau, t]$ we get

$$\int_{M_t} f^p d\mu_t + \frac{4(p-1)}{p} \int_{\tau'}^t \int_{M_t} |\nabla f^{\frac{p}{2}}|^2 d\mu_t dt \le \left(\beta p + \frac{1}{\tau' - \tau}\right) \int_{\tau}^{T_0} \int_{M_t} f^p d\mu_t dt. \tag{5}$$

For the integral $\int_{\tau'}^{T_0} \int_{M_t} f^{p(1+\frac{2}{n})} d\mu_t dt$, by Schwarz inequality and Sobolev inequality in Lemma 2.2, we have

$$\begin{aligned} \int_{\tau'}^{T_0} \int_{M_t} f^{p(1+\frac{2}{n})} d\mu_t dt &\le \int_{\tau'}^{T_0} \left(\int_{M_t} f^p d\mu_t\right)^{\frac{2}{n}} \left(\int_{M_t} f^{\frac{np}{n-2}} d\mu_t\right)^{\frac{n-2}{n}} dt \\ &\le \max_{t \in [\tau', T_0]} \left(\int_{M_t} f^p d\mu_t\right)^{\frac{2}{n}} \int_{\tau'}^{T_0} \left(\int_{M_t} f^{\frac{np}{n-2}} d\mu_t\right)^{\frac{n-2}{n}} dt \\ &\le \left(\beta p + \frac{1}{\tau' - \tau}\right)^{\frac{2}{n}} \left(\int_{\tau}^{T_0} \int_{M_t} f^p d\mu_t dt\right)^{\frac{2}{n}} \\ &\quad \times \int_{\tau'}^{T_0} \bigg[\frac{4(n-1)C^2(n)(1+s)}{(n-2)^2} \int_{M_t} |\nabla f^{\frac{p}{2}}|^2 d\mu_t \\ &\quad + \frac{n}{2}\beta C^2(n)\left(1 + \frac{1}{s}\right) \int_{M_t} f^p d\mu_t\bigg] dt. \end{aligned}$$

For the third factor on the right hand side, we have from (5)

$$\begin{aligned}
&\int_{\tau'}^{T_0}\left[\frac{4(n-1)C^2(n)(1+s)}{(n-2)^2}\int_{M_t}|\nabla f^{\frac{p}{2}}|^2 d\mu_t + \frac{n}{2}\beta C^2(n)\left(1+\frac{1}{s}\right)\int_{M_t} f^p d\mu_t\right]dt\\
&\le \frac{4(n-1)C^2(n)(1+s)}{(n-2)^2}\int_{\tau'}^{T_0}\int_{M_t}|\nabla f^{\frac{p}{2}}|^2 d\mu_t dt\\
&\quad+\frac{n}{2}\beta C^2(n)\left(1+\frac{1}{s}\right)\int_{\tau'}^{T_0}\left[\left(\beta p+\frac{1}{\tau'-\tau}\right)\int_{\tau}^{T_0}\int_{M_t} f^p d\mu_t dt\right]dt\\
&\le \frac{(n-1)C^2(n)p(1+s)}{(n-2)^2(p-1)}\left(\beta p+\frac{1}{\tau'-\tau}\right)\int_{\tau}^{T_0}\int_{M_t} f^p d\mu_t dt\\
&\quad+\frac{n}{2}\beta C^2(n)T_0\left(1+\frac{1}{s}\right)\left(\beta p+\frac{1}{\tau'-\tau}\right)\int_{\tau}^{T_0}\int_{M_t} f^p d\mu_t dt\\
&=\left[\frac{(n-1)C^2(n)p(1+s)}{(n-2)^2(p-1)}+\frac{n}{2}\beta C^2(n)T_0\left(1+\frac{1}{s}\right)\right]\left(\beta p+\frac{1}{\tau'-\tau}\right)\int_{\tau}^{T_0}\int_{M_t} f^p d\mu_t dt.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\tau'}^{T_0}\int_{M_t} f^{p\left(1+\frac{2}{n}\right)} d\mu_t dt\\
&\le \left[\frac{(n-1)C^2(n)p(1+s)}{(n-2)^2(p-1)}+\frac{n}{2}\beta C^2(n)T_0\left(1+\frac{1}{s}\right)\right]\\
&\quad\times\left(\beta p+\frac{1}{\tau'-\tau}\right)^{1+\frac{2}{n}}\left(\int_{\tau}^{T_0}\int_{M_t} f^p d\mu_t dt\right)^{1+\frac{2}{n}}.
\end{aligned}\tag{6}$$

Put $L(p,t)=\int_t^{T_0}\int_{M_t} f^p d\mu_t dt$, $s=\frac{\left[\frac{2}{n}(p-1)T_0\beta\right]^{\frac{1}{2}}(n-2)}{[n(p-1)]^{\frac{1}{2}}}$, and $D_{n,q}=\frac{[(n-1)p]^{\frac{1}{2}}C(n)}{(n-2)(p-1)^{\frac{1}{2}}}$. The inequality in (6) can be rewritten as

$$L\left(p\left(1+\frac{2}{n}\right),\tau'\right)\le D_{n,q}^2\left(\beta p+\frac{1}{\tau'-\tau}\right)^{1+\frac{2}{n}} L(p,\tau)^{1+\frac{2}{n}}. \tag{7}$$

Now let $\mu=1+\frac{2}{n}$, $p_k=\frac{n+2}{2}\mu^k$ and $\tau_k=\left(1-\frac{1}{\mu^{k+1}}\right)t$. Then from (7) we obtain

$$L(p_{k+1},\tau_{k+1})^{\frac{1}{p_{k+1}}}\le D_{n,\frac{n+2}{2}}^{\Sigma_{i=0}^k\frac{2}{p_{i+1}}}\left(\frac{(n+2)\beta}{2}+\frac{n+2}{2t}\right)^{\Sigma_{i=0}^k\frac{1}{p_i}}\mu^{\Sigma_{i=0}^k\frac{i}{p_i}}L(p_0,\tau_0)^{\frac{2}{n+2}}. \tag{8}$$

As $k\to+\infty$, we conclude from (8) that

$$f(x,t)\le D_{n,\frac{n+2}{2}}^{\frac{2n}{n+2}}\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\left(\frac{n+2}{2}\beta+\frac{n+2}{2t}\right)\left(\int_0^{T_0}\int_{M_t} f^{\frac{n+2}{2}} d\mu_t dt\right)^{\frac{2}{n+2}}. \tag{9}$$

Therefore, for any $(x,t)\in M\times[\frac{T_0}{2},T_0]$, we get from (9)

$$H^2(x,t)\le C_2\left(\int_0^{T_0}\int_{M_t}|H|^{n+2}d\mu_t dt\right)^{\frac{2}{n+2}}, \tag{10}$$

where C_2 is a constant depending on n, T_0 and $\sup_{(x,t)\in M\times[0,T_0]}|A|$. Since $(x,t)\in M\times[\frac{T_0}{2},T_0]$ is arbitrary, it follows from (10) that

$$\max_{(x,t)\in M\times[\frac{T_0}{2},T_0]} H^2(x,t) \le C_2\left(\int_0^{T_0}\int_{M_t}|H|^{n+2}d\mu_t dt\right)^{\frac{2}{n+2}},$$

which is desired.

3. Mean curvature flow with finite total mean curvature. We are now in a position to prove our theorems.

Proof of Theorem 1.1. We only need to prove the theorem for $\alpha = n+2$ since by Hölder's inequality, $||H||_{\alpha,M\times[0,T)}<\infty$ for $\alpha>n+2$ implies $||H||_{n+2,M\times[0,T)}<\infty$. We argue by contradiction.

Suppose that the solution of the mean curvature flow cannot be extended over T. Then A becomes unbounded as $t\to T$. Since $h_{ij}\ge -C$, we get $\sum_{i,j}(h_{ij}+C)^2\le C_3[tr(h_{ij}+C)]^2$, where C_3 is a constant depending only on n. On one hand, $|A|^2$ is unbounded implies that $\sum_{i,j}(h_{ij}+C)^2$ is unbounded. On the other hand,

$$[tr(h_{ij}+C)]^2 = (H+nC)^2 = H^2+2nCH+n^2C^2.$$

Thus H^2 is unbounded. Namely,

$$\sup_{(x,t)\in M\times[0,T)} H^2(x,t)=\infty.$$

Choose an increasing time sequence $t^{(i)}$, $i=1,2,\cdots$, such that $\lim_{i\to\infty}t^{(i)}=T$. We take a sequence of points $x^{(i)}\in M$ satisfying

$$H^2(x^{(i)},t^{(i)}) = \max_{(x,t)\in M\times[0,t^{(i)})} H^2(x,t).$$

Then $\lim_{i\to\infty}H^2(x^{(i)},t^{(i)})=\infty$.

Putting $Q^{(i)}=H^2(x^{(i)},t^{(i)})$, we have $\lim_{i\to\infty}Q^{(i)}=\infty$. This together with $\lim_{i\to\infty}t^{(i)}=T>0$ implies that there exists a positive integer i_0 such that $Q^{(i)}t^{(i)}\ge 1$ and $Q^{(i)}\ge 1$ for $i\ge i_0$. For $i\ge i_0$ and $t\in[0,1]$, we define $F^{(i)}(t)=\left(Q^{(i)}\right)^{\frac{1}{2}}F\left(\frac{t-1}{Q^{(i)}}+t^{(i)}\right)$. Then the metric on M induced by $F^{(i)}(t)$ is $g^{(i)}(t)=Q^{(i)}g\left(\frac{t-1}{Q^{(i)}}+t^{(i)}\right)$, and $F^{(i)}(t):M^n\to\mathbb{R}^{n+1}$ is still a solution of the mean curvature flow on $t\in[0,1]$. Since F_t satisfies $h_{ij}\ge -C$ for $(x,t)\in M\times[0,T)$, we have

$$H^2_{(i)}(x,t)\le 1 \quad on \quad M\times[0,1],$$

$$h^{(i)}_{jk}\ge -\frac{C}{\sqrt{Q^{(i)}}} \quad on \quad M\times[0,1], \tag{11}$$

where $H_{(i)}$ and $A^{(i)}=h^{(i)}_{jk}$ are the mean curvature and the second fundamental form of $F^{(i)}(t)$, respectively. The inequality in (11) gives that $h^{(i)}_{jk}+\frac{C}{\sqrt{Q^{(i)}}}\ge 0$. Hence

$$h^{(i)}_{jk}+\frac{C}{\sqrt{Q^{(i)}}}\le tr\left(h^{(i)}_{jk}+\frac{C}{\sqrt{Q^{(i)}}}\right)\le H_{(i)}+\frac{nC}{\sqrt{Q^{(i)}}}. \tag{12}$$

The inequality in (12) implies that $h_{jk}^{(i)} \le H_{(i)} + \frac{(n-1)C}{\sqrt{Q^{(i)}}}$. Since $Q^{(i)} \ge 1$ when $i \ge i_0$, it follows that $|A^{(i)}| \le C_4$ for $i \ge i_0$ and $t \in [0,1]$, where C_4 is a positive constant independent of i.

Set $(M^{(i)}, g^{(i)}(t), x^{(i)}) = \left(M, Q^{(i)} g\left(\frac{t-1}{Q^{(i)}} + t^{(i)}\right), x^{(i)}\right)$, $t \in [0,1]$. From [2] we know that there is a subsequence of $(M^{(i)}, g^{(i)}(t), x^{(i)})$ converges to a Riemannian manifold $(\widetilde{M}, \widetilde{g}(t), \widetilde{x})$, and the corresponding subsequence of immersions $F^{(i)}(t)$ converges to an immersion $\widetilde{F}(t) : \widetilde{M} \to \mathbb{R}^{n+1}$. Also, it follows from Theorem 2.3 that for $i \ge i_0$,

$$\max_{(x,t)\in M^{(i)}\times[\frac{1}{2},1]} H_{(i)}^2(x,t) \le C_5 \left(\int_0^1 \int_{M_t} |H|_{(i)}^{n+2} d\mu_{g^{(i)}(t)} dt \right)^{\frac{2}{n+2}},$$

where C_5 is a constant independent of i. Hence

$$\begin{aligned}
\max_{(x,t)\in \widetilde{M}\times[\frac{1}{2},1]} \widetilde{H}^2(x,t) &\le \lim_{i\to\infty} C_5 \left(\int_0^1 \int_{M_t} |H|_{(i)}^{n+2} d\mu_{g^{(i)}(t)} dt \right)^{\frac{2}{n+2}} \\
&\le \lim_{i\to\infty} C_5 \left(\int_{t^{(i)}}^{t^{(i)}+(Q^{(i)})^{-1}} \int_{M_t} |H|_{(i)}^{n+2} d\mu_t dt \right)^{\frac{2}{n+2}} \\
&= 0. \qquad (13)
\end{aligned}$$

The equality in (13) holds because $\int_0^T \int_{M_t} H^{n+2} d\mu dt < +\infty$ and $\lim_{i\to\infty} (Q^{(i)})^{-1} = 0$. However, according to the choice of the points, we have

$$\widetilde{H}^2(\widetilde{x}, 1) = \lim_{i\to\infty} H_{(i)}^2(x^{(i)}, 1) = 1.$$

This is a contradiction. We complete the proof of Theorem 1.1.

With a similar method, we can prove Theorem 1.2.

Proof of Theorem 1.2. Since $H > 0$ at $t = 0$, there exists a positive constant C_6 such that $|A|^2 \le C_6 H^2$. The evolution equation of H in Lemma 2.1 implies that $H > 0$ is preserved along the mean curvature flow. By [7] we have the following evolution equation of $\frac{|A|^2}{H^2}$

$$\frac{\partial}{\partial t}\left(\frac{|A|^2}{H^2}\right) = \triangle\left(\frac{|A|^2}{H^2}\right) + \frac{2}{H}\left\langle \nabla H, \nabla\left(\frac{|A|^2}{H^2}\right)\right\rangle - \frac{2}{H^4}|H\nabla_i h_{jk} - \nabla_i H \cdot h_{jk}|^2. \qquad (14)$$

Using the maximum principle, we obtain from (14) that $|A|^2 \le C_6 H^2$ is preserved along the mean curvature flow.

It is sufficient to prove the theorem for $\alpha = n+2$. We still argue by contradiction. Suppose that the solution of the mean curvature flow cannot be extended over time T. Then $|A|^2$ is unbounded as $t \to T$. This implies that H^2 is also unbounded since $|A|^2 \le C_6 H^2$. Let $(x^{(i)}, t^{(i)})$, $Q^{(i)}$, $F^{(i)}(t)$, $g^{(i)}(t)$ and $(\widetilde{M}, \tilde{g}(t), \tilde{x})$ be the same as in the proof of Theorem 1.1. Let $A^{(i)}$ and $H_{(i)}$ be the second fundamental form and mean curvature of the immersion $F^{(i)}(t)$, respectively. Then we have $|A^{(i)}|^2 \le C_6 |H_{(i)}|^2$ for $(x,t) \in M \times [0,1]$, which implies that $A^{(i)}$ is bounded by a constant independent of i

for $t \in [0,1]$. It follows from Theorem 2.3 that

$$\max_{(x,t)\in M^{(i)}\times[\frac{1}{2},1]} H^2_{(i)}(x,t) \le C_7 \left(\int_0^1 \int_{M_t} |H|^{n+2}_{(i)} d\mu_{g^{(i)}(t)} dt\right)^{\frac{2}{n+2}},$$

where C_7 is a constant independent of i. By an argument similar to the proof of Theorem 1.1, we get a contradiction which completes the proof of Theorem 1.2.

Finally we would like to propose the following

Open Question. Can one generalize Theorems 1.1 and 1.2 to the case where F_t is the solution of mean curvature flow of closed submanifolds in a Riemannian manifold?

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