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IS THERE A NOTION OF WEAK MAXIMAL CONTACT IN CHARACTERISTIC $p > 0$?*

VINCENT COSSART†

Dedicated to Professor H. Hironaka

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Introduction. This article is a continuation of the conference made in the meeting "On the Resolution of Singularities" (December 2008) at the Research Institute for Mathematical Sciences (RIMS) of Kyoto and downloadable on their site.

The purpose of this conference is to give some hints about the proof of the theorem cited below and to show on two examples that the notion of maximal contact should be completely revised in the case of the positive characteristic.

Let us see the following theorem.

Theorem. (Cossart and Piltant) [CP1,CP2]. *Let k be a field of positive characteristic which is differentially finite over a perfect field k_0 and Z/k be a reduced quasiprojective scheme of dimension three with singular locus Σ. There exists a projective morphism $\pi : \tilde{Z} \to Z$, such that*
(i) $\tilde{Z}$ is regular.
(ii) π induces an isomorphim $\tilde{Z}\backslash\pi^{-1}(\Sigma) \simeq Z\backslash\Sigma$.
(iii) $\pi^{-1}(\Sigma) \subset \tilde{Z}$ is a divisor with strict normal crossings.

Such a $\tilde{Z}$ is called *a desingularization of Z*.

Two strategies. There are two main strategies to prove the existence of a desingularization.

The first one was initiated by O. Zariski in [Z]: he cutted the problem in two parts.
(1) Uniformization along a valuation,
(2) patching the uniformizations to get a desingularization.

This is the strategy used in [A1], [CP1, CP2], [Z].

The other strategy was initiated by H. Hironaka who introduced in [H] the key ideas and the fundamental techniques. This strategy is very fruitful in characteristic 0 and is followed by many others: Bierstone, Milman, Villamayor, Encinas, Hauser, Włodarczyk, Cutkosky, Temkin and all people I forget.

To simplify, we can say that the trick is to make a descending induction on the embedding dimension. At the very beginning, your singular variety Z is a closed subvariety of some regular variety W. Then, at every singular point $x \in Z$ there exists a closed *regular* subvariety $W_x \subset W$ which has maximal contact with Z at x. This notion is defined recursively (we are more precised in **III.2.1**):
(1) the Hilbert-Samuel stratum $HS - stratum(Z, x)$ of x is contained in W_x,

*Received June 16, 2010; accepted for publication May 20, 2011.
†Laboratoire de Mathématiques LMV UMR 8100, Université de Versailles, 45 avenue des États-Unis, 78035 Versailles cedex, France (cossart@math.uvsq.fr).

(2) if you blow up Z along any $Y \subset HS - stratum(Z, x)$,

$$Z' \subset W' \longrightarrow Z \subset W,$$

Y regular, and if there exists $x' \in Z'$ above x with the same Hilbert-Samuel function, then x' is on the strict transform $W_{x'}$ of W_x and $W_{x'}$ has maximal contact with Z' at x'.

Let us note that for any $x' \in Z'$ above x, the Hilbert-Samuel function $HS(Z', x')$ (or $HS(x')$ to simplify) of x' is smaller or equal to $HS(Z, x)$ [B], [Gi1].

In characteristic 0, such a W_x exists, of dimension at most $\dim(Z)$. Then the trick is to define a coefficient ideal or an "idealistic exponent" [H3], $Coeff(x, Z) \subset \mathcal{O}_{W_x,x}$, and to replace the problem of the desingularization of $I(Z) \subset \mathcal{O}_{W,x}$ by the desingularization of $Coeff(x, Z) \subset \mathcal{O}_{W_x,x}$. As $\dim(W_x) <\dim(W)$, you get the induction. We do not want to be more precised.

A negative result in characteristic $p > 0$. In [Gi2], Giraud proved that W_x exists in characteristic $p > 0$ with all the properties of maximal contact except one: W_x is in general *not regular*. Even worse, in the purely inseparable case (section **I.2** below), Giraud gets $W_x = Z$: the theory is empty. The descending induction fails completely.

A weaker notion of maximal contact? A new question arises (cf. **III.2.2**): in characteristic $p > 0$ can we find a weaker notion of maximal contact but strong enough to make some induction?

In the purely inseparable case (cf. **I.2**), Cossart and Piltant [CP2] use a new invariant called ω and it appears that $(HS(x), \omega(x))$ is upper semi-continuous along $\mathrm{Sing}(Z)$ for the lexicographical ordering. So, in the purely inseparable case, the question above may be precised: in (1)(2) replace $HS - stratum(Z, x)$ of x by $(HS, \omega) - stratum$ of x and $HS(x)$ (resp. $HS(x')$) by $(HS(x), \omega(x))$ (resp. $(HS(x'), \omega(x'))$). Call these new conditions (1')(2').

Is there a regular W_x with condition (1')(2') with $\dim(W_x) = \dim(Z)$?

In this article, we show that, even in dimension 3, the answer is **NO** (example **II**), there is another example in [CJS, appendix].

A new refined question arises (cf. **III.2.2**): in characteristic $p > 0$, in the very general case, can we find a suitable W_ν regular which has maximal contact "along the valuation ν", i.e. in (1)(2) above replace (2) by

(2") let ν a valuation of the field of functions of Z such that x is the center of ν in Z, if you blow up Z along any $Y \subset HS - stratum(Z, x)$ permissible for ω [CP2, chapter 1, II.5],

$$Z' \subset W' \longrightarrow Z \subset W,$$

Y regular, call x' the center of ν in Z', if $HS(x') = HS(x)$, then x' is on the strict transform W'_ν of W_ν and W'_ν has maximal contact along ν with Z' at x'.

If this turns out to be true, with $\dim(W_\nu) = \dim(Z)$, then we could at least solve the uniformization problem.

In this article, we show that, even in dimension 2, the answer is **NO** (example **III.2**).

Conclusion: for the moment, no reasonable notion of weak maximal contact is proposed.

Summary. In the first section we comment the proof of the theorem above and define the invariant ω (**I.2**).

In section **II**, we show by a first example that there is no maximal contact for the pair (HS, ω).

The worse example (= answer **NO** to the last question) is given in section **III**.

I. Main reduction in dimension 3. In [CP1], we prove that the proof may be reduced to the case of this theorem below.

THEOREM. *Let k be a field of positive characteristic which is differentially finite over a perfect field k_0, i.e. Ω^1_{k/k_0} has finite dimension.*

Let S be a regular local ring of dimension three, essentially of finite type over k and such that $K := QF(S)$ has transcendance degree 3 over k. Let $\bar{R}$ be an Artin-Schreier or purely inseparable singularity of dimension three over S.

Let $K := QF(S)$ and $L := QF(\bar{R})$ (in particular L/K is a finite field extension).

Then, each k-valuation μ of L dominating $\bar{R}$ and satisfying properties (i) and (ii) below has a local uniformization:
(i) μ has rank one and $\kappa(\mu)/\kappa(S)$ is algebraic;
(ii) μ is the unique extension of its restriction to K.

I.1. Notations. We let $R := S[X]_{(X,m_S)}$, X_0 =Spec$(R/(h)$, x_0 his closed point, $\mathfrak{M} = (X, m_S)$, and $k(x_0) = R/\mathfrak{M}$ is a finitely generated field extension of k. We denote by (u_1, u_2, u_3) a regular system of parameters (r.s.p. for short) of S, so $\mathfrak{M} = (X, u_1, u_2, u_3)$. We assume all along this text that h is irreducible over $S[X]$, i.e. that f is not of the form $\theta^p - \theta g^{p-1}$ for any $\theta \in S$.

If $g \neq 0$, such a singularity is called "Artin-Schreier", if $g = 0$, it is called "purely inseparable".

I.2 Purely inseparable case. Let us concentrate on the purely inseparable case which already contains enough difficulties.

To simplify, we take

$$h = X^p + f(u_1, u_2, u_3), \; f \in k[u_1, u_2, u_3],$$

$\dim_{k_0}(\Omega_{k/k_0}) < \infty$, k_0 perfect. We suppose that there is an exceptional divisor E which contains locally the singular locus of $h = 0$ and such that $\mathrm{I}(E)$ divises $u_1u_2u_3$. This can be achieved easily (see [CP1]). We suppose that $\mathrm{ord}_{x_0}(f) \geqslant p$ (else, the singularity is easily solved). We define $\mathcal{J}(f, E)$, the ideal generated by the coefficients of $df \in \Omega^1_{S/k_0}(\log E)$. We defined $H(x) = \prod_{div(u_i)\subset E} U_i^{a(i)}$, where $a(i)$ =$\mathrm{ord}_{u_i}(\mathcal{J}(f, E))$, then set

$$J(f, E) := H(x)^{-1}\mathcal{J}(f, E).$$

The main invariant is:

$$\omega(x) := \mathrm{ord}_x(J(f, E))$$

which, obviously does not depend on X.

I.2.1 Important remark. The case $\omega(x_0) = 0$ is easily solved: see [CP2] **II.4.4** to **II.4.7**.

It can be shown that

$$\omega(x_0) = 0 \;\Leftrightarrow\; f = \gamma M \;\mathrm{mod}S^p, \; M = \prod_{div(u_i)\subset E} u_i^{a(i)},$$

with γ invertible or parameter orthogonal to E and, if γ invertible, then M is not a p-power or $\bar{\gamma} \in k(x_0)$ is not a p-power, $k(x_0)$ is the residual field of the point x_0.

Villamayor's example. In his conference, O. Villamayor proposed the following example as a "terminal case".

$$X^2 + u^2 v,\ p = 2,\ E = \operatorname{div}(u).$$

The computations give:

$$\mathcal{J}(E) = (u^2),\ H(x) = u^2,\ J(E) = (1),\ \omega(x_0) = 0.$$

We agree with Orlando: it is a terminal case. Indeed, the case $\omega(x_0) = 0$ is easy to solve [CP2, **II.4.6**].

All this leads to:

I.2.2 Second reduction. Let $W := \operatorname{Spec}(S)$, find W', some iterated blowing up of W, where $x \in W'$, the center of the restriction of μ in W', verifies:

$$\omega(x) < \omega(x_0).$$

The problem is reduced to a problem of monomialization modulo p-powers in a smooth variety: the problem is more difficult, but the dimension of the ambient space drops of one.

I.2.3 Permissible centers. Now we forget X_0 and we work in W, E is a normal crossing divisor in W. We do not write here the definitions [CP2, chapter 1, II.5] of the permissible centers for ω. We just recall that, if Y, a closed subset of W is permissible at x for ω, then it is smooth at x and locally normal crossing with E, furthermore, **closed points are permissible**. Bad news: **permissible centers are not necessarily contained in the locus where ω is maximal.** We have the following propositions.

I.2.3.1 Proposition. *If Y is irreducible of generic point ξ and permissible at x for ω, then $\omega(\xi) \geqslant \omega(x) - 1$.*

I.2.3.2 Transformation laws. *Let Y be as above and $\pi : X' \to X'$ be the blowing up cenetred at Y. Then $J(f', E')$ is the weak transform of $J(f, E, Y)$ (defined below), where f' is the strict transform of f and E' the total transform of E.*

I.2.3.3 Jacobian adapted to Y. Let

$$\mathcal{D}(E) := \{D \in \operatorname{Der}_{k_0} \mathcal{O}_W \mid D(I(E)) \subset I(E)\},$$

$$\mathcal{D}(E, Y) := \{D \in \mathcal{D}(E) \mid D(I(Y)) \subset I(Y)\}.$$

Then

$$J(f, E, Y) := H(x)^{-1}(D(f), D \in \mathcal{D}(\mathcal{E}, \mathcal{Y}).$$

I.2.3.4 Example. In the case where $Y = x$ and E=div$(u_1 u_2)$, $k_0 = k$, we get

$$\mathcal{D}(E, Y) = (\frac{u_1 \partial}{\partial u_1}, \frac{u_2 \partial}{\partial u_2}, \mathfrak{M} \frac{\partial}{\partial u_3}).$$

I.2.3.5 Proposition. *With notations as above, if $x' \in W'$ is above x, then*

$$\omega(x') \leqslant \omega(x).$$

II. No maximal contact for (HS, ω). It is very well known that there is no maximal contact for the Hilbert-samuel function in characteristic $p > 0$ (see section **III**).

Optimist people may think that there may be maximal contact for the couple (HS, ω) where HS is the Hilbert-Samuel function. This is wrong. The following example shows that there cannot exist in X a surface Σ such that the Hilbert-Samuel function the strict transforms of Σ is constant at the points x_i above x with $(HS(x_0), \omega(x_0)) = (HS(x_i), \omega(x_i))$, here $HS(x)$ is just the local multiplicity which is equal to p.

II.1. Example in dimension 3. (joint work with O. Piltant). With the notations of **I.2**. We have a singularity in dimension 3 of equation:

$$h = X^p + u_1^a u_2^b (v^{p^e} + (u_2 - u_1)^{p^e} u_1^{p+2} + extra).$$

$extra \in S = k[u_1, u_2, v]$, of very big order, $E = \mathrm{div}(u_1 u_2)$, $a + b = 0 \bmod p$, $ab \neq 0 \bmod p$, $p \neq 2$. We suppose $k = k_0$ algebraically closed.

Computations give:

$$H(x_0) = u_1^a u_2^b, \; J(E) = (v^{p^e}) \bmod \mathfrak{M}^{1+p^e}, \; \omega(x_o) = p^e.$$

We blow up along the origin.

We take as new origin the point x_1 of parameters

$$(X/u_1, u_1, w := (u_2 - u_1)/u_1, v/u_1)$$

that we note (X, u_1, w, v), using an usual convention.

We get

$$h_1 = X^p + u_1^{a+b-p} (w + 1)^b (v^{p^e} + w^{p^e} u_1^{p+2} + u_1^A extra')$$

where h_1 is the strict transform of h, $A \in \mathbb{N}$, A very big. The reader sees that the set of points x' above x_0 with $(HS(x_0), \omega(x_0)) = (HS(x'), \omega(x'))$ is exactly $\mathrm{Proj}(X, V)$, where $V := \mathrm{in}_{x_0}(v)$, this implies that, **if Σ exists, its directrix [CJS,1.20] at x_0 is $\mathbf{V}(X, V)$.**

We change one variable: let $Y := X + u_1^{\frac{a+b-p}{p}}$, then

$$h_1 = Y^p + u_1^{a+b-p} (\gamma v^{p^e} w + w^{p^e} u_1^{p+2} + u_1^A extra'),$$

$$E_1 = \mathrm{div}(u_1), \; h(x_1) = u_1^{a+b-p}, \; J(E_1) = (v^{p^e}, w^{p^e} u_1^{p+2}) \bmod (u_1^A), \; \omega(x') = p^e.$$

We go on: we look at the sequences of blowing ups centered at closed points on the strict transform of (Y, v, w). We use the usual convention i.e. we note the parameters of x_{i+1} (Y, u_1, v, w) instead of $(Y/u_1^i, u_1, v/u_1^i, w/u_1^i)$.

We get at x_2:

$$h_2 = X^p + u_1^{1 \bmod(p)}(w+1)^b(v^{p^e}w + w^{p^e}u_1^{p+1} + u_1^A extra_2)$$

$$E_2 = \mathrm{div}(u_1),\ h(x_2) = u_1^{1 \bmod(p)},\ J(E_2) = (v^{p^e}, w^{p^e}u_1^{p+1}) \bmod (u_1^A),\ \omega(x') = p^e.$$

We get at x_{p+3}:

$$h_{p+3} = X^p + u_1^{2 \bmod(p)}(w+1)^b(v^{p^e}w + w^{p^e} + u_1^A extra_3)$$

$$E_{p+3}\mathrm{div}(u_1),\ h(x_3) = u_1^{2 \bmod(p)},\ J(E_3) = (v^{p^e}, w^{p^e}) \bmod (u_1^A),\ \omega(x') = p^e.$$

Things are looking well: the initial part of $J(E_{p+3})$ is (v^{p^e}, w^{p^e}), the directrix of the ideal $J(E_{p+3})$ has dimension 1: the equations are $v = w = 0$, the dimension was 2 for $J(E_i)$, $i < p+3$, the equation was $v = 0$.

We get at x_{p+4}:

$$h_{p+4} = X^p + u_1^{2 \bmod(p)}(w+1)^b(v^{p^e}wu_1 + w^{p^e} + u_1^A extra_4)$$

$$E_{p+4} = \mathrm{div}(u_1),\ h(x_4) = u_1^{2 \bmod(p)},\ J(E_4) = (v^{p^e}u_1, w^{p^e}) \bmod (u_1^A),\ \omega(x') = p^e.$$

The initial part of $J(E_{p+4})$ is (w^{p^e}), the directrix of the ideal $J(E_{p+4})$ has dimension 2: the equation is $w = 0$.

We get at x_{p+3+N}:

$$h_{p+3+N} = X^p + u_1^{2 \bmod(p)}(w+1)^b(v^{p^e}wu_1^N + w^{p^e} + u_1^A extra_{p+3+N})$$

$$E_{p+3+N} = \mathrm{div}(u_1),\ h(x_{p+3+N}) = u_1^{2 \bmod(p)},$$
$$J(E_{p+3+N}) = (v^{p^e}u_1^N, w^{p^e}) \bmod (u_1^A),\ \omega(x') = p^e.$$

The initial part of $J(E_{p+3+N})$ is (w^{p^e}), the directrix of the ideal $J(E_{p+3+N})$ has dimension 2: the equation is $w = 0$.

Let us blow up along x_{p+3+N}: it is easily seen that the set of points x' above x_{p+3+N} with $(HS(x_0), \omega(x_0)) = (HS(x'), \omega(x'))$ is exactly $\mathrm{Proj}(X, W)$, where $W := \mathrm{in}_{x_{p+3+N}}(w)$, this implies that, **if Σ exists, the directrix of its strict transform Σ_{p+3+N} at x_{p+3+N} is $\mathbf{V}(X, W)$.** This is impossible, by [CJS, 12.1, 12.3, 13.3], its ideal should be $(X, V) \bmod (U_1)$.

II.2. Conclusion. This kills the hope to have a maximal contact in "Giraud's sense" for (HS, ω), i.e. to find a surface Σ such that the strict drop of its local Hilbert-Samuel function would imply a strict drop of some invariant of the original singularity.

The question is: define another invariant, finer than (HS, ω) such that, for this invariant, you get a maximal contact in "Giraud's sense"... For the moment, there is no answer.

III. No maximal contact along a valuation. (joint work with U. Jannsen and S. Saito). As there is no generalization of ω to cases other than the inseparable case or the Artin-Schreier case, we may be interested in a notion of maximal contact "along a valuation".

III.1. Recall of the algorithm, dimension 2, hypersurface case. We follow Hironaka in [CGO, appendix]. X is a singular surface embedded in a 3-dimensional smooth k-variety Z, k is an algebraic closed field of characteristic $p > 0$. We suppose that the worse HS-stratum is a finite union of closed points, in this case the worse HS-stratum is the locus of maximal multiplicity $\mu(x)$, we call it $\mathrm{HS}(X)$

The algorithm is: blow up the locus of multiplicity $\geqslant \mu(x)$. This will stop.

More precisely, in an open neighbourhood $U \subset X$ of $x \in \mathrm{HS}(X)$, you blow up X, Z along x:

$$X_1 \subset Z_1 \longrightarrow X \subset Z.$$

One can show that there are three different cases [CJS, section 2].
(i) Either there is no point in X_1 near to x (no point with same multiplicity): STOP, the maximum multiplicity dropped strictly above x.
(ii) Either there are a finite number of closed points in X_1 near to x, then above U, blow up X_1, Z_1 along these points:

$$X_2 \subset Z_2 \longrightarrow X_1 \subset Z_1.$$

(iii) Or the set of points in X_1 near to x is a projective line D_1 then above U, blow up X_1, Z_1 along D_1:

$$X_2 \subset Z_2 \longrightarrow X_1 \subset Z_1,$$

either there is no point in X_2 near to x, either there are a finite number of closed points in X_2 near to x or the set of points in X_2 near to x is a projective line D_2 which projects isomorphically on D_1, then above U, blow up X_2, Z_2 along D_2.

In the latter case, above U, the algorithm creates a "fundamental sequence" [CJS, section 5], i.e. a finite sequence of blowing ups

$$X_m \subset Z_m \longrightarrow X_{m-1} \subset Z_{m-1} \longrightarrow \cdots X_2 \subset Z_2 \longrightarrow X_1 \subset Z_1 \longrightarrow X \subset Z,$$

where the center of the blowing up $X_i \subset Z_i \longrightarrow X_{i-1} \subset Z_{i-1}$ is a projective line D_{i-1} which projects isomorphically on D_1, $2 \leqslant i \leqslant m$ and either there is no point in X_m near to x, or there are a finite number of closed points in X_m near to x.

III.2 Maximal contact along a valuation. The example given in [CJS, section 15] shows that there is no maximal contact in positive characteristic. What is new in this paper is that there is no maximal contact along a valuation. Let us recall the definition of maximal contact [CJS, section 15].

III.2.1 Definition: hypersurface of maximal contact. *Let Z be an excellent regular scheme and $X \subset Z$ be a closed subscheme.*

A closed subscheme $W \subset Z$ is called to have maximal contact with X at $x \in X$ if the following conditions are satisfied:
(i) $x \in W$.
(ii) Take any sequence of permissible blowups (the Hilbert-Samuel function is constant along the regular center D_n, $n \geqslant 0$):

$$(1)\qquad \begin{array}{ccccccccccc} Z = & Z_0 & \xleftarrow{\pi_1} & Z_1 & \xleftarrow{\pi_2} & Z_2 & \leftarrow \dots \leftarrow & Z_{n-1} & \xleftarrow{\pi_n} & Z_n & \leftarrow \cdots \\ & \cup & & \cup & & \cup & & \cup & & \cup & \\ X = & X_0 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & X_2 & \leftarrow \dots \leftarrow & X_{n-1} & \xleftarrow{\pi_n} & X_n & \leftarrow \cdots \end{array}$$

where for any $n \geqslant 0$

$$
\begin{array}{ccccc}
Z_{n+1} & = & \mathrm{Bl}_{D_n}(Z_n) & \overset{\pi_{n+1}}{\leftarrow} & Z_n \\
 & & \cup & & \cup \\
X_{n+1} & = & \mathrm{Bl}_{D_n}(X_n) & \overset{\pi_{n+1}}{\leftarrow} & X_n
\end{array} \tag{2}
$$

and $D_n \subset X_n$ is permissible. Assume that there exists a sequence of points $x_n \in D_n$ ($n = 0, 1, \ldots$) such that $x_0 = x$ and x_n is near to x_{n-1} for all $n \geqslant 1$ (this means $HS(x_{n-1}) = HS(x_n)$ for all $n \geqslant 1$). Then $D_n \subset W_n$ for all $n \geqslant 0$, where W_n is strict transform W in Z_n.

Some optimist people asked us:

"Your definition of maximal contact is weaker than Hironaka's, but still too strong, for the uniformization problem, you just need the definition below. Have you an example where there is no maximal contact along a valuation?"

We found one just before Kyoto conference, in the surface-hypersurface case which is perfectly understood now ([HH], [CJS], [C2], [C3]).

III.2.2 Definition: maximal contact along a valuation. *In the definition above suppose X is a projective variety over a field k, let ν be a k-valuation, then on each X_n, ν has a center x_n.*

We say that a closed subscheme $W_\nu \subset Z$ has maximal contact with X along ν if, for the sequence (1) where x_n is the center of ν in $\mathbb{Z}_n$ for all $n \geqslant 0$, then $x_n \in W_n$ for all $n \geqslant 0$, where W_n is strict transform W_ν in Z_n.

Indeed if for any k-valuation ν there could exist a smooth W_ν satisfying **III.2.2**, life would be much easier (as says the guru Abhyankar) in desingularization theory: the uniformization problem should be solved.

III.2.3 The example.

$$p = 3$$

No exceptional divisor E, $E = \varnothing$, take

$$f := y^3 + u_2^2[(u_1^3 - u_2^2)(u_1^3 + u_2^2)^3 + u_1^N],\ N \neq 0 \bmod (p),\ N >> 0.$$

Let us recall some definitions.

III.2.3.1 Hironaka's polyhedrons. [H1] or [CJS, section 7] *In the case where*

$$f = y^m + \sum_{i,a,b,a+b>i,0\leqslant i\leqslant m} \lambda_{i,a,b} y^{m-i} u_1^a u_2^b,\ \lambda_{i,a,b} \in k,$$

$\Delta(f,u,y)$ is the convex hull spanned by $\{(\frac{a}{i}, \frac{b}{i}), \lambda_{i,a,b} \neq 0\} + {\mathbb{R}_{\geqslant 0}}^2$. Hironaka defines $\Delta(f,u)$ as:

$$\Delta(f,u) = \cap_{y, in_{\mathfrak{M}}(f)=Y^m} \Delta(f,u,y).$$

III.2.3.2 Notations. $\delta(f,u,t)$ =inf$\{a+b, (a,b) \in \Delta(f,u,t)\}$, $\delta(f,u)$ =inf$\{a+b, (a,b) \in \Delta(f,u)\}$.

We write sometimes $\delta(x)$ instead of $\delta(f,u)$ where x is the point of parameters (y,u_1,u_2), indeed, one can prove that $\delta(f,u)$ does not depend on (u_1,u_2).

In our example, we get

$$\delta(x) = 3 + 1/3$$

III.2.3.3 $\epsilon(x)$**.** *With the notations and hypotheses of* **III.2.3.1***, assume that there is an exceptional divisor E with components smooth and orthogonal to $y = 0$, then assume that*

$$E \subset \mathrm{div}(u_1u_2),$$

(we say E has "new components"), we define A_1 =inf$\{a|(a,b) \in \Delta(f,u)\}$, mutatis mutandis, we define A_2. Then

$$\epsilon(x) := \delta(x) - \sum_{\mathrm{div}(u_i)\subset E} A_i.$$

In our example, $E = \varnothing$, we get

$$\epsilon(x) = \delta(x) = 3 + 1/3$$

III.2.3.4 Change of $\epsilon(x)$**.** In the example, to avoid useless denominators, we will replace $\epsilon(x)$ by $3\epsilon(x)$ that we still denote by $\epsilon(x)$. From now on:

$$\epsilon(x) = 10.$$

III.2.3.5. *In the example, x is isolated in the HS-stratum.*

Indeed

$$\frac{\partial f}{\partial u_1} = u_1^{N-1}u_2^2.$$

So, if a curve is contained in the Hs-stratum at the beginning, $u_1^{N-1}u_2^2$ has order at least 2 along this curve which is contained in $\mathrm{div}(u_1)$ or $\mathrm{div}(u_2)$
(i) if it is contained in $\mathrm{div}(u_2)$, as $f = y^3 \bmod (u_2)$, the only possibility is the curve $\mathrm{V}(y,u_2)$ which does not fit
(ii) if it is contained in $\mathrm{div}(u_1)$, it should be $\mathrm{V}(z,u_1)$ with $y^3 + u_2^{10} \in (z,u_1)^3$

$$\frac{\partial y^3 + u_2^{10}}{\partial u_1} = u_2^9 \in (z,u_1)^2$$

so $z = u_2$, which does not fit.

III.2.3.6 Let us start Hironaka's algorithm. We blow up along the origin and take the point x_1 above of parameters
$y/u_1, u_1, u_2/u_1$ $\epsilon(x_1) = 6$
${y^{(1)}}^3 + u_1^7 {u_2^{(1)}}^2[(u_1 - {u_2^{(1)}}^2)(u_1 + {u_2^{(1)}}^2)^3 + u_1^{N-8}]$
The exceptional divisor is $\mathrm{div}(u_1)$ "new component". Following Hironaka, we make the "fundamental sequence": we get the point x_3 above of parameters

$$(y^{(3)}, u_1^{(3)}, u_2^{(3)}) := (y^{(1)}/u_1^2, u_1, u_2^{(1)}),$$

$f_3 := {y^{(3)}}^3 + u_1^{(3)}{u_2^{(3)}}^2[(u_1^{(3)} - {u_2^{(3)}}^2)(u_1^{(3)} + {u_2^{(3)}}^2)^3 + {u_1^{(3)}}^{N-8}]\ \epsilon(x_3) = 6$

We make again the "fundamental sequence": we first blow up along x_3, we look at the point x_4 above of parameters

$$(y^{(4)}, u_1^{(4)}, u_2^{(4)}) := (y^{(3)}/u_2^{(3)}, u_1^{(3)}/u_2^{(3)}, u_2^{(3)})$$

$f_4 := {y^{(4)}}^3 + u_1^{(4)}{u_2^{(4)}}^4[(u_2^{(4)} - u_2^{(4)})(u_2^{(4)} + u_2^{(4)})^3 + {u_1^{(4)}}^{N-8}{u_2^{(4)}}^{N-13}]$, $\epsilon(x_4) = 4$

$\mathrm{div}(u_1^{(4)}u_2^{(4)})$ is the exceptional divisor, both components are "new": we end this second "fundamental sequence" we look at the point x_5 above of parameters

$$(y^{(5)}, u_1^{(5)}, u_2^{(5)}) := (y^{(4)}/u_2^{(4)}, u_1^{(4)}, u_2^{(4)})$$

$f_5 := {y^{(5)}}^3 + u_1^{(5)}u_2^{(5)}[(u_1^{(5)} - u_2^{(5)})(u_1^{(5)} + u_2^{(5)})^3 + {u_1^{(5)}}^{N-8}{u_2^{(5)}}^{N-12}]$, $\epsilon(x_5) = 4$.

Following the algorithm, we blow up along the origin, above at the point of parameters

$z := y^{(5)}/u_1^{(5)}, v_1 := u_1^{(5)}, v := u_2^{(5)}/u_1^{(5)}+1$, let us see that the ϵ-invariant increases strictly: $\epsilon(x_6) = 5$ (kangaroo point as defined by H. Hauser [HH]).

III.2.3.7 Surprising computation. Exercise for the reader: compute $\omega(x_5)$ and $\omega(x_6)$ (cf. **I.2**). The main point is that $z^3+v_1^3(v-1)[(v+1)v^3+v_1^{2N-24}(v-1)^{N-12}]$ and $(2,1)$ a *solvable* vertex of $\Delta(f, z, v_1, v)$.

Let us solve it.

$$f = z^3 + v_1^3[(v^2-1)v^3 + v_1^{N-12}(v-1)^{N-11}]$$

$$w := z - v_1^2 v.$$

This gives

$$f = w^3 + v_1^3[v^5 + v_1^{N-12}(v-1)^{N-11}]$$

As $N \neq 0 \bmod (3)$, $\Delta(f, w, v_1, v)$ has two non solvable vertices (non integer coordinates)

$$(1, 5/3),\ (N/3 - 4, 0),$$

$\mathrm{div}(v_1)$ is the new component: $\epsilon(x_6) = 5 > \epsilon(x_5) = 4$.

III.2.3.8 No maximal contact on this example. I claim that, in this example, if we end the fundamental sequence at x_6 and add another fundamental sequence, there will be a point $x_9 \in X_9$ such that there exists no $t = y - \gamma$, $\gamma \in k[[u_1, u_2]]$ such that the x_i are on the strict transform of $\mathrm{div}(t)$, $0 \leqslant i \leqslant 8$. One can see that it implies that there is no smooth hypersurface $W \subset Z$ such that the x_i are on the strict transform of $W \subset Z$. We define x_7 as the point on the strict transform of $\mathrm{div}(v)$ in the bu along x_6. These points x, x_1,....,x_8,x_9 are near to each other: there is a valuation ν of the function field whose center on X_i is x_i, $0 \leqslant i \leqslant 9$: there is no maximal contact ALONG THE VALUATION ν. So no hope of maximal contact, even for the uniformization problem or Hironaka's game.

Suppose a smooth hypersurface $W_\nu \subset Z$ has maximal contact along ν, let us call $t = 0$ its equation in a neighbourhood of x. For short, we write W instead of W_ν

III.2.4 Proposition. [CJS, section 15] *Suppose $x = x_0$ isolated in its HS-stratum, then if there exists a smooth hypersurface $t = 0$ such that along a fundamental* *sequence starting at $x = x_0$ the x_j, $j \geqslant 0$ (resp. $j \geqslant i$) are on the strict transform of div(t), then*

$$t = y - \gamma,\ \gamma \in k[[u_1, u_2]],\ \lfloor \delta(f, u, t) \rfloor = \lfloor \delta(f, u) \rfloor.$$

As $\delta(x) = 3 + 1/3$, by **III.2.4**, $\gamma \in (u_1, u_2)^3$.

$$\gamma = P_3(u_1, u_2) + P_4(u_1, u_2) + \rho,\ \rho \in (u_1, u_2)^5,$$

$P_i(u_1, u_2) \in k[u_1, u_2]$, homogeneous of degree i or $= 0$, $i = 3, 4$.

$$f := t^3 + u_2^2[(u_1^3 - u_2^2)(u_1^3 + u_2^2)^3 + u_1^N] + P_3^3 + P_4^3 + \rho^3.$$

$f_1 = {t^{(1)}}^3 + u_1^7 {u_2^{(1)}}^2[(u_1 - {u_2^{(1)}}^2)(u_1 + {u_2^{(1)}}^2)^3 + u_1^{N-8}] + u_1^9 P_3(1, u_2^{(1)})^3 + u_1^9 P_4(1, u_2^{(1)})^3 + u_1^{12}\rho'$.

$f_3 := {t^{(3)}}^3 + u_1^{(3)} {u_2^{(3)}}^2[(u_1^{(3)} - {u_2^{(3)}}^2)(u_1^{(3)} + {u_2^{(3)}}^2)^3 + {u_1^{(3)}}^{N-8}] + P_3(1, u_2^{(3)})^3 + {u_1^{(3)}}^3 P_4(1, u_2^{(3)})^3 + {u_1^{(3)}}^6 \rho'^3$,

$\delta(x_3) = 2 + 1/3$, so by **III.2.4**, $P_3(1, u_2^{(3)}) \in (u_2^{(3)})^2$, let us denote $P_3(1, u_2^{(3)}) = a{u_2^{(3)}}^2 + b{u_2^{(3)}}^3$, $a, b \in k$.

$f_4 := {t^{(4)}}^3 + u_1^{(4)} {u_2^{(4)}}^4[(u_2^{(4)} - u_2^{(4)})(u_2^{(4)} + u_2^{(4)})^3 + {u_2^{(4)}}^{N-8} {u_2^{(4)}}^{N-13}] + a^3 {u_2^{(4)}}^3 + b^3 {u_2^{(4)}}^6 + {u_1^{(4)}}^3 P_4(1, u_2^{(4)})^3 + {u_1^{(4)}}^6 {u_2^{(4)}}^3 \rho''^3$.

As $\mathrm{V}(y^{(4)}, u_2^{(4)})$ is permissible, it is contained in $\mathrm{div}(t^{(4)})$, so $\mathrm{V}(y^{(4)}, u_2^{(4)}) = V(t^{(4)}, u_2^{(4)})$

$$P_4(1, u_2^{(4)}) = \lambda u_2^{(4)} + c{u_2^{(4)}}^2 + d{u_2^{(4)}}^3 + e{u_2^{(4)}}^4,\ \lambda, c, d, e \in k.$$

$f_5 := {t^{(5)}}^3 + u_1^{(5)} u_2^{(5)}[(u_1^{(5)} - u_2^{(5)})(u_1^{(5)} + u_2^{(5)})^3 + {u_1^{(5)}}^{N-8} {u_2^{(5)}}^{N-12}] + a^3 + b^3 {u_2^{(5)}}^3 + {u_1^{(5)}}^3(\lambda^3 + c^3 {u_2^{(5)}}^3 + d^3 {u_2^{(5)}}^6 + e^3 {u_2^{(5)}}^9) + {u_1^{(5)}}^9 \rho''^3$,

$\delta(f_5, u) = 2$, by **III.2.4**,

$$a = b = \lambda = 0,\ \rho' \text{ not invertible.}$$

$f_7 := {t^{(6)}}^3 + {v_1}^3(v-1)[(v+1)v^3 + v_1^{N-12}(v-1)^{N-12}] + {v_1}^3(c^3(v-1)^3 + d^3(v-1)^6 {v_1}^3 + e^3(v-1)^9 {v_1}^6) + {v_1}^6 \times$ *something*

$$f_7 = w^3 + v_1^3[v^5 + v_1^{N-12}(v-1)^{N-11}].$$

$\delta(x_7) = 2 + 2/3$.

We end the fundamental sequence, we get

$f_8 = t^3 + (v-1)[(v+1)v^3 + v_1^{N-12}(v-1)^{N-12}] + c^3(v-1)^3 + d^3(v-1)^6 {v_1}^3 + e^3(v-1)^9 {v_1}^6 + {v_1}^3 \times$ *something*

$$f_8 = w^3 + v^5 + v_1^{N-12}(v-1)^{N-11}$$

$\delta(x_8) = 5/3$, $c = 0$, else there is no point on the strict transform of $\mathrm{div}(t)$ and x_8 is, furthermore by **III.2.4**,

$$d = 0.$$

We go on: we blow up x_8 and we look at the near point x_9 on the strict transform of $v = 0$.

$$f_9 = w^3 + u^2v^5 + u^{N-15}(uv-1)^{N-11}$$

As x_9 is supposed to be on the strict transform of $\mathrm{div}(t)$,

$$\textit{something} \text{ is not invertible.}$$

$f_9 = t^3+(uv-1)[(uv+1)v^3+u^{N-15}(v-1)^{N-12}]+e^3(uv-1)^9u^3+u^3\times \textit{something}'$, $\textit{something}'$ is invertible or divisible by u^3.

In any case, as $\delta(x_8) = 7/3 > 2$, the monomial $(uv-1)(uv+1)v^3$ gives a contradiction with **III.2.4**.

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