COLLAPSE OF THE MEAN CURVATURE FLOW
FOR EQUIFOCAL SUBMANIFOLDS[*]

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Abstract. In this paper, we investigate the mean curvature flows having an equifocal submanifold in a symmetric space of compact type and its focal submanifolds as initial data. It is known that an equifocal submanifold of codimension greater than one in an irreducible symmetric space of compact type occurs as a principal orbit of a Hermann action. However, we investigate the flows conceptionally without use of this fact. The investigation is performed by investigating the mean curvature flows having the lifts of the submanifolds to an (infinite dimensional separable) Hilbert space through a Riemannian submersion as initial data .

Key words. Mean curvature flow, equifocal submanifold, isoparametric submanifold.

AMS subject classifications. 53C40, 53C35.

1. Introduction. Let f_t's ($t \in [0, T)$) be a one-parameter C^∞-family of immersions of a manifold M into a Riemannian manifold N, where T is a positive constant or $T = \infty$. Define a map $\widetilde{f} : M \times [0, T) \to N$ by $\widetilde{f}(x, t) = f_t(x)$ ($(x, t) \in M \times [0, T)$). If, for each $t \in [0, T)$, $\widetilde{f}_*((\frac{\partial}{\partial t})_{(x,t)})$ is the mean curvature vector of $f_t : M \hookrightarrow N$, then f_t's ($t \in [0, T)$) is called a mean curvature flow. In particular, if f_t's are embeddings, then we call $M_t := f_t(M)$'s ($0 \in [0, T)$) rather than f_t's ($0 \in [0, T)$) a mean curvature flow. Liu-Terng [LT] investigated the mean curvature flow having isoparametric submanifolds (or their focal submanifolds) in a Euclidean space as initial data and obtained the following facts.

FACT 1 ([LT]). *Let M be a compact isoparametric submanifold in a Euclidean space. Then the following statements* (i) *and* (ii) *hold:*

(i) *The mean curvature flow M_t having M as initial data collapses to a focal submanifold of M in finite time. If a focal map of M onto F is spherical, then the mean curvature flow M_t has type I singularity, that is,* $\lim_{t \to T-0} \max_{v \in S^\perp M_t} ||A_v^t||_\infty^2 (T - t) < \infty$, *where A_v^t is the shape operator of M_t for v, $||A_v^t||_\infty$ is the sup norm of A_v^t and $S^\perp M_t$ is the unit normal bundle of M_t.*

(ii) *For any focal submanifold F of M, there exists a parallel submanifold M' of M such that the mean curvature flow having M' as initial data collapses to F in finite time.*

FACT 2 ([LT]). *Let M be as in Fact 1, C be the Weyl domain of M at x_0 ($\in M$) and σ be a stratum of dimension greater than zero of ∂C. Then the following statements* (i) *and* (ii) *hold:*

(i) *For any focal submanifold F (of M) through σ, the maen curvature flow F_t having F as initial data collapses to a focal submanifold F' (of M) through $\partial \sigma$ in finite time. If the fibration of F onto F' is spherical, then the mean curvature flow F_t has type I singularity.*

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(ii) For any focal submanifold F (of M) through $\partial\sigma$, there exists a focal submanifold F' (of M) through σ such that the mean curvature flow F'_t having F' as initial data collapses to F in finite time.

As a generalized notion of compact isoparametric hypersurfaces in a sphere and a hyperbolic space, and a compact isoparametric submanifolds in a Euclidean space, Terng-Thorbergsson [TT] defined the notion of an equifocal submanifold in a symmetric space as a compact submanifold M satisfying the following three conditions:

(i) the normal holonomy group of M is trivial,

(ii) M has a flat section, that is, for each $x \in M$, $\Sigma_x := \exp^{\perp}(T_x^{\perp}M)$ is totally geodesic and the induced metric on Σ_x is flat, where $T_x^{\perp}M$ is the normal space of M at x and $\exp^{\perp}$ is the normal exponential map of M.

(iii) for each parallel normal vector field v of M, the focal radii of M along the normal geodesic γ_{v_x} (with $\gamma'_{v_x}(0) = v_x$) are independent of the choice of $x \in M$, where $\gamma'_{v_x}(0)$ is the velocity vector of γ_{v_x} at 0.

On the other hand, Heintze-Liu-Olmos [HLO] defined the notion of an isoparametric submanifold with flat section in a general Riemannian manifold as a submanifold M satisfying the above condition (i) and the following conditions (ii′) and (iii′):

(ii′) for each $x \in M$, there exists a neighborhood U_x of the zero vector (of $T_x^{\perp}M$) in $T_x^{\perp}M$ such that $\Sigma_x := \exp^{\perp}(U_x)$ is totally geodesic and the induced metric on Σ_x is flat,

(iii′) sufficiently close parallel submanifolds of M are CMC with respect to the radial direction.

In the case where the ambient space is a symmetric space G/K of compact type, they showed that the notion of an isoparametric submanifold with flat section coincides with that of an equifocal submanifold. The proof was performed by investigating its lift to $H^0([0,1], \mathfrak{g})$ through a Riemannian submersion $\pi \circ \phi$, where π is the natural projection of G onto G/K and ϕ is the parallel transport map for G (which is a Riemannian submersion of $H^0([0,1], \mathfrak{g})$ onto G ($\mathfrak{g}$:the Lie algebra of G)). Let M be an equifocal submanifold in G/K and v be a parallel normal vector field of M. The end-point map $\eta_v(: M \mapsto G/K)$ for v is defined by $\eta_v(x) = \exp^{\perp}(v_x)$ $(x \in M)$. Set $M_v := \eta_v(M)$. We call M_v a parallel submanifold of M when $\dim M_v = \dim M$ and a focal submanifold of M when $\dim M_v < \dim M$. The parallel submanifolds of M are equifocal. Let $f : M \times [0, T) \to G/K$ be the mean curvature flow having M as initial data. Then, it is shown that, for each $t \in [0, T)$, $f_t : M \hookrightarrow G/K$ is a parallel submanifold of M and hence it is equifocal (see Lemma 3.1). Fix $x_0 \in M$. Let $\widetilde{C} (\subset T_{x_0}^{\perp}M)$ be the fundamental domain containing the zero vector (of $T_{x_0}^{\perp}M$) of the Coxeter group (which acts on $T_{x_0}^{\perp}M$) of M at x_0 and set $C := \exp^{\perp}(\widetilde{C})$, where we note that $\exp^{\perp}|_{\widetilde{C}}$ is a diffeomorphism onto C. Without loss of generality, we may assume that G is simply connected. Set $\widetilde{M} := (\pi \circ \phi)^{-1}(M)$, which is an isoparametric submanifold in $H^0([0,1], \mathfrak{g})$. Fix $u_0 \in (\pi \circ \phi)^{-1}(x_0)$. The normal space $T_{x_0}^{\perp}M$ is identified with the normal space $T_{u_0}^{\perp}\widetilde{M}$ of $\widetilde{M}$ at u_0 through $(\pi \circ \phi)_{*u_0}$. Each parallel submanifold of M intersects with C at the only point and each focal submanifold of M intersects with ∂C at the only point, where ∂C is the boundary of C. Hence, for the mean curvature flow $f : M \times [0, T) \to G/K$ having M as initial data, each $M_t(:= f_t(M))$ intersects with C at the only point. Denote by $x(t)$ this intersection

point and define $\xi : [0, T) \to \widetilde{C} \ (\subset T_{x_0}^\perp M = T_{u_0}^\perp \widetilde{M})$ by $\exp^\perp(\xi(t)) = x(t) \ (t \in [0, T))$. Set $\widetilde{M}_t := (\pi \circ \phi)^{-1}(M_t) \ (t \in [0, T))$. It is shown that $\widetilde{M}_t \ (t \in [0, T))$ is the mean curvature flow having $\widetilde{M}$ as initial data because the mean curvature vector of $\widetilde{M}_t$ is the horizontal lift of that of M_t through $\pi \circ \phi$. By investigating $\xi : [0, T) \to T_{u_0}^\perp \widetilde{M}$, we obtain the following fact corresponding to Fact 1.

THEOREM A. *Let M be an equifocal submanifold in a symmetric space G/K of compact type. Then the following statements* (i) *and* (ii) *hold:*

(i) *If M is not minimal, then the mean curvature flow M_t having M as initial data collapses to a focal submanifold F of M in finite time. Furtheremore, if M is irreducible, the codimension of M is greater than one and if the fibration of M onto F is spherical, then the flow M_t has type I singularity.*

(ii) *For any focal submanifold F of M, there exists a parallel submanifold of M collapsing to F along the mean curvature flow and the set of all parallel submanifolds collapsing to F along the mean curvature flow is a one-parameter C^∞-family.*

REMARK 1.1. U. Christ ([Ch]) showed that all irreducible equifocal submanifolds of codimension greater than one on symmetric spaces of compact type are homogeneous and hence they occur as principal orbits of hyperpolar actions. On the other hand, A. Kollross ([Kol]) showed that all hyperpolar actions of cohomogeneity greater than one on irrdeucible symmetric spaces of compact type are orbit equivalent to Hermann actions on the space. Hence all equifocal submanifolds of codimension greater than one on irreducible symmetric spaces of compact type occurs as principal orbits of Hermann actions. Therefore they are classified completely. Hence we can show the statement of Theorem A by using this classification. However, it is very important to prove the statement conceptionally without the use of this classification. In fact, Liu-Terng ([LT]) prove the results in [LT] conceptionally without the use of the classification of isoparametric submanifolds in Euclidean spaces by J. Dadok ([D]). So, in this paper, we prove the statement of Theorem A conceptionally.

Also, we obtain the following fact corresponding to Fact 2 for the mean curvature flow having a focal submanifold of an equifocal submanifold as initial data.

THEOREM B. *Let M be as in the statement of Theorem A and σ be a stratum of dimension greater than zero of ∂C (which is a stratified space). Then the following statements* (i) *and* (ii) *hold:*

(i) *For any non-minimal focal submanifold F (of M) through σ, the maen curvature flow F_t having F as initial data collapses to a focal submanifold F' (of M) through $\partial\sigma$ in finite time. Furthermore, if M is irreducible, the codimension of M is greater than one and if the fibration of F onto F' is spherical, then the flow F_t has type I singularity.*

(ii) *For any focal submanifold F of M through $\partial\sigma$, there exists a focal submanifold of M through σ collapsing to F along the mean curvature flow, and the set of all focal submanifolds of M through σ collapsing to F along the mean curvature flow is a one-parameter C^∞-family.*

Since focal submanifolds of M through 0-dimensional stratums of ∂C are minimal, it follows from these theorems that M collapses to a minimal focal submanifold of M after finitely many times of collapses along the mean curvature flows.

$$M_t \underset{(t \to T_1)}{\longrightarrow} \underset{\text{non-min.}}{F^1}$$
$$F_t^1 \underset{(t \to T_2)}{\longrightarrow} \underset{\text{non-min.}}{F^2}$$
$$\ddots$$
$$F_t^{k-1} \underset{(t \to T_k)}{\longrightarrow} \underset{\text{min.}}{F^k}$$

$$\left(\begin{array}{l} F^1 \ : \ \text{a focal submanifold of } M \\ F^i \ : \ \text{a focal submanifold of } F^{i-1} \ (i = 2, \cdots, k) \end{array} \right)$$

According to the homogeneity theorem for an equifocal submanifold by Christ [Ch], all irreducible equifocal submanifolds of codimension greater than one in symmetric spaces of compact type are homogeneous. Hence, according to the result by Heintze-Palais-Terng-Thorbergsson [HPTT], they are principal orbits of hyperpolar actions. Furthermore, according to the classification by Kollross [Kol] of hyperpolar actions on irreducible symmetric spaces of compact type, all hyperpolar actions of cohomogeneity greater than one on the symmetric spaces are Hermann actions. Therefore, all equifocal submanifolds of codimension greater than one in irreducible symmetric spaces of compact type are principal orbits of Hermann actions. In the last section, we describe explicitly the mean curvature flows having orbits of Hermann actions of cohomogeneity two on irreducible symmetric spaces of compact type and rank two as initial data.

2. Preliminaries. In this section, we briefly review the quantities associated with an isoparametric submanifold in an (infinite dimensional separable) Hilbert space, which was introduced by Terng [T2]. Let M be an isoparametric submanifold in a Hilbert space V.

2.1. Principal curvatures, curvature normals and curvature distributions. Let E_0 and E_i ($i \in I$) be all the curvature distributions of M, where E_0 is defined by $(E_0)_x = \underset{v \in T_x^\perp M}{\cap} \operatorname{Ker} A_v$ ($x \in M$). For each $x \in M$, we have $T_x M = \overline{(E_0)_x \oplus \left(\underset{i \in I}{\oplus} (E_i)_x \right)}$, which is the common eigenspace decomposition of A_v's ($v \in T_x^\perp M$). Also, let λ_i ($i \in I$) be the principal curvatures of M, that is, λ_i is the section of the dual bundle $(T^\perp M)^*$ of $T^\perp M$ such that $A_v|_{(E_i)_x} = (\lambda_i)_x(v)\mathrm{id}$ holds for any $x \in M$ and any $v \in T_x^\perp M$, and $\mathbf{n}_i$ be the curvature normal corresponding to λ_i, that is, $\lambda_i(\cdot) = \langle \mathbf{n}_i, \cdot \rangle$.

2.2. The Coxeter group associated with an isoparametric submanifold. Denote by l_i^x the affine hyperplane $(\lambda_i)_x^{-1}(1)$ in $T_x^\perp M$. The focal set of M at x is equal to the sum $\underset{i \in I}{\cup}(x + l_i^x)$ of the affine hyperplanes $x + l_i^x$'s ($i \in I$) in the affine subspace $x + T_x^\perp M$ of V. Each affine hyperplane l_i^x is called a focal hyperplane of M at x. Let W be the group generated by the reflection R_i^x's ($i \in I$) with respect to l_i^x. This group is independent of the choice x of M up to group isomorphism. This group is called the Coxeter group associated with M. The fundamental domain of the Coxeter group containing the zero vector of $T_x^\perp M$ is given by $\{v \in T_x^\perp M \,|\, \lambda_i(v) < 1 \ (i \in I)\}$.

2.3. Principal curvatures of parallel submanifolds. Let M_w be the parallel submanifold of M for a (non-focal) parallel normal vector field w, that is, $M_w =$

$\eta_w(M)$, where η_w is the end-point map for w. Denote by A^w the shape tensor of M_w. This submanifold M_w also is isoparametric and $A_v^w|_{\eta_{w*}(E_i)_x} = \frac{(\lambda_i)_x(v)}{1-(\lambda_i)_x(w_x)}\mathrm{id}$ $(i \in I)$ for any $v \in T_{\eta_w(x)}^\perp M_w$, that is, $\frac{\lambda_i}{1-\lambda_i(w)}$'s $(i \in I)$ are the principal curvatures of M_w and hence $\frac{\mathbf{n}_i}{1-\lambda_i(w)}$'s $(i \in I)$ are the curvature normals of M_w, where we identify $T_{\eta_w(x)}^\perp M_w$ with $T_x^\perp M$.

Let M be a (general) submanifold in a Hilbert space V.

2.4. The mean curvature vector of a regularizable submanifold. Assume that M is regularizable in sense of [HLO], that is, for each normal vector v of M, the regularizable trace $\mathrm{Tr}_r A_v$ and $\mathrm{Tr}\, A_v^2$ exist, where $\mathrm{Tr}_r A_v$ is defined by $\mathrm{Tr}_r A_v := \sum_{i=1}^\infty (\mu_i^+ + \mu_i^-)$ $(\mu_1^- \le \mu_2^- \le \cdots \le 0 \le \cdots \le \mu_2^+ \le \mu_1^+ :$ the spectrum of A_v). Then the mean curvature vector H of M is defined by $\langle H, v \rangle = \mathrm{Tr}_r A_v$ $(\forall v \in T^\perp M)$.

2.5. The mean curvature flow for a regularizable submanifold. Let $f_t : M \hookrightarrow V$ $(0 \le t < T)$ be a C^∞-family of regularizable submanifold immersions into V. Denote by H_t the mean curvature vector of f_t. Define a map $F : M \times [0, T) \to V$ by $F(x, t) := f_t(x)$ $((x, t) \in M \times [0, T))$. If $\frac{\partial F}{\partial t} = H_t$ holds, then we call f_t $(0 \le t < T)$ the *mean curvature flow*. In particular, if f_t's are embeddings, then we set $M_t := f_t(M)$ and call M_t (rather than f_t) the *mean curvature flow*. Note that, for a given regularizable submanifold immersion $f : M \hookrightarrow V$, there does not necessarily exist the mean curvature flow having f as initial data in short time. Furthermore, we note that, for the restriction $f|_U$ of f to a relative compact domain U of M, the existence of the mean curvature flow having $f|_U$ as initial data in short time is not assured for the sake of absence of the infinite dimensional vector bundle version of the Hamilton's theorem (Theorem 5.1 of [Ha]) for the existence and the uniqueness of solutions of a certain kind of evolution equation (which includes the Ricci flow equation and the mean curvature flow equation (of finite dimensional case)) in short time.

Let M be an equifocal submanifold in a symmetric space G/K of compact type and set $\widetilde{M} := (\pi \circ \phi)^{-1}(M)$, where π is the natural projection of G onto G/K and $\phi : H^0([0, 1], \mathfrak{g}) \to G$ is the parallel transport map for G.

2.6. The mean curvature vector of the lifted submanifold. Denote by $\widetilde{H}$ (resp. H) the mean curvature vector of $\widetilde{M}$ (resp. M). Then $\widetilde{M}$ is a regularizable isoparametric submanifold and $\widetilde{H}$ is equal to the horizontal lift of H^L of H (see Lemma 5.2 of [HLO]).

3. Proofs of Theorems A and B. In this section, we prove Theorems A and B. Let M be an equifocal submanifold in a symmetric space G/K of compact type, $\pi : G \to G/K$ be the natural projection and ϕ be the parallel transport map for G. Set $\widetilde{M} := (\pi \circ \phi)^{-1}(M)$. Take $u_0 \in \widetilde{M}$ and set $x_0 := (\pi \circ \phi)(u_0)$. We identify $T_{x_0}^\perp M$ with $T_{u_0}^\perp \widetilde{M}$. Let $\widetilde{C}(\subset T_{u_0}^\perp \widetilde{M} = T_{x_0}^\perp M)$ be the fundamental domain of the Coxeter group of $\widetilde{M}$ at u_0 containing the zero vector $\mathbf{0}$ of $T_{u_0}^\perp \widetilde{M}(= T_{x_0}^\perp M)$ and set $C := \exp^\perp(\widetilde{C})$, where $\exp^\perp$ is the normal exponential map of M. Denote by H (resp. $\widetilde{H}$) the mean curvature vector of M (resp. $\widetilde{M}$). The mean curvature vector H and $\widetilde{H}$ are a parallel normal vector field of M and $\widetilde{M}$, respectively. Let w be a parallel

normal vector field of M and w^L be the horizontal lift of w to $H^0([0,1],\mathfrak{g})$, which is a parallel normal vector field of $\widetilde{M}$. Denote by M_w (resp. $\widetilde{M}_{w^L}$) the parallel (or focal) submanifold $\eta_w(M)$ (resp. $\eta_{w^L}(\widetilde{M})$) of M (resp. $\widetilde{M}$), where η_w (resp. η_{w^L}) is the end-point map for w (resp. w^L). Then we have $\widetilde{M}_{w^L} = (\pi \circ \phi)^{-1}(M_w)$. Denote by H^w (resp. $\widetilde{H}^{w^L}$) the mean curvature vector of M_w (resp. $\widetilde{M}_{w^L}$). Define a vector field X on $\widetilde{C}(\subset T_{u_0}^{\perp}\widetilde{M} = T_{x_0}^{\perp}M)$ by $X_w := (\widetilde{H}^{\widetilde{w}})_{u_0+w}$ $(w \in \widetilde{C})$, where $\widetilde{w}$ is the parallel normal vector field of $\widetilde{M}$ with $\widetilde{w}_{u_0} = w$. Let $\xi : (-S,T) \to \widetilde{C}$ be the maximal integral curve of X with $\xi(0) = \mathbf{0}$. Note that S and T are possible be equal to ∞. Let $\widetilde{\xi(t)}$ be the parallel normal vector field of M with $\widetilde{\xi(t)}_{x_0} = \xi(t)$.

LEMMA 3.1. *The family* $\widetilde{M}_{\widetilde{\xi(t)}^L}$ *($0 \le t < T$) is the mean curvature flow having* $\widetilde{M}$ *as initial data and* $M_{\widetilde{\xi(t)}}$ *($0 \le t < T$) is the mean curvature flow having* M *as initial data.*

Proof. Fix $t_0 \in [0,T)$. Define a flow $F : \widetilde{M} \times [0,T) \to H^0([0,1],\mathfrak{g})$ by $F(u,t) := \eta_{\widetilde{\xi(t)}^L}(u)$ $((u,t) \in \widetilde{M} \times [0,T))$ and $f_t : \widetilde{M} \to H^0([0,1],\mathfrak{g})$ $(0 \le t < T)$ by $f_t(u) := F(u,t)$ $(u \in \widetilde{M})$. Here we note that $f_t(\widetilde{M}) = \widetilde{M}_{\widetilde{\xi(t)}^L}$. For simplicity, denote by $\widetilde{H}^{t_0}$ the mean curvature vector of $\widetilde{M}_{\widetilde{\xi(t_0)}^L}$. It is easy to show that $F_*((\frac{\partial}{\partial t})_{(\cdot,t_0)})$ is a parallel normal vector field of $\widetilde{M}_{\widetilde{\xi(t_0)}^L}$ and that $F_*((\frac{\partial}{\partial t})_{(u_0,t_0)}) = (\widetilde{H}^{t_0})_{f_{t_0}(u_0)}$. On the other hand, since $\widetilde{M}_{\widetilde{\xi(t_0)}^L}$ is isoparametric, $\widetilde{H}^{t_0}$ is also a parallel normal vector field of $\widetilde{M}_{\widetilde{\xi(t_0)}^L}$. Hence we have $F_*((\frac{\partial}{\partial t})_{(\cdot,t_0)}) = \widetilde{H}^{t_0}$. Therefore, it follows from the arbitrariness of t_0 that $\widetilde{M}_{\widetilde{\xi(t)}^L}$ $(0 \le t < T)$ is the mean curvature flow having $\widetilde{M}$ as initial data. Define a flow $\overline{F} : M \times [0,T) \to G/K$ by $\overline{F}(x,t) := \eta_{\widetilde{\xi(t)}}(x)$ $((x,t) \in M \times [0,T))$ and $\bar{f}_t : M \to G/K$ $(0 \le t < T)$ by $\bar{f}_t(x) := \overline{F}(x,t)$ $(x \in M)$. Here we note that $\bar{f}_t(M) = M_{\widetilde{\xi(t)}}$. For simplicity, denote by H^t the mean curvature vector of $M_{\widetilde{\xi(t)}}$. Fix $t_0 \in [0,T)$. Since $\widetilde{M}_{\widetilde{\xi(t_0)}^L} = (\pi \circ \phi)^{-1}(M_{\widetilde{\xi(t_0)}})$, we have $(H^{t_0})^L = \widetilde{H}^{t_0}$. On the other hand, we have $\overline{F}_*((\frac{\partial}{\partial t})_{(\cdot,t_0)})^L = F_*((\frac{\partial}{\partial t})_{(\cdot,t_0)})(= \widetilde{H}^{t_0})$. Hence we have $\overline{F}_*((\frac{\partial}{\partial t})_{(\cdot,t_0)}) = H^{t_0}$. Therefore, it follows from the arbitrariness of t_0 that $M_{\widetilde{\xi(t)}}$ $(0 \le t < T)$ is the mean curvature flow having M as initial data. $\square$

Proof of Theorem A. Clearly we suffice to show the statement of Theorem A in the case where M is full. Hence, in the sequel, we assume that M is full. Denote by Λ the set of all principal curvatures of $\widetilde{M}$. Set $r := \operatorname{codim} M$. It is shown that the set of all focal hyperplanes of $\widetilde{M}$ is given as the sum of finite pieces of infinite parallel families consisting of hyperplanes in $T_{u_0}^{\perp}\widetilde{M}$ which arrange at equal intervals. Let $\{l_{aj} \,|\, j \in \mathbb{Z}\}$ $(1 \le a \le \bar{r})$ be the finite pieces of infinite parallel families consisting of hyperplanes in $T_{u_0}^{\perp}\widetilde{M}$. Since l_{aj}'s $(j \in \mathbb{Z})$ arrange at equal intervals, we can express as $\Lambda = \bigcup_{a=1}^{\bar{r}} \{\frac{\lambda_a}{1+b_a j} \,|\, j \in \mathbf{Z}\}$, where λ_a's and b_a's are parallel sections of $(T^{\perp}\widetilde{M})^*$ and positive constants greater than one, respectively, which are defined by $((\lambda_a)_{u_0})^{-1}(1+b_a j) = l_{aj}$. For simplicity, we set $\lambda_{aj} := \frac{\lambda_a}{1+b_a j}$. Denote by $\mathbf{n}_{aj}$ and E_{aj} the curvature normal and the curvature distribution corresponding to λ_{aj}, respectively. The fundamental domain $\widetilde{C}$ of the Coxeter group of $\widetilde{M}$ at u_0 is given

by

$$\widetilde{C} = \{w \in T_{u_0}^{\perp}\widetilde{M} \mid \lambda_a(w) < 1 \ (1 \le a \le \bar{r})\}.$$

It is shown that, for each a, $\lambda_{a,2j}$'s $(j \in \mathbb{Z})$ have the same multiplicity and so are also $\lambda_{a,2j+1}$'s $(j \in \mathbb{Z})$. Denote by m_a^e and m_a^o the multiplicities of $\lambda_{a,2j}$ and $\lambda_{a,2j+1}$, respectively. Take a parallel normal vector field w of $\widetilde{M}$ with $w_{u_0} \in \widetilde{C}$. Denote by $\widetilde{A}^w$ (resp. $\widetilde{H}^w$) the shape tensor (resp. the mean curvature vector) of the parallel submanifold $\widetilde{M}_w$. Since $\widetilde{A}_v^w|_{\eta_{w*}(E_{aj})_u} = \frac{(\lambda_{aj})_u(v)}{1-(\lambda_{aj})_u(w_u)}$ id $(v \in T_u^{\perp}\widetilde{M})$, we have

$$\mathrm{Tr}_r\widetilde{A}_v^w = \sum_{a=1}^{\bar{r}} \left(\sum_{j\in\mathbb{Z}} \frac{m_a^e(\lambda_{a,2j})_u(v)}{1-(\lambda_{a,2j})_u(w_u)} + \sum_{j\in\mathbb{Z}} \frac{m_a^o(\lambda_{a,2j+1})_u(v)}{1-(\lambda_{a,2j+1})_u(w_u)} \right)$$
$$= \sum_{a=1}^{\bar{r}} \left(m_a^e \cot\frac{\pi}{2b_a}(1-(\lambda_a)_u(w_u)) - m_a^o \tan\frac{\pi}{2b_a}(1-(\lambda_a)_u(w_u)) \right) \frac{\pi}{2b_a}(\lambda_a)_u(v),$$

where we use the relation $\cot\frac{\theta}{2} = \sum_{j\in\mathbb{Z}} \frac{2}{\theta+2j\pi}$. Therefore we have

$$(3.1) \qquad \widetilde{H}^w = \sum_{a=1}^{\bar{r}} \left(m_a^e \cot\frac{\pi}{2b_a}(1-\lambda_a(w)) - m_a^o \tan\frac{\pi}{2b_a}(1-\lambda_a(w)) \right) \frac{\pi}{2b_a}\mathbf{n}_a,$$

where $\mathbf{n}_a$ is the curvature normal corresponding to λ_a. Denote by $\widetilde{\sigma}_a$ $(1 \le a \le \bar{r})$ the maximal dimensional stratum $(\lambda_a)_{u_0}^{-1}(1)$ of $\partial\widetilde{C}$. Fix $a_0 \in \{1,\cdots,\bar{r}\}$. Take $w_0 \in \widetilde{\sigma}_{a_0}$ and and $w_0' \in \widetilde{C}$ near w_0 such that $w_0 - w_0'$ is normal to $\widetilde{\sigma}_{a_0}$. Set $w_0^{\varepsilon} := \varepsilon w_0' + (1-\varepsilon)w_0$ for $\varepsilon \in (0,1)$. Then we have $\lim_{\varepsilon\to+0}(\lambda_{a_0})_{u_0}(w_0^{\varepsilon}) = 1$ and $\sup_{0<\varepsilon<1}(\lambda_a)_{u_0}(w_0^{\varepsilon}) < 1$ for each $a \in \{1,\cdots,\bar{r}\} \setminus \{a_0\}$. Hence we have $\lim_{\varepsilon\to+0}\cot\frac{\pi}{2b_{a_0}}(1-(\lambda_{a_0})_{u_0}(w_0^{\varepsilon})) = \infty$ and $\sup_{0<\varepsilon<1}\cot(1-(\lambda_a)_{u_0}(w_0^{\varepsilon})) < \infty$ $(a \in \{1,\cdots,\bar{r}\} \setminus \{a_0\})$. Therefore, we have $\lim_{\varepsilon\to+0}\frac{X_{w_0^{\varepsilon}}}{||X_{w_0^{\varepsilon}}||}$ is the outward unit normal vector of $\widetilde{\sigma}_{a_0}$. Also we have $\lim_{\varepsilon\to+0}||X_{w_0^{\varepsilon}}|| = \infty$. From these facts, X is as in Fig. 1 on a sufficiently small collar neighborhood of $\widetilde{\sigma}_{a_0}$.
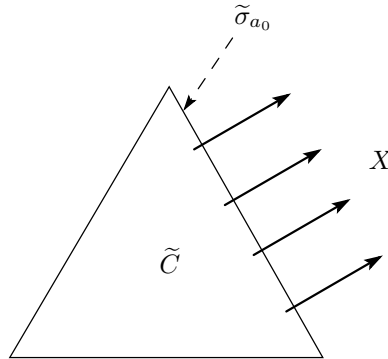
Fig. 1.

Define a function ρ over $\widetilde{C}$ by

$$\rho(w) := -\sum_{a=1}^{\bar{r}} \left(m_a^e \log \sin \frac{\pi}{2b_a}(1 - (\lambda_a)_{u_0}(w)) \right.$$
$$\left. + m_a^o \log \cos \frac{\pi}{2b_a}(1 - (\lambda_a)_{u_0}(w)) \right) \qquad (w \in \widetilde{C}).$$

Let $(x_1, \cdots, x_r)$ be the Euclidean coordinate of $T_{u_0}^{\perp}\widetilde{M}$. For simplicity, set $\partial_i := \frac{\partial}{\partial x_i}$ $(i = 1, \cdots, r)$. Then it follows from the definition of X and (3.1) that $(\partial_i \rho)(w) = \langle X_w, \partial_i \rangle$ $(w \in \widetilde{C}, \ i = 1, \cdots, r)$, that is, $\operatorname{grad} \rho = X$. Also we have

$$(\partial_i \partial_j \rho)(w) = \sum_{a=1}^{\bar{r}} \left(\frac{m_a^e}{\sin^2 \frac{\pi}{2b_a}(1 - (\lambda_a)_{u_0}(w))} + \frac{m_a^o}{\cos^2 \frac{\pi}{2b_a}(1 - (\lambda_a)_{u_0}(w))} \right)$$
$$\times \frac{\pi^2}{4b_a^2}(\lambda_a)_{u_0}(\partial_i)(\lambda_a)_{u_0}(\partial_j).$$

It follows from this relation that ρ is downward convex. Also it is shown that $\rho(w) \to \infty$ as $w \to \partial \widetilde{C}$. Hence we see that ρ has the only minimal point. Denote by w_0 this minimal point. It is clear that $X_{w_0} = 0$. From these facts and the fact that X is as in Fig. 1 on a sufficiently small collar neighborhood of each maximal dimensional stratum $\widetilde{\sigma}$ of $\partial \widetilde{C}$, ρ and X are as in Fig. 2.
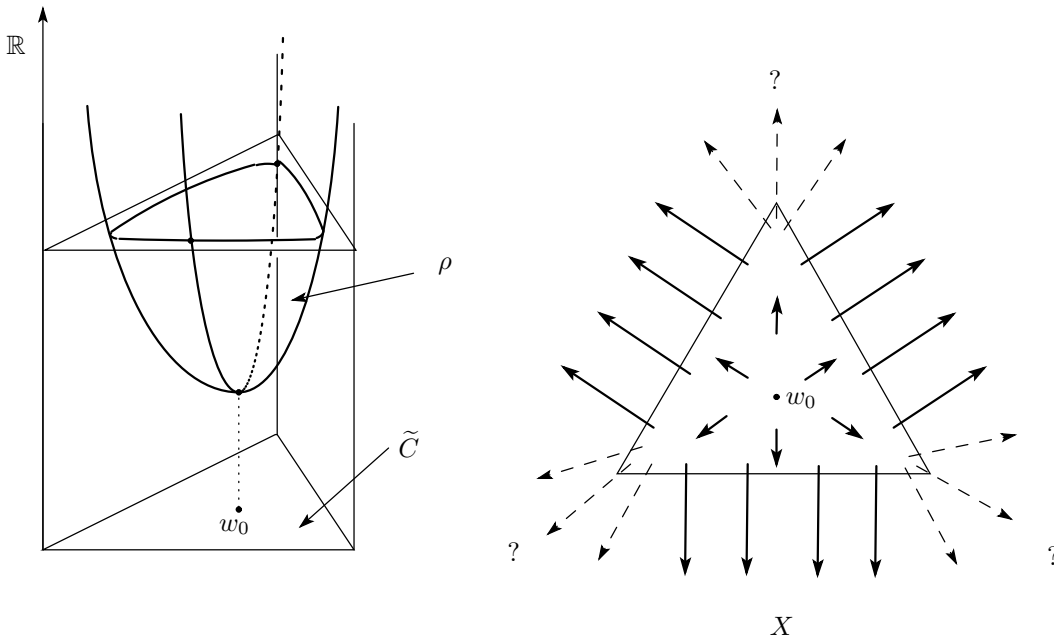
Fig. 2.

Let W be the Coxeter group of $\widetilde{M}$ at u_0 and W_0 the isotropy group of W at $\mathbf{0}$. Also, let Γ be the lattice of translations of W and set $\Gamma^* := \{\omega \in (T_{u_0}^{\perp}\widetilde{M})^* \,|\, \omega(\Gamma) \subset \mathbf{Z}\}$. Also, let $C^{\triangle}(T_{u_0}^{\perp}\widetilde{M})^W$ be the space of all finite sums of W-invariant eigenfunctions

of $\triangle$, where $\triangle$ is the Laplace operator of $T_{u_0}^{\perp}\widetilde{M}$. Then we can show

$$C^{\triangle}(T_{u_0}^{\perp}\widetilde{M})^W \otimes \mathbb{C} = \{f = \sum_{\omega \in \Gamma^*} a_{\omega} e^{2\pi\sqrt{-1}\omega} \,|\, a_{\omega} \in \mathbb{C} \ (a_{\omega} = 0 \text{ except for finite } \omega)$$
$$\& \ f : W - \text{invariant}\}.$$

Hence, according to Chapter VI,§ 3, Theorem 1 of [B], we have $C^{\triangle}(T_{u_0}^{\perp}\widetilde{M})^W \otimes \mathbb{C} = \mathbb{C}[\phi_1, \cdots, \phi_r]$ (polynomial ring) for some $\phi_1, \cdots, \phi_r (\in C^{\triangle}(T_{u_0}^{\perp}\widetilde{M})^W)$. Here we note that $r = \dim T_{u_0}^{\perp}\widetilde{M}$ because G/K is irreducible and hence M is full and irreducible. By reordering $\phi_1, \cdots, \phi_r$ suitably, it is shown that $\{\text{Re}\,\phi_1, \cdots, \text{Re}\,\phi_{r_1+r_2}, \text{Im}\,\phi_{r_1+1}, \cdots, \text{Im}\,\phi_{r_1+r_2}\}$ is a base of $C^{\triangle}(T_{u_0}^{\perp}(\widetilde{M}))^W$ (see proof of Theorem 7.6 of [HLO]), where $\text{Re}(\cdot)$ (resp. $\text{Im}(\cdot)$) is the real part (resp. the imaginary part) of $(\cdot)$, and r_1 and r_2 are positive integers with $r_1 + 2r_2 = r$. Denote by $\{\psi_1, \cdots, \psi_r\}$ this base for simplicity. Set $\Psi := (\psi_1, \cdots, \psi_r)$, which is a C^{∞}-map from $T_{u_0}^{\perp}\widetilde{M}$ onto $\mathbb{R}^r$. It is shown that Ψ is injective and that $\Psi|_{\overline{\widetilde{C}}}$ is a homeomorphism of $\overline{\widetilde{C}}$ onto $\Psi(\overline{\widetilde{C}})$, where $\overline{\widetilde{C}}$ is the closure of $\widetilde{C}$. Set $\xi_w(t) := \psi_t(w)$ and $\bar{\xi}_w(t) := \Psi(\psi_t(w))$, where $w \in \widetilde{C}$ and $\{\psi_t\}$ is the local one parameter transformation group of X. Also, we set $\xi_w^i(t) := x_i(\xi_w(t))$ and $\bar{\xi}_w^i(t) := y_i(\bar{\xi}_w(t))$, where $(y_1, \cdots, y_r)$ is the natural coordinate of $\mathbb{R}^r$. Then we have

$$(\bar{\xi}_w^i)'(t) = \langle \text{grad}(y_i \circ \Psi)_{\xi_w(t)}, X_{\xi_w(t)} \rangle$$
$$= \sum_{a=1}^{\bar{r}} \left(\left(m_a^e \cot(\frac{\pi}{2b_a}(1 - (\lambda_a)_{u_0}(\xi_w(t)))) - m_a^o \tan(\frac{\pi}{2b_a}(1 - (\lambda_a)_{u_0}(\xi_w(t)))) \right) \right.$$
$$\left. \times \frac{\pi}{2b_a}(\lambda_a)_{u_0}(\text{grad}(y_i \circ \Psi)_{\xi_w(t)}) \right).$$

Let f_i be the W-invariant C^{∞}-function over $T_{u_0}^{\perp}\widetilde{M}$ such that

$$f_i(v) := \sum_{a=1}^{\bar{r}} \left(\left(m_a^e \cot(\frac{\pi}{2b_a}(1 - (\lambda_a)_{u_0}(v))) - m_a^o \tan(\frac{\pi}{2b_a}(1 - (\lambda_a)_{u_0}(v))) \right) \right.$$
$$\left. \times \frac{\pi}{2b_a}(\lambda_a)_{u_0}(\text{grad}(y_i \circ \Psi)_v) \right)$$

for all $v \in W \cdot \widetilde{C}$. It is easy to show that such a W-invariant C^{∞}-function exists uniquely. Hence, we can describe f_i as $f_i = Y_i \circ \Psi$ in terms of some C^{∞}-function Y_i over $\mathbb{R}^r$. Set $Y := (Y_1, \cdots, Y_r)$, which is regarded as a C^{∞}-vector field on $\mathbb{R}^r$. Then we have $Y_{\Psi(w)} = \Psi_*(X_w)$ $(w \in \widetilde{C})$, that is, $Y|_{\Psi(\widetilde{C})} = \Psi_*(X)$. Also we can show that $Y|_{\partial\Psi(\widetilde{C})}$ has no zero point. From these facts and the fact that X is as in Fig. 2, we see that the flow of X starting any point of $\widetilde{C}$ other than w_0 (w_0 : the only zero point of X) converges to a point of $\partial\widetilde{C}$ in finite time, and that, furthermore, for each point of $\partial\widetilde{C}$, there exists a unique flow of X converging to the point. Since M is not minimal by the assumption, we have $X_{\mathbf{0}} \neq 0$, that is, $\mathbf{0} \neq w_0$. Hence we have $T < \infty$ and $\lim_{t \to T} \xi(t) \in \partial\widetilde{C}$, where $\xi(t) = \psi_t(\mathbf{0})$ and T is the supremum of the domain of ξ. Set $w_1 := \lim_{t \to T-0} \xi(t)$. Therefore, since $M_t = M_{\widetilde{\xi(t)}}$, the mean curvature flow M_t collapses to the focal submanifold $F := M_{\widetilde{w}_1}$ in time T, where $\widetilde{w}_1$ is the parallel normal vector field of M with $(\widetilde{w}_1)_{x_0} = w_1$. Also, M_t's $(-S < t < T)$ are all of parallel submanifolds of M collapsing to F along the mean curvature flow, where

$-S$ is the infimum of the domain of ξ. Thus the first-half part of the statement (i) and the statement (ii) are shown. Next we shall show the second-half part of the statement (i). Assume that M is irreducible and the codimension of M is greater than one and that the fibration of M onto F is spherical. Set $\widetilde{M} := (\pi \circ \phi)^{-1}(M)$ and $\widetilde{F} := (\pi \circ \phi)^{-1}(F)$. Since the fibration of M onto F is spherical, $\widetilde{F}$ passes through a highest dimensional stratum $\widetilde{\sigma}$ of $\partial \widetilde{C}$. Let a_0 be the element of $\{1, \cdots, \bar{r}\}$ with $\widetilde{\sigma} \subset (\lambda_{a_0})_{u_0}^{-1}(1)$. Set $\widetilde{M}_t := (\pi \circ \phi)^{-1}(M_t)$ $(t \in [0, T))$, which is the mean curvature flow having $\widetilde{M}$ as initial data. Denote by A^t (resp. $\widetilde{A}^t$) the shape tensor of M_t (resp. $\widetilde{M}_t$). Then, since $\widetilde{M}_t$ is the parallel submanifold of $\widetilde{M}$ for $\widetilde{\xi(t)}^L$, we have

$$\operatorname{Spec} \widetilde{A}_v^t \setminus \{0\} = \{\frac{(\lambda_{aj})_{u_0}(v)}{1 - (\lambda_{aj})_{u_0}(\xi(t))} \,|\, a = 1, \cdots, \bar{r}, \ j \in \mathbb{Z}\}$$

for each $v \in T_{u_0+\xi(t)}^{\perp} \widetilde{M}_t = T_{u_0}^{\perp} \widetilde{M}$. Since $\lim_{t \to T-0} \xi(t) \in (\lambda_{a_0})_{u_0}^{-1}(1)$ and $\lim_{t \to T-0} \xi(t) \notin (\lambda_a)_{u_0}^{-1}(1)$ $(a \in \{1, \cdots, \bar{r}\} \setminus \{a_0\})$, we have $\lim_{t \to T-0} (\lambda_{a_0})_{u_0}(\xi(t)) = 1$ and $\lim_{t \to T-0} (\lambda_a)_{u_0}(\xi(t)) < 1$ $(a \neq a_0)$. From these facts, $\xi'(t) = (\widetilde{H}^{\widetilde{\xi(t)}^L})_{u_0+\xi(t)}$ and (3.1), we have

$$
\begin{aligned}
&\lim_{t \to T-0} ||\widetilde{A}_v^t||_\infty^2 (T - t) \\
(3.2) \quad =\ &\lim_{t \to T-0} \frac{(\lambda_{a_0})_{u_0}(v)^2}{(1 - (\lambda_{a_0})_{u_0}(\xi(t)))^2} (T - t) \\
=\ &\frac{1}{2} (\lambda_{a_0})_{u_0}(v)^2 \lim_{t \to T-0} \frac{1}{(1 - (\lambda_{a_0})_{u_0}(\xi(t)))(\lambda_{a_0})_{u_0}(\xi'(t))} \\
=\ &\frac{(\lambda_{a_0})_{u_0}(v)^2}{2 m_{a_0}^e ||(\mathbf{n}_{a_0})_{u_0}||^2}.
\end{aligned}
$$

Hence we have

$$\lim_{t \to T-0} \max_{v \in S_{u_0+\xi(t)}^{\perp} \widetilde{M}_t} ||\widetilde{A}_v^t||_\infty^2 (T - t) = \frac{1}{2 m_{a_0}^e}.$$

Thus the mean curvature flow $\widetilde{M}_t$ has type I singularity. Set $\bar{v}_t := (\pi \circ \phi)_{*u_0+\xi(t)}(v)$ and let $\{\lambda_1^t, \cdots, \lambda_n^t\}$ $(\lambda_1^t \leq \cdots \leq \lambda_n^t)$ (resp. $\{\mu_1^t, \cdots, \mu_n^t\}$ $(0 \leq \mu_1^t \leq \cdots \leq \mu_n^t)$) be all the eigenvalues of $A_{\bar{v}_t}^t$ (resp. $R(\cdot, \bar{v}_t)\bar{v}_t$), where $n := \dim M$. Since M is an irreducible equifocal submanifold of codimension greater than one by the assumption, it is homogeneous by the homogeneity theorem of Christ (see [Ch]) and hence it is a principal orbit of a Hermann action by the result of Heintze-Palais-Terng-Thorbergsson (see [HPTT]) and the classification of hyperpolar actions by Kollross (see [Kol]). Furthermore, M and its parallel submanifolds are curvature-adapted by the result of Goertsches-Thorbergsson (see [GT]). Therefore, $A_{\bar{v}_t}^t$ and $R(\cdot, \bar{v}_t)\bar{v}_t$ commute and hence we have

$$\sum_{i=1}^n \sum_{j=1}^n \left(\operatorname{Ker}(A_{\bar{v}_t}^t - \lambda_i^t \operatorname{id}) \cap \operatorname{Ker}(R(\cdot, \bar{v}_t)\bar{v}_t - \mu_j^t \operatorname{id}) \right) = T_{(\pi \circ \phi)(u_0+\xi(t))} M_t.$$

Set $\bar{E}_{ij}^t := \operatorname{Ker}(A_{\bar{v}_t}^t - \lambda_i^t \operatorname{id}) \cap \operatorname{Ker}(R(\cdot, \bar{v}_t)\bar{v}_t - \mu_j^t \operatorname{id})$ $(i, j \in \{1, \cdots, n\})$ and $I_t :=$

$\{(i,j) \in \{1, \cdots, n\}^2 \,|\, \bar{E}_{ij}^t \neq \{0\}\}$. For each $(i,j) \in I_t$, we have

$$\operatorname{Spec}(\widetilde{A}_v^t|_{(\pi\circ\phi)_*^{-1}(\bar{E}_{ij}^t)}) = \begin{cases} \left\{ \dfrac{\sqrt{\mu_j^t}}{\arctan \frac{\sqrt{\mu_j^t}}{\lambda_i^t} + k\pi} \,\Big|\, k \in \mathbb{Z} \right\} & (\mu_j^t \neq 0) \\[2mm] \{\lambda_i^t\} & (\mu_j^t = 0) \end{cases}$$

in terms of Proposition 3.2 of [Koi1] and hence

$$||\widetilde{A}_v^t||_\infty = \max\left(\left\{ \frac{\sqrt{\mu_j^t}}{\arctan \frac{\sqrt{\mu_j^t}}{|\lambda_i^t|}} \,\Big|\, (i,j) \in I_t \text{ s.t. } \mu_j^t \neq 0 \right\} \cup \{|\lambda_i^t|\,|\,(i,j) \in I_t \text{ s.t. } \mu_j^t = 0\} \right).$$

It is clear that $\sup\limits_{0 \leq t < T} \mu_n^t < \infty$. If $\lim\limits_{t \to T-0} |\lambda_i^t| = \infty$, then we have

$$\lim_{t \to T-0} \left(\frac{\sqrt{\mu_j^t}}{\arctan \frac{\sqrt{\mu_j^t}}{\lambda_i^t}} \right) / \lambda_i^t = 1. \text{ Hence we have}$$

$$\lim_{t \to T-0} ||\widetilde{A}_v^t||_\infty^2 (T - t)$$
$$= \max\left\{ \lim_{t \to T-0} (\lambda_i^t)^2 (T - t) \,|\, i = 1, \cdots, n \right\}$$
$$= \lim_{t \to T-0} \max\{(\lambda_i^t)^2 (T - t) \,|\, i = 1, \cdots, n\}$$
$$= \lim_{t \to T-0} ||A_{\bar{v}_t}^t||_\infty^2 (T - t),$$

which together with (3.2) deduces

$$\lim_{t \to T-0} ||A_{\bar{v}_t}^t||_\infty^2 (T - t) = \frac{(\lambda_{a_0})_{u_0}(v)^2}{2m_{a_0}^e ||(\mathbf{n}_{a_0})_{u_0}||^2}.$$

Therefore we obtain

$$\lim_{t \to T-0} \max_{v \in S^\perp_{\exp^\perp(\xi(t))} M_t} ||A_v^t||_\infty^2 (T - t) = \frac{1}{2m_{a_0}^e} < \infty.$$

Thus the mean curvature flow M_t has type I singularity. $\square$

Next we prove Theorem B.

Proof of Theorem B. For simplicity, set $I := \{1, \cdots, \bar{r}\}$. Let $\widetilde{\sigma}$ be a stratum of dimension greater than zero of $\partial\widetilde{C}$ and $I_{\widetilde{\sigma}} := \{a \in I \,|\, \widetilde{\sigma} \subset (\lambda_a)_{u_0}^{-1}(1)\}$. Let $w_1 \in \widetilde{\sigma}$. Denote by F (resp. $\widetilde{F}$) the focal submanifold of M (resp. $\widetilde{M}$) for $\widetilde{w}_1$ (resp. $\widetilde{w}_1^L$). Assume that F is not minimal. Then, since $\operatorname{Ker}\eta_{\widetilde{w}_1*} = \bigoplus\limits_{a \in I_{\widetilde{\sigma}}} (E_{a0})_{u_0}$, we have

$$T_{u_0+w_1}\widetilde{F} = \left(\bigoplus_{a \in I \setminus I_{\widetilde{\sigma}}} \bigoplus_{j \in \mathbf{Z}} \eta_{\widetilde{w}_1*}((E_{aj})_{u_0}) \right) \oplus \left(\bigoplus_{a \in I_{\widetilde{\sigma}}} \bigoplus_{j \in \mathbf{Z} \setminus \{0\}} \eta_{\widetilde{w}_1*}((E_{aj})_{u_0}) \right).$$

Also we have

$$T^\perp_{u_0+w_1}\widetilde{F} = \left(\bigoplus_{a \in I_{\widetilde{\sigma}}} (E_{a0})_{u_0} \right) \oplus T^\perp_{u_0}\widetilde{M},$$

where we identify $T_{u_0+w_1} H^0([0,1], \mathfrak{g})$ with $T_{u_0} H^0([0,1], \mathfrak{g})$. For $v \in T_{u_0}^{\perp} \widetilde{M} (\subset T_{u_0+w_1}^{\perp} \widetilde{F})$, we have

$$\widetilde{A}_v^{\widetilde{w}_1^L}|_{\eta_{\widetilde{w}_1 *}((E_{aj})_{u_0})} = \frac{(\lambda_{aj})_{u_0}(v)}{1 - (\lambda_{aj})_{u_0}(w_1)} \mathrm{id} \quad ((a,j) \in ((I \setminus I_{\widetilde{\sigma}}) \times \mathbf{Z}) \cup (I_{\widetilde{\sigma}} \times (\mathbf{Z} \setminus \{0\}))).$$

Hence we have

$$\mathrm{Tr}_r \widetilde{A}_v^{\widetilde{w}_1^L}$$
$$= \sum_{a \in I \setminus I_{\widetilde{\sigma}}} \left(\sum_{j \in \mathbf{Z}} \frac{m_a^e (\lambda_{a,2j})_{u_0}(v)}{1 - (\lambda_{a,2j})_{u_0}(w_1)} + \sum_{j \in \mathbf{Z}} \frac{m_a^o (\lambda_{a,2j+1})_{u_0}(v)}{1 - (\lambda_{a,2j+1})_{u_0}(w_1)} \right)$$
$$+ \sum_{a \in I_{\widetilde{\sigma}}} \left(\sum_{j \in \mathbf{Z} \setminus \{0\}} \frac{m_a^e (\lambda_{a,2j})_{u_0}(v)}{1 - (\lambda_{a,2j})_{u_0}(w_1)} + \sum_{j \in \mathbf{Z}} \frac{m_a^o (\lambda_{a,2j+1})_{u_0}(v)}{1 - (\lambda_{a,2j+1})_{u_0}(w_1)} \right)$$
$$= \sum_{a \in I \setminus I_{\widetilde{\sigma}}} \left(m_a^e \cot \frac{\pi}{2b_a} (1 - (\lambda_a)_{u_0}(w_1)) - m_a^o \tan \frac{\pi}{2b_a} (1 - (\lambda_a)_{u_0}(w_1)) \right) \frac{\pi}{2b_a} (\lambda_a)_{u_0}(v),$$

that is, the $T_{u_0}^{\perp} \widetilde{M}$-component $((\widetilde{H}^{\widetilde{w}_1^L})_{u_0+w_1})_{T_{u_0}^{\perp} \widetilde{M}}$ of $(\widetilde{H}^{\widetilde{w}_1^L})_{u_0+w_1}$ is equal to

$$\sum_{a \in I \setminus I_{\widetilde{\sigma}}} \left(m_a^e \cot \frac{\pi}{2b_a} (1 - (\lambda_a)_{u_0}(w_1)) - m_a^o \tan \frac{\pi}{2b_a} (1 - (\lambda_a)_{u_0}(w_1)) \right) \frac{\pi}{2b_a} (\mathbf{n}_a)_{u_0}.$$

Denote by $\Phi_{u_0+w_1}$ the normal holonomy group of $\widetilde{F}$ at $u_0 + w_1$ and L_{u_0} be the focal leaf through u_0 for $\widetilde{w}_1$. Since $L_{u_0} = \Phi_{u_0+w_1} \cdot u_0$, there exists $\mu \in \Phi_{u_0+w_1}$ such that $\mu(T_{u_0}^{\perp} \widetilde{M}) = T_{u_1}^{\perp} \widetilde{F}$ for any point u_1 of L_{u_0}. On the other hand, since $\widetilde{F}$ has constant principal curvatures in the sense of [HOT], $(\widetilde{H}^{\widetilde{w}_1^L})_{u_0+w_1}$ is $\Phi_{u_0+w_1}$-invariant. Hence we have $(\widetilde{H}^{\widetilde{w}_1^L})_{u_0+w_1} \in \bigcap_{u \in L_{u_0}} T_u^{\perp} \widetilde{M}$, where we note that $\bigcap_{u \in L_{u_0}} T_u^{\perp} \widetilde{M}$ contains $\widetilde{\sigma}$ as an open subset. Therefore, we obtain

$$(3.3) \quad (\widetilde{H}^{\widetilde{w}_1^L})_{u_0+w_1} = \sum_{a \in I \setminus I_{\widetilde{\sigma}}} \left(m_a^e \cot \frac{\pi}{2b_a} (1 - (\lambda_a)_{u_0}(w_1)) \right.$$
$$\left. - m_a^o \tan \frac{\pi}{2b_a} (1 - (\lambda_a)_{u_0}(w_1)) \right) \frac{\pi}{2b_a} (\mathbf{n}_a)_{u_0} \quad (\in T\widetilde{\sigma}).$$

Define a tangent vector field $X^{\widetilde{\sigma}}$ on $\widetilde{\sigma}$ by $X_w^{\widetilde{\sigma}} := (\widetilde{H}^{\widetilde{w}^L})_{u_0+w}$ $(w \in \widetilde{\sigma})$. Let $\xi : (-S, T) \to \widetilde{\sigma}$ be the maximal integral curve of $X^{\widetilde{\sigma}}$ with $\xi(0) = w_1$. Define a function $\rho_{\widetilde{\sigma}}$ over $\widetilde{\sigma}$ by

$$\rho_{\widetilde{\sigma}}(w) := - \sum_{a \in I \setminus I_{\widetilde{\sigma}}} \left(m_a^e \log \sin \frac{\pi}{2b_a} (1 - (\lambda_a)_{u_0}(w)) \right.$$
$$\left. + m_a^o \log \cos \frac{\pi}{2b_a} (1 - (\lambda_a)_{u_0}(w)) \right) \quad (w \in \widetilde{\sigma}).$$

It follows from the definition of $X^{\widetilde{\sigma}}$ and (3.3) that $\mathrm{grad}\, \rho_{\widetilde{\sigma}} = X^{\widetilde{\sigma}}$. Also we can show that $\rho_{\widetilde{\sigma}}$ is downward convex and that $\rho_{\widetilde{\sigma}}(w) \to \infty$ as $w \to \partial \widetilde{\sigma}$. Hence we see that $\rho_{\widetilde{\sigma}}$ has the only minimal point. Denote by w_0 this minimal point. It is clear that $X_{w_0}^{\widetilde{\sigma}} = 0$. Also, by imitating the proof of Theorem A, we can show that the flow of $X^{\widetilde{\sigma}}$ starting

any point of $\widetilde{\sigma}$ other than w_0 converges to a point of $\partial\widetilde{\sigma}$ in finite time, and that, furthermore, for each point of $\partial\widetilde{\sigma}$, there exists a unique flow of $X^{\widetilde{\sigma}}$ converging to the point. Since F is not minimal by the assumption, we have $X^{\widetilde{\sigma}}_{w_1} \neq 0$, that is, $w_1 \neq w_0$. Hence we have $T < \infty$ and $\lim\limits_{t\to T-0} \xi(t) \in \partial\widetilde{\sigma}$. Set $w_2 := \lim\limits_{t\to T-0} \xi(t)$. Therefore, since $F_t = M_{\widetilde{\xi(t)}}$, the mean curvature flow F_t collapses to the lower dimensional focal submanifold $F' := M_{\widetilde{w_2}}$ in time T, where $\widetilde{w}_2$ is the parallel normal vector field of M with $(\widetilde{w}_2)_{x_0} = w_2$. Also, F_t's $(-S < t < T)$ are all of focal submanifolds of M through $\widetilde{\sigma}$ collapsing to F' along the mean curvature flow. Thus the first-half part of the statement (i) and the statement (ii) are shown. Also, by imitating the proof of Theorem A, we can show the second-half part of the statement (i). $\square$

4. Hermann actions of cohomogeneity two. According to the homogeneity theorem for an equifocal submanifold in a symmetric space of compact type by Christ ([Ch]), equifocal submanifolds of codimension greater than one in an irreducible compact type symmetric space are homogeneous. Hence, according to the result by Heintze-Palais-Terng-Thorbergsson ([HPTT]), they occur as principal orbits of hyperpolar actions on the symmetric space. Furthermore, by using the classification of hyperpolar actions on irreducible compact type symmetric spaces by Kollross ([Kol]), we see that they occur as principal orbits of Hermann actions on the symmetric spaces. We have only to analyze the vector field X defined in the previous secton to analyze the mean curvature flows having parallel submanifolds of an equifocal submanifold M as initial data. Also, we have only to analyze the vector fields $X^{\widetilde{\sigma}}$'s ($\widetilde{\sigma}$: a simplex of $\partial\widetilde{C}$) defined in the proof of Theorem B to analyze the mean curvature flows having focal submanifolds of M as initial data. In this section, we shall explicitly describe the vector field X defined for principal orbits of all Hermann actions of cohomogeneity two on all irreducible symmetric spaces of compact type and rank two (see Table 3). Let G/K be a symmetric space of compact type and H be a symmetric subgroup of G. Also, let θ be an involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$ and τ be an invloution of G with $(\mathrm{Fix}\,\tau)_0 \subset H \subset \mathrm{Fix}\,\tau$, where $\mathrm{Fix}\,\theta$ (resp. $\mathrm{Fix}\,\tau$) is the fixed point group of θ (resp. τ) and $(\mathrm{Fix}\,\theta)_0$ (resp. $(\mathrm{Fix}\,\tau)_0$) is the identity component of $\mathrm{Fix}\,\theta$ (resp. $\mathrm{Fix}\,\tau$). In the sequel, we assume that $\tau\circ\theta = \theta\circ\tau$. Set $L := \mathrm{Fix}(\theta\circ\tau)$. Denote by the same symbol θ (resp. τ) the involution of the Lie algebra $\mathfrak{g}$ of G induced from θ (resp. τ). Set $\mathfrak{k} := \mathrm{Ker}(\theta-\mathrm{id})$, $\mathfrak{p} := \mathrm{Ker}(\theta+\mathrm{id})$, $\mathfrak{h} := \mathrm{Ker}(\tau-\mathrm{id})$ and $\mathfrak{q} := \mathrm{Ker}(\tau+\mathrm{id})$. The space $\mathfrak{p}$ is identified with $T_{eK}(G/K)$. From $\theta\circ\tau = \tau\circ\theta$, we have $\mathfrak{p} = \mathfrak{p}\cap\mathfrak{h}+\mathfrak{p}\cap\mathfrak{q}$. Take a maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{p} \cap \mathfrak{q}$ and let $\mathfrak{p} = \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) + \sum\limits_{\beta\in\triangle'_+} \mathfrak{p}_\beta$ be the root space decomposition with respect to $\mathfrak{b}$, where $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$ is the centralizer of $\mathfrak{b}$ in $\mathfrak{p}$, $\triangle'_+$ is the positive root system of $\triangle' := \{\beta \in \mathfrak{b}^* \,|\, \exists X(\neq 0) \in \mathfrak{p} \text{ s.t. } \mathrm{ad}(b)^2(X) = -\beta(b)^2X \ (\forall b \in \mathfrak{b})\}$ under some lexicographic ordering of $\mathfrak{b}^*$ and $\mathfrak{p}_\beta := \{X \in \mathfrak{p} \,|\, \mathrm{ad}(b)^2(X) = -\beta(b)^2X \ (\forall b \in \mathfrak{b})\}$ ($\beta \in \triangle'_+$). Also, let $\triangle'^V_+ := \{\beta \in \triangle'_+ \,|\, \mathfrak{p}_\beta \cap \mathfrak{q} \neq \{0\}\}$ and $\triangle'^H_+ := \{\beta \in \triangle'_+ \,|\, \mathfrak{p}_\beta \cap \mathfrak{h} \neq \{0\}\}$. Then we have $\mathfrak{q} = \mathfrak{b} + \sum\limits_{\beta\in\triangle'^V_+} (\mathfrak{p}_\beta \cap \mathfrak{q})$ and $\mathfrak{h} = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{b}) + \sum\limits_{\beta\in\triangle'^H_+} (\mathfrak{p}_\beta \cap \mathfrak{h})$, where $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{b})$ is the centralizer of $\mathfrak{b}$ in $\mathfrak{h}$. The orbit $H(eK)$ is a reflective submanifold and it is isometric to the symmetric space $H/H\cap K$ (equipped with a metric scaled suitably). Also, $\exp^\perp(T^\perp_{eK}(H(eK)))$ is also a reflective submanifold and it is isometric to the symmetric space $L/H\cap K$ (equipped with a metric scaled suitably), where $\exp^\perp$ is the normal exponential map of $H(eK)$. The system $\triangle'^V := \triangle'^V_+ \cup (-\triangle'^V_+)$ is the root

system of $L/H \cap K$. Define a subset $\widetilde{C}$ of $\mathfrak{b}$ by

$$\widetilde{C} := \{b \in \mathfrak{b} \mid 0 < \beta < \pi \,(\forall\, \beta \in \triangle'^V_+), \; -\frac{\pi}{2} < \beta < \frac{\pi}{2} \,(\forall\, \beta \in \triangle'^H_+)\}.$$

Set $C := \mathrm{Exp}(\widetilde{C})$, where Exp is the exponential map of G/K at eK. Let $P(G, H \times K) := \{g \in H^1([0,1], G) \mid (g(0), g(1)) \in H \times K\}$, where $H^1([0,1], G)$ is the Hilbert Lie group of all H^1-paths in G. This group acts on $H^0([0,1], \mathfrak{g})$ as gauge action. The orbits of the $P(G, H \times K)$-action are the inverse images of orbits of the H-action by $\pi \circ \phi$. The set $\Sigma := \mathrm{Exp}(\mathfrak{b})$ is a section of the H-action and $\mathfrak{b}$ is a section of the $P(G, H \times K)$-action on $H^0([0,1], \mathfrak{g})$, where $\mathfrak{b}$ is identified with the horizontal lift of $\mathfrak{b}$ to the zero element $\hat{0}$ of $H^0([0,1], \mathfrak{g})$ ($\hat{0}$: the constant path at the zero element 0 of $\mathfrak{g}$). The set $\widetilde{C}$ is the fundamental domain of the Coxeter group of a principal $P(G, H \times K)$-orbit and each prinicipal H-orbit meets C at one point and each singular H-orbit meets ∂C at one point. The focal set of the principal orbit $P(G, H \times K) \cdot Z_0$ ($Z_0 \in \widetilde{C}$) consists of the hyperplanes $\beta^{-1}(j\pi)$'s ($\beta \in \triangle'^V_+ \setminus \triangle'^H_+$, $j \in \mathbb{Z}$), $\beta^{-1}((j + \frac{1}{2})\pi)$'s ($\beta \in \triangle'^H_+ \setminus \triangle'^V_+$, $j \in \mathbb{Z}$), $\beta^{-1}(\frac{j\pi}{2})$'s ($\beta \in \triangle'^V_+ \cap \triangle'^H_+$, $j \in \mathbb{Z}$) in $\mathfrak{b}(= T^\perp_{Z_0}(P(G, H \times K) \cdot Z_0))$. Denote by $\exp^G$ the exponential map of G. Note that $\pi \circ \exp^G |_{\mathfrak{p}} = \mathrm{Exp}$. Let $Y_0 \in \widetilde{C}$ and $M(Y_0) := H(\mathrm{Exp}(Y_0))$. Then we have $T^\perp_{\mathrm{Exp}(Y_0)} M(Y_0) = (\exp^G(Y_0))_*(\mathfrak{b})$. Denote by A^{Y_0} the shape tensor of $M(Y_0)$. Take $v \in T^\perp_{\mathrm{Exp}(Y_0)} M(Y_0)$ and set $\bar{v} := (\exp^G(Y_0))^{-1}_*(v)$. By scaling the metric of G/K by a suitable positive constant, we have

$$(4.1) \qquad A^{Y_0}_v |_{\exp^G(Y_0)_*(\mathfrak{p}_\beta \cap \mathfrak{q})} = -\frac{\beta(\bar{v})}{\tan \beta(Y_0)} \mathrm{id} \quad (\beta \in \triangle'^V_+)$$

and

$$(4.2) \qquad A^{Y_0}_v |_{\exp^G(Y_0)_*(\mathfrak{p}_\beta \cap \mathfrak{h})} = \beta(\bar{v}) \tan \beta(Y_0) \mathrm{id} \quad (\beta \in \triangle'^H_+).$$

Set $m^V_\beta := \dim(\mathfrak{p}_\beta \cap \mathfrak{q})$ ($\beta \in \triangle'^V_+$) and $m^H_\beta := \dim(\mathfrak{p}_\beta \cap \mathfrak{h})$ ($\beta \in \triangle'^H_+$). Set $\widetilde{M}(Y_0) := (\pi \circ \phi)^{-1}(M(Y_0))(= P(G, H \times K) \cdot Y_0)$. We can show $(\pi \circ \phi)(Y_0) = \mathrm{Exp}(Y_0)$. Denote by $\widetilde{A}^{Y_0}$ the shape tensor of $\widetilde{M}(Y_0)$. According to Proposition 3.2 of [Koi1], we have

$$\mathrm{Spec}(\widetilde{A}^{Y_0}_{\bar{v}} |_{(\pi \circ \phi)^{-1}_{*Y_0}(\exp^G(Y_0)_*(\mathfrak{p}_\beta \cap \mathfrak{q}))}) \setminus \{0\} = \{\frac{-\beta(\bar{v})}{\beta(Y_0) + j\pi} \mid j \in \mathbb{Z}\} \quad (\beta \in \triangle'^V_+),$$

$$\mathrm{Spec}(\widetilde{A}^{Y_0}_{\bar{v}} |_{(\pi \circ \phi)^{-1}_{*Y_0}(\exp^G(Y_0)_*(\mathfrak{p}_\beta \cap \mathfrak{h}))}) \setminus \{0\} = \{\frac{-\beta(\bar{v})}{\beta(Y_0) + (j + \frac{1}{2})\pi} \mid j \in \mathbb{Z}\} \quad (\beta \in \triangle'^H_+),$$

and

$$\mathrm{Spec}(\widetilde{A}^{Y_0}_{\bar{v}} |_{(\pi \circ \phi)^{-1}_{*Y_0}(\exp^G(Y_0)_*(\mathfrak{z}_\mathfrak{h}(\mathfrak{b})))}) = \{0\}.$$

Hence the set $\mathcal{PC}_{\widetilde{M}(Y_0)}$ of all principal curvatures of $\widetilde{M}(Y_0)$ is given by

$$\mathcal{PC}_{\widetilde{M}(Y_0)} = \{\frac{-\widetilde{\beta}}{\beta(Y_0) + j\pi} \mid \beta \in \triangle'^V_+, \; j \in \mathbb{Z}\} \cup \{\frac{-\widetilde{\beta}}{\beta(Y_0) + (j + \frac{1}{2})\pi} \mid \beta \in \triangle'^H_+, \; j \in \mathbb{Z}\},$$

where $\widetilde{\beta}$ is the parallel section of $(T^\perp \widetilde{M}(Y_0))^*$ with $\widetilde{\beta}_{u_0} = \beta \circ \exp^G(Y_0)^{-1}_*$. Also, we can show that the multiplicity of $\frac{-\widetilde{\beta}}{\beta(Y_0) + j\pi}$ ($\beta \in \triangle'^V_+$) is equal to m^V_β and that of

$\frac{-\widetilde{\beta}}{\beta(Y_0)+(j+\frac{1}{2})\pi}$ ($\beta \in \triangle'^H_+$) is equal to m^H_β. Define $\lambda^{Y_0}_\beta$ and $b^{Y_0}_\beta$ ($\beta \in \triangle'_+$) by

$$
(\lambda^{Y_0}_\beta, b^{Y_0}_\beta) := \begin{cases} (\dfrac{-\widetilde{\beta}}{\beta(Y_0)}, \dfrac{\pi}{\beta(Y_0)}) & (\beta \in \triangle'^V_+ \setminus \triangle'^H_+) \\[2ex] (\dfrac{-\widetilde{\beta}}{\beta(Y_0)+\frac{\pi}{2}}, \dfrac{\pi}{\beta(Y_0)+\frac{\pi}{2}}) & (\beta \in \triangle'^H_+ \setminus \triangle'^V_+) \\[2ex] (\dfrac{-\widetilde{\beta}}{\beta(Y_0)}, \dfrac{\pi}{2\beta(Y_0)}) & (\beta \in \triangle'^V_+ \cap \triangle'^H_+). \end{cases}
$$

Then we have $\frac{-\widetilde{\beta}}{\beta(Y_0)+j\pi} = \frac{\lambda^{Y_0}_\beta}{1+jb^{Y_0}_\beta}$ when $\beta \in \triangle'^V_+ \setminus \triangle'^H_+$, $\frac{-\widetilde{\beta}}{\beta(Y_0)+(j+\frac{1}{2})\pi} = \frac{\lambda^{Y_0}_\beta}{1+jb^{Y_0}_\beta}$ when $\beta \in \triangle'^H_+ \setminus \triangle'^V_+$ and $(\frac{-\widetilde{\beta}}{\beta(Y_0)+j\pi}, \frac{-\widetilde{\beta}}{\beta(Y_0)+(j+\frac{1}{2})\pi}) = (\frac{\lambda^{Y_0}_\beta}{1+2jb^{Y_0}_\beta}, \frac{\lambda^{Y_0}_\beta}{1+(2j+1)b^{Y_0}_\beta})$ when $\beta \in \triangle'^V_+ \cap \triangle'^H_+$. That is, we have

$$
\mathcal{PC}_{\widetilde{M}(Y_0)} = \{\frac{\lambda^{Y_0}_\beta}{1+jb^{Y_0}_\beta} \mid \beta \in \triangle'_+, j \in \mathbb{Z}\}.
$$

Denote by $m_{\beta j}$ the multiplicity of $\frac{\lambda_\beta}{1+jb_\beta}$. Then we have

$$
m_{\beta,2j} = \begin{cases} m^V_\beta & (\beta \in \triangle'^V_+ \setminus \triangle'^H_+) \\ m^H_\beta & (\beta \in \triangle'^H_+ \setminus \triangle'^V_+) \\ m^V_\beta & (\beta \in \triangle'^V_+ \cap \triangle'^H_+), \end{cases} \qquad m_{\beta,2j+1} = \begin{cases} m^V_\beta & (\beta \in \triangle'^V_+ \setminus \triangle'^H_+) \\ m^H_\beta & (\beta \in \triangle'^H_+ \setminus \triangle'^V_+) \\ m^H_\beta & (\beta \in \triangle'^V_+ \cap \triangle'^H_+), \end{cases}
$$

where $j \in \mathbb{Z}$. Denote by $\widetilde{H}^{Y_0}$ the mean curvature vector of $\widetilde{M}(Y_0)$ and $\mathbf{n}^{Y_0}_\beta$ the curvature normal corresponding to $\lambda^{Y_0}_\beta$. Define $\beta^\sharp (\in \mathfrak{b})$ by $\beta(\cdot) = \langle \beta^\sharp, \cdot \rangle$ and let $\widetilde{\beta^\sharp}^{Y_0}$ be the parallel normal vector field of $\widetilde{M}(Y_0)$ with $(\widetilde{\beta^\sharp}^{Y_0})_{Y_0} = \beta^\sharp$, where we identify $\mathfrak{b}$ with $T^\perp_{Y_0}\widetilde{M}(Y_0)$. From (3.1) (the case of $w=0$), we have

$$
\begin{aligned}
(4.3) \qquad \widetilde{H}^{Y_0} &= \sum_{\beta \in \triangle'^V_+} m^V_\beta \cot \frac{\pi}{2b^{Y_0}_\beta} \cdot \frac{\pi}{2b^{Y_0}_\beta} \mathbf{n}^{Y_0}_\beta \\
&\quad - \sum_{\beta \in \triangle'^H_+} m^H_\beta \tan \frac{\pi}{2b^{Y_0}_\beta} \cdot \frac{\pi}{2b^{Y_0}_\beta} \mathbf{n}^{Y_0}_\beta \\
&= -\sum_{\beta \in \triangle'^V_+} m^V_\beta \cot \beta(Y_0) \widetilde{\beta^\sharp}^{Y_0} + \sum_{\beta \in \triangle'^H_+} m^H_\beta \tan \beta(Y_0) \widetilde{\beta^\sharp}^{Y_0}.
\end{aligned}
$$

Define a tangent vector field X on $\widetilde{C}$ by assigning $(\widetilde{H}^{Y_0})_{Y_0} (\in T^\perp_{Y_0}\widetilde{M}(Y_0) = \mathfrak{b}(\subset V))$ to each $Y_0 \in \widetilde{C}$. From (4.3), we have

$$
(4.4) \qquad X_{Y_0} = -\sum_{\beta \in \triangle'^V_+} m^V_\beta \cot \beta(Y_0) \beta^\sharp + \sum_{\beta \in \triangle'^H_+} m^H_\beta \tan \beta(Y_0) \beta^\sharp.
$$

By using this description, we can explicitly describe this vector field X for all Hermann actions of cohomogeneity two on all irreducible symmetric spaces of compact type and rank two. All Hermann actions of cohomogeneity two on all irreducible symmetric spaces of compact type and rank two are given in Table 1. The systems $\triangle'^V_+$ and $\triangle'^H_+$

for the Hermann actions are given in Table 2 and the explicit descriptions of X for the Hermann actions are given in Table 3. In Table 1, $H^* \curvearrowright G^*/K$ implies the dual action of $H \curvearrowright G/K$ and $L^*/H \cap K$ is the dual of $L/H \cap K$. In Table 2, $\{\alpha, \beta, \alpha + \beta\}$ implies a positive root system of the root system of $(\mathfrak{a}_2)$-type $(\alpha = (2,0), \beta = (-1, \sqrt{3}))$, $\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$ implies a positive root system of the root system of $(\mathfrak{b}_2)(=(\mathfrak{c}_2))$-type $(\alpha = (1,0), \beta = (-1,1))$ and $\{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta\}$ implies a positive root system of the root system of $(\mathfrak{g}_2)$-type $(\alpha = (2\sqrt{3}, 0), \beta = (-\sqrt{3}, 1))$. In Table 1 $\sim$ 3, ρ_i $(i = 1, \cdots, 16)$ imply automorphisms of G and $(\cdot)^2$ implies the product Lie group $(\cdot) \times (\cdot)$ of a Lie group $(\cdot)$. In Table 2, $\underset{(m)}{\alpha}$ and so on imply that the multiplicity of α is equal to m.

$H \curvearrowright G/K$	$H^* \curvearrowright G^*/K$	$L^*/H \cap K$
$\rho_1(SO(3)) \curvearrowright SU(3)/SO(3)$	$SO_0(1,2) \curvearrowright SL(3,\mathbb{R})/SO(3)$	$(SL(2,\mathbb{R})/SO(2)) \times \mathbb{R}$
$SO(6) \curvearrowright SU(6)/Sp(3)$	$SO^*(6) \curvearrowright SU^*(6)/Sp(3)$	$SL(3,\mathbb{C})/SU(3)$
$\rho_2(Sp(3)) \curvearrowright SU(6)/Sp(3)$	$Sp(1,2) \curvearrowright SU^*(6)/Sp(3)$	$(SU^*(4)/Sp(2)) \times U(1)$
$SO(q+2) \curvearrowright$ $SU(q+2)/S(U(2) \times U(q))$	$SO_0(2,q) \curvearrowright$ $SU(2,q)/S(U(2) \times U(q))$	$SO_0(2,q)/SO(2) \times SO(q)$
$S(U(j+1) \times U(q-j+1)) \curvearrowright$ $SU(q+2)/S(U(2) \times U(q))$	$S(U(1,j) \times U(1,q-j)) \curvearrowright$ $SU(2,q)/S(U(2) \times U(q))$	$(SU(1,j)/S(U(1) \times U(j))) \times$ $(SU(1,q-j)/S(U(1) \times U(q-j)))$
$SO(j+1) \times SO(q-j+1) \curvearrowright$ $SO(q+2)/SO(2) \times SO(q)$	$SO(1,j) \times SO(1,q-j) \curvearrowright$ $SO(2,q)/SO(2) \times SO(q)$	$(SO_0(1,j)/SO(j)) \times$ $(SO_0(1,q-j)/SO(q-j))$
$SO(4) \times SO(4) \curvearrowright$ $SO(8)/U(4)$	$SO^*(4) \times SO^*(4) \curvearrowright$ $SO^*(8)/U(4)$	$SU(2,2)/S(U(2) \times U(2))$
$\rho_3(SO(4) \times SO(4)) \curvearrowright$ $SO(8)/U(4)$	$SO(4,\mathbb{C}) \curvearrowright SO^*(8)/U(4)$	$SO(4,\mathbb{C})/SO(4)$
$\rho_4(U(4)) \curvearrowright SO(8)/U(4)$	$U(2,2) \curvearrowright SO^*(8)/U(4)$	$(SO^*(4)/U(2)) \times (SO^*(4)/U(2))$
$SO(4) \times SO(6) \curvearrowright$ $SO(10)/U(5)$	$SO^*(4) \times SO^*(6) \curvearrowright$ $SO^*(10)/U(5)$	$SU(2,3)/S(U(2) \times U(3))$
$SO(5) \times SO(5) \curvearrowright$ $SO(10)/U(5)$	$SO^{(}5,\mathbb{C}) \curvearrowright SO^*(10)/U(5)$	$SO(5,\mathbb{C})/SO(5)$
$\rho_5(U(5)) \curvearrowright SO(10)/U(5)$	$U(2,3) \curvearrowright SO^*(10)/U(5)$	$(SO^*(4)/U(2)) \times (SO^*(6)/U(3))$
$SO(2)^2 \times SO(3)^2 \curvearrowright$ $(SO(5) \times SO(5))/SO(5)$	$SO(2,\mathbb{C}) \times SO(3,\mathbb{C}) \curvearrowright$ $SO(5,\mathbb{C})/SO(5)$	$SO_0(2,3)/SO(2) \times SO(3)$
$\rho_6(SO(5)) \curvearrowright$ $(SO(5) \times SO(5))/SO(5)$	$SO_0(2,3) \curvearrowright SO(5,\mathbb{C})/SO(5)$	$(SO(2,\mathbb{C})/SO(2))$ $\times (SO(3,\mathbb{C})/SO(3))$
$\rho_7(U(2)) \curvearrowright Sp(2)/U(2)$	$U(1,1) \curvearrowright Sp(2,\mathbb{R})/U(2)$	$(Sp(1,\mathbb{R})/U(1))$ $\times (Sp(1,\mathbb{R})/U(1))$
$SU(q+2) \curvearrowright$ $Sp(q+2)/Sp(2) \times Sp(q)$	$SU(2,q) \curvearrowright$ $Sp(2,q)/Sp(2) \times Sp(q)$	$SU(2,q)/S(U(2) \times U(q))$

TABLE 1.

$H \curvearrowright G/K$	$H^* \curvearrowright G^*/K$	$L^*/H \cap K$
$U(4) \curvearrowright$ $Sp(4)/Sp(2) \times Sp(2)$	$U^*(4) \curvearrowright$ $Sp(2,2)/Sp(2) \times Sp(2)$	$Sp(2,\mathbb{C})/Sp(2)$
$Sp(j+1) \times Sp(q-j+1) \curvearrowright$ $Sp(q+2)/Sp(2) \times Sp(q)$	$Sp(1,j) \times Sp(1,q-j) \curvearrowright$ $Sp(2,q)/Sp(2) \times Sp(q)$	$(Sp(1,j)/Sp(1) \times Sp(j)) \times$ $(Sp(1,q-j)/Sp(1) \times Sp(q-j))$
$SU(2)^2 \cdot SO(2)^2 \curvearrowright$ $(Sp(2) \times Sp(2))/Sp(2)$	$SL(2,\mathbb{C}) \cdot SO(2,\mathbb{C}) \curvearrowright$ $Sp(2,\mathbb{C})/Sp(2)$	$Sp(2,\mathbb{R})/U(2)$
$\rho_8(Sp(2)) \curvearrowright$ $(Sp(2) \times Sp(2))/Sp(2)$	$Sp(2,\mathbb{R}) \curvearrowright Sp(2,\mathbb{C})/Sp(2)$	$(SL(2,\mathbb{C})/SU(2))$ $\times(SO(2,\mathbb{C})/SO(2))$
$\rho_9(Sp(2)) \curvearrowright$ $(Sp(2) \times Sp(2))/Sp(2)$	$Sp(1,1) \curvearrowright Sp(2,\mathbb{C})/Sp(2)$	$(Sp(1,\mathbb{C})/Sp(1))$ $\times(Sp(1,\mathbb{C})/Sp(1))$
$Sp(4) \curvearrowright E_6/Spin(10) \cdot U(1)$	$Sp(2,2) \curvearrowright E_6^{-14}/Spin(10) \cdot U(1)$	$Sp(2,2)/Sp(2) \times Sp(2)$
$SU(6) \cdot SU(2) \curvearrowright$ $E_6/Spin(10) \cdot U(1)$	$SU(2,4) \cdot SU(2) \curvearrowright$ $E_6^{-14}/Spin(10) \cdot U(1)$	$SU(2,4)/S(U(2) \times U(4))$
$\rho_{10}(SU(6) \cdot SU(2)) \curvearrowright$ $E_6/Spin(10) \cdot U(1)$	$SU(1,5) \cdot SL(2,\mathbb{R}) \curvearrowright$ $E_6^{-14}/Spin(10) \cdot U(1)$	$SO^*(10)/U(5)$
$\rho_{11}(Spin(10) \cdot U(1)) \curvearrowright$ $E_6/Spin(10) \cdot U(1)$	$SO^*(10) \cdot U(1) \curvearrowright$ $E_6^{-14}/Spin(10) \cdot U(1)$	$(SU(1,5)/S(U(1) \times U(5)))$ $\times(SL(2,\mathbb{R})/SO(2))$
$\rho_{12}(Spin(10) \cdot U(1)) \curvearrowright$ $E_6/Spin(10) \cdot U(1)$	$SO_0(2,8) \cdot U(1) \curvearrowright$ $E_6^{-14}/Spin(10) \cdot U(1)$	$SO_0(2,8)/SO(2) \times SO(8)$
$Sp(4) \curvearrowright E_6/F_4$	$Sp(1,3) \curvearrowright E_6^{-26}/F_4$	$SU^*(6)/Sp(3)$
$\rho_{13}(F_4) \curvearrowright E_6/F_4$	$F_4^{-20} \curvearrowright E_6^{-26}/F_4$	$(SO_0(1,9)/SO(9)) \times U(1)$
$\rho_{14}(SO(4)) \curvearrowright G_2/SO(4)$	$SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \curvearrowright G_2^2/SO(4)$	$SO(4)/SO(2) \times SO(2)$
$\rho_{15}(SO(4)) \curvearrowright G_2/SO(4)$	$\rho_{15}^*(SO(4)) \curvearrowright G_2^2/SO(4)$	$(SL(2,\mathbb{R})/SO(2))$ $\times(SL(2,\mathbb{R})/SO(2))$
$\rho_{16}(G_2) \curvearrowright (G_2 \times G_2)/G_2$	$G_2^2 \curvearrowright G_2^{\mathbf{C}}/G_2$	$(SL(2,\mathbb{C})/SU(2))$ $\times(SL(2,\mathbb{C})/SU(2))$
$SU(2)^4 \curvearrowright (G_2 \times G_2)/G_2$	$SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \curvearrowright G_2^{\mathbf{C}}/G_2$	$G_2^2/SO(4)$

TABLE 1 (CONTINUED).

$H \curvearrowright G/K$	$\triangle_+ = \triangle'_+$	$\triangle'^V_+$	$\triangle'^H_+$
$\rho_1(SO(3)) \curvearrowright SU(3)/SO(3)$	$\{\underset{(1)}{\alpha}, \underset{(1)}{\beta}, \underset{(1)}{\alpha+\beta}\}$	$\{\underset{(1)}{\alpha}\}$	$\{\underset{(1)}{\beta}, \underset{(1)}{\alpha+\beta}\}$
$SO(6) \curvearrowright SU(6)/Sp(3)$	$\{\underset{(4)}{\alpha}, \underset{(4)}{\beta}, \underset{(4)}{\alpha+\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\beta}, \underset{(2)}{\alpha+\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\beta}, \underset{(2)}{\alpha+\beta}\}$
$\rho_2(Sp(3)) \curvearrowright SU(6)/Sp(3)$	$\{\underset{(4)}{\alpha}, \underset{(4)}{\beta}, \underset{(4)}{\alpha+\beta}\}$	$\{\underset{(4)}{\alpha}\}$	$\{\underset{(4)}{\beta}, \underset{(4)}{\alpha+\beta}\}$
$SO(q+2) \curvearrowright$ $SU(q+2)/S(U(2)\times U(q))$ $(q>2)$	$\{\underset{(2q-4)}{\alpha}, \underset{(2)}{\beta}, \underset{(2q-4)}{\alpha+\beta},$ $\underset{(2)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(q-2)}{\alpha}, \underset{(1)}{\beta}, \underset{(q-2)}{\alpha+\beta},$ $\underset{(1)}{2\alpha+\beta}\}$	$\{\underset{(q-2)}{\alpha}, \underset{(1)}{\beta}, \underset{(q-2)}{\alpha+\beta},$ $\underset{(1)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$
$S(U(j+1)\times U(q-j+1)) \curvearrowright$ $SU(q+2)/S(U(2)\times U(q))$ $(q>2)$	$\{\underset{(2q-4)}{\alpha}, \underset{(2)}{\beta}, \underset{(2q-4)}{\alpha+\beta},$ $\underset{(2)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(2j-2)}{\alpha}, \underset{(2q-2j-2)}{\alpha+\beta},$ $\underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(2q-2j-6)}{\alpha}, \underset{(2)}{\beta},$ $\underset{(2j-2)}{\alpha+\beta}, \underset{(2)}{2\alpha+\beta}\}$
$S(U(2)\times U(2)) \curvearrowright$ $SU(4)/S(U(2)\times U(2))$ (non-isotropy gr. act.)	$\{\underset{(2)}{\alpha}, \underset{(1)}{\beta}, \underset{(2)}{\alpha+\beta},$ $\underset{(1)}{2\alpha+\beta}\}$	$\{\underset{(1)}{\alpha}, \underset{(1)}{\alpha+\beta}\}$	$\{\underset{(1)}{\alpha}, \underset{(1)}{\beta},$ $\underset{(1)}{\alpha+\beta}, \underset{(1)}{2\alpha+\beta}\}$
$SO(j+1)\times SO(q-j+1) \curvearrowright$ $SO(q+2)/SO(2)\times SO(q)$	$\{\underset{(q-2)}{\alpha}, \underset{(1)}{\beta}, \underset{(q-2)}{\alpha+\beta},$ $\underset{(1)}{2\alpha+\beta}\}$	$\{\underset{(j-1)}{\alpha}, \underset{(q-j-1)}{\alpha+\beta}\}$	$\{\underset{(q-j-1)}{\alpha}, \underset{(1)}{\beta},$ $\underset{(j-1)}{\alpha+\beta}, \underset{(1)}{2\alpha+\beta}\}$
$SO(4)\times SO(4) \curvearrowright$ $SO(8)/U(4)$	$\{\underset{(4)}{\alpha}, \underset{(1)}{\beta}, \underset{(4)}{\alpha+\beta},$ $\underset{(1)}{2\alpha+\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\beta}, \underset{(2)}{\alpha+\beta},$ $\underset{(1)}{2\alpha+\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\alpha+\beta}\}$
$\rho_3(SO(4)\times SO(4)) \curvearrowright$ $SO(8)/U(4)$	$\{\underset{(4)}{\alpha}, \underset{(1)}{\beta}, \underset{(4)}{\alpha+\beta},$ $\underset{(1)}{2\alpha+\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\alpha+\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(1)}{\beta}, \underset{(2)}{\alpha+\beta},$ $\underset{(1)}{2\alpha+\beta}\}$
$\rho_4(U(4)) \curvearrowright SO(8)/U(4)$	$\{\underset{(4)}{\alpha}, \underset{(1)}{\beta}, \underset{(4)}{\alpha+\beta},$ $\underset{(1)}{2\alpha+\beta}\}$	$\{\underset{(1)}{\alpha}, \underset{(1)}{\alpha+\beta}\}$	$\{\underset{(3)}{\alpha}, \underset{(1)}{\beta}, \underset{(3)}{\alpha+\beta},$ $\underset{(1)}{2\alpha+\beta}\}$
$SO(4)\times SO(6) \curvearrowright$ $SO(10)/U(5)$	$\{\underset{(4)}{\alpha}, \underset{(4)}{\beta}, \underset{(4)}{\alpha+\beta},$ $\underset{(4)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\beta}, \underset{(2)}{\alpha+\beta},$ $\underset{(2)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\beta}, \underset{(2)}{\alpha+\beta},$ $\underset{(2)}{2\alpha+\beta}\}$
$SO(5)\times SO(5) \curvearrowright$ $SO(10)/U(5)$	$\{\underset{(4)}{\alpha}, \underset{(4)}{\beta}, \underset{(4)}{\alpha+\beta},$ $\underset{(4)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\beta}, \underset{(2)}{\alpha+\beta},$ $\underset{(2)}{2\alpha+\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\beta}, \underset{(2)}{\alpha+\beta},$ $\underset{(2)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$

TABLE 2.

$H \curvearrowright G/K$	$\triangle_+ = \triangle'_+$	$\triangle'^V_+$	$\triangle'^H_+$
$\rho_5(U(5)) \curvearrowright SO(10)/U(5)$	$\{\alpha_{(4)}, \beta_{(4)}, \alpha+\beta_{(4)},\ 2\alpha+\beta_{(4)}, 2\alpha_{(1)}, 2\alpha+2\beta_{(1)}\}$	$\{\alpha_{(4)}, 2\alpha_{(1)}, 2\alpha+2\beta_{(1)}\}$	$\{\beta_{(4)}, \alpha+\beta_{(4)}, 2\alpha+\beta_{(4)}\}$
$SO(2)^2 \times SO(3)^2 \curvearrowright$ $(SO(5) \times SO(5))/SO(5)$	$\{\alpha_{(2)}, \beta_{(2)}, \alpha+\beta_{(2)},\ \{2\alpha+\beta\}_{(2)}$	$\{\alpha_{(1)}, \beta_{(1)}, \alpha+\beta_{(1)},\ 2\alpha+\beta_{(1)}\}$	$\{\alpha_{(1)}, \beta_{(1)}, \alpha+\beta_{(1)},\ 2\alpha+\beta_{(1)}\}$
$\rho_6(SO(5)) \curvearrowright$ $(SO(5) \times SO(5))/SO(5)$	$\{\alpha_{(2)}, \beta_{(2)}, \alpha+\beta_{(2)},\ \{2\alpha+\beta\}_{(2)}$	$\{\alpha_{(2)}\}$	$\{\beta_{(2)}, \alpha+\beta_{(2)}, 2\alpha+\beta_{(2)}\}$
$\rho_7(U(2)) \curvearrowright Sp(2)/U(2)$	$\{\alpha_{(1)}, \beta_{(1)}, \alpha+\beta_{(1)},\ \{2\alpha+\beta\}_{(1)}$	$\{\alpha_{(1)}, \alpha+\beta_{(1)}\}$	$\{\beta_{(1)}, 2\alpha+\beta_{(1)}\}$
$SU(q+2) \curvearrowright$ $Sp(q+2)/Sp(2) \times Sp(q)$ $(q > 2)$	$\{\alpha_{(4q-8)}, \beta_{(4)}, \alpha+\beta_{(4q-8)},\ 2\alpha+\beta_{(4)}, 2\alpha_{(3)}, 2\alpha+2\beta_{(3)}\}$	$\{\alpha_{(2q-4)}, \beta_{(2)}, \alpha+\beta_{(2q-4)}\ 2\alpha+\beta_{(2)}, 2\alpha_{(1)}, 2\alpha+2\beta_{(1)}\}$	$\{\alpha_{(2q-4)}, \beta_{(2)}, \alpha+\beta_{(2q-4)}\ 2\alpha+\beta_{(2)}, 2\alpha_{(2)}, 2\alpha+2\beta_{(2)}\}$
$SU(4) \curvearrowright$ $Sp(4)/Sp(2) \times Sp(2)$	$\{\alpha_{(4)}, \beta_{(3)}, \alpha+\beta_{(3)},\ 2\alpha+\beta_{(4)}\}$	$\{\alpha_{(2)}, \beta_{(1)}, \alpha+\beta_{(1)}\ 2\alpha+\beta_{(1)}\}$	$\{\alpha_{(2)}, \beta_{(1)}, \alpha+\beta_{(1)}\ 2\alpha+\beta_{(3)}\}$
$U(4) \curvearrowright$ $Sp(4)/Sp(2) \times Sp(2)$	$\{\alpha_{(4)}, \beta_{(3)}, \alpha+\beta_{(3)},\ 2\alpha+\beta_{(4)}\}$	$\{\alpha_{(2)}, \beta_{(2)}, \alpha+\beta_{(2)}\ 2\alpha+\beta_{(2)}\}$	$\{\alpha_{(2)}, \beta_{(1)}, \alpha+\beta_{(1)}\ 2\alpha+\beta_{(2)}\}$
$Sp(j+1) \times Sp(q-j+1) \curvearrowright$ $Sp(q+2)/Sp(2) \times Sp(q)$ $(q > 2)$	$\{\alpha_{(4q-8)}, \beta_{(4)}, \alpha+\beta_{(4q-8)},\ 2\alpha+\beta_{(4)}, 2\alpha_{(3)}, 2\alpha+2\beta_{(3)}\}$	$\{\alpha_{(2j-4)}, 2\alpha_{(3)}, \alpha+\beta_{(4q-4j-4)}\ 2\alpha+2\beta_{(3)}\}$	$\{\alpha_{(4q-4j-4)}, \beta_{(4)}, \alpha+\beta_{(4j-4)}\ 2\alpha+\beta_{(4)}\}$
$Sp(2) \times Sp(2) \curvearrowright$ $Sp(4)/Sp(2) \times Sp(2)$	$\{\alpha_{(4)}, \beta_{(3)}, \alpha+\beta_{(3)},\ 2\alpha+\beta_{(4)}\}$	$\{\alpha_{(3)}, \alpha+\beta_{(3)}\}$	$\{\alpha_{(1)}, \beta_{(3)}, 2\alpha+\beta_{(4)}\}$
$SU(2)^2 \cdot SO(2)^2 \curvearrowright$ $(Sp(2) \times Sp(2))/Sp(2)$	$\{\alpha_{(2)}, \beta_{(2)}, \alpha+\beta_{(2)},\ 2\alpha+\beta_{(2)}\}$	$\{\alpha_{(1)}, \beta_{(1)}, \alpha+\beta_{(1)},\ 2\alpha+\beta_{(1)}\}$	$\{\alpha_{(1)}, \beta_{(1)}, \alpha+\beta_{(1)},\ 2\alpha+\beta_{(1)}\}$
$\rho_8(Sp(2)) \curvearrowright$ $(Sp(2) \times Sp(2))/Sp(2)$	$\{\alpha_{(2)}, \beta_{(2)}, \alpha+\beta_{(2)},\ 2\alpha+\beta_{(2)}\}$	$\{\alpha_{(2)}, \alpha+\beta_{(2)}\}$	$\{\beta_{(2)}, 2\alpha+\beta_{(2)}\}$

TABLE 2 (CONTINUED).

$H \curvearrowright G/K$	$\triangle_+ = \triangle'_+$	$\triangle'^V_+$	$\triangle'^H_+$
$\rho_9(Sp(2)) \curvearrowright$ $(Sp(2) \times Sp(2))/Sp(2)$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\beta}, \underset{(2)}{\alpha+\beta}, \underset{(2)}{2\alpha+\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\alpha+\beta}\}$	$\{\underset{(2)}{\beta}, \underset{(2)}{2\alpha+\beta}\}$
$Sp(4) \curvearrowright$ $E_6/Spin(10)\cdot U(1)$	$\{\underset{(8)}{\alpha}, \underset{(6)}{\beta}, \underset{(9)}{\alpha+\beta}, \underset{(5)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(4)}{\alpha}, \underset{(3)}{\beta}, \underset{(3)}{\alpha+\beta}, \underset{(4)}{2\alpha+\beta}\}$	$\{\underset{(4)}{\alpha}, \underset{(3)}{\beta}, \underset{(6)}{\alpha+\beta}, \underset{(1)}{2\alpha+\beta}\}$
$SU(6) \cdot SU(2) \curvearrowright$ $E_6/Spin(10)\cdot U(1)$	$\{\underset{(8)}{\alpha}, \underset{(6)}{\beta}, \underset{(9)}{\alpha+\beta}, \underset{(5)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(4)}{\alpha}, \underset{(2)}{\beta}, \underset{(4)}{\alpha+\beta}, \underset{(2)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(4)}{\alpha}, \underset{(4)}{\beta}, \underset{(5)}{\alpha+\beta}, \underset{(3)}{2\alpha+\beta}\}$
$\rho_{10}(SU(6) \cdot SU(2)) \curvearrowright$ $E_6/Spin(10)\cdot U(1)$	$\{\underset{(8)}{\alpha}, \underset{(6)}{\beta}, \underset{(9)}{\alpha+\beta}, \underset{(5)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(4)}{\alpha}, \underset{(4)}{\beta}, \underset{(4)}{\alpha+\beta}, \underset{(4)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(4)}{\alpha}, \underset{(2)}{\beta}, \underset{(5)}{\alpha+\beta}, \underset{(1)}{2\alpha+\beta}\}$
$\rho_{11}(Spin(10) \cdot U(1)) \curvearrowright$ $E_6/Spin(10)\cdot U(1)$	$\{\underset{(8)}{\alpha}, \underset{(6)}{\beta}, \underset{(9)}{\alpha+\beta}, \underset{(5)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(8)}{\alpha}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(6)}{\beta}, \underset{(9)}{\alpha+\beta}, \underset{(5)}{2\alpha+\beta}\}$
$\rho_{12}(Spin(10) \cdot U(1)) \curvearrowright$ $E_6/Spin(10)\cdot U(1)$	$\{\underset{(8)}{\alpha}, \underset{(6)}{\beta}, \underset{(9)}{\alpha+\beta}, \underset{(5)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$	$\{\underset{(6)}{\alpha}, \underset{(1)}{\beta}, \underset{(6)}{\alpha+\beta}, \underset{(1)}{2\alpha+\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(5)}{\beta}, \underset{(3)}{\alpha+\beta}, \underset{(4)}{2\alpha+\beta}, \underset{(1)}{2\alpha}, \underset{(1)}{2\alpha+2\beta}\}$
$Sp(4) \curvearrowright E_6/F_4$	$\{\underset{(8)}{\alpha}, \underset{(8)}{\beta}, \underset{(8)}{\alpha+\beta}\}$	$\{\underset{(4)}{\alpha}, \underset{(4)}{\beta}, \underset{(4)}{\alpha+\beta}\}$	$\{\underset{(4)}{\alpha}, \underset{(4)}{\beta}, \underset{(4)}{\alpha+\beta}\}$
$\rho_{13}(F_4) \curvearrowright E_6/F_4$	$\{\underset{(8)}{\alpha}, \underset{(8)}{\beta}, \underset{(8)}{\alpha+\beta}\}$	$\{\underset{(8)}{\alpha}\}$	$\{\underset{(8)}{\beta}, \underset{(8)}{\alpha+\beta}\}$
$\rho_{14}(SO(4)) \curvearrowright$ $G_2/SO(4)$	$\{\underset{(1)}{\alpha}, \underset{(1)}{\beta}, \underset{(1)}{\alpha+\beta}, \underset{(1)}{2\alpha+\beta}, \underset{(1)}{3\alpha+\beta}, \underset{(1)}{3\alpha+2\beta}\}$	$\{\underset{(1)}{\alpha}, \underset{(1)}{3\alpha+2\beta}\}$	$\{\underset{(1)}{\beta}, \underset{(1)}{\alpha+\beta}, \underset{(1)}{2\alpha+\beta}, \underset{(1)}{3\alpha+\beta}\}$
$\rho_{15}(SO(4)) \curvearrowright$ $G_2/SO(4)$	$\{\underset{(1)}{\alpha}, \underset{(1)}{\beta}, \underset{(1)}{\alpha+\beta}, \underset{(1)}{2\alpha+\beta}, \underset{(1)}{3\alpha+\beta}, \underset{(1)}{3\alpha+2\beta}\}$	$\{\underset{(1)}{\alpha}, \underset{(1)}{3\alpha+2\beta}\}$	$\{\underset{(1)}{\beta}, \underset{(1)}{\alpha+\beta}, \underset{(1)}{2\alpha+\beta}, \underset{(1)}{3\alpha+\beta}\}$
$\rho_{16}(G_2) \curvearrowright$ $(G_2 \times G_2)/G_2$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\beta}, \underset{(2)}{\alpha+\beta}, \underset{(2)}{2\alpha+\beta}, \underset{(2)}{3\alpha+\beta}, \underset{(2)}{3\alpha+2\beta}\}$	$\{\underset{(2)}{\alpha}, \underset{(2)}{3\alpha+2\beta}\}$	$\{\underset{(2)}{\beta}, \underset{(2)}{\alpha+\beta}, \underset{(2)}{2\alpha+\beta}, \underset{(2)}{3\alpha+\beta}\}$
$SU(2)^4 \curvearrowright$ $(G_2 \times G_2)/G_2$	$\{\underset{(2)}{\alpha}, \underset{(2)}{\beta}, \underset{(2)}{\alpha+\beta}, \underset{(2)}{2\alpha+\beta}, \underset{(2)}{3\alpha+\beta}, \underset{(2)}{3\alpha+2\beta}\}$	$\{\underset{(1)}{\alpha}, \underset{(1)}{\beta}, \underset{(1)}{\alpha+\beta}, \underset{(1)}{2\alpha+\beta}, \underset{(1)}{3\alpha+\beta}, \underset{(1)}{3\alpha+2\beta}\}$	$\{\underset{(1)}{\alpha}, \underset{(1)}{\beta}, \underset{(1)}{\alpha+\beta}, \underset{(1)}{2\alpha+\beta}, \underset{(1)}{3\alpha+\beta}, \underset{(1)}{3\alpha+2\beta}\}$

TABLE 2 (CONTINUED^2).

$H \curvearrowright G/K$	$X \qquad (\widetilde{C})$
$\rho_1(SO(3)) \curvearrowright SU(3)/SO(3)$	$X_{(x_1,x_2)} = (\tan(x_1 + \sqrt{3}x_2) - 2\cot 2x_1 + \tan(x_1 - \sqrt{3}x_2),$ $\sqrt{3}\tan(x_1 + \sqrt{3}x_2) - \sqrt{3}\tan(x_1 - \sqrt{3}x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > \frac{1}{\sqrt{3}}x_1 - \frac{\pi}{2\sqrt{3}}, x_2 < -\frac{1}{\sqrt{3}}x_1 + \frac{\pi}{2\sqrt{3}})$
$SO(6) \curvearrowright SU(6)/Sp(3)$	$X_{(x_1,x_2)} = (-4\cot 2x_1 - 2\cot(x_1 - \sqrt{3}x_2) - 2\cot(x_1 + \sqrt{3}x_2)$ $+4\tan 2x_1 + 2\tan(x_1 - \sqrt{3}x_2) + 2\tan(x_1 + \sqrt{3}x_2),$ $2\sqrt{3}\cot(x_1 - \sqrt{3}x_2) - 2\sqrt{3}\cot(x_1 + \sqrt{3}x_2)$ $-2\sqrt{3}\tan(x_1 - \sqrt{3}x_2) + 2\sqrt{3}\tan(x_1 + \sqrt{3}x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > \frac{1}{\sqrt{3}}x_1, x_2 < -\frac{1}{\sqrt{3}}x_1 + \frac{\pi}{2\sqrt{3}})$
$\rho_2(Sp(3)) \curvearrowright SU(6)/Sp(3)$	$X_{(x_1,x_2)} = (-8\cot 2x_1 + 4\tan(x_1 - \sqrt{3}x_2) + 4\cot(x_1 + \sqrt{3}x_2),$ $4\sqrt{3}\tan(x_1 + \sqrt{3}x_2) - 4\sqrt{3}\tan(x_1 - \sqrt{3}x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > \frac{1}{\sqrt{3}}x_1 - \frac{\pi}{2\sqrt{3}}, x_2 < -\frac{1}{\sqrt{3}}x_1 + \frac{\pi}{2\sqrt{3}})$
$SO(q+2) \curvearrowright$ $SU(q+2)/S(U(2) \times U(q))$ $(q > 2)$	$X_{(x_1,x_2)} = (-(q-2)\cot x_1 + \cot(x_1 - x_2) - \cot(x_1 + x_2)$ $+(q-2)\tan x_1 + \tan(x_1 - x_2) + \tan(x_1 + x_2) + 2\tan 2x_1,$ $\cot(x_1 - x_2) - (q-2)\cot x_2 - \cot(x_1 + x_2)$ $-\tan(x_1 - x_2) + (q-2)\tan x_2 + \tan(x_1 + x_2) + 2\tan 2x_2)$ $(\widetilde{C} : x_1 > 0, x_2 > x_1, x_2 < \frac{\pi}{4})$
$SO(4) \curvearrowright$ $SU(4)/S(U(2) \times U(2))$	$X_{(x_1,x_2)} = (-\cot x_1 + \tan x_1 + \tan(x_1 - x_2) + \tan(x_1 + x_2),$ $-\cot x_2 - \tan(x_1 - x_2) + \tan x_2 + \tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$
$S(U(j+1) \times U(q-j+1)) \curvearrowright$ $SU(q+2)/S(U(2) \times U(q))$ $(q > 2)$	$X_{(x_1,x_2)} = (-2(j-1)\cot x_1 - 2\cot 2x_1 + 2(q-j-1)\tan x_1$ $+2\tan(x_1 - x_2) + 2\tan(x_1 + x_2),$ $-2(q-j-1)\cot x_2 - 2\cot 2x_2 - 2\tan(x_1 - x_2)$ $+2(j-1)\tan x_2 + 2\tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$
$S(U(2) \times U(2)) \curvearrowright$ $SU(4)/S(U(2) \times U(2))$ (non-isotropy gr. act.)	$X_{(x_1,x_2)} = (-\cot x_1 + \tan x_1 + \tan(x_1 - x_2) + \tan(x_1 + x_2),$ $-\cot x_2 - \tan(x_1 - x_2) + \tan x_2 + \tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$
$SO(j+1) \times SO(q-j+1) \curvearrowright$ $SO(q+2)/SO(2) \times SO(q)$ $(q > 2)$	$X_{(x_1,x_2)} = (-(j-1)\cot x_1 + (q-j-1)\tan x_1$ $+\tan(x_1 - x_2) + \tan(x_1 + x_2),$ $-(q-j-1)\cot x_2 - \tan(x_1 - x_2)$ $+(j-1)\tan x_2 + \tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$

TABLE 3.

$H \curvearrowright G/K$	$X \qquad (\widetilde{C})$
$SO(4) \times SO(4) \curvearrowright$ $SO(8)/U(4)$	$X_{(x_1,x_2)} = (-2\cot x_1 - \cot(x_1 - x_2)$ $-2\cot(x_1 + x_2) + 2\tan x_1,$ $\cot(x_1 - x_2) - 2\cot x_2$ $-2\cot(x_1 + x_2) + 2\tan x_2)$ $(\widetilde{C} \,:\, x_1 > 0, x_2 > 0, x_2 < x_1, x_1 + x_2 < \pi)$
$\rho_3(SO(4) \times SO(4)) \curvearrowright$ $SO(8)/U(4)$	$X_{(x_1,x_2)} = (-2\cot x_1 + 2\tan x_1$ $+\tan(x_1 - x_2) + \tan(x_1 + x_2),$ $-2\cot x_2 - \tan(x_1 - x_2)$ $+2\tan x_2 + \tan(x_1 + x_2))$ $(\widetilde{C} \,:\, x_1 > 0, x_2 < 0, x_1 + x_2 < \frac{\pi}{2})$
$\rho_4(U(4)) \curvearrowright SO(8)/U(4)$	$X_{(x_1,x_2)} = (-\cot x_1 + 3\tan x_1$ $+\tan(x_1 - x_2) + \tan(x_1 + x_2),$ $-\cot x_2 - \tan(x_1 - x_2)$ $+3\tan x_2 + \tan(x_1 + x_2))$ $(\widetilde{C} \,:\, x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$
$SO(4) \times SO(6) \curvearrowright$ $SO(10)/U(5)$	$X_{(x_1,x_2)} = 2(-\cot x_1 - \cot(x_1 - x_2) - \cot(x_1 + x_2)$ $-\cot 2x_1 + \tan x_1$ $+\tan(x_1 - x_2) + \tan(x_1 + x_2),$ $\cot(x_1 - x_2) - \cot x_2 - \cot(x_1 + x_2)$ $-\cot 2x_2 - \tan(x_1 - x_2)$ $+\tan x_2 + \tan(x_1 + x_2))$ $(\widetilde{C} \,:\, x_1 > 0, x_2 > x_1, x_1 + x_2 < \frac{\pi}{2})$
$SO(5) \times SO(5) \curvearrowright$ $SO(10)/U(5)$	$X_{(x_1,x_2)} = 2(-\cot x_1 - \cot(x_1 - x_2) - \cot(x_1 + x_2)$ $+\tan x_1 + \tan(x_1 - x_2)$ $+\tan(x_1 + x_2) + \tan 2x_1,$ $\cot(x_1 - x_2) - \cot x_2 - \cot(x_1 + x_2)$ $-\tan(x_1 - x_2) + \tan x_2$ $+\tan(x_1 + x_2) + \tan 2x_2)$ $(\widetilde{C} \,:\, x_1 > 0, x_2 > 0, x_2 > x_1, x_2 < \frac{\pi}{4})$
$\rho_5(U(5)) \curvearrowright SO(10)/U(5)$	$X_{(x_1,x_2)} = 2(-2\cot x_1 - \cot 2x_1$ $+2\tan(x_1 - x_2) + 2\tan(x_1 + x_2),$ $-\cot 2x_2 - 2\tan(x_1 - x_2)$ $+2\tan(x_1 + x_2) + 2\tan x_2)$ $(\widetilde{C} \,:\, x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$

TABLE 3 (CONTINUED).

$H \curvearrowright G/K$	$X \qquad (\widetilde{C})$
$SO(2)^2 \times SO(3)^2 \curvearrowright$ $(SO(5) \times SO(5))/SO(5)$	$X_{(x_1,x_2)} = (-\cot x_1 - \cot(x_1 - x_2) - \cot(x_1 + x_2)$ $+ \tan x_1 + \tan(x_1 - x_2) + \tan(x_1 + x_2),$ $\cot(x_1 - x_2) - \cot x_2 - \cot(x_1 + x_2)$ $- \tan(x_1 - x_2) + \tan x_2 + \tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, x_2 > x_1, x_1 + x_2 < \frac{\pi}{2})$
$\rho_6(SO(5)) \curvearrowright$ $(SO(5) \times SO(5))/SO(5)$	$X_{(x_1,x_2)} = 2(-\cot x_1 + \tan(x_1 - x_2) + \tan(x_1 + x_2),$ $- \tan(x_1 - x_2) + \tan x_2 + \tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > x_1 - \frac{\pi}{2}, x_1 + x_2 < \frac{\pi}{2})$
$\rho_7(U(2)) \curvearrowright Sp(2)/U(2)$	$X_{(x_1,x_2)} = (-\cot x_1 + \tan(x_1 - x_2) + \tan(x_1 + x_2),$ $- \cot x_2 - \tan(x_1 - x_2) + \tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$
$U(q + 2) \curvearrowright$ $Sp(q + 2)/Sp(2) \times Sp(q)$ $(q > 2)$	$X_{(x_1,x_2)} = (-(2q - 4)\cot x_1 - 2\cot(x_1 - x_2) - 2\cot(x_1 + x_2)$ $-2\cot 2x_1 + (2q - 4)\tan x_1 + 2\tan(x_1 - x_2)$ $+2\tan(x_1 + x_2) + 4\tan 2x_1,$ $2\cot(x_1 - x_2) - (2q - 4)\cot x_2 - 2\cot(x_1 + x_2)$ $-2\cot 2x_2 - 2\tan(x_1 - x_2) + (2q - 4)\tan x_2$ $+2\tan(x_1 + x_2) + 4\tan 2x_2)$ $(\widetilde{C} : x_1 > 0, x_2 > x_1, x_2 < \frac{\pi}{4})$
$SU(4) \curvearrowright$ $Sp(4)/Sp(2) \times Sp(2)$	$X_{(x_1,x_2)} = (-2\cot x_1 - \cot(x_1 - x_2) - \cot(x_1 + x_2)$ $2\tan x_1 + 2\tan(x_1 - x_2) + 3\tan(x_1 + x_2),$ $\cot(x_1 - x_2) - 2\cot x_2 - \cot(x_1 + x_2)$ $-2\tan(x_1 - x_2) + \tan x_2 + 3\tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > x_1, x_1 + x_2 < \frac{\pi}{2})$
$U(4) \curvearrowright$ $Sp(4)/Sp(2) \times Sp(2)$	$X_{(x_1,x_2)} = (-2\cot x_1 - 2\cot(x_1 - x_2) - 2\cot(x_1 + x_2)$ $2\tan x_1 + \tan(x_1 - x_2) + 2\tan(x_1 + x_2),$ $2\cot(x_1 - x_2) - 2\cot x_2 - 2\cot(x_1 + x_2)$ $-\tan(x_1 - x_2) + \tan x_2 + 2\tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > x_1, x_1 + x_2 < \frac{\pi}{2})$
$Sp(j + 1) \times Sp(q - j + 1) \curvearrowright$ $Sp(q + 2)/Sp(2) \times Sp(q)$ $(q > 2)$	$X_{(x_1,x_2)} = (-4(j - 1)\cot x_1 - 6\cot 2x_1$ $+4(q - j - 1)\tan x_1 + 4\tan(x_1 - x_2) + 4\tan(x_1 + x_2),$ $-4(q - j - 1)\cot x_2 - 6\cot 2x_2 - 4\tan(x_1 - x_2)$ $+4(j - 1)\tan x_2 + 4\tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$

TABLE 3 (CONTINUED²).

$H \curvearrowright G/K$	$X \qquad (\widetilde{C})$
$Sp(2) \times Sp(2) \curvearrowright$ $Sp(4)/Sp(2) \times Sp(2)$	$X_{(x_1,x_2)} = (-3\cot x_1 + \tan x_1$ $+3\tan(x_1 - x_2) + 4\tan(x_1 + x_2),$ $-3\cot x_2 - 3\tan(x_1 - x_2) + 4\tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$
$SU(2)^2 \cdot SO(2)^2 \curvearrowright$ $(Sp(2) \times Sp(2))/Sp(2)$	$X_{(x_1,x_2)} = (-\cot x_1 - \cot(x_1 - x_2) - \cot(x_1 + x_2)$ $+\tan x_1 + \tan(x_1 - x_2) + \tan(x_1 + x_2),$ $\cot(x_1 - x_2) - \cot x_2 - \cot(x_1 + x_2)$ $-\tan(x_1 - x_2) + \tan x_2 + \tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > x_1, x_1 + x_2 < \frac{\pi}{2})$
$\rho_8(Sp(2)) \curvearrowright$ $(Sp(2) \times Sp(2))/Sp(2)$	$X_{(x_1,x_2)} = 2(-\cot x_1 + \tan(x_1 - x_2) + \tan(x_1 + x_2),$ $-\cot x_2 - \tan(x_1 - x_2) + \tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$
$\rho_9(Sp(2)) \curvearrowright$ $(Sp(2) \times Sp(2))/Sp(2)$	$X_{(x_1,x_2)} = 2(-\cot x_1 + \tan(x_1 - x_2) + \tan(x_1 + x_2),$ $-\cot x_2 - \tan(x_1 - x_2) + \tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$
$Sp(4) \curvearrowright$ $E_6/Spin(10) \cdot U(1)$	$X_{(x_1,x_2)} = (-4\cot x_1 - 3\cot(x_1 - x_2) - 4\cot(x_1 + x_2)$ $+4\tan x_1 + 3\tan(x_1 - x_2) + \tan(x_1 + x_2),$ $3\cot(x_1 - x_2) - 3\cot x_2 - 4\cot(x_1 + x_2)$ $-3\tan(x_1 - x_2) + 6\tan x_2 + \tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > x_1, x_1 + x_2 < \frac{\pi}{2})$
$SU(6) \cdot SU(2) \curvearrowright$ $E_6/Spin(10) \cdot U(1)$	$X_{(x_1,x_2)} = (-4\cot x_1 - 2\cot(x_1 - x_2) - 2\cot(x_1 + x_2)$ $-2\cot 2x_1 + 4\tan x_1$ $+4\tan(x_1 - x_2) + 3\tan(x_1 + x_2),$ $2\cot(x_1 - x_2) - 4\cot x_2 - 2\cot(x_1 + x_2)$ $-2\cot 2x_2 - 4\tan(x_1 - x_2)$ $+5\tan x_2 + 3\tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > x_1, x_1 + x_2 < \frac{\pi}{2})$
$\rho_{10}(SU(6) \cdot SU(2)) \curvearrowright$ $E_6/Spin(10) \cdot U(1)$	$X_{(x_1,x_2)} = (-4\cot x_1 - 4\cot(x_1 - x_2) - 4\cot(x_1 + x_2)$ $-2\cot 2x_1 + 4\tan x_1$ $+2\tan(x_1 - x_2) + \tan(x_1 + x_2),$ $4\cot(x_1 - x_2) - 4\cot x_2 - 4\cot(x_1 + x_2)$ $-2\cot 2x_2 - 2\tan(x_1 - x_2) + 5\tan x_2$ $+\tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > x_1, x_1 + x_2 < \frac{\pi}{2})$

TABLE 3 (CONTINUED[3]).

$H \curvearrowright G/K$	$X \qquad (\widetilde{C})$
$\rho_{11}(Spin(10) \cdot U(1)) \curvearrowright$ $E_6/Spin(10) \cdot U(1)$	$X_{(x_1,x_2)} = (-8\cot x_1 - 2\cot 2x_1$ $+6\tan(x_1 - x_2) + 5\tan(x_1 + x_2),$ $-2\cot 2x_2 - 6\tan(x_1 - x_2)$ $+9\tan x_2 + 5\tan(x_1 + x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, x_1 + x_2 < \frac{\pi}{2})$
$\rho_{12}(Spin(10) \cdot U(1)) \curvearrowright$ $E_6/Spin(10) \cdot U(1)$	$X_{(x_1,x_2)} = (-6\cot x_1 - \cot(x_1 - x_2) - \cot(x_1 + x_2)$ $2\tan x_1 + 5\tan(x_1 - x_2)$ $+4\tan(x_1 + x_2) + 2\tan 2x_1,$ $\cot(x_1 - x_2) - 6\cot x_2 - \cot(x_1 + x_2)$ $-5\tan(x_1 - x_2) + 3\tan x_2$ $+4\tan(x_1 + x_2) + 2\tan 2x_2)$ $(\widetilde{C} : x_1 > 0, x_2 < \frac{\pi}{4}, x_2 > x_1)$
$Sp(4) \curvearrowright E_6/F_4$	$X_{(x_1,x_2)} = (-8\cot 2x_1 - 4\cot(x_1 - \sqrt{3}x_2) - 4\cot(x_1 + \sqrt{3}x_2)$ $+8\tan 2x_1 + 4\tan(x_1 - \sqrt{3}x_2) + 4\tan(x_1 + \sqrt{3}x_2),$ $4\sqrt{3}\cot(x_1 - \sqrt{3}x_2) - 4\sqrt{3}\cot(x_1 + \sqrt{3}x_2)$ $-4\sqrt{3}\tan(x_1 - \sqrt{3}x_2) + 4\sqrt{3}\tan(x_1 + \sqrt{3}x_2))$ $(\widetilde{C} : 0 < x_1 < \frac{\pi}{4}, \frac{x_1}{\sqrt{3}} < x_2 < \frac{x_1}{\sqrt{3}} + \frac{\pi}{2\sqrt{3}}, -\frac{x_1}{\sqrt{3}} < x_2 < -\frac{x_1}{\sqrt{3}} + \frac{\pi}{2\sqrt{3}})$
$\rho_{13}(F_4) \curvearrowright E_6/F_4$	$X_{(x_1,x_2)} = (-16\cot 2x_1 + 8\tan(x_1 - \sqrt{3}x_2) + 8\tan(x_1 + \sqrt{3}x_2),$ $-8\sqrt{3}\tan(x_1 - \sqrt{3}x_2) + 8\sqrt{3}\tan(x_1 + \sqrt{3}x_2))$ $(\widetilde{C} : x_1 > 0, x_1 - \sqrt{3}x_2 < \frac{\pi}{2}, x_1 + \sqrt{3}x_2 < \frac{\pi}{2})$
$\rho_{14}(SO(4)) \curvearrowright G_2/SO(4)$	$X_{(x_1,x_2)} = (-2\cot 2x_1 + 3\tan(3x_1 - \sqrt{3}x_2) + \tan(x_1 - \sqrt{3}x_2)$ $+\tan(x_1 + \sqrt{3}x_2) + 3\tan(3x_1 + \sqrt{3}x_2),$ $-2\sqrt{3}\cot 2\sqrt{3}x_2 - \sqrt{3}\tan(3x_1 - \sqrt{3}x_2) - \sqrt{3}\tan(x_1 - \sqrt{3}x_2)$ $+\sqrt{3}\tan(x_1 + \sqrt{3}x_2) + \sqrt{3}\tan(3x_1 + \sqrt{3}x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, \sqrt{3}x_1 + x_2 < \frac{\pi}{2\sqrt{3}})$
$\rho_{15}(SO(4)) \curvearrowright G_2/SO(4)$	$X_{(x_1,x_2)} = (-2\cot 2x_1 + 3\tan(3x_1 - \sqrt{3}x_2) + \tan(x_1 - \sqrt{3}x_2)$ $+\tan(x_1 + \sqrt{3}x_2) + 3\tan(3x_1 + \sqrt{3}x_2),$ $-2\sqrt{3}\cot 2\sqrt{3}x_2 - \sqrt{3}\tan(3x_1 - \sqrt{3}x_2) - \sqrt{3}\tan(x_1 - \sqrt{3}x_2)$ $+\sqrt{3}\tan(x_1 + \sqrt{3}x_2) + \sqrt{3}\tan(3x_1 + \sqrt{3}x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, \sqrt{3}x_1 + x_2 < \frac{\pi}{2\sqrt{3}})$

TABLE 3 (CONTINUED[4]).

$H \curvearrowright G/K$	X $\qquad$ $(\widetilde{C})$
$\rho_{16}(G_2) \curvearrowright (G_2 \times G_2)/G_2$	$X_{(x_1,x_2)} = 2(-2\cot 2x_1 + 3\tan(3x_1 - \sqrt{3}x_2) + \tan(x_1 - \sqrt{3}x_2)$ $+ \tan(x_1 + \sqrt{3}x_2) + 3\tan(3x_1 + \sqrt{3}x_2),$ $-2\sqrt{3}\cot 2\sqrt{3}x_2 - \sqrt{3}\tan(3x_1 - \sqrt{3}x_2) - \sqrt{3}\tan(x_1 - \sqrt{3}x_2)$ $+\sqrt{3}\tan(x_1 + \sqrt{3}x_2) + \sqrt{3}\tan(3x_1 + \sqrt{3}x_2))$ $(\widetilde{C} : x_1 > 0, x_2 > 0, \sqrt{3}x_1 + x_2 < \frac{\pi}{2\sqrt{3}})$
$SU(2)^4 \curvearrowright (G_2 \times G_2)/G_2$	$X_{(x_1,x_2)} = (-2\cot 2x_1 - 3\cot(3x_1 - \sqrt{3}x_2) - \cot(x_1 - \sqrt{3}x_2)$ $- \cot(x_1 + \sqrt{3}x_2) - 3\cot(3x_1 + \sqrt{3}x_2) + 2\tan 2x_1$ $+3\tan(3x_1 - \sqrt{3}x_2) + \tan(x_1 - \sqrt{3}x_2)$ $+ \tan(x_1 + \sqrt{3}x_2) + 3\tan(3x_1 + \sqrt{3}x_2),$ $\sqrt{3}\cot(3x_1 - \sqrt{3}x_2) + \sqrt{3}\cot(x_1 - \sqrt{3}x_2) - \sqrt{3}\cot(x_1 + \sqrt{3}x_2)$ $-\sqrt{3}\cot(3x_1 + \sqrt{3}x_2) - 2\sqrt{3}\cot 2\sqrt{3}x_2 - \sqrt{3}\tan(3x_1 - \sqrt{3}x_2)$ $-\sqrt{3}\tan(x_1 - \sqrt{3}x_2) + \sqrt{3}\tan(x_1 + \sqrt{3}x_2)$ $+\sqrt{3}\tan(3x_1 + \sqrt{3}x_2) + 2\sqrt{3}\tan 2\sqrt{3}x_2)$ $(\widetilde{C} : x_1 > 0, x_2 < \frac{\pi}{4\sqrt{3}}, x_2 > \sqrt{3}x_1)$

TABLE 3 (CONTINUED[5]).

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