ASIAN J. MATH.
Vol. 14, No. 4, pp. 529–580, December 2010

003

ESTIMATES FOR THE HEAT KERNEL ON DIFFERENTIAL FORMS ON RIEMANNIAN SYMMETRIC SPACES AND APPLICATIONS *

N. LOHOUÉ† AND S. MEHDI‡

Abstract. We prove upper bounds estimates for the large time behavior of the heat kernel and for the resolvent of the form Laplacian on Riemannian symmetric spaces, and we obtain $L^{2+\epsilon}$-estimates for its resolvent on locally symmetric spaces. We deduce lower bounds for the bottom of the spectrum of the form Laplacian and some results on the vanishing of the L^2-cohomology of locally symmetric spaces.

Key words. Heat kernel, Form Laplacian, Riemannian symmetric space, locally symmetric space, Plancherel formula, Paley-Wiener theorem, L^2- cohomology, principal series representation, isotropy representation.

AMS subject classifications. Primary 22E46, 43A85, 53C35, 58J35, 58J50; Secondary 22E40, 57T15

1. Introduction. In the last decades the heat kernel has become a fundamental and powerful tool, subject of a rich and vast literature, reflecting its universality and formidable efficiency: Atiyah-Singer index theory, K-theory, spectral geometry, zeta and theta functions, L^2-invariants, anomalies, quantum gravity, ... (see e.g. [4], [7], [28], [37]). Despite these tremendous advances, explicit estimates for the asymptotics of the heat kernel and computation of related L^2-invariants are not available in general. However, for a large class of Riemannian manifolds, representation theoretic techniques may be used to obtain estimates for the asymptotics, compute L^2-invariants and derive some results on the L^2-cohomology.

More precisely, let G be a non compact connected semisimple Lie group with finite center and K a maximal compact subgroup of G. The homogeneous space G/K is naturally equipped with a structure of a non compact Riemannian symmetric manifold, the metric being induced by the Killing form of G. A finite dimensional representation (τ, E) of K induces a homogeneous vector bundle $\mathcal{E}$ over G/K. The group G acts by left translations on the Hilbert space $L^2(G/K, \mathcal{E})$ of square integrable sections of $\mathcal{E}$. Let

$$D : L^2(G/K, \mathcal{E}) \to L^2(G/K, \mathcal{E})$$

be a G-invariant selfadjoint positive elliptic operator, i.e D commutes with the action of G on $L^2(G/K, \mathcal{E})$. Denote by $P_t = e^{-tD}$ the fundamental solution of the heat equation

$$\begin{cases} DP_t = -\frac{\partial}{\partial t}P_t, \quad t > 0 \\ P_0 = \delta \end{cases}$$

where δ is the Dirac function. In particular, for each ψ in $L^2(G/K, \mathcal{E})$, the convolution

*Received May 30, 2009; accepted for publication October 4, 2010.
†UMR 8628–CNRS and Département de mathématiques, Université Paris Sud–Orsay, France (Noel.Lohoue@math.upsud.fr).
‡UMR 7122–CNRS and Département de mathématiques, Université Paul Verlaine–Metz, France (mehdi@univ-metz.fr).

product $P_t \star \psi$ is a solution of the heat equation in $L^2(G/K, \mathcal{E})$

$$\begin{cases} D\phi_t = -\frac{\partial}{\partial t}\phi_t, & t > 0 \\ \phi_0 = \psi. \end{cases}$$

It is known (see [6]) that the *heat operator* $P_t = e^{-tD}$ is a bounded operator on $L^2(G/K, \mathcal{E})$ so that

$$(P_t f)(g) = \int_G \mathcal{P}_t(g, g') f(g') dg', \quad \forall f \in L^2(G/K, \mathcal{E})$$

where

$$\mathcal{P}_t : G \times G \to \mathrm{End}(E)$$

is the *heat kernel*, and $\mathrm{End}(E)$ denotes the vector space of complex endomorphisms of E.

On the other hand, a torsion free discrete subgroup Γ of G acts on the left on G/K, so that the double coset space $\Gamma\backslash G/K$ is a locally symmetric Riemannian manifold. Except otherwise stated, we assume that Γ is not of finite covolume in G. Since G/K is simply connected, it is the universal cover of $\Gamma\backslash G/K$ and

$$\Gamma \simeq \Pi_1(\Gamma\backslash G/K).$$

The bundle $\mathcal{E}$ can be pushed down to a bundle over $\Gamma\backslash G/K$ and we let $L^2(\Gamma\backslash G/K, \mathcal{E})$ denote the corresponding Hilbert space of square integrable sections. In particular, the operator D drops down to $\Gamma\backslash G/K$ and defines a *locally invariant* selfadjoint positive elliptic operator

$$\widetilde{D} : L^2(\Gamma\backslash G/K, \mathcal{E}) \to L^2(\Gamma\backslash G/K, \mathcal{E}).$$

Write respectively $\widetilde{P}_t$ and $\widetilde{\mathcal{P}}_t$ for the corresponding heat operator and heat kernel. In this setting a certain pair $(D, \mathcal{E})$ will be distinguished. This particular pair, which we will focus on, may be thought of as a fundamental model for the general theory.

Consider the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$$

of $\mathfrak{g}$, where $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the complexification of the Lie algebra of G (resp. K) and $\mathfrak{s}$ is a complex vector subspace of $\mathfrak{g}$ satisfying the bracket relations

$$[\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s} \quad \text{and} \quad [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}.$$

The adjoint representation of G induces a finite dimensional representation σ_ℓ of K on the exterior product

$$V_\ell = \Lambda^\ell \mathfrak{s}$$

of $\mathfrak{s}$, for $\ell = 0, \cdots, \dim(\mathfrak{s})$, known as the isotropy representation. It should be noted that the decomposition of σ_ℓ into irreducible components is not known in general, except for (real) rank one groups [13]. We have made some explicit computations for real groups with (real) rank two and for complex groups. These results are described in the appendix. From now on we choose the representation (τ, E) to be the isotropy representation of K. In particular the Hilbert space $L^2(G/K, \mathcal{E})$ (resp. $L^2(\Gamma\backslash G/K, \mathcal{E})$) identifies naturally with the space $L^2(G/K, \mathcal{V}_\ell)$ (resp. $L^2(\Gamma\backslash G/K, \mathcal{V}_\ell)$) of square integrable ℓ-forms on G/K (resp. $\Gamma\backslash G/K$). Then we take D (resp. $\widetilde{D}$) to be the Hodge-de Rham Laplacian Δ_ℓ (resp. $\widetilde{\Delta}_\ell$) acting on ℓ-forms, with corresponding heat kernel $\mathcal{P}_t^\ell$ (resp. $\widetilde{\mathcal{P}}_t^\ell$). Related to the large time behavior of the heat kernel, there are several interesting L^2-invariants of $\Gamma\backslash G/K$ which can be computed explicitly [33]. One of these invariants will bear some special interest to us. Let $\Delta_\ell^\perp$ be the restriction of Δ_ℓ to the orthogonal complement of $\mathrm{Ker}(\Delta_\ell)$ in $L^2(G/K, \mathcal{V}_\ell)$. The Γ-trace of the corresponding heat kernel $\mathcal{P}_t^{\ell,\perp}$ is defined as follows:

$$\mathrm{Tr}_\Gamma\big(\mathcal{P}_t^{\ell,\perp}\big) = \int_{\mathcal{F}} \mathrm{Tr}\big(\mathcal{P}_t^{\ell,\perp}(x,x)\big)\,dx$$

where $\mathcal{F} \subset G/K$ is a fundamental domain for the action of Γ on G/K. Then the ℓth Novikov-Shubin invariant of $\Gamma\backslash G/K$ is given by

$$a_\ell(\Gamma\backslash G/K) = \sup\{b \in \mathbf{R}_+ \mid \mathrm{Tr}_\Gamma\big(\mathcal{P}_t^{\ell,\perp}\big) \overset{t\to+\infty}{=} O(t^{-\frac{b}{2}})\}. \tag{1.1}$$

This value (possibly infinite), which does not depend on Γ, nor on the Riemannian metric on $\Gamma\backslash G/K$, measures the asymptotic behavior of the spectral density function of Δ_ℓ at 0. Roughly speaking the ℓth Novikov-Shubin invariant measures the thickness of the spectrum of Δ_ℓ near 0. Using a Plancherel formula for differential forms, we have computed, for Γ of finite covolume, explicitly these invariants in [26] (see [33] for a more complete discussion on L^2-invariants of locally symmetric spaces):

$$a_\ell(\Gamma\backslash G/K) = \begin{cases} \mathrm{rk}_{\mathbf{C}}(G) - \mathrm{rk}_{\mathbf{C}}(K) \text{ if } \ell \in I(G;K) \text{ and } \mathrm{rk}_{\mathbf{C}}(G) > \mathrm{rk}_{\mathbf{C}}(K) \\ \\ \infty^+ \text{ otherwise} \end{cases}$$

where $\mathrm{rk}_{\mathbf{C}}(G)$ (resp. $\mathrm{rk}_{\mathbf{C}}(K)$) denotes the complex rank of G (resp. K) and $I(G;K)$ is the interval $\big[\frac{1}{2}\dim_{\mathbf{R}}(G/K) - \frac{1}{2}\big(\mathrm{rk}_{\mathbf{C}}(G) - \mathrm{rk}_{\mathbf{C}}(K)\big), \frac{1}{2}\dim_{\mathbf{R}}(G/K) + \frac{1}{2}\big(\mathrm{rk}_{\mathbf{C}}(G) - \mathrm{rk}_{\mathbf{C}}(K)\big)\big]$.

In the sequel we shall focus on the continuous part of the heat kernel. More precisely, let $\mathcal{P}_t^{\ell,\perp}$ be the heat kernel associated with $\Delta_\ell^\perp$, i.e the projection of $\mathcal{P}_t^\ell$ onto the restriction of Δ_ℓ to the orthogonal complement of $\mathrm{Ker}(\Delta_\ell)$ in $L^2(G/K, \mathcal{V}_\ell)$. It turns out that by a result of Borel (2.7), $\mathcal{P}_t^\ell$ and $\mathcal{P}_t^{\ell,\perp}$ coincide when $\mathrm{rk}_{\mathbf{C}}(G) > \mathrm{rk}_{\mathbf{C}}(K)$ or when $\mathrm{rk}_{\mathbf{C}}(G) = \mathrm{rk}_{\mathbf{C}}(K)$ and $\ell \neq \frac{1}{2}\dim_{\mathbf{R}}(G/K)$.

We now turn to the statement of our main results.

THEOREM 1 (Theorem 3.1). *For all $\epsilon \in]0,1[$ there exist two positive numbers a_ϵ and A_ϵ such that, for all $g \in G$ and $t \in \mathbf{R}$ satisfying $\| g \| > A_\epsilon$ and $t > 1$, we have*

$$\| \mathcal{P}_t^{\ell,\perp}(g) \| \leq a_\epsilon e^{-t\lambda_\ell(G/K)} \Phi_0(g) e^{-\frac{1-\epsilon}{(1+2\epsilon)^2}\frac{\|g\|^2}{4t}} t^{-\epsilon\frac{r+z}{2}}$$

where $\lambda_\ell(G/K)$ is the bottom of the spectrum of Δ_ℓ, Φ_0 is the Harish-Chandra spherical function on G, r is the minimal dimension of non trivial split components of cuspidal parabolic subgroups of G and z is the minimum of the orders of zero of the Harish-Chandra $\mathbf{c}$-functions corresponding to the conjugacy classes of proper cuspidal parabolic subgroups of G.

The strategy of the proof is

- use the expression of $\mathcal{P}_t^{\ell,\perp}$ derived from the Plancherel formula for square integrable ℓ-forms on G/K,
- following an idea of Alexopoulos and Lohoué [2], we decompose the scalar product $\langle p_t^{\ell,\perp}(g)\eta\,,\beta\rangle_{\Lambda^\ell\mathfrak{s}^C}$, $g\in G$, in two pieces $\varphi_{1,\epsilon}(g)$ and $\varphi_{2,\epsilon}(g)$ with support depending on ϵ, observing that we can choose η and β in the same irreducible component of σ_ℓ,
- combine a recent result of van den Ban and Souaifi on the proof by Delorme of a Paley-Wiener Theorem on the group G, to see that $\varphi_{2,\epsilon}$ is smooth and compactly supported, whose support does not contain g,
- estimate $\varphi_{1,\epsilon}$.

A link between the power of t in the above estimate and the ℓth Novikov-Shubin invariant of $\Gamma\backslash G/K$ is provided by Corollary 3.10.

Theorem 2 (Theorem 4.1). *For all $\epsilon\in]0,1[$, there exist two positive numbers b_ϵ and B_ϵ such that, for all $g\in G$ satisfying $\| g\|> B_\epsilon$, we have*

$$\| (\Delta_\ell-\mu)^{-1}(g)\|\leq b_\epsilon\Phi_0(g)e^{-(1-\epsilon)\tau_{\mu,\ell}(G/K)\|g\|}$$

where μ is a complex number in the resolvent set of Δ_ℓ and $\tau_{\mu,\ell}(G/K)$ is some positive real number depending on μ and on the bottom of the spectrum of Δ_ℓ.

The main steps of the proof are
- estimate the convolution product $(\Delta_\ell-\mu)^{-1}\star P_{\epsilon_0}^\ell$ for ϵ_0 sufficiently small,
- prove that the constants involved in our estimates do not depend on ϵ_0,
- take the limit $\epsilon_0\to 0$.

In the case of functions, i.e when $\ell=0$, sharp estimates for the heat kernel $\mathcal{P}_t^\ell$ and the resolvent of Δ_ℓ were obtained by J.-P. Anker and L. Ji in [3].

Theorem 3 (Theorem 5.9). *Assume that Γ is of finite covolume in G. Then for all complex number μ with positive imaginary part and element $\dot g\in\Gamma\backslash G$, there exists a positive number ϵ such that $\big(\widetilde{\Delta}_\ell-\mu\big)^{-k}(\dot g,\cdot)$ belongs to $L^{2+\epsilon}(\Gamma\backslash G/K,\mathcal{E}nd(\Lambda^\ell\mathfrak{s}))$, for all integer $k>\dfrac{1}{4}\dim_{\mathbf{R}}(G/K)$.*

The main lines of the proof are
- recall, by a result of A. Borel and H. Garland, that the kernel $\mathrm{Ker}(\widetilde{\Delta}_\ell)$ of $\widetilde{\Delta}_\ell$ is finite dimensional,
- use a result of N. Lohoué on the stability of L^p-cohomology around 2 to show that the orthogonal projection $T_\ell: L^2(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})\to\mathrm{Ker}(\widetilde{\Delta}_\ell)$ is a bounded operator on $L^{2+\epsilon}(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})$ for some positive real number ϵ,
- use a Stein interpolation theorem to prove that $\big(\widetilde{\Delta}_\ell-\mu\big)^{-1}$ is a bounded operator on $L^p(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})$ for $p\in[2,2+\epsilon]$ and $\mathrm{Im}(\mu)$ sufficiently large,
- analyze the generic terms occuring in the kth power of $\big(\widetilde{\Delta}_\ell-\mu\big)^{-1}$, with $k>\dfrac{1}{4}\dim_{\mathbf{R}}(G/K)$.

Note that this theorem, combined with Proposition 5.5, generalizes a result of R. Miatello and N. Wallach proved for functions, i.e when $\ell=0$, in the case where G has real rank one (Theorem 3.4 in [30]).

Theorem 4 (Theorem 6.1). *Let $\beta_\ell(\Gamma\backslash G/K)$ be the bottom of the spectrum of $\widetilde{\Delta}_\ell$ and $\delta(\Gamma)$ the critical exponent of Γ. We assume that Γ is of infinite covolume in G. Let ρ be the half sum of positive restricted $\mathfrak{g}$-roots and ρ_{min} the minimum of the* *values $\rho(X)$, $\| X \|= 1$, taken on the closure of a positive Weyl chamber in $\mathfrak{s}$. If we assume that $\lambda_\ell(G/K)$ does not vanish, then we have*

(i) *if* $\delta(\Gamma) \leq \rho_{min}$ *then* $\beta_\ell(\Gamma\backslash G/K) \geq \lambda_\ell(G/K)$,

(ii) *if* $\rho_{min} \leq \delta(\Gamma) \leq \| \rho \| + \sqrt{\lambda_\ell(G/K)}$ *then* $\beta_\ell(\Gamma\backslash G/K) \geq \lambda_\ell(G/K) - \big(\delta(\Gamma) - \rho_{min}\big)^2$, and

(iii) *if* $| \delta(\Gamma) - \rho_{min} | \leq \| \rho \| < \delta(\Gamma)$ *and* $\lambda_\ell(G/K) \geq \big(\delta(\Gamma) - \rho_{min}\big)^2$ *then* $\beta_\ell(\Gamma\backslash G/K) \geq \lambda_\ell(G/K) - \big(\delta(\Gamma) - \rho_{min}\big)^2$.

The main idea of the proof is

- use Poincaré series to deduce, from the previous theorem, an estimate for the resolvent of $\widetilde{\Delta}_\ell$,
- combine this estimate with some recent result of E. Leuzinger on $\beta_0(\Gamma\backslash G/K)$.

An immediate consequence on the L^2-cohomology of $\Gamma\backslash G/K$ can be deduced (see Section 2.13 for definitions).

Corollary (Corollary 6.5). *The (reduced or unreduced) L^2-cohomology group of degree ℓ of $\Gamma\backslash G/K$ is trivial in the following cases*:

(i) $\delta(\Gamma) \leq \rho_{min}$,

(ii) $\rho_{min} \leq \delta(\Gamma) \leq \| \rho \| + \sqrt{\lambda_\ell(G/K)}$ *and* $\sqrt{\lambda_\ell(G/K)} > \delta(\Gamma) - \rho_{min}$,

(iii) $| \delta(\Gamma) - \rho_{min} | \leq \| \rho \| < \delta(\Gamma)$ *and* $\sqrt{\lambda_\ell(G/K)} > | \delta(\Gamma) - \rho_{min} |$.

In particular, in these cases, the kernel of $\widetilde{\Delta}_\ell$ is reduced to $\{0\}$.

Analogous results for hyperbolic manifolds were obtained by G. Caron and E. Pedon in [13].

Our paper is organized as follows: in Section 2, we fix notations, recall some facts and give a representation theoretic description of the bottom of the spectrum of Δ_ℓ (proposition 2.32). Section 3 (resp. Section 4) is devoted to the proof of upper bounds estimates for the large time behavior of the heat kernel (resp. resolvent) of Δ_ℓ. Section 5 contains a proof of an $L^{2+\epsilon}$-estimate for the resolvent of $\widetilde{\Delta}_\ell$. In Section 6, we give lower bounds for the bottom of the spectrum of $\widetilde{\Delta}_\ell$ and we deduce some results on the vanishing of the (reduced or unreduced) L^2-cohomology of $\Gamma\backslash G/K$. Finally we have gathered in the appendix some computations on the bottom of the spectrum of Δ_ℓ. The main results in this paper were announced without proof in [25].

Acknowledgements. We thank Erik van den Ban for providing us with some new insights on Delorme's Paley-Wiener theorem which helped us to fill a gap in a first version of the paper. The second named author is indebted to Martin Olbrich for useful conversations. We also thank the referee for comments and suggestions that helped us improve the paper.

2. Preliminaries.

2.1. Roots, decompositions and norms. Let G be a non compact connected semisimple real Lie group with finite center and Lie algebra $\mathfrak{g}_0$. Fix a Cartan involution Θ of G and let K be the corresponding maximal compact subgroup of G with Lie algebra $\mathfrak{k}_0$. We shall drop the subscript 0 for the complexification. Let θ be the Cartan involution of $\mathfrak{g}_0$ derived from Θ and let

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$$

be the associated Cartan decomposition. There is a finite number s of conjugacy classes of θ-stable Cartan subalgebras in $\mathfrak{g}_0$, so we fix an element $\mathfrak{h}_{i,0}$ in each class and we put

$$\mathfrak{a}_{i,0} = \mathfrak{h}_{i,0} \cap \mathfrak{s}_0 \text{ and } \mathfrak{t}_{i,0} = \mathfrak{h}_{i,0} \cap \mathfrak{k}_0.$$

Let Δ_i be the set of $\mathfrak{g}$-roots relative to $\mathfrak{h}_i$ and fix a system of positive roots $\Delta_i^+ \subset \Delta_i$. Write Σ_i for the set of restricted roots, i.e the set of $\mathfrak{g}_0$-roots with respect to $\mathfrak{a}_{i,0}$. Choose a system $\Sigma_i^+ \subset \Sigma_i$ of positive restricted roots with the compatibility condition

$$\Big(\alpha \in \Delta_i^+ \text{ and } \alpha|_{\mathfrak{a}_{i,0}} \neq 0\Big) \implies \alpha|_{\mathfrak{a}_{i,0}} \in \Sigma_i^+.$$

As usual write

$$\rho_i = \frac{1}{2} \sum_{\alpha \in \Delta_i^+} \alpha$$

for the half-sum of positive roots and

$$\rho_{\mathfrak{a}_{i,0}} = \frac{1}{2} \sum_{\alpha \in \Sigma_i^+} m_\alpha \alpha$$

for the half-sum of positive restricted roots counted with their multiplicities, i.e $m_\alpha = \dim(\mathfrak{g}_0)_\alpha$, where $(\mathfrak{g}_0)_\alpha$ denotes the root space corresponding to α. The subset Σ_i^{++} of Σ_i^+ will denote the set of positive indivisible restricted roots

$$\Sigma_i^{++} = \{\alpha \in \Sigma_i^+ \mid \frac{1}{2}\alpha \notin \Sigma_i^+\}.$$

Write $\mathrm{rk}_{\mathbf{C}}(G)$ and $\mathrm{rk}_{\mathbf{C}}(K)$ for the complex ranks of G and K respectively. Denote by W_i the Weyl group associated with Δ_i and by $\mid W_i \mid$ its order. Let

$$\mathfrak{n}_{i,0} = \sum_{\alpha \in \Sigma_i^+} (\mathfrak{g}_0)_\alpha \text{ and } \mathfrak{m}_{i,0} = \mathfrak{t}_{i,0} + \sum_{\beta \in \Delta_i, \beta|_{\mathfrak{a}_{i,0}} = 0} (\mathfrak{g}_0)_\beta.$$

Write $(M_i)_e$, A_i and N_i for the analytic subgroups of G with Lie algebra $\mathfrak{m}_{i,0}, \mathfrak{a}_{i,0}$ and $\mathfrak{n}_{i,0}$ respectively. There exists a unique Θ-stable subgroup M_i of G such that the centralizer of $\mathfrak{a}_{i,0}$ in G is $M_i A_i$. The subgroup

$$P_i = M_i A_i N_i$$

is a cuspidal parabolic subgroup of G, in particular the discrete series $(\widehat{M_i})_d$ of M_i is not empty. We may describe, in this way, the set of all conjugacy classes of cuspidal parabolic subgroups of G with Lie algebra

$$\mathfrak{p}_{i,0} = \mathfrak{m}_{i,0} \oplus \mathfrak{a}_{i,0} \oplus \mathfrak{n}_{i,0}.$$

Observe that the group G itself is cuspidal if, and only if, the discrete series $\widehat{G}_d$ of G is not empty, i.e $\mathrm{rk}_{\mathbf{C}}(G) = \mathrm{rk}_{\mathbf{C}}(K)$.

We shall drop the subscript i and simply write $\mathfrak{a}_0$ for a maximal abelian subspace of $\mathfrak{s}_0$, $\mathfrak{a}$ its complexification, Σ the set of restricted roots, $\rho_{\mathfrak{a}_0}$ the half-sum of positive restricted roots and $P = MAN$ the corresponding (minimal) parabolic subgroup of G. The real rank $\mathrm{rk}_{\mathbf{R}}(G)$ of G is the dimension of $\mathfrak{a}_0$. The Iwasawa decomposition of G is

$$G = KAN$$

where any element g of G can be written in a unique way as

$$g = k(g)e^{a(g)}n(g).$$

Moreover our choice of Σ^+ fixes a positive Weyl chamber $\mathfrak{a}_0^+$ in $\mathfrak{a}_0$ which defines the following Cartan decomposition of G

$$G = K \exp(\overline{\mathfrak{a}_0^+})K$$

where any element g of G can be written as

$$g = k_1(g)e^{a^+(g)}k_2(g), \tag{2.1}$$

$\overline{\mathfrak{a}_0^+}$ being the closure of $\mathfrak{a}_0^+$. Note that the component $a^+(g)$ of g is uniquely determined, whereas the K-components $k_1(g)$ and $k_2(g)$ are not.

The Killing form of $\mathfrak{g}$

$$\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbf{C},\ (X, Y) \mapsto \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$$

defines the following G-invariant inner product on $\mathfrak{g}$

$$\langle X\,, Y\rangle = -\mathcal{K}(X, \theta(Y))$$

which in turn induces a Riemannian structure on the symmetric space G/K, whose tangent space at the origin eK is identified with $\mathfrak{s}$. In particular, this enables us to identify $\mathfrak{g}$ with its vector dual $\mathfrak{g}^\star$, as well as subspaces of $\mathfrak{g}$ with subspaces in $\mathfrak{g}^\star$. We shall denote by the same symbol $||\ ||$ the induced norms on $\mathfrak{g}$ and $\mathfrak{g}^\star$, as well as the norm on G defined by

$$|| \, g \, || = || \, a^+(g) \, || \,.$$

In particular one has

$$|| \, g^{-1} \, || = || \, g \, || \text{ and } || \, kgk' \, || = || \, g \, || \text{ for all } g \in G \text{ and } k, k' \in K.$$

2.2. Principal series representations. Fix a proper parabolic subgroup $P_i = M_iA_iN_i$ of G. Let δ_i be a discrete series representation of M_i in some Hilbert space V_{δ_i} equipped with an M_i-invariant scalar product $\langle\,,\rangle_{V_{\delta_i}}$ and an induced norm $||\ ||_{V_{\delta_i}}$. Let α_i be a linear form on $\mathfrak{a}_i$. The principal series representation of G associated with the data P_i, δ_i and α_i is the induced representation

$$\pi_{P_i,\delta_i,\alpha_i} \stackrel{def.}{=} \mathrm{Ind}_{P_i}^G \left(\delta_i \otimes e^{\alpha_i + \rho_{\mathfrak{a}_{i,0}}} \otimes 1\right)$$

of G in some Hilbert space $\mathcal{H}_{P_i,\delta_i,\alpha_i}$. More precisely, write $V_{\delta_i}^\infty$ for the space of smooth vectors in δ_i and consider the vector space $\mathcal{H}^\infty_{P_i,\delta_i,\alpha_i}$ of $V_{\delta_i}^\infty$-valued smooth functions on G satisfying the equivariance relation

$$f(gman) = e^{-(\alpha_i + \rho_{\mathfrak{a}_{i,0}})(a)}\delta_i(m)^{-1}(f(g))\ \ \forall g \in G,\ m \in M_i,\ a \in A_i,\ n \in N_i$$

equipped with the scalar product

$$\langle \phi_1\,, \phi_2\rangle_{\mathcal{H}_{P_i,\delta_i,\alpha_i}} = \int_K \langle \phi_1(k)\,, \phi_2(k)\rangle_{V_{\delta_i}} dk.$$

Then the Hilbert space $\mathcal{H}_{P_i,\delta_i,\alpha_i}$ is the completion of $\mathcal{H}^{\infty}_{P_i,\delta_i,\alpha_i}$ with respect to this norm on which G acts by left translations. If α_i is imaginary then $\pi_{P_i,\delta_i,\alpha_i}$ is a unitary representation (Chapter VII of [20]). The module of smooth vectors of the principal series representation $\pi_{P_i,\delta_i,\alpha_i}$ of G has a realization in the space $C^\infty(K;\delta_i)$ of smooth V_{δ_i}-valued maps $\phi : K \longrightarrow V_{\delta_i}$ on K transforming under the rule

$$\phi(km) = \delta_i^{-1}(m)\phi(k) \quad \forall k \in K,\ m \in K \cap M_i.$$

If $g \in G$ decomposes under $G = KM_iA_iN_i$ as

$$g = \kappa(g)\mu(g)e^{H(g)}n$$

then the action of G on $C^\infty(K;\delta_i)$ is given by

$$\pi_{P_i,\delta_i,\alpha_i}(g)\phi(k) = e^{-(\alpha_i+\rho_{\mathfrak{a}_{i,0}})(H(g^{-1}k))}\delta_i(\kappa(g^{-1}k))^{-1}\phi(\kappa(g^{-1}k)). \tag{2.2}$$

This is known as the *compact picture realization* of the principal series representation $\pi_{P_i,\delta_i,\alpha_i}$ (chapter VII of [20]).

2.3. L^p-integrable differential forms on G/K. The group K acts on $\mathfrak{s}$ by the restriction of the (linear extension of the) adjoint action Ad of G. This action induces a representation σ_ℓ of K on the exterior product $V_\ell = \wedge^\ell \mathfrak{s}$

$$\sigma_\ell(k)(v_1 \wedge v_2 \wedge \cdots \wedge v_\ell) = \mathrm{Ad}(k)v_1 \wedge \mathrm{Ad}(k)v_2 \cdots \wedge \mathrm{Ad}(k)v_\ell, \quad \ell \geq 1,$$
$$\sigma_0(k)v = v, \quad v \in \mathbf{C},$$

known as the isotropy representation. Observe that, by Hodge isomorphism, σ_ℓ and $\sigma_{\dim(\mathfrak{s})-\ell}$ are equivalent for all $0 \leq \ell \leq \dim(\mathfrak{s})$. The isotropy representation is not irreducible in general, and its explicit decomposition into irreducibles

$$(\sigma_\ell, V_\ell) \simeq \bigoplus (\sigma_\ell^j, V_\ell^j) \tag{2.3}$$

is still an open problem. We fix a K-invariant scalar product $\langle\, ,\rangle_{\Lambda^\ell \mathfrak{s}}$ on V_ℓ such that

$$\langle V_\ell^i, V_\ell^j \rangle = \{0\} \text{ if } i \neq j.$$

The isotropy representation defines a homogeneous vector bundle $\mathcal{V}_\ell$ over G/K and we let $L^p(G/K, \mathcal{V}_\ell)$ be the space of its L^p-sections, i.e the L^p-integrable ℓ-forms on G/K, with $p \in \mathbf{N}^\star$. Naturally there is an action, by left translations, of the group G on $L^p(G/K, \mathcal{V}_\ell)$. More precisely, the tensor product $L^p(G) \otimes \wedge^\ell \mathfrak{s}$ is equipped with an action of G and of K given respectively by $L \otimes \mathbb{1}$ and $R \otimes \sigma_\ell$, where $L^p(G)$ is the space of L^p-integrable complex functions on G and L (resp. R) is the left (resp. right) translation by G. In particular, we obtain an isomorphism of G-modules

$$L^p(G/K, \mathcal{V}_\ell) \simeq \big(L^p(G) \otimes \wedge^\ell \mathfrak{s}\big)^K,$$

where $\big(L^p(G) \otimes \wedge^\ell \mathfrak{s}\big)^K$ denotes the subspace of K-invariant vectors, equipped with the natural norm

$$\| \phi \|_{L^p(G/K,\mathcal{V}_\ell)} = \Big(\int_G \| \phi(g) \|^p_{\Lambda^\ell \mathfrak{s}}\, dg \Big)^{\frac{1}{p}}.$$

When $\ell = 0$, i.e in the case of functions, we will simply write $L^p(G/K)$. Next the group K acts on the vector space $\mathrm{End}(\Lambda^\ell \mathfrak{s})$ of complex endomorphisms of $\Lambda^\ell \mathfrak{s}$ as follows

$$\widetilde{\sigma}_\ell(k)(T) = \sigma_\ell(k) \circ T \circ \sigma_\ell(k)^{-1}, \ \ \forall k \in K, \ \forall T \in \mathrm{End}(\Lambda^\ell \mathfrak{s}).$$

This representation induces a homogeneous vector bundle $\mathcal{E}nd(\Lambda^\ell \mathfrak{s})$ over G/K and $L^p(G/K, \mathcal{E}nd(\Lambda^\ell \mathfrak{s}))$ will denote the space of L^p-sections. We also have an isomorphism of G-modules

$$L^p(G/K, \mathcal{E}nd(\Lambda^\ell \mathfrak{s})) \simeq \Big(L^p(G) \otimes \mathrm{End}(\Lambda^\ell \mathfrak{s})\Big)^K,$$

for the K-action $R \otimes \widetilde{\sigma}_\ell$, with the norm

$$\| \phi \|_{L^p(G/K, \mathcal{E}nd(\Lambda^\ell \mathfrak{s}))} = \Big(\int_G \| \phi(g) \|^p_{\mathrm{End}(\Lambda^\ell \mathfrak{s})} \, dg \Big)^{\frac{1}{p}}.$$

In the case where $p = +\infty$, these definitions are adapted as usual.

2.4. Laplacian on square integrable differential forms on G/K. The Killing form of $\mathfrak{g}$ induces a sequence of G-equivariant maps

$$\mathrm{End}(\mathfrak{g}) \overset{\text{canonical}}{\longrightarrow} \mathfrak{g} \otimes \mathfrak{g}^\star \overset{\text{Killing}}{\longrightarrow} \mathfrak{g} \otimes \mathfrak{g} \overset{\text{injection}}{\hookrightarrow} T(\mathfrak{g}) \overset{\text{quotient}}{\hookrightarrow} \mathcal{U}(\mathfrak{g}),$$

where $\mathrm{End}(\mathfrak{g})$ denotes the vector space of complex endomorphisms of $\mathfrak{g}$, $\mathfrak{g}^\star$ the vector dual of $\mathfrak{g}$, $T(\mathfrak{g})$ the tensor algebra of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let Ω_G be the image in (the center of) $\mathcal{U}(\mathfrak{g})$ of the identity. Any element $A = A_1 \cdot A_2 \cdot \dots \cdot A_p$ of $\mathcal{U}(\mathfrak{g})$ defines a differential operator $\tilde{A}$ on G

$$\tilde{A} = \tilde{A}_1 \circ \dots \circ \tilde{A}_p \text{ where } (\tilde{A}_j f)(g) = \frac{d}{dt}_{|t=0} f(\exp(-tA_j)g) \, \forall f \in C^\infty(G), \ g \in G.$$

In particular, the G-invariant differential operator $\widetilde{\Omega}_G$ on G is the Casimir operator of G. Similarly we define the Casimir operators $\widetilde{\Omega}_K$ of K and $\widetilde{\Omega}_{M_i}$ of M_i. The representation $\pi_{P_i,\delta_i,\alpha_i}$ defines, by differentiation, an action of $\mathcal{U}(\mathfrak{g})$ on the smooth vectors of $\mathcal{H}_{P_i,\delta_i,\alpha_i}$. We will denote this action by the same symbol $\pi_{P_i,\delta_i,\alpha_i}$. It is known that $\widetilde{\Omega}_G$ acts as a scalar operator on the (smooth vectors of the) principal series representation $\pi_{P_i,\delta_i,\alpha_i}$ (see Proposition 8.22 of [20])

$$\pi_{P_i,\delta_i,\alpha_i}(\widetilde{\Omega}_G) = \omega_{\delta_i,\alpha_i} \mathrm{Id}$$

with

$$\omega_{P_i,\delta_i,\alpha_i} = \| \mathrm{char}(\delta_i) \|^2 + \| \alpha_i \|^2 - \| \rho_i \|^2 = \delta_i(\widetilde{\Omega}_{M_i}) + \| \alpha_i \|^2 - \| \rho_{\mathfrak{a}_i} \|^2, \tag{2.4}$$

where $\mathrm{char}(\delta_i)$ denotes the infinitesimal character of δ_i. If we let

$$Q_\ell = \int_K R(k) \otimes \sigma_\ell(k) dk$$

be the projection of $L^2(G) \otimes \wedge^\ell \mathfrak{s}$ onto the subspace $\Big(L^2(G) \otimes \wedge^\ell \mathfrak{s}\Big)^K$ of K-invariant vectors, then the Laplacian Δ_ℓ acting on square integrable ℓ-forms on G/K is defined by

$$\Delta_\ell \circ Q_\ell = -Q_\ell \circ (\widetilde{\Omega}_G \otimes \mathrm{Id}_{\wedge^\ell \mathfrak{s}}). \tag{2.5}$$

On the other hand, the spectrum $sp(\Delta_\ell)$ of Δ_ℓ decomposes as a discrete spectrum $sp_d(\Delta_\ell)$ and continuous spectrum $sp_c(\Delta_\ell)$

$$sp(\Delta_\ell) = sp_c(\Delta_\ell) \cup sp_d(\Delta_\ell). \tag{2.6}$$

It is well known that (see Theorem A and B in [8], and Proposition 1.2 in [33])

<u>if $\mathrm{rk}_{\mathbf{C}}(G) > \mathrm{rk}_{\mathbf{C}}(K)$:</u>

$\mathrm{Ker}(\Delta_\ell) = \{0\}$ for all ℓ,

$0 \in sp(\Delta_\ell) \Leftrightarrow 2\ell \in \Big[\dim(G/K) + \mathrm{rk}_{\mathbf{C}}(K) - \mathrm{rk}_{\mathbf{C}}(G), \dim(G/K) + \mathrm{rk}_{\mathbf{C}}(G) - \mathrm{rk}_{\mathbf{C}}(K)\Big]$,

<u>if $\mathrm{rk}_{\mathbf{C}}(G) = \mathrm{rk}_{\mathbf{C}}(K)$:</u>

$\mathrm{Ker}(\Delta_{\frac{1}{2}\dim_{\mathbf{R}}(G/K)})$ is infinite dimensional, and $\mathrm{Ker}(\Delta_\ell) = \{0\}$ if $\ell \neq \frac{1}{2}\dim_{\mathbf{R}}(G/K)$,

$0 \in sp(\Delta_{\frac{1}{2}\dim_{\mathbf{R}}(G/K)})$, and $sp(\Delta_\ell)$ is strictly bounded away from zero if $\ell \neq \frac{1}{2}\dim_{\mathbf{R}}(G/K)$.

For the harmonic ℓ-forms, writing

$$\widehat{G}(\sigma_\ell) = \{\pi \in \widehat{G}_d \mid \mathrm{char}(\pi) = \mathrm{char}(1\!\!1_G)\}$$

where $1\!\!1_G$ denotes the trivial representation of G, we have

$$\mathrm{Ker}(\Delta_\ell) = \begin{cases} \sum_{\pi\in\widehat{G}(\sigma_\ell)} \mathcal{H}_\pi \text{ if } \mathrm{rk}_{\mathbf{C}}(G) = \mathrm{rk}_{\mathbf{C}}(K) \text{ and } \ell = \frac{1}{2}\dim_{\mathbf{R}}(G/K), \\ \{0\} \text{ otherwise.} \end{cases} \tag{2.7}$$

In other words, $\mathrm{Ker}(\Delta_\ell)$ is either reduced to $\{0\}$ or is infinite dimensional. Observe that the number $\dim_{\mathbf{R}}(G/K) + \mathrm{rk}_{\mathbf{C}}(K) - \mathrm{rk}_{\mathbf{C}}(G)$ is positive and $\dim_{\mathbf{R}}(G/K) + \mathrm{rk}_{\mathbf{C}}(G) - \mathrm{rk}_{\mathbf{C}}(K)$ is always even. A throughout discussion on the spectrum of Δ_ℓ for more general manifolds is given in [27].

2.5. Plancherel formula for square integrable differential forms on G/K. Following (2.6) the space of square integrable ℓ-forms decomposes under the action of G into a continuous part and a discrete part

$$L^2(G/K,\mathcal{V}_\ell) = L^2(G/K,\mathcal{V}_\ell)_c \oplus L^2(G/K,\mathcal{V}_\ell)_d.$$

The Harish-Chandra Plancherel formula decomposes the continuous part of the biregular representation $L\otimes R$ of G as

$$L^2(G)_c \simeq \sum_{i,\ \dim(\mathfrak{a}_i)>0}\ \sum_{\delta_i\in(\widehat{M_i})_d} \int_{\mathfrak{a}^\star_{i,0}}^{\widehat{\oplus}} \mathcal{H}_{\delta_i,\sqrt{-1}\nu_i} \widehat{\otimes} \mathcal{H}^\star_{P_i,\delta_i,\sqrt{-1}\nu_i} \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i,$$

where $\widehat{\oplus}$ (resp. $\widehat{\otimes}$) denotes the Hilbert sum (resp. product), $d\nu_i$ a Lebesgue measure on $\mathfrak{a}^\star_{i,0}$ and, for a fixed $\delta_i \in (\widehat{M_i})_d$, the function $\mathbf{c}_{\delta_i}$ is the Plancherel density. It is known that $\mathbf{c}_{\delta_i}$ is a non-negative continuous function with polynomial growth on $\mathfrak{a}^\star_{i,0}$ (Theorem 19 of [18]), i.e there exist a positive real number b_i and a non negative integer $N_i \in 4\mathbf{N}$ (both depending on δ_i) such that

$$\mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) \leq b_i\big(1+ \|\ \nu_i\ \|^2\ \big)^{\frac{N_i}{4}}. \tag{2.8}$$

We have

$$L^2(G)_c \otimes \Lambda^\ell \mathfrak{s} \simeq \sum_{i,\ \dim(\mathfrak{a}_i)>0} \sum_{\delta_i \in (\widehat{M}_i)_d} \int_{\mathfrak{a}^\star_{i,0}}^{\widehat{\oplus}} \mathcal{H}_{\delta_i,\sqrt{-1}\nu_i} \widehat{\otimes} \Big(\mathcal{H}^\star_{P_i,\delta_i,\sqrt{-1}\nu_i} \otimes \Lambda^\ell \mathfrak{s} \Big) \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i$$

so that

$$\begin{aligned} &\big(L^2(G)_c \otimes \Lambda^\ell \mathfrak{s}\big)^K \\ \text{(2.9)}\quad &\simeq \sum_{i,\ \dim(\mathfrak{a}_i)>0} \sum_{\delta_i \in (\widehat{M}_i)_d} \int_{\mathfrak{a}^\star_{i,0}}^{\widehat{\oplus}} \mathcal{H}_{\delta_i,\sqrt{-1}\nu_i} \widehat{\otimes} \big(\mathcal{H}^\star_{P_i,\delta_i,\sqrt{-1}\nu_i} \otimes \Lambda^\ell \mathfrak{s} \big)^K \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i. \end{aligned}$$

Next the Frobenius reciprocity Theorem implies that the restriction $\mathrm{Ind}_{M_i\cap K}^K(\delta_i\mid_{M_i\cap K})$ of $\pi_{\delta_i,\sqrt{-1}\nu_i}$ to K does not depend on ν_i. If we let $\mathcal{H}_{\delta_i}$ be the completion of the complex vector space $C^\infty(K;\delta_i)$ with respect to the norm given by

$$\| f \|_{\mathcal{H}_{\delta_i}} = \Big(\int_K \| f(k) \|^2_{V_{\delta_i}} \, dk \Big)^{1/2},$$

then the restriction to K is an isometry of $\mathcal{H}_{P_i,\delta_i,\sqrt{-1}\nu_i}$ onto the K-module $\mathcal{H}_{\delta_i}$. Moreover the complex vector space $\big(\mathcal{H}^*_{\delta_i,\sqrt{-1}\nu_i} \otimes \Lambda^\ell \mathfrak{s}\big)^K$ is isomorphic, as a K-module, to the space $\mathrm{Hom}_K(\mathcal{H}_{\delta_i}, \Lambda^\ell \mathfrak{s})$ of K-equivariant homomorphisms from $\mathcal{H}_{\delta_i}$ onto $\mathcal{V}_l$, so that (2.9) becomes

$$L^2(G/K,\mathcal{V}_\ell)_c \simeq \sum_{i,\ \dim(\mathfrak{a}_i)>0} \sum_{\delta_i \in (\widehat{M}_i)_d} \int_{\mathfrak{a}^\star_{i,0}}^{\widehat{\oplus}} \mathcal{H}_{\delta_i,\sqrt{-1}\nu_i} \widehat{\otimes} \mathrm{Hom}_K(\mathcal{H}_{\delta_i}, \Lambda^\ell \mathfrak{s}^\star) \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i.$$

In particular the only principal series representations $\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}$ of G appearing in the above decomposition are those satisfying $\mathrm{Hom}_{M_i\cap K}(\sigma_\ell,\delta_i) \neq \{0\}$. We deduce that (see Section 5 of [26])

$$\begin{aligned} &L^2(G/K,\mathcal{V}_\ell)_c \\ &\simeq \sum_{i,\ \dim(\mathfrak{a}_i)>0} \sum_{\delta_i \in \widehat{M}_i(\sigma_\ell)} \int_{\mathfrak{a}^\star_{i,0}}^{\widehat{\oplus}} \mathcal{H}_{P_i,\delta_i,\sqrt{-1}\nu_i} \widehat{\otimes} \Big(\mathcal{H}^\star_{P_i,\delta_i,\sqrt{-1}\nu_i} \otimes \Lambda^\ell \mathfrak{s} \Big)^K \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i, \end{aligned}$$

where

$$\text{(2.10)}\qquad \widehat{M}_i(\sigma_\ell) \overset{def.}{=} \Big\{ \delta \in (\widehat{M}_i)_d \mid \mathrm{Hom}_{M_i\cap K}(\sigma_\ell,\delta) \neq \{0\} \Big\}.$$

Similarly for the discrete part, we deduce, from (2.7), that

$$L^2(G/K,\mathcal{V}_\ell)_d = \begin{cases} \sum_{\pi\in\widehat{G}(\sigma_\ell)} \mathcal{H}_\pi \otimes \mathrm{Hom}_K(\mathcal{H}_\pi, \Lambda^\ell \mathfrak{s}^\star) \text{ if } \mathrm{rk}_{\mathbf{C}}(G) = \mathrm{rk}_{\mathbf{C}}(K) \\ \qquad\qquad\qquad\qquad \text{and } \ell = \frac{1}{2}\dim_{\mathbf{R}}(G/K), \\ \{0\} \text{ otherwise.} \end{cases}$$

Therefore the Plancherel formula for square integrable ℓ-forms is given by

(2.11)

$$\begin{aligned}
&L^2(G/K,\mathcal{V}_\ell)\\
&\simeq \sum_{i,\ \dim(\mathfrak{a}_i)>0}\ \sum_{\delta_i\in\widehat{M}_i(\sigma_\ell)} \int_{\mathfrak{a}^\star_{i,0}}^{\widehat{\oplus}} \mathcal{H}_{P_i,\delta_i,\sqrt{-1}\nu_i}\widehat{\otimes}\Big(\mathcal{H}^\star_{P_i,\delta_i,\sqrt{-1}\nu_i}\otimes\Lambda^\ell\mathfrak{s}\Big)^K \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i)d\nu_i\\
&+\sum_{\pi\in\widehat{G}(\sigma_\ell)}\mathcal{H}_\pi\otimes\mathrm{Hom}_K(\mathcal{H}_\pi,\Lambda^\ell\mathfrak{s}^\star)\\
&\quad\text{if } \mathrm{rk}_{\mathbf{C}}(G)=\mathrm{rk}_{\mathbf{C}}(K) \text{ and } \ell=\frac{1}{2}\dim_{\mathbf{R}}(G/K),\\
&L^2(G/K,\mathcal{V}_\ell)\\
&\simeq \sum_{i,\ \dim(\mathfrak{a}_i)>0}\ \sum_{\delta_i\in\widehat{M}_i(\sigma_\ell)} \int_{\mathfrak{a}^\star_{i,0}}^{\widehat{\oplus}} \mathcal{H}_{P_i,\delta_i,\sqrt{-1}\nu_i}\widehat{\otimes}\Big(\mathcal{H}^\star_{P_i,\delta_i,\sqrt{-1}\nu_i}\otimes\Lambda^\ell\mathfrak{s}\Big)^K \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i)d\nu_i\\
&\quad\text{if } \mathrm{rk}_{\mathbf{C}}(G)\neq\mathrm{rk}_{\mathbf{C}}(K) \text{ or } \ell\neq\frac{1}{2}\dim_{\mathbf{R}}(G/K).
\end{aligned}$$

2.6. Spherical Fourier transform and inverse Fourier transform. Consider the decomposition (2.3) of the isotropy representation and write pr_j for the corresponding projection

$$\mathrm{pr}_j : V_\ell \to V_\ell^j.$$

Let $f : G \to \Lambda^\ell\mathfrak{s}$ be a compactly supported smooth map which is K-equivariant, i.e

$$f(gk) = \sigma_\ell^{-1}(k)f(g),\ \ \forall g\in G,\ k\in K.$$

We decompose f as the sum

$$f=\sum_j f^j$$

of K-equivariant maps, where

$$f^j = \mathrm{pr}_j\circ f.$$

Similarly to (2.10), define for each j the set

$$\widehat{M_i}(\sigma_\ell^j) \overset{def.}{=} \Big\{\delta\in(\widehat{M_i})_d \mid \mathrm{Hom}_{M_i\cap K}(\sigma_\ell^j,\delta)\neq\{0\}\Big\}.$$

For δ_i, j and ℓ fixed, let $\{T^{\ell,j}_{\delta_i,r}\}_{r\geq 1}$ be an orthonormal basis of the (finite dimensional) complex vector space $\mathrm{Hom}_K(\mathcal{H}_{\delta_i},V_\ell^j)$ with respect to the usual scalar product

$$\langle B\,,C\rangle = \frac{1}{\dim(V_\ell^j)}\mathrm{Tr}(B^\star C)$$

where $B^\star$ denotes the adjoint of B. Define the maps

$$T^{\ell,j}_{\delta_i} = \sum_r T^{\ell,j}_{\delta_i,r} \text{ and } T^{\ell,j\star}_{\delta_i} = \sum_r T^{\ell,j\star}_{\delta_i,r}.$$

Now the Fourier transform $\widehat{f^j}$ of f^j is the map

$$\widehat{f^j} : \widehat{M}_i(\sigma_\ell^j) \times \sqrt{-1}\mathfrak{a}_{i,0}^\star \to \mathcal{H}_{P_i,\delta_i,\sqrt{-1}\nu_i} \times \mathrm{Hom}_K(\mathcal{H}_{\delta_i}, V_\ell^j)$$

defined by

$$\widehat{f^j}(\delta_i, \sqrt{-1}\nu_i) = \frac{1}{\dim(V_\ell^j)} \int_G \pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(g) \circ T_{\delta_i}^{\ell,j\star}(f^j(g)) \otimes T_{\delta_i}^{\ell,j} dg. \tag{2.12}$$

The inverse Fourier transform is given by

$$\begin{aligned} &f^j(g) \\ &= \sum_i \sum_{\delta_i \in \widehat{M_i}(\sigma_\ell^j)} \frac{1}{\mid W_i \mid} \frac{1}{\dim(V_\ell^j)} \int_{\mathfrak{a}_{i,0}^\star} \Phi_{\delta_i}^j \Big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(g^{-1}) \otimes 1_{\mathrm{Hom}_K(\mathcal{H}_{\delta_i}, V_\ell^j)} \big(\widehat{f^j}(\delta_i, \sqrt{-1}\nu_i)\big)\Big) \\ &\qquad \times \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i \end{aligned} \tag{2.13}$$

where

$$\Phi_{\delta_i}^j : \mathcal{H}_{P_i,\delta_i,\sqrt{-1}\nu_i} \otimes \mathrm{Hom}_K(\mathcal{H}_{\delta_i}, V_\ell^j) \to V_\ell^j$$

is the contraction map. The Fourier transform of f is $\widehat{f} = \sum_j \widehat{f^j}$. It should be noted that when G has a non empty discrete series, then $P_i = G$ for some i, with $A_i = N_i = \{e\}$ and $\mathbf{c}_{\delta_i}(0) > 0$.

Remark 2.14. *When $\ell = 0$, $f = f^j : G \to \mathbf{C}$ is a compactly supported complex-valued function on G. In this case, one has*

$$\begin{aligned} &\Phi_{\delta_i}^j \Big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(g^{-1}) \otimes 1_{Hom_K(\mathcal{H}_{\delta_i}, V_\ell^j)} \big(\widehat{f^j}(\delta_i, \sqrt{-1}\nu_i)\big)\Big) \\ &= Tr\Big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(g^{-1}) \circ \widehat{f^j}(\delta_i, \sqrt{-1}\nu_i)\Big) \end{aligned}$$

so that our formulas (2.12) and (2.13) reduce to

$$\widehat{f}(\delta_i, \sqrt{-1}\nu_i) = \int_G \pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(g) f(g) dg \tag{2.15}$$

and

$$f(g) = \sum_i \sum_{\delta_i \in (\widehat{M}_i)_d} \frac{1}{\mid W_i \mid} \int_{\mathfrak{a}_{i,0}^\star} Tr\Big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(g^{-1}) \circ \widehat{f}(\delta_i, \sqrt{-1}\nu_i)\Big) \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i$$

which are respectively the Harish-Chandra Fourier transform and inverse Fourier transform for complex-valued functions on G [18].

2.7. Spherical functions on G. In the sequel, it will be useful to write the Fourier transform in term of some spherical functions on G. From (2.12) we have

$$\begin{aligned} &\Big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(g^{-1}) \otimes 1_{\mathrm{Hom}_K(\mathcal{H}_{\delta_i}, V_\ell^j)} \big(\widehat{f^j}(\delta_i, \sqrt{-1}\nu_i)\big)\Big) \\ &= \frac{1}{\dim(V_\ell^j)} \int_G \pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(g') T_{\delta_i}^{\ell,j\star}(f^j(gg')) \otimes T_{\delta_i}^{\ell,j} dg' \end{aligned}$$

so that

$$\begin{aligned}&\Phi^j_{\delta_i}\Big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(g^{-1})\otimes 1_{\mathrm{Hom}_K(\mathcal{H}_{\delta_i},V^j_\ell)}\big(\widehat{f^j}(\delta_i,\sqrt{-1}\nu_i)\big)\Big)\\&=\frac{1}{\dim(V^j_\ell)}\int_G \Psi^{\ell,j}_{\delta_i,\sqrt{-1}\nu_i}(g')(f^j(gg'))dg'\\&=\frac{1}{\dim(V^j_\ell)}(\Psi^{\ell,j}_{\delta_i,\sqrt{-1}\nu_i}\star f^j)(g),\end{aligned}$$

where $\Psi^{\ell,j}_{\delta_i,\sqrt{-1}\nu_i}$ is the $\mathrm{End}(V^j_\ell)$-valued function on G defined by

$$\Psi^{\ell,j}_{\delta_i,\nu_i}(g)=T^{\ell,j}_{\delta_i}\circ\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(g)\circ T^{\ell,j\star}_{\delta_i}\ \ \forall g\in G.$$

$\Psi^{\ell,j}_{\delta_i,\nu_i}$ is an $\mathrm{End}(V^j_\ell)$-valued σ^j_ℓ-spherical function on G satisfying the following property [11]

$$\widetilde{\Omega}_G\Psi^{\ell,j}_{\delta_i,\nu_i}=\omega_{P_i,\delta_i,\sqrt{-1}\nu_i}\Psi^{\ell,j}_{\delta_i,\nu_i}.$$

Remark 2.16. *If F is an $End(V^j_l)$-valued function on G, we define the Fourier transform of F as the map*

$$\begin{aligned}\widehat{M}_i(\sigma^j_\ell)\times\sqrt{-1}\mathfrak{a}^\star_{i,0}&\to \mathit{End}(V^j_\ell)\\(\delta_i,\sqrt{-1}\nu_i)\quad&\mapsto\widehat{F}(\delta_i,\sqrt{-1}\nu_i)=\int_G\Psi^{\ell,j}_{\delta_i,\sqrt{-1}\nu_i}(g)\circ F(g)dg.\end{aligned}$$

The Harish-Chandra spherical function Φ_λ on G associated with $\lambda\in\mathfrak{a}^\star$ is the function defined by (see Chapter VII of [20])

$$\Phi_\lambda(g)=\int_K e^{-(\lambda+\rho_{\mathfrak{a}_0})(\log(a(g^{-1}k)))}dk. \tag{2.17}$$

By Proposition 7.4 of [20], for all K-finite vectors u,v in $\mathcal{H}_{\delta_i}$ there exists a positive real number d_i such that for all $g\in G$

$$|\langle\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(g)u\,,v\rangle_{\mathcal{H}_{P_i,\delta_i,\sqrt{-1}\nu_i}}|\le d_i\Phi_0(g)\,\|\,u\,\|_{\mathcal{H}_{\delta_i}}\|\,v\,\|_{\mathcal{H}_{\delta_i}}\,. \tag{2.18}$$

Moreover the following estimate of the spherical function Φ_0 will be useful (see Proposition 2.2.12 in [3]). There exists a positive number C such that, for all $g\in G$

$$\Phi_0(g)\le C\Big(\Pi_{\alpha\in\Sigma^{++}}\big(1+\alpha(a^+(g))\big)\Big)e^{-\rho_{\mathfrak{a}_0}(a^+(g))}. \tag{2.19}$$

2.8. On Delorme's Paley-Wiener Theorem. We recall a recent result of P. Delorme on the Paley-Wiener theorem of Arthur [16]. We first start with the notion of successive partial derivatives of principal series representations of G introduced by Delorme. For this we will follow the description of van den Ban and Souaifi given in [5]. Let V be a Fréchet space and $\mathcal{V}_0$ be a finite dimensional real vector space with complexification $\mathcal{V}$. For any $\eta\in\mathcal{V}^\star$ and any holomorphic map $\Phi:\mathcal{V}^\star\longrightarrow\mathrm{End}(V)$, one defines the derivative $\Phi^{(\eta)}$ of Φ along η as the following holomorphic map

$$\frac{\partial}{\partial\eta}\Phi\overset{def.}{=}\Phi^{(\eta)}:\mathcal{V}^\star\overset{holo.}{\longrightarrow}\mathrm{End}(V\oplus V)$$

with

$$\Phi^{(\eta)}(\lambda)(v_1, v_2) = \Big(\phi(\lambda)v_1 + \frac{d}{dz}(\phi(\lambda + z\eta)v_2)\mid_{z=0}, \phi(\lambda)v_2\Big). \tag{2.20}$$

By iteration, for any finite sequence $\eta = (\eta_1, \eta_2, \cdots, \eta_N) \in \mathcal{V}^\star$, one defines the successive derivative $\Phi^{(\eta)}$ of Φ along η as the map

$$\Phi^{(\eta)} : \mathcal{V}^\star \xrightarrow{holo.} \mathrm{End}(V^{(\eta)})$$

with

$$\Phi^{(\eta)} = \Big(\cdots\big(\Phi^{(\eta_N)}\big)^{(\eta_{N-1})}\cdots\Big)^{(\eta_1)} \tag{2.21}$$

where $V^{(\eta)}$ is the direct sum of 2^N copies of V.

Recall, from section 2.1, that $\mathfrak{a}_0$ is a maximal abelian subspace in $\mathfrak{s}_0$ and $A = \exp(\mathfrak{a}_0)$ is the analytic subgroup of G with Lie algebra $\mathfrak{a}_0$. Denote by $\mathcal{P}(A)$ the set of cuspidal parabolic subgroups of G containing A. The set $\mathcal{P}(A)$ is finite and each element P of $\mathcal{P}(A)$ has Langlands decomposition $P = M_P A_P N_P$ where M_P is reductive, A_P abelian and N_P nilpotent with Lie algebras $\mathfrak{m}_{P,0}$, $\mathfrak{a}_{P,0}$ and $\mathfrak{n}_{P,0}$ respectively. Recall that $\rho_{\mathfrak{a}_{P,0}}$ is the half sum of positive roots in $\mathfrak{g}_0$ relative to $\mathfrak{a}_{P,0}$ counted with their multiplicities, for some fixed positive system for $\mathfrak{a}_{P,0}$-roots in $\mathfrak{g}_0$. Recall that we drop the subscript 0 for the complexification of real vector spaces. We shall write P_0 for the (standard) minimal parabolic subgroup. Given $\delta \in (\widehat{M_P})_d$, write $\mathrm{End}\big(C^\infty(K, \delta)\big)$ for the vector space of endomorphisms of $C^\infty(K, \delta)$ and define the map

$$\pi_{P,\delta,(\cdot)} : G \longrightarrow \Big(\mathfrak{a}_P^\star \longrightarrow \mathrm{End}\big(C^\infty(K, \delta)\big)\Big),\;\; g \mapsto \Big(\lambda \mapsto \pi_{P,\delta,\lambda}(g)\Big)$$

where $\pi_{P,\delta,\lambda}$ denotes the principal series representation of G associated with P, δ and λ. Here we use the realization of the principal series in the compact picture described in (2.2). Let $\mathcal{D}$ for the set of 4-tuples $\xi = (P, \delta, \lambda, \eta)$, where $P \in \mathcal{P}(A)$, $\delta \in (\widehat{M_P})_d$, $\lambda \in \mathfrak{a}_P^\star$ and η is a finite sequence in $\mathfrak{a}_P^\star$. We shall simply write $\mathcal{D}_{P_0}$ for the set of 4-tuples $\xi = (P, \delta, \lambda, \eta)$ with $P = P_0$. Given $\xi = (P, \delta, \lambda, \eta) \in \mathcal{D}$, we define the partial derivative along η of the principal series representation $\pi_{P,\delta,\lambda}$ as the G-representation defined by the following map

$$\pi_\xi : G \to \mathrm{End}\Big(C^\infty(K, \delta)^{(\eta)}\Big),\; g \mapsto \pi_{P,\delta,(\cdot)}(g)^{(\eta)}(\lambda) \tag{2.22}$$

where we have used the notation in (2.20) and (2.21) with $\mathcal{V}_0 = \mathfrak{a}_{P,0}$ and $V = C^\infty(K, \delta)$. Let $\mathcal{C}(\mathfrak{a}_P^\star)$ be the vector space of complex functions on $\mathfrak{a}_P^\star$ and $\mathcal{O}(\mathfrak{a}_P^\star)$ the vector space of complex valued holomorphic functions on $\mathfrak{a}_P^\star$. Write $\mathcal{S}(P; \delta)$ for the space of bi-K-finite elements of $\mathrm{End}\big(C^\infty(K, \delta)\big)$. Then, for an element

$$\phi \in \bigoplus_{P\in\mathcal{P}(A)} \bigoplus_{\delta\in(\widehat{M_P})_d} \Big(\mathcal{O}(\mathfrak{a}_P^\star) \otimes \mathcal{S}(P; \delta)\Big), \tag{2.23}$$

we define in a similar way $\phi_\xi \in \mathrm{End}\big(C^\infty(K, \delta)^{(\eta)}\big)$. Given a finite sequence $\xi = (\xi_1, \xi_2, \cdots, \xi_N)$ of elements in $\mathcal{D}$, we define

$$\pi_\xi = \pi_{\xi_1} \oplus \pi_{\xi_2} \oplus \cdots \oplus \pi_{\xi_N} \text{ and } \phi_\xi = \phi_{\xi_1} \oplus \phi_{\xi_2} \oplus \cdots \oplus \phi_{\xi_N}.$$

We can now state Delorme's intertwining conditions for a map ϕ as in (2.23).

(i) For each finite sequence $\xi \in \mathcal{D}^N$, the map ϕ_ξ preserves all invariant subspaces of π_ξ, and

(ii) for any two finite sequences $\xi^1 \in \mathcal{D}^{N_1}$ and $\xi^2 \in \mathcal{D}^{N_2}$, and any two sequences of closed invariant subspaces $U_j \subset V_j$ for π_{ξ^j}, the induced maps $\widetilde{\phi}_{\xi^j} \in \mathrm{End}\big(V_j/U_j\big)$ are intertwined by all intertwining operators $T : V_1/U_1 \longrightarrow V_2/U_2$.

There is a third condition. In the compact picture realization of principal series representations (2.2), each compactly supported smooth function $f \in C_c^\infty(G)$ has an operator valued Fourier transform defined by (see (2.15))

$$\widehat{f}(P,\delta,\lambda) \overset{def.}{=} \pi_{P,\delta,\lambda}(f) = \int_G f(g)\pi_{P,\delta,\lambda}(g)dg \in \mathrm{End}\big(C^\infty(K;\delta)\big). \tag{2.24}$$

In particular, if f is bi-K-finite then $\widehat{f}(P,\delta,\lambda)$ belongs to $\mathcal{S}(P;\delta)$. Define the pre-Paley-Wiener space $PW_P^{\mathrm{pre}}(G,K,r)$ associated with the parabolic $P \in \mathcal{P}(A)$ as the space of maps $\phi \in \bigoplus_{\delta\in(\widehat{M_P})_d} \mathcal{O}(\mathfrak{a}_P^\star) \otimes \mathcal{S}(P;\delta)$ for which there exists a number $r > 0$ and for every $n > 0$ a number $C_n > 0$ such that

(iii)$_r$ $\| \widehat{\phi}(P,\delta,\lambda) \| \leq C_n(1+ \mid \lambda \mid)^{-n} e^{r|\mathrm{Re}(\lambda)|}$ for all δ and λ,

where $\mathrm{Re}(\lambda)$ denotes the real part of λ. Then Delorme's Paley-Wiener space is the vector space defined by (Définition 3 in [16])

$$\begin{aligned}\mathcal{PW}_r(G,K) = \Big\{\phi \in \oplus_{P\in\mathcal{P}(A)} \oplus_{\delta\in(\widehat{M_P})_d} \mathcal{O}(\mathfrak{a}_P^\star) \otimes \mathcal{S}(P;\delta) \mid\ \phi \text{ statisfies}\\ \text{conditions } (i),\ (ii) \text{ and } (iii)_r\Big\}.\end{aligned} \tag{2.25}$$

Finally, we can now state the Paley-Wiener theorem proved by Delorme.

Paley-Wiener Theorem (Théorème 2 of [16]). Let $D_r(G)$ be the space of complex-valued functions on G which are compactly supported in the closed ball in G of radius r and center the neutral element of G. The map $D_r(G) \to \mathcal{PW}_r(G,K)$, $f \mapsto \widehat{f}$ is a topological isomorphism of Fréchet spaces.

Unfortunately Delorme's intertwining conditions, especially condition (ii), are not easy to check, even in particular cases. It turns out that, using recent results of van den Ban and Souaifi (Lemmas 4.1 and 4.2 in [5]), one can reduce considerably these intertwining conditions enabling us to use Delorme's Paley-Wiener theorem in our specific situation. Since we shall make an essential use of van den Ban and Souaifi's observation, and for the convenience of the reader, we include the proof of this reduction.

Proposition 2.26. [5] *Let ϕ be a map as in* (2.23). *Then one has*

(1) the map ϕ satisfies Delorme's intertwining conditions if, and only if, it statisfies condition (i).

(2) The following assertions are equivalent

(a) ϕ satisfies (i) for each finite sequence of data in $\mathcal{D}$.

(b) ϕ satisfies (i) for each finite sequence of data in $\mathcal{D}_{P_0}$.

Proof. For (1), let ξ^j, π_{ξ^j}, U_j and V_j be as in (ii), for $j = 1,2$. Let $T : V_1/U_1 \longrightarrow V_2/U_2$ be an intertwining operator. In particular T is equivariant and the graph of T is an invariant subspace of $V_1/U_1 \oplus V_2/U_2$. Since ϕ statisfies (i), the map $\widetilde{\phi}_{\xi^1} \oplus \widetilde{\phi}_{\xi^2}$ preserves the graph of T, i.e $T \circ \widetilde{\phi}_{\xi^1} = \widetilde{\phi}_{\xi^2} \circ T$.

For (2), $(a) \Rightarrow (b)$ is obvious. For the other direction, one proceeds in several steps.

• Fix a 4-tuple $\xi = (P, \delta, \lambda_0, \eta) \in \mathcal{D}$, where $P \in \mathcal{P}(A)$ is a cuspidal parabolic subgroup of G containing A with Langlands decomposition $P = M_P A_P N_P$. The group M_P is a real reductive subgroup of G with Cartan decomposition $M_P = \big(M_P \cap K\big)\exp(\mathfrak{m}_{P,0} \cap \mathfrak{s}_0)$. Let $\mathfrak{a}'_{P,0}$ be a maximal abelian subspace of $\mathfrak{m}_{P,0} \cap \mathfrak{s}_0$ so that

$$\mathfrak{a}_0 = \mathfrak{a}_{P,0} \oplus \mathfrak{a}'_{P,0}. \tag{2.27}$$

• By the subrepresentation theorem, there exist a (minimal) parabolic subgroup Q'_P of M_P with Langlands decomposition

$$Q'_P = M'_P A'_P N'_P$$

where $A'_P = \exp(\mathfrak{a}'_{P,0})$, a unitary irreducible representation $\sigma \in \widehat{M'_P}$ of M'_P and linear form $\mu \in \mathfrak{a}'^{\star}_P$ on $\mathfrak{a}'_P$ such that

$$\delta \simeq \text{ subrepresentation of } \mathrm{Ind}_{Q'_P}^{M_P} \sigma \otimes \mu \otimes 1.$$

• There exists a minimal parabolic subgroup Q_P of G containing A such that $Q_P \in \mathcal{P}(A)$ and

$$Q_P \cap M_P = Q'_P.$$

Then, using induction by stages, one obtains that:

$$\begin{aligned}\mathrm{Ind}_P^G \delta \otimes \lambda_0 \otimes 1 &\simeq \text{ subrepresentation of } \mathrm{Ind}_P^G\Big(\mathrm{Ind}_{Q'_P}^{M_P} \sigma \otimes \mu \otimes 1\Big) \otimes \lambda_0 \otimes 1 \\ &\simeq \text{ subrepresentation of } \mathrm{Ind}_{Q_P}^G \sigma \otimes (\lambda_0 + \mu) \otimes 1\end{aligned}$$

where we have identified, using (2.27), $\mathfrak{a}^\star_P$ and $\mathfrak{a}^\star_{P,0}$ with subspaces of $\mathfrak{a}^\star$.

• For the successive derivatives, we deduce that

$$\pi^{(\eta)}_{P,\delta,\lambda_0} \simeq \text{ subrepresentation of } \pi^{(\eta)}_{Q_P,\sigma,\lambda_0+\mu}.$$

In other words, if we define ξ' to be the 4-tuple $(Q_P, \sigma, \lambda_0 + \mu, \eta) \in \mathcal{D}$, we have

$$\pi_\xi \simeq \text{ subrepresentation of } \pi_{\xi'}. \tag{2.28}$$

• On the other hand, the parabolic subgroups P and Q_P of G are conjugate under the Weyl group of G with respect to A, i.e there exists $w \in N_K(\mathfrak{a}_0)$ such that $Q_P = w^{-1} P_0 w$. This induces an intertwining operator from $\pi_{Q_P,\sigma,\lambda_0+\mu}$ to $\pi_{P_0, w\cdot\sigma, w\cdot(\lambda_0+\mu)}$ and implies, for the successive derivatives, that

$$\pi^{(\eta)}_{Q_P,\sigma,\lambda_0+\mu} \simeq \pi^{(w\cdot\eta)}_{P_0,w\cdot\sigma,w\cdot(\lambda_0+\mu)}$$

where w acts on η componentwise. Defining the 4-tuple $\xi_0 = (P_0, w\cdot\sigma, w\cdot(\lambda_0+\mu), w\cdot\eta) \in \mathcal{D}_{P_0}$ and using (2.28), we deduce that

$$\pi_\xi \simeq \text{ subrepresentation of } \pi_{\xi_0}.$$

• By additivity, this extends to the case where ξ is a finite sequence $(\xi_1, \xi_2, \cdots, \xi_N) \in \mathcal{D}^N$ which proves that $(b) \Rightarrow (a)$. ☐

2.9. On the bottom of the spectrum of Δ_ℓ. From Kuga's formula (see Proposition 2.5 of [10]), one has

$$\pi_{P_i,\delta_i,\alpha_i}(\Delta_\ell) = -\pi_{P_i,\delta_i,\alpha_i}(\widetilde{\Omega}_G) \tag{2.29}$$

independently of the degree ℓ. Moreover it is known from [38] that

$$\delta_i \in \widehat{M_i}(\sigma_\ell) \Longrightarrow || \operatorname{char}(\delta_i) || \leq || \rho_i || \tag{2.30}$$

and the equality holds only if $\mathfrak{h}_i$ is maximally compact (i.e $\mathfrak{g}$ does not have real roots relative to $\mathfrak{h}_i$). Since there is a discrete number of irreducible unitary representations δ_i of M occurring in $L^2(M_i)$ with Harish-Chandra parameter contained in the closed ball defined by $|| \operatorname{char}(\delta_i) || \leq || \rho_i ||$, the set $\widehat{M_i}(\sigma_\ell)$ is finite. We define the real number

$$\lambda_\ell(G/K) = \inf\Big\{ -\omega_{P_i,\delta_i,\sqrt{-1}\nu_i} \mid \delta_i \in \widehat{M_i}(\sigma_\ell),\ \nu_i \in \mathfrak{a}^\star_{i,0},\ 1 \leq i \leq s\Big\}. \tag{2.31}$$

In particular we have

(i) $\lambda_\ell(G/K) \geq 0$ for all ℓ (by(2.30)),

(ii) $\lambda_\ell(G/K) = \inf\Big\{ || \rho_i ||^2 - || \operatorname{char}(\delta_i) ||^2 \mid \delta_i \in \widehat{M_i}(\sigma_\ell),\ 1 \leq i \leq s\Big\}$ (by(2.30)), and

(iii) $\lambda_\ell(G/K) = 0 \Leftrightarrow 2\ell \in \Big[\dim_{\mathbf{R}}(G/K) + \operatorname{rk}_{\mathbf{C}}(K) - \operatorname{rk}_{\mathbf{C}}(G), \dim_{\mathbf{R}}(G/K) + \operatorname{rk}_{\mathbf{C}}(G) - \operatorname{rk}_{\mathbf{C}}(K)\Big]$ (by (2.7)).

The link with the bottom of the spectrum of Δ_ℓ is given by the following proposition.

PROPOSITION 2.32. *The number $\lambda_\ell(G/K)$ equals the bottom of the spectrum of Δ_ℓ.*

Proof. Let μ_ℓ be the bottom of the spectrum of Δ_ℓ. By the Plancherel theorem (2.11) and Kuga's formula (2.29), we know that

$$\lambda_\ell(G/K) \leq \mu_\ell.$$

Assume that $\lambda_\ell(G/K) < \mu_\ell$ and let φ be a smooth real function with compact support in the interval $[\lambda_\ell(G/K), \mu_\ell]$. Then we have

$$\varphi(\Delta_\ell) \equiv 0$$

where

$$\varphi(\Delta_\ell) = \int_{-\infty}^{+\infty} \widehat{\varphi}(t) e^{\sqrt{-1}t\Delta_\ell} dt.$$

Now we choose $\delta_i \in \widehat{M_i}(\sigma_\ell)$ and $\nu_i \in \mathfrak{a}^\star_{i,0}$ such that

$$\lambda_\ell(G/K) < || \rho_i ||^2 - || \operatorname{char}(\delta_i) ||^2 + || \nu_i ||^2 < \mu_\ell$$

and we pick a non zero ℓ-form f in $L^2(G/K, \mathcal{V}_\ell)$ such that

$$\widehat{f}(\delta_i, \sqrt{-1}\nu_i) \neq 0$$

and $\widehat{f}(\delta_i, \cdot)$ is continuous on $\sqrt{-1}\mathfrak{a}^{\star}_{i,0}$. Then we deduce that

$$\widehat{\varphi(\Delta_\ell)f}(\delta_i, \sqrt{-1}\nu_i) = \varphi(\| \rho_i \|^2 - \| \operatorname{char}(\delta_i) \|^2 + \| \nu_i \|^2)\widehat{f}(\delta_i, \sqrt{-1}\nu_i) \neq 0$$

which is absurd. □

In the case of functions, i.e when $\ell = 0$, it is not difficult to check, using the Plancherel formula for functions, that

$$\lambda_0(G/K) = \| \rho_{\mathfrak{a}_0} \|^2 .$$

However the bottom of the spectrum of Δ_ℓ is not known in general (see the appendix for computations of $\lambda_\ell(G/K)$ in some examples).

2.10. Heat kernel for differential forms. Recall that the Laplacian on G is the negative elliptic differential operator Δ on G defined by

$$\Delta = \widetilde{\Omega}_G - 2\widetilde{\Omega}_K.$$

We denote by $P_t = e^{t\Delta}$ the fundamental solution of the corresponding heat equation on G

$$\Delta\phi_t = \frac{\partial}{\partial t}\phi_t.$$

It is well known that

$$(P_t f)(g_0) = \int_G p_t(g_0^{-1}g)f(g)dg \ \ \forall f \in L^2(G),\ g_0 \in G$$

where $p_t \in L^2(G)\cap C^\infty(G)$ is the heat kernel on G [6]. Similarly we may consider the heat equation for differential forms on G/K

$$\Delta_\ell\phi_t = -\frac{\partial}{\partial t}\phi_t$$

and the corresponding fundamental solution

$$P_t^\ell = e^{-t\Delta_\ell}.$$

The operator

$$P_t^\ell : L^2(G/K, \mathcal{V}_\ell) \to L^2(G/K, \mathcal{V}_\ell)$$

is a smoothing pseudo-differential operator commuting with the representation π_ℓ of G. Actually we have

$$(P_t^\ell\phi)(g_0) = \int_G p_t^\ell(g_0^{-1}g)(\phi(g))dg \ \ \forall\phi \in L^2(G/K, \mathcal{V}_\ell),\ g_0 \in G$$

where

$$p_t^\ell : G \to \operatorname{End}(\Lambda^\ell\mathfrak{s})$$

is a smooth map satisfying the covariance property

$$p_t^\ell(kgk') = \sigma_\ell(k)^{-1} \circ p_t^\ell(g) \circ \sigma_\ell(k')^{-1} \ \ \forall g \in G,\ k, k' \in K. \tag{2.33}$$

We shall refer to p_t^ℓ as the heat kernel of ℓ-forms on G/K (see Section 2 of [6]). It is easy to see from (2.5) that

$$e^{-t\Delta_\ell} \circ Q_\ell = Q_\ell \circ (e^{t\Delta} \otimes e^{2t\widetilde{\Omega}_K}),$$

and

$$p_t^\ell(g) = \int_{K\times K} p_t(k^{-1}gk')\sigma_\ell(k)e^{2t\widetilde{\Omega}_K}\sigma_\ell(k')^{-1}dkdk'. \tag{2.34}$$

Similarly, for each irreducible component σ_ℓ^j of the isotropy representation, we define the heat kernel $p_t^{\ell,j}$

$$p_t^{\ell,j}(g)\circ \mathrm{pr}_j = \mathrm{pr}_j \circ p_t^\ell(g), \quad \forall g \in G.$$

Write $p_t^{\ell,\perp}$ for the heat kernel corresponding to the projection $\Delta_\ell^\perp$ of Δ_ℓ on the orthogonal complement of $\mathrm{Ker}(\Delta_\ell)$ in $L^2(G/K, \mathcal{V}_\ell)$. The heat kernel $p_t^{\ell,j,\perp}$ is defined accordingly. From the Plancherel formula for differential form (2.11), one can deduce an explicit formula for $p_t^{\ell,j,\perp}$ using spherical Fourier transform (see (8.6) of [26])

$$\begin{aligned} &p_t^{\ell,j,\perp}(g) \\ &= \sum_{i,\ \dim(\mathfrak{a}_i)>0}\ \sum_{\delta_i\in\widehat{M}_i(\sigma_\ell^j)} \frac{1}{\mid W_i\mid}\frac{1}{\dim(V_\ell^j)}\int_{\mathfrak{a}_{i,0}^\star} e^{t\omega_{P_i,\delta_i,\sqrt{-1}\nu_i}}\Psi_{\delta_i,\nu_i}^{\ell,j}(g)\mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i)d\nu_i. \end{aligned} \tag{2.35}$$

Moreover one can check that

$$\begin{aligned} \Delta_\ell p_t^{\ell,j} &= -\frac{\partial}{\partial t}p_t^{\ell,j} \\ \widehat{p_t^{\ell,j}}(\delta_i, \sqrt{-1}\nu_i) &= e^{t\omega_{P_i,\delta_i,\sqrt{-1}\nu_i}}\mathrm{Id}_{V_\ell^j} \\ p_t^{\ell,j}\star p_{t'}^{\ell,j} &= p_{t+t'}^{\ell,j} \ \text{ for all } \ t,t'>0. \end{aligned}$$

It should be noted that the continuous heat kernel $p_t^{\ell,\perp}$ coincides with the full heat kernel p_t^ℓ whenever $\mathrm{rk}_{\mathbf{C}}(G) > \mathrm{rk}_{\mathbf{C}}(K)$ or $\ell \neq \frac{1}{2}\dim(G/K)$.

In the sequel, we shall use the following basic estimates of the heat kernel h_t for functions on G/K (Chapter V of [14] for $0<t<1$ and Section 3 of [24] for $t\geq 1$). There exist positive constants C_1 and C_2 such that, for all $g\in G$, we have

$$h_t(gK) \leq C_1 t^{-\frac{1}{2}\dim_{\mathbf{R}}(G/K)}e^{-\frac{||g||^2}{4t}} \ \text{ for } \ 0<t<1, \tag{2.36}$$

and

$$h_t(gK) \leq C_2 t^{-\frac{1}{2}\mathrm{rk}_{\mathbf{R}}(G)-|\Sigma^{++}|}e^{-||\rho_{\mathfrak{a}_0}||^2 t} \ \text{ for } \ t\geq 1. \tag{2.37}$$

On the other hand, there is a well known relation between h_t and the heat kernel p_t^ℓ on ℓ-forms. Indeed, there exists two positive numbers α_ℓ and C_3 such that, for all $t>0$ and $g\in G$ (Lemme 2.4 in [22]), one has

$$||\, p_t^\ell(g)\, ||_{\mathrm{End}(\Lambda^\ell\mathfrak{s})} \leq C_3 e^{t\alpha_\ell}h_t(gK). \tag{2.38}$$

The heat operator on functions on G/K will be denoted by H_t.

2.11. Laplacian on square integrable differential forms on locally symmetric spaces. If Γ is a torsion free discrete subgroup of G, it acts on the left on G/K, so that the double coset space $\Gamma\backslash G/K$ is locally symmetric. Except otherwise stated, we do not assume that Γ is of finite covolume in G. Since G/K is simply connected, it is the universal cover of $\Gamma\backslash G/K$ and $\Gamma \simeq \Pi_1(\Gamma\backslash G/K)$. Moreover, given a Haar measure on G, there exists a unique measure $d\nu$ on $\Gamma\backslash G$ such that

$$\int_G f(g)dg = \int_{\Gamma\backslash G} \Big[\sum_{\gamma\in\Gamma} f(\gamma g)\Big] d\nu(\Gamma g) \tag{2.39}$$

for all compactly supported function f on G. A smooth ℓ-form on $\Gamma\backslash G/K$ may be viewed as a smooth $\Lambda^\ell\mathfrak{s}$-valued function ϕ on G satisfying the relation

$$\phi(\gamma gk) = \sigma_\ell(k)^{-1}\phi(g), \;\; \forall g \in g, \; k \in K, \; \gamma \in \Gamma.$$

Write $C_0^\infty(\Gamma\backslash G/K, \mathcal{V}_\ell)$ for the complex vector space of compactly supported smooth ℓ-forms on $\Gamma\backslash G/K$. Similarly, we define the vector space $L^p(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$ of L^p-integrable ℓ-forms on $\Gamma\backslash G/K$, equipped with the norm

$$|| \phi ||_{L^p(\Gamma\backslash G/K,\mathcal{V}_\ell)} = \Big(\int_{\Gamma\backslash G} || \phi(\Gamma g) ||^p_{\Lambda^\ell\mathfrak{s}} \, d\nu(\Gamma g)\Big)^{\frac{1}{p}}$$

When $\ell = 0$, i.e in the case of functions, we shall simply write $C_0^\infty(\Gamma\backslash G/K)$ and $L^2(\Gamma\backslash G/K)$. The space $L^p(\Gamma\backslash G/K, \mathcal{E}nd(\Lambda^\ell\mathfrak{s}))$ of $\mathrm{End}(\Lambda^\ell\mathfrak{s})$-valued L^p functions on $\Gamma\backslash G/K$ is defined accordingly, with the norm

$$|| \phi ||_{L^p(\Gamma\backslash G/K,\mathcal{E}nd(\Lambda^\ell\mathfrak{s}))} = \Big(\int_{\Gamma\backslash G} || \phi(g) ||^p_{\mathrm{End}(\Lambda^\ell\mathfrak{s})} \, dg\Big)^{\frac{1}{p}}.$$

Write

$$d_\ell : C_0^\infty(\Gamma\backslash G/K, \mathcal{V}_\ell) \to C_0^\infty(\Gamma\backslash G/K, \mathcal{V}_{\ell+1})$$

for the exterior differential and

$$d_\ell^\star : C_0^\infty(\Gamma\backslash G/K, \mathcal{V}_{\ell+1}) \to C_0^\infty(\Gamma\backslash G/K, \mathcal{V}_\ell)$$

for its adjoint. Then, the *locally invariant* positive elliptic operator

$$\widetilde{\Delta}_\ell = d_\ell^\star d_\ell + d_{\ell-1}d_{\ell-1}^\star$$

is the Laplacian on compactly supported smooth ℓ-forms on $\Gamma\backslash G/K$. This differential operator has a unique selfadjoint extension to $L^2(\Gamma\backslash G/K, \mathcal{V}_\ell)$ which will be also denoted by the same symbol $\widetilde{\Delta}_\ell$. In particular, we may also consider the heat equation on $\Gamma\backslash G/K$

$$\widetilde{\Delta}_\ell \phi_t = -\frac{\partial}{\partial t}\phi_t.$$

We shall write $\widetilde{P}_t^\ell$ for its fundamental solution and $\widetilde{p}_t^\ell$ for the corresponding heat kernel. The estimate (2.38) is still true if we replace p_t^ℓ (resp. h_t) by $\widetilde{p}_t^\ell$ (resp. $\widetilde{h}_t^\ell$) where $\widetilde{h}_t^\ell$ (resp. $\widetilde{H}_t$) denotes the heat kernel (resp. heat operator) on functions on $\Gamma\backslash G/K$. Similarly we define $\widetilde{p}_t^{\ell,\perp}$ and $\widetilde{h}_t^{\ell,\perp}$.

2.12. On the bottom of the spectrum of $\widetilde{\Delta}_\ell$. Let $\beta_\ell(\Gamma\backslash G/K)$ be the bottom of the L^2-spectrum of $\widetilde{\Delta}_\ell$:

$$\beta_\ell(\Gamma\backslash G/K) = \inf\big\{\langle\widetilde{\Delta}_\ell f\,,f\rangle_{L^2(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s}^\star)} \mid f\in C_0^\infty(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s}^\star),\ \|\, f\,\|_{L^2(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s}^\star)}=1\big\}. \tag{2.40}$$

In the case of functions, i.e $\ell=0$, some estimates for $\beta_0(\Gamma\backslash G/K)$ have been proved by E. Leuzinger [21]. More precisely, let $D(\ ,\ )$ be the Riemannian distance on G/K induced by the Killing form. The Poincaré series associated with Γ is given, for all $x,x'\in G/K$ and $s\in\mathbf{R}$, by

$$\mathcal{P}_s(x,x')=\sum_{\gamma\in\Gamma}e^{-sD(x,\gamma x')}.$$

Then the critical exponent of Γ is the real number $\delta(\Gamma)$ defined as follows. For all $x,x'\in G$, $P_s(x,x')$ converges for $s>\delta(\Gamma)$ and diverges for $s<\delta(\Gamma)$, i.e

$$\delta(\Gamma)=\inf\Big\{s\mid\sum_{\gamma\in\Gamma}e^{-sD(x,\gamma x')}<+\infty\Big\}.$$

It is not difficult to check that (see Section 2.2 of [21])

$$0\le\delta(\Gamma)\le 2\,\|\,\rho_{\mathfrak{a}_0}\,\|\,.$$

When Γ is a lattice in G, it is known that (see Theorem 7.4 of [2])

$$\delta(\Gamma)=2\,\|\,\rho_{\mathfrak{a}_0}\,\|\,.$$

In the general case, we have the following result.

Theorem (E. Leuzinger, Section 1 of [21]). Let G be a semisimple Lie group without compact factors and with trivial center. Let ρ_{min} be the positive real number defined by

$$\rho_{min}=\inf\Big\{\rho_{\mathfrak{a}_0}(X)\mid X\in\overline{\mathfrak{a}_0^+},\ \|\,X\,\|=1\Big\}.$$

If Γ is a torsion free discrete subgroup of G, the following estimates hold.

(i) If $\delta(\Gamma)\in\big[0\ ,\ \rho_{min}\big]$ then $\beta_0(\Gamma\backslash G/K)=\|\,\rho_{\mathfrak{a}_0}\,\|^2$,
(ii) if $\delta(\Gamma)\in\big[\rho_{min}\ ,\ \|\,\rho_{\mathfrak{a}_0}\,\|\,\big]$ then $\|\,\rho_{\mathfrak{a}_0}\,\|^2-(\delta(\Gamma)-\rho_{min})^2\le\beta_0(\Gamma\backslash G/K)\le\|\,\rho_{\mathfrak{a}_0}\,\|^2$, and
(iii) if $\delta(\Gamma)\in\big[\,\|\,\rho_{\mathfrak{a}_0}\,\|\,,\ 2\,\|\,\rho_{\mathfrak{a}_0}\,\|\,\big]$ then $Max\big\{0;\|\,\rho_{\mathfrak{a}_0}\,\|^2-(\delta(\Gamma)-\rho_{min})^2\big\}\le\beta_0(\Gamma\backslash G/K)\le\|\,\rho_{\mathfrak{a}_0}\,\|^2-(\delta(\Gamma)-\|\,\rho_{\mathfrak{a}_0}\,\|)^2$.

2.13. L^2-cohomology on $\Gamma\backslash G/K$. We do not assume that Γ is of finite covolume in G. Let $W_{2,\ell}$ be the vector subspace of $L^2(\Gamma\backslash G/K,\mathcal{V}_\ell)$ defined, for $\ell\ge0$, by

$$W_{2,\ell}=\{\omega\in L^2(\Gamma\backslash G/K,\mathcal{V}_\ell)\mid\|\,d_\ell\omega\,\|_{L^2(\Gamma\backslash G/K,\mathcal{V}_\ell)}<+\infty\}.$$

It is easy to check that the kernel $\mathrm{Ker}(d_\ell)$ of d_ℓ is a subspace of $W_{2,\ell}$ which is closed in $L^2(\Gamma\backslash G/K,\mathcal{V}_\ell)$. However, the image $\mathrm{Im}(d_{\ell-1})$ of $d_{\ell-1}$ need not be closed in

$L^2(\Gamma\backslash G/K, \mathcal{V}_\ell)$. We are therefore led to define the *unreduced* L^2-cohomology group $H^{(\ell)}(\Gamma\backslash G/K)$ of degree ℓ of $\Gamma\backslash G/K$

$$H^{(\ell)}(\Gamma\backslash G/K) = \mathrm{Ker}(d_\ell)/\mathrm{Im}(d_{\ell-1})$$

and the *reduced* L^2-cohomology group $\overline{H}^{(\ell)}(\Gamma\backslash G/K)$ of degree ℓ of $\Gamma\backslash G/K$

$$\overline{H}^{(\ell)}(\Gamma\backslash G/K) = \mathrm{Ker}(d_\ell)/\overline{\mathrm{Im}(d_{\ell-1})},$$

where $\overline{\mathrm{Im}(d_{\ell-1})}$ denotes the closure of $\mathrm{Im}(d_{\ell-1})$ in $L^2(\Gamma\backslash G/K, \mathcal{V}_\ell)$. Observe that there is a natural surjection

$$H^{(\ell)}(\Gamma\backslash G/K) \longrightarrow \overline{H}^{(\ell)}(\Gamma\backslash G/K)$$

and

$$\overline{H}^{(\ell)}(\Gamma\backslash G/K) \simeq \mathrm{Ker}(d_\ell) \cap \overline{\mathrm{Im}(d_{\ell-1})}^{\perp} = \mathrm{Ker}(\widetilde{\Delta}_\ell)$$

where $\overline{\mathrm{Im}(d_{\ell-1})}^{\perp}$ denotes the orthogonal of $\overline{\mathrm{Im}(d_{\ell-1})}$ in $L^2(\Gamma\backslash G/K, \mathcal{V}_\ell)$. When Γ is cocompact, unreduced and reduced L^2-cohomologies coincide.

3. Estimates for the heat kernel.

THEOREM 3.1. *Let G be a non compact connected semisimple real Lie group with finite center and K a maximal compact subgroup of G. Let $\lambda_\ell(G/K)$ be the bottom of the spectrum of Δ_ℓ and Φ_0 the Harish-Chandra spherical function on G. Put $r = \inf_i \Big\{ \dim_{\mathbf{R}}(\mathfrak{a}_{i,0}) > 0 \Big\}$ and $z = \inf_i \Big\{$order of zero of $\mathbf{c}_{\delta_i}$ at $\nu_i = 0,\ \delta_i \in \widehat{M_i}(\sigma_\ell)\Big\}$. Then, for all $\epsilon \in]0,1[$, there exist two positive numbers a_ϵ and A_ϵ such that*

$$\| p_t^{\ell,\perp}(g) \|_{End(\Lambda^\ell \mathfrak{s})} \leq a_\epsilon e^{-t\lambda_\ell(G/K)} \Phi_0(g) e^{-\frac{1-\epsilon}{(1+2\epsilon)^2}\frac{\|g\|^2}{4t}} t^{-\epsilon\frac{r+z}{2}}$$

for all $g \in G$ and $t \in \mathbf{R}$ satisfying $\| g \| > A_\epsilon$ and $t > 1$.

Proof. Throughout the proof the symbols B_j and C_j will denote positive real numbers.

Step 1: *we reduce the problem.* Since the Casimir operator $\widetilde{\Omega}_K$ of K acts a a scalar operator on each irreducible component σ_ℓ^j of σ_ℓ and, any two irreducible components are orthogonal with respect to the scalar product $\langle\,,\rangle_{\Lambda^\ell\mathfrak{s}}$, we may assume that η and β belong to the same irreducible component σ_ℓ^j for some j. Then, using the Cartan decomposition (2.1), we have

$$\begin{aligned}\langle p_t^\ell(g)\eta\,,\omega\rangle_{\Lambda^\ell\mathfrak{s}} &= \langle p_t^{\ell,j}(g)\eta\,,\beta\rangle_{\Lambda^\ell\mathfrak{s}}\\ &= \langle p_t^{\ell,j}(k_1(g)e^{a^+(g)}k_2(g))\eta\,,\beta\rangle_{\Lambda^\ell\mathfrak{s}}\\ &= \langle p_t^{\ell,j}(e^{a^+(g)})\sigma_\ell^j(k_2(g))\eta\,,\sigma_\ell^j(k_1(g))^{-1}\beta\rangle_{\Lambda^\ell\mathfrak{s}} \text{ by (2.33).}\end{aligned}$$

Therefore it is enough to consider $\langle p_t^{\ell,j,\perp}(a)\eta\,,\beta\rangle_{\Lambda^\ell\mathfrak{s}}$ for $a \in \exp(\overline{\mathfrak{a}_0^+})$ and $\eta, \beta \in V_\ell^j$. In particular, from (2.35), we obtain

$$\begin{aligned}&\langle p_t^{\ell,j,\perp}(a)\eta\,,\beta\rangle_{\Lambda^\ell\mathfrak{s}}\\ &= \sum_{i,\ \dim(\mathfrak{a}_i)>0}\ \sum_{\delta_i\in\widehat{M_i}(\sigma_\ell^j)} \frac{1}{\mid W_i\mid}\frac{1}{\dim(V_\ell^j)}\int_{\mathfrak{a}_{i,0}^\star} e^{t\omega_{P_i,\delta_i,\sqrt{-1}\nu_i}}\langle\Psi^{\ell,j}_{\delta_i,\nu_i}(a)\eta\,,\beta\rangle_{\Lambda^\ell\mathfrak{s}}\mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i)d\nu_i.\end{aligned}$$

Put $\eta^{\ell}_{i,j} = T^{\ell,j\star}_{\delta_i}\eta$ and $\beta^{\ell}_{i,j} = T^{\ell,j\star}_{\delta_i}\beta$ so that $\eta^{\ell}_{i,j}$ and $\beta^{\ell}_{i,j}$ are two K-finite vectors in $\mathcal{H}_{\delta_i}$ with

$$\int_{\mathfrak{a}^{\star}_{i,0}} e^{t\omega_{P_i,\delta_i,\sqrt{-1}\nu_i}} \langle \Psi^{\ell,j}_{\delta_i,\nu_i}(a)\eta\,,\beta\rangle_{\Lambda^{\ell}\mathfrak{s}} \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i)d\nu_i$$
$$= \int_{\mathfrak{a}^{\star}_{i,0}} e^{t\omega_{P_i,\delta_i,\sqrt{-1}\nu_i}} \langle \pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(a)\eta^{\ell}_{i,j}\,,\beta^{\ell}_{i,j}\rangle_{\Lambda^{\ell}\mathfrak{s}} \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i)d\nu_i.$$

Define the operator

$$\eta^{\ell}_{i,j}\otimes\beta^{\ell}_{i,j} : \mathcal{H}_{\delta_i}\to\mathcal{H}_{\delta_i}, f\mapsto \langle \eta^{\ell}_{i,j}\,,f\rangle_{\mathcal{H}_{\delta_i}}\beta^{\ell}_{i,j}$$

and write ξ^t and $\xi^t_{\delta_i}$ for the complex functions

$$\xi^t : \widehat{M_i}(\sigma^j_\ell)\times\mathfrak{a}^{\star}_i\to\mathbf{C},\ (\delta_i,\alpha_i)\mapsto e^{t\omega_{P_i,\delta_i,\alpha_i}}$$
$$\xi^t_{\delta_i} : \mathfrak{a}^{\star}_i\to\mathbf{C},\ \alpha_i\mapsto e^{t\omega_{P_i,\delta_i,\alpha_i}}.$$

Consider the $\mathrm{End}(\mathcal{H}_{\delta_i})$-valued map

$$\phi^t_i : \widehat{M_i}(\sigma^j_\ell)\times\mathfrak{a}^{\star}_i\to\mathrm{End}(\mathcal{H}_{\delta_i}),\ (\delta_i,\alpha_i)\mapsto\xi^t(\delta_i,\alpha_i)\eta^{\ell}_{i,j}\otimes\beta^{\ell}_{i,j}.$$

Following an idea of Alexopoulos and Lohoué [2], we decompose this map in two pieces. For this we fix a smooth function $\zeta:\mathbf{R}\to\mathbf{R}$ such that $\zeta(\tau)=1$ if $\mid\tau\mid>1$, and ζ vanishes in an neighborhood of the origin. Then, for $\epsilon\in]0,1[$, define the function

$$\zeta_a : \mathfrak{a}^{\star}_i\to\mathbf{R},\ y\mapsto\zeta\Big((1+\epsilon)\frac{\|y\|}{\|a\|}\Big)$$

and write

$$\phi^t_i = \phi^t_{i,a} + \widetilde{\phi}^t_{i,a} \tag{3.2}$$

with

$$\phi^t_{i,a}(\delta_i,\alpha_i) = (\xi^t_{\delta_i}\star\widehat{\zeta_a})(\alpha_i)\ \eta^{\ell}_{i,j}\otimes\beta^{\ell}_{i,j}$$
$$\widetilde{\phi}^t_{i,a}(\delta_i,\alpha_i) = (\xi^t_{\delta_i}\star\widehat{1-\zeta_a})(\alpha_i)\ \eta^{\ell}_{i,j}\otimes\beta^{\ell}_{i,j}.$$

Here it should be noted that the convolution and the Fourier transform are defined on $\mathfrak{a}_i^\star$, in particular only the parabolic subgroup $P_i = M_i A_i N_i$ is involved. We have

$$\begin{aligned}
&\langle p_t^{\ell,j,\perp}(a)\eta\,,\beta\rangle_{\Lambda^{\ell}\mathfrak{s}}\\
=&\sum_{i,\ \dim(\mathfrak{a}_i)>0}\ \sum_{\delta_i\in\widehat{M}_i(\sigma_\ell^j)}\frac{1}{|\,W_i\,|}\frac{1}{\dim(V_\ell^j)}\\
&\times\int_{\mathfrak{a}_{i,0}^\star} e^{t\omega_{P_i,\delta_i,\sqrt{-1}\nu_i}}\langle\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(a)\eta_{i,j}^\ell\,,\beta_{i,j}^\ell\rangle_{\mathcal{H}_{\delta_i}}\mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i)d\nu_i\\
=&\sum_{i,\ \dim(\mathfrak{a}_i)>0}\ \sum_{\delta_i\in\widehat{M}_i(\sigma_\ell^j)}\frac{1}{|\,W_i\,|}\frac{1}{\dim(V_\ell^j)}\\
&\times\int_{\mathfrak{a}_{i,0}^\star} e^{t\omega_{P_i,\delta_i,\sqrt{-1}\nu_i}}\operatorname{Tr}\Big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(a)\circ(\eta_{i,j}^\ell\otimes\beta_{i,j}^\ell)\Big)\mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i)d\nu_i\\
=&\sum_{i,\ \dim(\mathfrak{a}_i)>0}\ \sum_{\delta_i\in\widehat{M}_i(\sigma_\ell^j)}\frac{1}{|\,W_i\,|}\frac{1}{\dim(V_\ell^j)}\int_{\mathfrak{a}_{i,0}^\star}\operatorname{Tr}\Big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(a)\circ\phi_i^t(\delta_i,\sqrt{-1}\nu_i)\Big)\mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i)d\nu_i\\
=&\sum_{i,\ \dim(\mathfrak{a}_i)>0}\ \sum_{\delta_i\in\widehat{M}_i(\sigma_\ell^j)}\frac{1}{|\,W_i\,|}\frac{1}{\dim(V_\ell^j)}\int_{\mathfrak{a}_{i,0}^\star}\operatorname{Tr}\Big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(a)\circ\phi_{i,a}^t(\delta_i,\sqrt{-1}\nu_i)\Big)\mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i)d\nu_i\\
+&\sum_{i,\ \dim(\mathfrak{a}_i)>0}\ \sum_{\delta_i\in\widehat{M}_i(\sigma_\ell^j)}\frac{1}{|\,W_i\,|}\frac{1}{\dim(V_\ell^j)}\int_{\mathfrak{a}_{i,0}^\star}\operatorname{Tr}\Big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(a)\circ\widetilde{\phi}_{i,a}^t(\delta_i,\sqrt{-1}\nu_i)\Big)\mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i)d\nu_i.
\end{aligned}\tag{3.3}$$

Observe that

$$\widetilde{\phi}_{i,a}^t(\delta_i,\alpha_i)=(\xi_{\delta_i}^t\star\widehat{1-\zeta_a})(\alpha_i)\xi_{\delta_i}^{-t}(\alpha_i)\phi_i^t(\delta_i,\alpha_i).\tag{3.4}$$

Consider the global maps

$$\Phi^t=\Phi_a^t+\widetilde{\Phi}_a^t\in\bigoplus_{P_i\in\mathcal{P}(A)}\ \bigoplus_{\delta_I\in(\widehat{M}_i)_d}\mathcal{C}(\mathfrak{a}_i^\star)\otimes\operatorname{End}\big(C^\infty(K,\delta_i)\big)\tag{3.5}$$

where

$$\begin{aligned}
\Phi_a^t(P_i,\delta_i,\alpha_i)&=\phi_{i,a}^t(\delta_i,\alpha_i)\\
\widetilde{\Phi}_a^t(P_i,\delta_i,\alpha_i)&=\widetilde{\phi}_{i,a}^t(\delta_i,\alpha_i)
\end{aligned}$$

Step 2: *we apply Delorme's Paley-Wiener theorem to* $\widetilde{\Phi}_a^t$.

$\star$ We start with the pre-Paley-Wiener condition $(\mathrm{iii})_r$ in Section 2.8 for some r which will be specified below. For $u,v\in\mathcal{H}_{\delta_i}^\infty$, one has

$$\langle\widetilde{\phi}_{i,a}^t(\delta_i,\sqrt{-1}\nu_i)u\,,v\rangle_{\mathcal{H}_{\delta_i}}=\widehat{\psi}_{i,a}^t(\sqrt{-1}\nu_i)$$

where the function

$$\psi_{i,a}^t:\sqrt{-1}\mathfrak{a}_{i,0}^\star\to\mathbf{C},\sqrt{-1}\nu_i\mapsto\xi^t(\delta_i,\sqrt{-1}\nu_i)(1-\zeta_a)(\sqrt{-1}\nu_i)\langle\eta_{ij}^\ell\,,u\rangle_{\mathcal{H}_{\delta_i}}\langle\beta_{ij}^\ell\,,v\rangle_{\mathcal{H}_{\delta_i}}$$

is supported in the closed ball in $\mathfrak{a}_{i,0}^\star$ of radius $R_\epsilon=\dfrac{1}{1+\epsilon}\,||\,a\,||$ and center the origin. Now the classical Paley-Wiener theorem implies that

- $\widetilde{\phi}^t_{i,a}$ extends to an entire function on $\mathfrak{a}_i^\star$, in particular the map $\tilde{\Phi}^t_a$ is holomorphic in the variable α_i, and
- for all integer $N \in \mathbf{N}$, we have

$$\sup_{\alpha_i \in \mathfrak{a}_i^\star} \big(1+ \| \alpha_i \|^2 \big)^N e^{-R_\epsilon \|\mathrm{Re}(\alpha_i)\|} \mid \widehat{\psi}^t_{i,a}(\delta_i, \alpha_i) \mid < +\infty.$$

Therefore, since $\widehat{M}_i(\sigma^j_\ell)$ is a finite set, one has for all $\theta_1, \theta_2 \in \mathcal{U}(\mathfrak{k})$

$$\sup_{\alpha_i \in \mathfrak{a}_i^\star, \delta_i \in \widehat{M}_i(\sigma^j_\ell)} \big(1+ \| \alpha_i \|^2 + \| \mathrm{char}(\delta_i) \|^2 \big)^N e^{-R_\epsilon \|\mathrm{Re}(\alpha_i)\|}$$
$$\times \| \pi_{\delta_i,\alpha_i}(\theta_1) \widetilde{\phi}^t_{i,a}(\delta_i, \alpha_i) \pi_{\delta_i,\alpha_i}(\theta_2) \|_{L_c(\mathcal{H}_{\delta_i,\alpha_i})} < +\infty,$$

where

$$\begin{aligned} &\| \pi_{\delta_i,\alpha_i}(\theta_1) \widetilde{\phi}^t_{i,a}(\delta_i, \alpha_i) \pi_{\delta_i,\alpha_i}(\theta_2) \|_{L_c(\mathcal{H}_{\delta_i,\alpha_i})} \\ = &\sup_{\|\omega_1\|=\|\omega_2\|=1} \mid \langle \pi_{\delta_i,\alpha_i}(\theta_1) \widetilde{\phi}^t_{i,a}(\delta_i, \alpha_i) \pi_{\delta_i,\alpha_i}(\theta_2)\omega_1 \,, \omega_2 \rangle_{\mathcal{H}_{\delta_i}} \mid \\ = &\sup_{\|\omega_1\|=\|\omega_2\|=1} \mid \langle \widetilde{\phi}^t_{i,a}(\delta_i, \alpha_i) \pi_{\delta_i,\alpha_i}(\theta_2)\omega_1 \,, \pi_{\delta_i,\alpha_i}(\theta_1)\omega_2 \rangle_{\mathcal{H}_{\delta_i}} \mid . \end{aligned}$$

Then, by bi-K-equivariance (2.33) of the heat kernel, we deduce that

$$\widetilde{\Phi}^t_a \in \bigoplus_{P_i \in \mathcal{P}(A)} \bigoplus_{\delta_i \in (\widehat{M}_i)_d} \mathcal{O}(\mathfrak{a}_i^\star) \otimes \mathrm{End}\big(\mathcal{S}(K, \delta_i)\big)$$

and $\widetilde{\Phi}^t_a$ satisfies pre-Paley-Wiener condition (iii)$_r$ for $r = R_\epsilon$.

⋆ We turn now to the intertwining conditions (i) and (ii). By (3.3) we see that the map Φ^t is actually the Fourier transform $\Phi^t = \widehat{h}_t$ of the complex-valued function

$$h_t : G \to \mathbf{C},\ g \mapsto \langle p^{\ell,j,\perp}_t(g^{-1})\eta \,, \beta \rangle_{\Lambda^\ell \mathfrak{s}}.$$

Fix a basis $\{X_j\}$ of $\mathfrak{g}_0$ such that $\alpha_i = \sum_j \alpha_{i,j} \langle X_j \,, \cdot \rangle$ and choose a sequence $\{f_n\}$ of compactly supported smooth functions on G with supports in some balls in G such that (see [36])

$\lim_{n\to+\infty} f_n(g) = 1$, $\forall g \in G$, and, $\exists C > 0$ such that $\forall k_j \in \mathbf{N}$, $\forall g \in G$, $\mid (L(X_{j_1})^{k_1} \cdots L(X_{j_p})^{k_p} f_n)(g) \mid \leq C$ and $\lim_{n\to+\infty} (L(X_{j_1})^{k_1} \cdots L(X_{j_p})^{k_p} f_n)(g) = 0$.

Then, we have for $u \in \mathcal{H}^\infty_{\delta_i}$

$$\begin{aligned} &\| \widehat{f_n h_t}(P_i, \delta_i, \alpha_i)u - \widehat{h}_t(P_i, \delta_i, \alpha_i)u \|_{\mathcal{H}_{\delta_i}} \\ &= \| \int_G \big(f_n(g) - 1\big) h_t(g) \pi_{P_i,\delta_i,\alpha_i}(g) u dg \| \\ &\leq \int_G \mid f_n(g) - 1 \mid \mid \langle p^{\ell,j,\perp}_t(g^{-1})\eta \,, \beta \rangle_{\Lambda^\ell \mathfrak{s}} \mid \| \pi_{P_i,\delta_i,\alpha_i}(g) u \|_{\mathcal{H}_{P_i,\delta_i,\alpha_i}} dg. \end{aligned}$$

Now, since the Casimir operator $\widetilde{\Omega}_K$ acts on V^j_ℓ by a non-negative scalar $\sigma^j_\ell(\widetilde{\Omega}_K)$, we deduce from (2.34) that

$$p^{\ell,j}_t(g^{-1}) = e^{2t\sigma^j_\ell(\widetilde{\Omega}_K)} \int_{K\times K} p_t(k^{-1}g^{-1}k') \sigma_\ell(k) \sigma_\ell(k')^{-1} dk dk'.$$

But it is not difficult to check that (see Lemma 8 of [34])

$$\mid \int_K p_t(k^{-1}g^{-1}k')dkdk' \mid \leq C_1 \mid p_{t+1}(g^{-1}) \mid \quad \forall g \in G,\ t > 1$$

and it is also well known that (see Theorem 1 of [32])

$$\mid p_{t+1}(g^{-1}) \mid \leq C_2 e^{-\frac{||g||^2}{4(t+1)}} \quad \forall g \in G,\ t > 1.$$

Therefore we have

$$|| \, p_t^{\ell,j,\perp}(g^{-1}) \, ||_{\mathrm{End}(V_\ell^j)} \leq C_3 e^{2t\sigma_\ell^j(\widetilde{\Omega}_K)} e^{-\frac{||g||^2}{4(t+1)}} \quad \forall g \in G,\ t > 1.$$

On the other hand, one has (see Proposition 7.15 of [20])

$$|| \, \pi_{P_i,\delta_i,\alpha_i}(g)u \, ||_{\mathcal{H}_{\delta_i,\alpha_i}} \leq C_4 e^{||\mathrm{Re}(\alpha_i)|| \, ||g||}.$$

Since the function $g \mapsto e^{-\frac{||g||^2}{4(t+1)} + ||\mathrm{Re}(\alpha_i)|| \, ||g||}$ belongs to $L^1(G)$, we may apply the Lebesgue convergence theorem to see that

$$\lim_{n \to +\infty} || \, \widehat{f_n h_t}(P_i, \delta_i, \alpha_i)u - \widehat{h}_t(P_i, \delta_i, \alpha_i)u \, ||_{\mathcal{H}_{P_i,\delta_i,\alpha_i}} = 0.$$

Similarly, we have

$$\begin{aligned}
& || \, \pi_{P_i,\delta_i,\alpha_i}(X_j)\widehat{f_n h_t}(P_i, \delta_i, \alpha_i)u - \pi_{P_i,\delta_i,\alpha_i}(X_j)\widehat{h}_t(P_i, \delta_i, \alpha_i)u \, ||_{\mathcal{H}_{P_i,\delta_i,\alpha_i}} \\
&= || \int_G h_t(g)\big(f_n(g) - 1\big)\pi_{P_i,\delta_i,\alpha_i}(X_j)\pi_{P_i,\delta_i,\alpha_i}(g)udg \, ||_{\mathcal{H}_{P_i,\delta_i,\alpha_i}} \\
&= || \int_G h_t(g)\big(f_n(g) - 1\big)\frac{d}{ds} \mid_{s=0} \pi_{P_i,\delta_i,\alpha_i}(\exp(sX_j)g)udg \, ||_{\mathcal{H}_{P_i,\delta_i,\alpha_i}} \\
&= || \int_G \frac{d}{ds} \mid_{s=0} h_t(\exp(-sX_j)g)\big(f_n(\exp(-sX_j)g) - 1\big)\pi_{P_i,\delta_i,\alpha_i}(g)udg \, ||_{\mathcal{H}_{P_i,\delta_i,\alpha_i}} \\
&\leq \int_G \mid (L(X_j)h_t)(g)(f_n(g) - 1) \mid || \, \pi_{P_i,\delta_i,\alpha_i}(g)u \, ||_{\mathcal{H}_{P_i,\delta_i,\alpha_i}} \, dg \\
&+ \int_G \mid h_t(g) \mid \mid (L(X_j)f_n)(g) \mid || \, \pi_{P_i,\delta_i,\alpha_i}(g)u \, ||_{\mathcal{H}_{P_i,\delta_i,\alpha_i}} \, dg,
\end{aligned}$$

and, by iteration, the Lebesgue convergence theorem shows that

$$|| \, (\pi_{P_i,\delta_i,\alpha_i}(X_{j_1})^{k_1} \cdots \pi_{P_i,\delta_i,\alpha_i}(X_{j_p})^{k_p} \big(\widehat{f_n h_t}(P_i, \delta_i, \alpha)u - \widehat{h}_t(P_i, \delta_i, \alpha)u\big) \, ||_{\mathcal{H}_{P_i,\delta_i,\alpha_i}} \overset{\lim_{n \to +\infty}}{=} 0.$$

Therefore, since $\widehat{f_n h_t}$ is compactly supported, it satisfies intertwining condition (i) so that if V is a closed G-invariant subspace of $\mathcal{H}_{P_i,\delta_i,\alpha_i}$, then, we have that $\widehat{f_n h_t}(P_i, \delta_i, \alpha_i)V \subset V$. Taking the limit $n \to +\infty$ and using (3.4), we get that

$$\widetilde{\Phi}_a^t(P_i, \delta_i, \alpha_i)V \subset V.$$

Next let $V_1 \times V_2$ be a closed invariant subspace for $\frac{\partial}{\partial \lambda_k}\pi_{P_i,\delta_i,\alpha_i}$. Since $\widehat{f_n h_t}$ satisfies intertwining condition (i), the partial derivative $\frac{\partial}{\partial \lambda_k}\widehat{f_n h_t}$ leaves $V_1 \times V_2$ invariant, using definition (2.20). Observing that

$$\begin{aligned}&\Big[\frac{\partial}{\partial\lambda_k}\widehat{f_n h_t}(P_i,\delta_i,\alpha_i)-\frac{\partial}{\partial\lambda_k}\widehat{h}_t(P_i,\delta_i,\alpha_i)\Big](v_1,v_2)\\&=\Big(\widehat{f_n h_t}(P_i,\delta_i,\alpha_i)v_1+\alpha_{i,k}\widehat{f_n h_t}(P_i,\delta_i,\alpha_i)v_2,\widehat{f_n h_t}(P_i,\delta_i,\alpha_i)v_2\Big)\\&-\Big(\widehat{h}_t(P_i,\delta_i,\alpha_i)v_1+\alpha_{i,k}\widehat{h}_t(P_i,\delta_i,\alpha_i)v_2,\widehat{h}_t(P_i,\delta_i,\alpha_i)v_2\Big),\ \ \forall(v_1,v_2)\in V_1\times V_2\end{aligned}$$

and using the same arguments as above, we deduce that $\frac{\partial}{\partial\lambda_k}\Phi^t$ leaves $V_1\times V_2$ invariant, i.e

$$\begin{aligned}&\Big[\frac{\partial}{\partial\lambda_k}\Phi^t(P_i,\delta_i,\alpha_i)\Big](v_1,v_2)\\&=\Big(\langle\eta^\ell_{i,j}\,,v_1\rangle_{\mathcal{H}_{\delta_i}}+2t\alpha_{i,k}\langle\eta^\ell_{i,j}\,,v_2\rangle_{\mathcal{H}_{\delta_i}}\Big)(\beta^\ell_{i,j},0)\\&+\langle\eta^\ell_{i,j}\,,v_2\rangle_{\mathcal{H}_{\delta_i}}(0,\beta^\ell_{i,j})\in V_1\times V_2,\ \ \forall(v_1,v_2)\in V_1\times V_2.\end{aligned}\tag{3.6}$$

On the other hand, we have, for all $(v_1,v_2)\in V_1\times V_2$

$$\begin{aligned}&\Big[\frac{\partial}{\partial\lambda_k}\widetilde{\Phi}^t_a(P_i,\delta_i,\alpha_i)\Big](v_1,v_2)\\&=\Big(\big(\xi^t_{\delta_i}\star\widehat{(1-\zeta_a)}\big)(\alpha_i)\langle\eta^\ell_{i,j}\,,v_1\rangle_{\mathcal{H}_{\delta_i}}+\frac{\partial}{\partial\lambda_k}\big(\xi^t_{\delta_i}\star\widehat{(1-\zeta_a)}\big)(\alpha_i)\langle\eta^\ell_{i,j}\,,v_2\rangle_{\mathcal{H}_{\delta_i}}\Big)(\beta^\ell_{i,j},0)\\&+\big(\xi^t_{\delta_i}\star\widehat{(1-\zeta_a)}\big)(\alpha_i)\langle\eta^\ell_{i,j}\,,v_2\rangle_{\mathcal{H}_{\delta_i}}(0,\beta^\ell_{i,j}).\end{aligned}$$

In particular, since both $\xi^t_{\delta_i}$ and $\xi^t_{\delta_i}\star\widehat{(1-\zeta_a)}$ are radial functions, $\frac{\partial}{\partial\lambda_k}\widetilde{\Phi}^t_a(P_i,\delta_i,0)$ leaves $V_1\times V_2$ invariant. Indeed, when $\alpha_i=0$, one has

$$\begin{aligned}&\Big[\frac{\partial}{\partial\lambda_k}\widetilde{\Phi}^t_a(P_i,\delta_i,0)\Big](v_1,v_2)\\&=\big(\xi^t_{\delta_i}\star\widehat{(1-\zeta_a)}\big)(0)\xi^{-t}_{\delta_i}(0)\Big[\frac{\partial}{\partial\lambda_k}\Phi^t(P_i,\delta_i,0)\Big](v_1,v_2)\in V_1\times V_2.\end{aligned}$$

Actually, if $\langle\eta^\ell_{ij}\,,v_2\rangle_{\mathcal{H}_{\delta_i}}=0$, and α_i need not be trivial, then

$$\begin{aligned}&\Big[\frac{\partial}{\partial\lambda_k}\widetilde{\Phi}^t_a(P_i,\delta_i,\alpha_i)\Big](v_1,v_2)\\&=\big(\xi^t_{\delta_i}\star\widehat{(1-\zeta_a)}\big)(\alpha_i)\xi^{-t}_{\delta_i}(\alpha_i)\Big[\frac{\partial}{\partial\lambda_k}\Phi^t(P_i,\delta_i,\alpha_i)\Big](v_1,v_2)\in V_1\times V_2.\end{aligned}$$

If both $\alpha_i\neq 0$ and $\langle\eta^\ell_{i,j}\,,v_2\rangle_{\mathcal{H}_{\delta_i}}\neq 0$, then, applying (3.6) for two distinct $t_1\neq 0$ and $t_2\neq 0$, we see that both vectors $(\beta^\ell_{i,j},0)$ and $(0,\beta^\ell_{i,j})$ belong to $V_1\times V_2$ which implies that $\Big[\frac{\partial}{\partial\lambda_k}\widetilde{\Phi}^t_a(P_i,\delta_i,\alpha_i)\Big](v_1,v_2)$ belongs to $V_1\times V_2$. Therefore the first partial derivative of $\widetilde{\Phi}^t_a$ leaves invariant any closed invariant subspace $V_1\times V_2$ for $\frac{\partial}{\partial\lambda_k}\pi_{P_i,\delta_i,\alpha_i}$, i.e

$$\Big[\frac{\partial}{\partial\lambda_k}\widetilde{\Phi}^t_a(i,\delta_i,\alpha_i)\Big]V_1\times V_2\subset V_1\times V_2.$$

Next let $(V_1\times V_2)\times(V_3\times V_4)$ be a closed invariant subspace of $\dfrac{\partial^2}{\partial\lambda_k\partial\lambda_r}\pi_{P_i,\delta_i,\alpha_i}$. Using the same argument as above we show that $\dfrac{\partial^2}{\partial\lambda_k\partial\lambda_r}\Phi^t$ leaves the subspace

$(V_1 \times V_2) \times (V_3 \times V_4)$ invariant, i.e

$$
\begin{aligned}
&\Big(\langle \eta_{ij}^{\ell}, v_1\rangle_{\mathcal{H}_{\delta_i}} + 2t\alpha_{i,r}\langle \eta_{ij}^{\ell}, v_2\rangle_{\mathcal{H}_{\delta_i}} + 2t\alpha_{i,k}\langle \eta_{ij}^{\ell}, v_3\rangle_{\mathcal{H}_{\delta_i}} \\
&+4t^2\alpha_{i,k}\alpha_{i,r}\langle \eta_{ij}^{\ell}, v_4\rangle_{\mathcal{H}_{\delta_i}} + 2t\delta_{jk}\langle \eta_{ij}^{\ell}, v_4\rangle_{\mathcal{H}_{\delta_i}}\Big)\big((\beta_{ij}^{\ell},0),(0,0)\big)\\
&+\Big(\langle \eta_{ij}^{\ell}, v_2\rangle_{\mathcal{H}_{\delta_i}} + 2t\alpha_{i,k}\langle \eta_{ij}^{\ell}, v_4\rangle_{\mathcal{H}_{\delta_i}}\Big)\big((0,\beta_{ij}^{\ell}),(0,0)\big)\\
&+\Big(\langle \eta_{ij}^{\ell}, v_3\rangle_{\mathcal{H}_{\delta_i}} + 2t\alpha_{i,r}\langle \eta_{ij}^{\ell}, v_4\rangle_{\mathcal{H}_{\delta_i}}\Big)\big((0,0),(\beta_{ij}^{\ell},0)\big)\\
&+\langle \eta_{ij}^{\ell}, v_4\rangle_{\mathcal{H}_{\delta_i}}\big((0,0),(0,\beta_{ij}^{\ell})\big)\\
&\in (V_1 \times V_2) \times (V_3 \times V_4),
\end{aligned}
$$

for all $((v_1,v_2),(v_3,v_4)) \in (V_1 \times V_2) \times (V_3 \times V_4)$, $\alpha \in \mathfrak{a}^\star$ and $t > 0$. On the other hand, we have, by definition

$$
\begin{aligned}
&\frac{\partial^2}{\partial\lambda_k\partial\lambda_r}\widetilde{\Phi}_a^t((v_1,v_2),(v_3,v_4))\\
&=\Big(\big(\xi_{\delta_i}^t \star \widehat{(1-\zeta_a)}\big)(\alpha_i)\langle \eta_{ij}^{\ell}, v_1\rangle_{\mathcal{H}_{\delta_i}} + \frac{\partial}{\partial\lambda_r}\big(\xi_{\delta_i}^t \star \widehat{(1-\zeta_a)}\big)(\alpha_i)\langle \eta_{ij}^{\ell}, v_2\rangle_{\mathcal{H}_{\delta_i}}\\
&+\frac{\partial}{\partial\lambda_k}\big(\xi_{\delta_i}^t \star \widehat{(1-\zeta_a)}\big)(\alpha_i)\langle \eta_{ij}^{\ell}, v_3\rangle_{\mathcal{H}_{\delta_i}}\\
&+\frac{\partial^2}{\partial\lambda_k\partial\lambda_r}\big(\xi_{\delta_i}^t \star \widehat{(1-\zeta_a)}\big)(\alpha_i)\langle \eta_{ij}^{\ell}, v_4\rangle_{\mathcal{H}_{\delta_i}}\Big)\big((\beta_{ij}^{\ell},0),(0,0)\big)\\
&+\Big(\big(\xi_{\delta_i}^t \star \widehat{(1-\zeta_a)}\big)(\alpha_i)\langle \eta_{ij}^{\ell}, v_2\rangle_{\mathcal{H}_{\delta_i}} + \frac{\partial}{\partial\lambda_k}\big(\xi_{\delta_i}^t \star \widehat{(1-\zeta_a)}\big)(\alpha_i)\langle \eta_{ij}^{\ell}, v_4\rangle_{\mathcal{H}_{\delta_i}}\Big)\big((0,\beta_{ij}^{\ell}),(0,0)\big)\\
&+\Big(\big(\xi_{\delta_i}^t \star \widehat{(1-\zeta_a)}\big)(\alpha_i)\langle \eta_{ij}^{\ell}, v_3\rangle_{\mathcal{H}_{\delta_i}} + \frac{\partial}{\partial\lambda_r}\big(\xi_{\delta_i}^t \star \widehat{(1-\zeta_a)}\big)(\alpha_i)\langle \eta_{ij}^{\ell}, v_4\rangle_{\mathcal{H}_{\delta_i}}\Big)\big((0,0),(\beta_{ij}^{\ell},0)\big)\\
&+\Big(\langle \eta_{ij}^{\ell}, v_3\rangle_{\mathcal{H}_{\delta_i}} + 2t\alpha_{i,r}\langle \eta_{ij}^{\ell}, v_4\rangle_{\mathcal{H}_{\delta_i}}\Big)\big((0,0),(\beta_{ij}^{\ell},0)\big)\\
&+\big(\xi_{\delta_i}^t \star \widehat{(1-\zeta_a)}\big)(\alpha_i)\langle \eta_{ij}^{\ell}, v_4\rangle_{\mathcal{H}_{\delta_i}}\big((0,0),(0,\beta_{ij}^{\ell})\big)
\end{aligned}
$$

Then, as in the case of the degree one above, by considering different values of α_i and t, we deduce that

$$
\begin{aligned}
&\frac{\partial^2}{\partial\lambda_k\partial\lambda_r}\Phi^t(P_i,\delta_i,\alpha_i)((v_1,v_2),(v_3,v_4)) \in (V_1 \times V_2) \times (V_3 \times V_4)\\
&\Longrightarrow \frac{\partial^2}{\partial\lambda_k\partial\lambda_r}\widetilde{\Phi}_a^t(P_i,\delta_i,\alpha_i)((v_1,v_2),(v_3,v_4)) \in (V_1 \times V_2) \times (V_3 \times V_4).
\end{aligned}
$$

In a similar way, if W is a closed invariant subspace for the successive partial derivative $\frac{\partial^{|q|}}{\partial\lambda_{j_1}^{q_1}\cdots\partial\lambda_{j_p}^{q_p}}\pi_{\delta,\lambda}$ of $\pi_{\delta,\lambda}$, we show that the successive partial derivative $\frac{\partial^{|q|}}{\partial\lambda_{j_1}^{q_1}\cdots\partial\lambda_{j_p}^{q_p}}\Phi^t$ leaves W invariant. And so does the corresponding derivative of $\widetilde{\Phi}_a^t$

$$
\Big[\frac{\partial^{|q|}}{\partial\lambda_{j_1}^{q_1}\cdots\partial\lambda_{j_p}^{q_p}}\widetilde{\Phi}_a^t\Big]W \subset W.
$$

In other words, the map $\widetilde{\Phi}_a^t$ satisfies intertwining condition (i), when P_i is a minimal parabolic subgroup of G in $\mathcal{P}(A)$. It should be noted that the fact both $\eta_{i,j}^{\ell}$ and $\beta_{i,j}^{\ell}$ do not depend on α_i and t is essential in our arguments. Therefore, by Proposition 2.26, the map $\widetilde{\Phi}_a^t$ actually satisfies both Delorme's intertwining conditions (i) and (ii). In particular, the global map $\widetilde{\Phi}_a^t$ belongs to Delorme's Paley-Wiener space $\mathcal{PW}_{R_\epsilon}(G,K)$ defined in (2.25). We apply Delorme's Paley-Wiener theorem to see that there exists a complex-valued function μ on G which is supported in the closed ball in G of radius $R_\epsilon = \dfrac{1}{1+\epsilon} \mid\mid a \mid\mid$ and center the neutral element such that

$$\sum_i \sum_{\delta_i \in \widehat{M}_i(\sigma_\ell^j)} \frac{1}{\mid W_i \mid} \frac{1}{\dim(V_\ell^j)} \int_{\mathfrak{a}_{i,0}^\star} \mathrm{Tr}\Big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(a) \circ \widetilde{\phi}_{i,a}^t(\delta_i, \sqrt{-1}\nu_i)\Big) \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i$$
$$= \mu(a^{-1})$$
$$= 0.$$

Step 3: *we estimate the term in (3.3) involving* Φ_a^t. Recall that $\phi_{i,a}^t(\delta_i, \alpha_i) = (\xi_{\delta_i}^t \star \widehat{\zeta_a})(\alpha_i)\, \eta_{i,j}^\ell \otimes \beta_{i,j}^\ell$ so that, writing $\varphi_t(\nu_i) = e^{-t||\nu_i||^2}$, (3.3) implies that

$$\begin{aligned}
&\mid \langle p_t^{\ell,j,\perp}(a)\eta\,, \beta\rangle_{\Lambda^\ell \mathfrak{s}} \mid \\
&= \mid \sum_i \sum_{\delta_i \in \widehat{M}_i(\sigma_\ell^j)} \frac{1}{\mid W_i \mid} \frac{1}{\dim(V_\ell^j)} \\
&\quad \times \int_{\mathfrak{a}_{i,0}^\star} \langle \pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(a)\eta_{i,j}^\ell\,, \beta_{i,j}^\ell\rangle_{\mathcal{H}_{\delta_i}} (\xi^{\delta_i} \star \widehat{\zeta_a})(\sqrt{-1}\nu_i) \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i \mid \\
&\leq B_1 e^{-t\lambda_\ell(G/K)} \Phi_0(a) \mid\mid \eta \mid\mid_{\Lambda^\ell \mathfrak{s}} \mid\mid \beta \mid\mid_{\Lambda^\ell \mathfrak{s}} \sup_i \int_{\mathfrak{a}_{i,0}^\star} \mid (\varphi_t \star \widehat{\zeta_a})(\nu_i) \mid^2 (1+ \mid\mid \nu_i \mid\mid^2)^{N_i} d\nu_i \\
&\quad \text{(by (2.8), (2.18) and the Cauchy-Schwartz inequality)} \\
&\leq B_2 e^{-t\lambda_\ell(G/K)} \Phi_0(a) \mid\mid \eta \mid\mid_{\Lambda^\ell \mathfrak{s}} \mid\mid \beta \mid\mid_{\Lambda^\ell \mathfrak{s}} \sup_i \Big(\int_{\mathfrak{a}_{i,0}^\star} \sum_{|q| \leq N_i} \mid (D^q \widehat{\varphi}_t \zeta_a)(z) \mid^2 dz \Big)^{\frac{1}{2}} \\
&\quad \Big(\text{by equivalence with the Sobolev norm, where} \\
&\quad D^q = \frac{\partial^{|q|}}{\partial x_1^{q_1} \cdots \partial x_p^{q_p}} \text{ and } \mid q \mid= q_1 + \cdots + q_p\Big) \\
&= B_2 e^{-t\lambda_\ell(G/K)} \Phi_0(a) \mid\mid \eta \mid\mid_{\Lambda^\ell \mathfrak{s}} \mid\mid \beta \mid\mid_{\Lambda^\ell \mathfrak{s}} \sup_i \Big(\int_{||z|| \geq \frac{1}{1+\epsilon}||a||} \sum_{|q| \leq N_i} \mid (D^q \widehat{\varphi}_t \zeta_a)(z) \mid^2 dz \Big)^{\frac{1}{2}} \\
&\leq B_3 e^{-t\lambda_\ell(G/K)} \Phi_0(a) \mid\mid \eta \mid\mid_{\Lambda^\ell \mathfrak{s}} \mid\mid \beta \mid\mid_{\Lambda^\ell \mathfrak{s}}\, e^{-\frac{||a||^2}{4(1+\epsilon)^2 t}} \text{ if } t > 1 \\
&\quad \Big(\text{since } \widehat{\varphi}_t(z) = (2t)^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} e^{-\frac{||z||^2}{4t}}\Big)
\end{aligned}$$

and therefore

$$\mid\mid p_t^{\ell,j,\perp}(a) \mid\mid_{\mathrm{End}(\Lambda^\ell \mathfrak{s})} \leq b_{\epsilon,j} e^{-t\lambda_\ell(G/K)} \Phi_0(a) e^{-\frac{||a||^2}{4(1+\epsilon)^2 t}} \tag{3.7}$$

for some positive constant $b_{\epsilon,j}$ depending on both ϵ and j. Similarly we have

$$\begin{aligned}
&\mid \langle p_t^{\ell,j,\perp}(a)\eta\,, \beta\rangle_{\Lambda^\ell \mathfrak{s}} \mid \\
&\quad \leq B_4 e^{-t\lambda_\ell(G/K)} \Phi_0(a) \mid\mid \eta \mid\mid_{\Lambda^\ell \mathfrak{s}} \mid\mid \beta \mid\mid_{\Lambda^\ell \mathfrak{s}} \sup_i \int_{\mathfrak{a}_{i,0}^\star} e^{-t||\nu_i||^2} \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i.
\end{aligned} \tag{3.8}$$

Now recall from (2.8) that there is a polynomial P_{δ_i} such that $\mathbf{c}_{\delta_i} \leq P_{\delta_i}$. Writing $P_{\delta_i} = \sum_{k\geq 0} a_k \nu_i^k$, we have

$$\begin{aligned}
&\int_{\mathfrak{a}^\star_{i,0}} e^{-t||\nu_i||^2} P_{\delta_i}(\nu_i) d\nu_i \\
&= \sum_{k\geq 0} a_k \int_{\mathfrak{a}^\star_{i,0}} e^{-t||\nu_i||^2} \nu_i^k d\nu_i \\
&= \sum_{k\geq 0} a_k \int_{||\nu_i||=1} \nu_i^k d\nu_i \int_0^{+\infty} e^{-tr^2} r^{k+\dim_{\mathbf{R}}(\mathfrak{a}_{i,0})-1} dr \\
&\quad \text{(using spherical coordinates)} \\
&= t^{-\frac{\dim_{\mathbf{R}}(\mathfrak{a}_{i,0})}{2}} \sum_{k\geq 0} t^{-\frac{k}{2}} a_k \int_{||\nu_i||=1} \nu_i^k d\nu_i \int_0^{+\infty} e^{-r^2} r^{k+\dim_{\mathbf{R}}(\mathfrak{a}_{i,0})-1} dr.
\end{aligned}$$

Then we deduce from (3.8) that

$$(3.9) \qquad || p_t^{\ell,j,\perp}(a) ||_{\mathrm{End}(\Lambda^\ell \mathfrak{s})} \leq B_5 e^{-t\lambda_\ell(G/K)} t^{-\frac{z+r}{2}} \Phi_0(a) \text{ if } t > 1$$

where $z = \inf_i \left\{ \text{order of zero of } \mathbf{c}_{\delta_i} \text{ at } \nu_i = 0, \quad \delta_i \in \widehat{M_i}(\sigma_\ell) \right\}$ and $r = \inf_i \left\{ \dim_{\mathbf{R}}(\mathfrak{a}_{i,0}) > 0 \right\}$. Combining (3.7) and (3.9) we obtain

$$\begin{aligned}
|| p_t^{\ell,j,\perp}(a) ||_{\mathrm{End}(\Lambda^\ell \mathfrak{s})} &= || p_t^{\ell,j,\perp}(a) ||^{1-\frac{\epsilon}{2}}_{\mathrm{End}(\Lambda^\ell \mathfrak{s})} || p_t^{\ell,j,\perp}(a) ||^{\frac{\epsilon}{2}}_{\mathrm{End}(\Lambda^\ell \mathfrak{s})} \\
&\leq c_{\epsilon,j} e^{-t\lambda_\ell(G/K)} \Phi_0(a) e^{-\frac{1-\frac{\epsilon}{2}}{(1+\epsilon)^2} \frac{||a||^2}{4t}} t^{-\frac{\epsilon}{2}\frac{z+r}{2}} \text{ if } t > 1
\end{aligned}$$

for some positive constant $c_{\epsilon,j}$ depending on both ϵ and j. □

When G has an empty discrete series, i.e $\mathrm{rk}_{\mathbf{C}}(G) > \mathrm{rk}_{\mathbf{C}}(K)$, the positive integer r equals $\mathrm{rk}_{\mathbf{C}}(G) - \mathrm{rk}_{\mathbf{C}}(K)$ and the previous theorem may be restated as follows.

Corollary 3.10. *Under the assumptions of Theorem 3.1, if G does not have discrete series representations then, for all $\epsilon \in]0,1[$, there exist two positive numbers A_ϵ and a_ϵ such that*

$$(3.11) \qquad || p_t^{\ell,\perp}(g) ||_{End(\Lambda^\ell \mathfrak{s})} \leq a_\epsilon e^{-t\lambda_\ell(G/K)} \Phi_0(g) e^{-\frac{1-\epsilon}{(1+2\epsilon)^2} \frac{||g||^2}{4t}} t^{-\epsilon \frac{z + rk_{\mathbf{C}}(G) - rk_{\mathbf{C}}(K)}{2}}$$

for all $g \in G$ and $t \in \mathbf{R}$ satisfying $|| g ||> A_\epsilon$ and $t > 1$.

If G has an empty discrete series and is such that $z = 0$ (as it is the case for the hyperbolic groups $G = SO_e(2n+1,1)$, with $n \geq 1$), the exponent of t in the estimate (3.11) has a nice geometric meaning. Indeed, in this case the (positive) integer $\mathrm{rk}_{\mathbf{C}}(G) - \mathrm{rk}_{\mathbf{C}}(K)$ is the ℓth Novikov-Shubin invariant $a_\ell(\Gamma\backslash G/K)$ of the locally symmetric space $\Gamma\backslash G/K$, where Γ is a torsion free discrete subgroup of G of finite covolume (see (1.1)).

4. Estimates for the resolvent.

Theorem 4.1. *Let G be a non compact connected semisimple real Lie group with finite center and K a maximal compact subgroup of G. For a complex number*

μ with real part $Re(\mu)$ and imaginary part $Im(\mu)$ satisfying either $Im(\mu) \neq 0$ or $Re(\mu) < \lambda_\ell(G/K)$, define the positive number

$$\tau_{\mu,\ell}(G/K) = \sqrt{\frac{\lambda_\ell(G/K) - Re(\mu) + \sqrt{\Big(\lambda_\ell(G/K) - Re(\mu)\Big)^2 + Im(\mu)^2}}{2}}.$$

Then, for all $\epsilon \in]0,1[$, there exist two positive numbers b_ϵ and B_ϵ such that

$$|| \, (\Delta_\ell - \mu)^{-1}(g) \, || \leq b_\epsilon \Phi_0(g) e^{-(1-\epsilon)\tau_{\mu,\ell}(G/K)||g||} \tag{4.2}$$

for all $g \in G$ satisfying $|| \, g \, || > B_\epsilon$.

Proof. We follow the same strategy (and notation) as in the proof of the above theorem. Throughout the proof the symbols B_j and C_j will denote positive real numbers.

Step 1: Write R^ℓ_μ for the resolvent operator $(\Delta_\ell - \mu)^{-1}$ of Δ_ℓ. For $\epsilon_0 > 0$ define $R^\ell_{\mu,\epsilon_0} = R^\ell_\mu \star P^\ell_{\epsilon_0}$, and the following functions

$$\phi^{\epsilon_0}_{1,i}(\nu_i) = e^{-\epsilon_0||\nu_i||^2}, \quad \phi^{\epsilon_0}_{2,i}(\nu_i) = \frac{1}{-\omega_{P_i,\delta_i,\sqrt{-1}\nu_i} - \mu},$$
$$\text{and} \;\; \phi^{\epsilon_0}_{0,i}(\nu_i) = \phi^{\epsilon_0}_{2,i}(\nu_i) e^{\epsilon_0 \omega_{P_i,\delta_i,\sqrt{-1}\nu_i}}.$$

Then we have, for all $a \in \exp(\overline{\mathfrak{a}^+})$ and $\eta, \beta \in V^j_\ell$

$$\begin{aligned}
&\langle R^\ell_{\mu,\epsilon_0}(a)\eta \, , \beta\rangle_{\Lambda^\ell\mathfrak{s}} \\
&= \sum_i \sum_{\delta_i \in \widehat{M_i}(\sigma^j_\ell)} \frac{1}{| \, W_i \, |} \frac{1}{\dim(V^j_\ell)} \int_{\mathfrak{a}^\star_{i,0}} \phi^{\epsilon_0}_{0,i}(\nu_i) \langle \Psi^{\ell,j}_{\delta_i,\nu_i}(a)\eta \, , \beta\rangle_{\Lambda^\ell\mathfrak{s}} \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i \\
&= \sum_i \sum_{\delta_i \in \widehat{M_i}(\sigma^j_\ell)} \frac{1}{| \, W_i \, |} \frac{1}{\dim(V^j_\ell)} \int_{\mathfrak{a}^\star_{i,0}} \mathrm{Tr}\big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i} \circ \phi^{\epsilon_0}_i(\delta_i, \sqrt{-1}\nu_i)\big) \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i \\
&\quad \text{where } \phi^{\epsilon_0}_i(\delta_i, \alpha_i) = \phi^{\epsilon_0}_{0,i}(\nu_i) \eta^\ell_{i,j} \otimes \beta^\ell_{i,j} \\
&= \sum_i \sum_{\delta_i \in \widehat{M_i}(\sigma^j_\ell)} \frac{1}{| \, W_i \, |} \frac{1}{\dim(V^j_\ell)} \int_{\mathfrak{a}^\star_{i,0}} \mathrm{Tr}\big(\pi_{P_i,\delta_i,\sqrt{-1}\nu_i} \circ \phi^{\epsilon_0}_{i,a}(\delta_i, \sqrt{-1}\nu_i)\big) \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i \\
&\quad \text{using Delorme Theorem with } \phi^{\epsilon_0}_{i,a}(\delta_i, \alpha_i) = \big(\phi^{\epsilon_0}_{0,i} \star \zeta_a\big)(\nu_i) \eta^\ell_{i,j} \otimes \beta^\ell_{i,j} \\
&\quad \text{and } \zeta_a \text{ is defined as in the proof of Theorem 3.1 above,} \\
&= \sum_i \sum_{\delta_i \in \widehat{M_i}(\sigma^j_\ell)} \frac{1}{| \, W_i \, |} \frac{1}{\dim(V^j_\ell)} e^{\epsilon_0\big(||\mathrm{char}(\delta_i)||^2 - ||\rho_i||^2\big)} \\
&\quad \times \int_{\mathfrak{a}^\star_{i,0}} \langle \pi_{P_i,\delta_i,\sqrt{-1}\nu_i}(a) \eta^\ell_{i,j} \, , \beta^\ell_{i,j}\rangle_{\mathcal{H}_{\delta_i}} (\varphi_{\epsilon_0} \star \widehat{\zeta_a})(\nu_i) \mathbf{c}_{\delta_i}(\sqrt{-1}\nu_i) d\nu_i \\
&\quad \text{where } \varphi_{\epsilon_0} = \varphi^{\epsilon_0}_{1,i} \varphi_{2,i}
\end{aligned}$$

so that

$$
\begin{aligned}
& \mid \langle R^{\ell}_{\mu,\epsilon_0}(a)\eta\,,\beta\rangle_{\Lambda^{\ell}\mathfrak{s}} \mid \\
& \leq B_1 e^{-\epsilon_0\lambda_\ell(G/K)}\Phi_0(a) \mid\mid \eta \mid\mid_{\Lambda^\ell\mathfrak{s}} \mid\mid \beta \mid\mid_{\Lambda^\ell\mathfrak{s}} \sup_i \int_{\mathfrak{a}^\star_{i,0}} \mid (\varphi_{\epsilon_0} \star \widehat{\zeta_a})(\nu_i) \mid^2 (1+ \mid\mid \nu_i \mid\mid^2)^{N_i} d\nu_i \\
& \quad \big(\text{by (2.8), (2.18) and the Cauchy-Schwartz inequality}\big) \\
& \leq B_2 e^{-\epsilon_0\lambda_\ell(G/K)}\Phi_0(a) \sup_i \Big(\int_{\mathfrak{a}_{i,0}^\star} \sum_{|k_1|+|k_2|\leq N_i} \mid (D^{k_1}(\widehat{\varphi}^{\epsilon_0}_{1,i} \star \widehat{\varphi}_{2,i}) D^{k_2}\zeta_a)(z) \mid^2 dz \Big)^{\frac{1}{2}} \\
& \quad \big(\text{by equivalence with the Sobolev norm, where} \\
& \quad D^q = \frac{\partial^{|q|}}{\partial x_1^{q_1} \cdots \partial x_{q_p}^{q_p}} \text{ and } \mid q \mid = q_1 + \cdots + q_p\big).
\end{aligned}
\tag{4.3}
$$

Step 2: Recall that $\widehat{\varphi^{\epsilon_0}_{1,i}}(z) = (2\epsilon_0)^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} e^{-\frac{||z||^2}{4\epsilon_0}}$. Define the following family of complex numbers $\{\tau_i\}$ by

$$\tau_i^2 = \mid\mid \mathrm{char}(\delta_i) \mid\mid^2 - \mid\mid \rho_i \mid\mid^2 + \mu.$$

Note that our assumptions on μ implies that the imaginary part of τ_i is positive. We deduce that

$$\widehat{\varphi}_{2,i}(z) = \sqrt{-1}\frac{\pi}{4}(\mid\mid z \mid\mid^{-\dim(\mathfrak{a}_{i,0})+2} \big(\tau_i \mid\mid z \mid\mid\big)^{\frac{\dim(\mathfrak{a}_{i,0})}{2}-1} H^{(1)}_{\frac{\dim(\mathfrak{a}_{i,0})}{2}-1}(\tau_i \mid\mid z \mid\mid) \tag{4.4}$$

where $H^{(1)}_\alpha$ denotes the Bessel-Neumann function (see p. 65 of [29]). Indeed assume that $\tau_i = \sqrt{-1} r_i$ is imaginary with $r_i > 0$, then

$$
\begin{aligned}
\widehat{\varphi}_{2,i}(z) &= \int_{\mathfrak{a}^\star_{i,0}} \frac{e^{\sqrt{-1}\langle z\,,\nu_i\rangle}}{\mid\mid \nu_i \mid\mid^2 + r_i^2} d\nu_i \\
&= \int_{\mathfrak{a}^\star_{i,0}} e^{\sqrt{-1}\langle z\,,\nu_i\rangle} \int_0^{+\infty} e^{-t\left(||\nu_i||^2 + r_i^2\right)} dt d\nu_i \\
&= \int_0^{+\infty} e^{-tr_i^2} \int_{\mathfrak{a}^\star_{i,0}} e^{-t||\nu_i||^2} e^{\sqrt{-1}\langle z\,,\nu_i\rangle} d\nu_i dt \\
&= 2^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} \int_0^{+\infty} e^{-tr_i^2} t^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} e^{-\frac{||z||^2}{4t}} dt \\
&= \Big(\frac{r_i}{\mid\mid z \mid\mid}\Big)^{\frac{\dim(\mathfrak{a}_{i,0})}{2}-1} K_{\frac{\dim(\mathfrak{a}_{i,0})}{2}-1}(r_i \mid\mid z \mid\mid) \\
&\quad \big(\text{where } K_\alpha \text{ denotes the modified Bessel function, see p. 85 of [29]}\big), \\
&= \sqrt{-1}\frac{\pi}{4}\Big(\frac{\sqrt{-1}r_i}{\mid\mid z \mid\mid}\Big)^{\frac{\dim(\mathfrak{a}_{i,0})}{2}-1} H^{(1)}_{\frac{\dim(\mathfrak{a}_{i,0})}{2}-1}(\sqrt{-1} r_i \mid\mid z \mid\mid) \\
&= \sqrt{-1}\frac{\pi}{4} \mid\mid z \mid\mid^{-\dim(\mathfrak{a}_{i,0})+2} \big(\sqrt{-1} r_i \mid\mid z \mid\mid\big)^{\frac{\dim(\mathfrak{a}_{i,0})}{2}-1} H^{(1)}_{\frac{\dim(\mathfrak{a}_{i,0})}{2}-1}(\sqrt{-1} r_i \mid\mid z \mid\mid),
\end{aligned}
$$

since (see p. 67 of [29])

$$K_\alpha(z) = \frac{1}{2}\sqrt{-1}\pi e^{\sqrt{-1}\frac{\pi}{2}\alpha} H^{(1)}_\alpha(z e^{-\sqrt{-1}\frac{\pi}{2}}). \tag{4.5}$$

Now (4.4) follows by analytic continuation.

Step 3: We shall prove that there exists a positive constant A_i that does not depend on ϵ_0 so that

$$| (D^{k_1}(\widehat{\varphi}^{\epsilon_0}_{1,i} \star \widehat{\varphi}_{2,i})(z) | \leq A_i e^{-\mathrm{Im}(\tau_i)||z||} \tag{4.6}$$

for $|| z ||$ sufficiently large. Indeed fix a smooth function ζ such that $\zeta(z) = 1$ if $|| z ||> 1$ and $\zeta(z) = 0$ if z belongs to some neighborhood of 0. Write

$$\widehat{\varphi}_{2,i} = \widehat{\varphi}^1_{2,i} + \widehat{\varphi}^2_{2,i} \tag{4.7}$$

where $\widehat{\varphi}^1_{2,i} = (1-\zeta)\widehat{\varphi}_{2,i}$ and $\widehat{\varphi}^2_{2,i} = \zeta\widehat{\varphi}_{2,i}$. It turns out that

$$D^{k_1}\widehat{\varphi}^{\epsilon_0}_{1,i}(z-y) = P_{k_1}(z-y)e^{-\frac{||z-y||^2}{4\epsilon_0}}$$

where $P_{k_1}(z-y)$ is a polynomial in $z-y$ with term of highest degree equal to $(2\epsilon_0)^{-|k_1|-\frac{\dim(\mathfrak{a}_{i,0})}{2}}(z_1-y_1)^{k_1^1}\cdots(z_{\dim(\mathfrak{a}_{i,0})}-y_{\dim(\mathfrak{a}_{i,0})})^{k_1^{\dim(\mathfrak{a}_{i,0})}}$ with $| k_1 |= k_1^1+\cdots+k_1^{\dim(\mathfrak{a}_{i,0})}$. We obtain successively

$$\begin{aligned}
&\sup_{||y||\leq 1} | D^{k_1}\widehat{\varphi}^{\epsilon_0}_{1,i}(z-y) | \\
&\leq 2^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}}\epsilon_0^{-|k_1|-\frac{\dim(\mathfrak{a}_{i,0})}{2}} || z ||^{|k_1|} e^{-\frac{(||z||-1)^2}{4\epsilon_0}} \\
&\quad \big(\text{since } 2 || z ||\geq || z-y ||\geq || z || -1 \text{ for } || z || \text{ sufficiently large}\big) \\
&\leq 2^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} || z ||^{-|k_1|} \epsilon_0^{-|k_1|-\frac{\dim(\mathfrak{a}_{i,0})}{2}} || z ||^{2|k_1|} e^{-\frac{||z||^2}{16\epsilon_0}} \\
&\quad \big(\text{since } || z || -1 \geq \frac{|| z ||}{2} \text{ for } || z ||\geq 2\big) \\
&\leq 2^{-|k_1|-\frac{\dim(\mathfrak{a}_{i,0})}{2}}\epsilon_0^{-|k_1|-\frac{\dim(\mathfrak{a}_{i,0})}{2}} || z ||^{2|k_1|} e^{-\frac{||z||^2}{16\epsilon_0}} \\
&\quad \big(\text{for } || z ||\geq 2\big) \\
&= 2^{-|k_1|-\frac{\dim(\mathfrak{a}_{i,0})}{2}} w^{|k_1|}e^{-\frac{w}{32}}\epsilon_0^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} e^{-\frac{||z||^2}{32\epsilon_0}} \\
&\quad \big(\text{writing } w = \frac{|| z ||^2}{\epsilon_0}\big) \\
&\leq B_1 2^{-|k_1|-\frac{\dim(\mathfrak{a}_{i,0})}{2}}\epsilon_0^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} e^{-\frac{||z||^2}{32\epsilon_0}} \\
&\quad \big(\text{for some positive number } B_1 \text{ that does not depend on } \epsilon_0\big) \\
&= B_1 2^{-|k_1|-\frac{\dim(\mathfrak{a}_{i,0})}{2}}\epsilon_0^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} e^{-\frac{||z||^2}{64\epsilon_0}} e^{-\frac{||z||^2}{64\epsilon_0}} \\
&\leq B_2 2^{-|k_1|-\frac{\dim(\mathfrak{a}_{i,0})}{2}} e^{-\frac{||z||^2}{64\epsilon_0}} \\
&\quad \big(\text{for some positive number } B_2 \text{ that does not depend on } \epsilon_0 \text{ and for } || z ||\geq 1\big) \\
&\leq B_2 2^{-|k_1|-\frac{\dim(\mathfrak{a}_{i,0})}{2}}(4r)^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} e^{-||z||^2} \\
&\quad \big(\text{for } \epsilon_0 \text{ sufficiently small}\big).
\end{aligned}$$

In particular we have, for $\| z \| \geq \mathrm{Im}(\tau_i)$

$$
\begin{aligned}
& | (D^{k_1}(\widehat{\varphi}^{\epsilon_0}_{1,i} \star \widehat{\varphi}^{1}_{2,i})(z) | = | (D^{k_1}(\widehat{\varphi}^{\epsilon_0}_{1,i}) \star \widehat{\varphi}^{1}_{2,i})(z) | \\
& \qquad \leq \sup_{\|y\| \leq 1} | D^{k_1}\widehat{\varphi}^{\epsilon_0}_{1,i}(z-y) | \int_{\|y\|\leq 1} | \widehat{\varphi}^{1}_{2,i})(z) | \, dy \\
& \qquad \leq B_3 e^{-\mathrm{Im}(\tau_i)\|z\|} \qquad (4.8) \\
& \qquad \big(\text{for some positive number } B_3 \text{ that does not depend on } \epsilon_0\big).
\end{aligned}
$$

Step 4: We turn now to $D^{k_1}(\widehat{\varphi}^{\epsilon_0}_{1,i} \star \widehat{\varphi}^{2}_{2,i})$. Combining the following relation satisfied by the modified Bessel functions (see p. 67 of [29])

$$
\big(\frac{d}{zdz}\big)^m [z^{-\gamma} K_\gamma(z)] = (-1)^m z^{-\gamma-m} K_{\gamma+m}(z), \quad m = 1, 2, 3, \cdots
$$

along with (4.4), (4.5) and the asymptotics of K_γ given on p. 139 of [29], we see that there exists a positive number C_1 such that

$$
| (D^{k_1}\widehat{\varphi}^{2}_{2,i})(z) | = | (D^{k_1}\zeta\widehat{\varphi}_{2,i})(z) | \leq C_1 e^{-\mathrm{Im}(\tau_i)\|z\|}
$$

for $\| z \|$ sufficiently large. Hence one has

$$
\begin{aligned}
| D^{k_1}(\widehat{\varphi}^{\epsilon_0}_{1,i} \star \widehat{\varphi}^{2}_{2,i})(z) | &= | \widehat{\varphi}^{\epsilon_0}_{1,i} \star (D^{k_1}\widehat{\varphi}^{2}_{2,i})(z) | \\
&\leq C_1 \int_{\mathfrak{a}^\star_{i,0}} (2\epsilon_0)^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} e^{-\mathrm{Im}(\tau_i)\|z-y\|} e^{-\frac{\|y\|^2}{4\epsilon_0}} dy \\
&= C_1 2^{\frac{\dim(\mathfrak{a}_{i,0})}{2}} \Big(\int_{\|z\|\geq\|2\sqrt{\epsilon_0}w\|} e^{-\mathrm{Im}(\tau_i)\|z-2\sqrt{\epsilon_0}w\|} e^{-\|w\|^2} dw \\
&+ \int_{\|z\|\leq\|2\sqrt{\epsilon_0}w\|} e^{-\mathrm{Im}(\tau_i)\|z-2\sqrt{\epsilon_0}w\|} e^{-\|w\|^2} dw \Big) \\
&\quad \big(\text{writing } w = \frac{y}{2\sqrt{\epsilon_0}}\big).
\end{aligned}
$$

Now we have

$$
\begin{aligned}
& \int_{\|z\|\geq\|2\sqrt{\epsilon_0}w\|} e^{-\mathrm{Im}(\tau_i)\|z-2\sqrt{\epsilon_0}w\|} e^{-\|w\|^2} dw \\
& \leq e^{-\mathrm{Im}(\tau_i)\|z\|} \int_{\|z\|\geq\|2\sqrt{\epsilon_0}w\|} e^{\mathrm{Im}(\tau_i)\|2\sqrt{\epsilon_0}w\|} e^{-\|w\|^2} dw \\
& \leq e^{-\mathrm{Im}(\tau_i)\|z\|} \int_{\mathfrak{a}^\star_{i,0}} e^{\mathrm{Im}(\tau_i)\|u\|} e^{-\frac{\|u\|^2}{4\epsilon_0}} 2^{-\dim(\mathfrak{a}_{i,0})} \epsilon_0^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} du \\
& \quad \big(\text{writing } u = 2\sqrt{\epsilon_0}w\big) \\
& = e^{-\mathrm{Im}(\tau_i)\|z\|} \int_{\|u\|\geq \mathrm{Im}(\tau_i)} e^{\mathrm{Im}(\tau_i)\|u\|} e^{-\frac{\|u\|^2}{4\epsilon_0}} 2^{-\dim(\mathfrak{a}_{i,0})} \epsilon_0^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} du \\
& + e^{-\mathrm{Im}(\tau_i)\|z\|} \int_{\|u\|\leq \mathrm{Im}(\tau_i)} e^{\mathrm{Im}(\tau_i)\|u\|} e^{-\frac{\|u\|^2}{4\epsilon_0}} 2^{-\dim(\mathfrak{a}_{i,0})} \epsilon_0^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}} du
\end{aligned}
$$

with

$$\begin{aligned}
&\int_{||u||\geq \mathrm{Im}(\tau_i)} e^{\mathrm{Im}(\tau_i)||u||}e^{-\frac{||u||^2}{4\epsilon_0}}2^{-\dim(\mathfrak{a}_{i,0})}\epsilon_0^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}}du\\
&=\int_{||u||\geq \mathrm{Im}(\tau_i)} e^{\mathrm{Im}(\tau_i)||u||}e^{-\frac{||u||^2}{8\epsilon_0}}2^{-\dim(\mathfrak{a}_{i,0})}\epsilon_0^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}}e^{-\frac{||u||^2}{8\epsilon_0}}du\\
&\leq \sup_{||u||\geq \mathrm{Im}(\tau_i)} e^{(\mathrm{Im}(\tau_i)-||u||)||u||}\int_{\mathfrak{a}^\star_{i,0}} 2^{-\dim(\mathfrak{a}_{i,0})}\epsilon_0^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}}e^{-\frac{||u||^2}{8\epsilon_0}}du\\
&= \sup_{||u||\geq \mathrm{Im}(\tau_i)} e^{(\mathrm{Im}(\tau_i)-||u||)||u||}\int_{\mathfrak{a}^\star_{i,0}} 2^{\frac{\dim(\mathfrak{a}_{i,0})}{2}}e^{-||w||^2}dw\\
&\quad\big(\text{writing } u=2^{\frac{3}{2}}\sqrt{\epsilon_0}w\big)\\
&\leq C_2
\end{aligned}$$

and

$$\begin{aligned}
&\int_{||u||\leq \mathrm{Im}(\tau_i)} e^{\mathrm{Im}(\tau_i)||u||}e^{-\frac{||u||^2}{4\epsilon_0}}2^{-\dim(\mathfrak{a}_{i,0})}\epsilon_0^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}}du\\
&\leq e^{(\mathrm{Im}(\tau_i))^2}\int_{\mathfrak{a}^\star_{i,0}} 2^{-\dim(\mathfrak{a}_{i,0})}\epsilon_0^{-\frac{\dim(\mathfrak{a}_{i,0})}{2}}e^{-\frac{||u||^2}{8\epsilon_0}}du\\
&= e^{(\mathrm{Im}(\tau_i))^2}\int_{\mathfrak{a}^\star_{i,0}} e^{-||v||^2}dv\\
&\quad\big(\text{writing } u=2\sqrt{\epsilon_0}v\big)\\
&\leq C_3
\end{aligned}$$

so that

$$\int_{||z||\geq||2\sqrt{\epsilon_0}w||} e^{-\mathrm{Im}(\tau_i)||z-2\sqrt{\epsilon_0}w||}e^{-||w||^2}dw \leq C_4e^{-\mathrm{Im}(\tau_i)||z||}$$

for $|| z ||$ sufficiently large. On the other hand, we have

$$\begin{aligned}
&\int_{||z||\leq||2\sqrt{\epsilon_0}w||} e^{-\mathrm{Im}(\tau_i)||z-2\sqrt{\epsilon_0}w||}e^{-||w||^2}dw\\
&\leq e^{-\mathrm{Im}(\tau_i)||z||}\int_{||z||\leq||2\sqrt{\epsilon_0}w||} e^{2\mathrm{Im}(\tau_i)(||z||-||\sqrt{\epsilon_0}w||)}e^{-||w||^2}dw\\
&\leq e^{-\mathrm{Im}(\tau_i)||z||}e^{2\mathrm{Im}(\tau_i)||z||}e^{-\frac{||z||^2}{4}}\\
&\leq C_4e^{-\mathrm{Im}(\tau_i)||z||}
\end{aligned}$$

and thus

$$| (D^{k_1}(\widehat{\varphi}^{\epsilon_0}_{1,i}\star\widehat{\varphi}^2_{2,i})(z) |\leq C_5e^{-\mathrm{Im}(\tau_i)||z||} \tag{4.9}$$

for $|| z ||$ sufficiently large. Now (4.6) follows from (4.7), (4.8) and (4.9).

Step 5: Finally we deduce that

$$
\begin{aligned}
&\Big(\int_{\mathfrak{a}^\star_{i,0}} \mid (D^{k_1}(\widehat{\varphi}^{\epsilon_0}_{1,i}\star\widehat{\varphi}_{2,i})D^{k_2}\zeta_a(z)\mid^2 dz\Big)^{\frac12}\\
&\le \Big(A_i\int_{||z||\ge\frac{||a||}{1+\epsilon}} e^{-2\mathrm{Im}(\tau_i)||z||}dz\Big)^{\frac12}\\
&= A_i^{\frac12}\Big(\int_{||z||\ge\frac{||a||}{1+\epsilon}} e^{-2(1-\epsilon)\mathrm{Im}(\tau_i)||z||}e^{-2\epsilon\mathrm{Im}(\tau_i)||z||}dz\Big)^{\frac12}\\
&\le A_i^{\frac12}e^{-\frac{1-\epsilon}{1+\epsilon}\mathrm{Im}(\tau_i)||a||}\Big(\int_{||z||\ge\frac{||a||}{1+\epsilon}} e^{-2\epsilon\mathrm{Im}(\tau_i)||z||}dz\Big)^{\frac12}\\
&= D_{\epsilon,i}e^{-\frac{1-\epsilon}{1+\epsilon}\mathrm{Im}(\tau_i)||a||}\\
&\quad\big(\text{for some positive number } D_{\epsilon,i} \text{ depending on } \epsilon\big)\\
&\le D_{\epsilon,i}e^{-(1-2\epsilon)\mathrm{Im}(\tau_i)||a||}.
\end{aligned}
$$

In particular, for $|| a ||$ sufficiently large, (4.3) can be rewritten as follows

$$
\begin{aligned}
\mid\langle R^\ell_{\mu,\epsilon_0}\eta\,,\beta\rangle\mid &\le B_2e^{-\epsilon_0\lambda_\ell(G/K)}\Phi_0(a)\sup_i D_{\epsilon,i}e^{-(1-2\epsilon)\mathrm{Im}(\tau_i)||a||}\\
&\le D_\epsilon e^{-\epsilon_0\lambda_\ell(G/K)}\Phi_0(a)e^{-(1-2\epsilon)||a||\inf_i\{\mathrm{Im}(\tau_i)\}}\\
&\quad\text{(for some positive number } D_\epsilon \text{ depending on } \epsilon).
\end{aligned}
$$

The theorem follows by taking the limit $\epsilon_0\to 0$, since D_ϵ does not depend on ϵ_0, and by observing that

$$
\mathrm{Im}(\tau_i)=\sqrt{\frac{||\,\rho_i\,||^2-||\,\mathrm{char}(\delta_i)\,||^2-\mathrm{Re}(\mu)+\sqrt{\Big(||\,\rho_i\,||^2-||\,\mathrm{char}(\delta_i)\,||^2-\mathrm{Re}(\mu)\Big)^2+\mathrm{Im}(\mu)^2}}{2}}.
$$

□

5. $L^{2+\epsilon}$**-estimate for the resolvent of** $\widetilde{\Delta}_\ell$**.** We start with some properties of the resolvent. Let Γ be a torsion free discrete subgroup of G. The resolvent operator $R^\ell_\mu=(\Delta_\ell-\mu)^{-1}$ (resp. $\widetilde{R}^\ell_\nu=(\widetilde{\Delta}_\ell-\nu)^{-1}$), where μ (resp. ν) is a complex number in the resolvent set of Δ_ℓ (resp. $\widetilde{\Delta}_\ell$), is a kernel operator. Given a positive integer k, define (whenever the integrals converge) the maps

$$
(g_1,g_2)\in G\times G\mapsto \mathcal{R}^\ell_{\mu,k}(g_1,g_2)=\frac{1}{(k-1)!}\int_0^{+\infty} t^{k-1}e^{\mu t}p^\ell_t(g_1,g_2)dt
$$

and

$$
(\dot g_1,\dot g_2)\in\Gamma\backslash G\times\Gamma\backslash G\mapsto \widetilde{\mathcal{R}}^\ell_{\nu,k}(\dot g_1,\dot g_2)=\frac{1}{(k-1)!}\int_0^{+\infty} t^{k-1}e^{\nu t}\widetilde{p}^\ell_t(\dot g_1,\dot g_2)dt.
$$

When $k=1$, we shall simply write $\mathcal{R}^\ell_\mu$ and $\widetilde{\mathcal{R}}^\ell_\nu$ in stead of $\mathcal{R}^\ell_{\mu,1}$ and $\widetilde{\mathcal{R}}^\ell_{\nu,1}$ so that

$$
\widetilde{\mathcal{R}}^\ell_\nu(\dot g_1,\dot g_2)=\sum_{\gamma\in\Gamma}\mathcal{R}^\ell_\nu(g_1^{-1}\gamma g_2).
$$

In the case of functions, i.e when $\ell=0$, write S_μ for the resolvent $(\Delta_0-\mu)^{-1}$ of the Laplacian Δ_0 on G/K. The following estimate of $\mathcal{S}_\mu$, for $||\,g\,||>1$ and μ real such that $0\le\mu<||\,\rho_{\mathfrak{a}_0}\,||^2$ is due to Anker and Ji (Theorem 4.2.2 in [3]):

$$
\mathcal{S}_\mu(g)\asymp C\mu\Phi_0(g)e^{-||g||\sqrt{||\rho_{\mathfrak{a}_0}||^2-\mu}} \qquad (5.1)
$$

for some positive constant C_μ. Here $f_1 \asymp f_2$ means that there exist two real numbers C and C' such that $0 < C \leq \dfrac{f_1(g)}{f_2(g)} \leq C'$ for $\| g \| > 1$. We will denote by $\widetilde{S}_\nu$ the resolvent $(\widetilde{\Delta}_0 - \nu)^{-1}$ of the Laplacian $\widetilde{\Delta}_0$ on $\Gamma\backslash G/K$ so that

$$S_\nu(\dot{g}_1, \dot{g}_2) = \sum_{\gamma\in\Gamma} S_\nu(g_1^{-1}\gamma g_2).$$

The following two propositions are well known, however we were not able to find a precise reference for their proofs. Therefore, for the convenience of the reader, we shall provide a proof.

Proposition 5.2. *(1) Assume k is a positive integer and μ a negative real number. Let $1 \leq p \leq +\infty$ be an integer and write p' for the conjugate of p, i.e $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have*

(i) $\mathcal{R}^\ell_{\mu,k}(g,\cdot)$, for all $g \in G$, belongs to $L^p(G/K, \mathcal{E}nd(\Lambda^\ell\mathfrak{s}))$ outside any ball centered at g with finite radius, provided $\mu + \alpha_\ell - \dfrac{\| \rho_{\mathfrak{a}_0} \|^2}{p'} < 0$.

(ii) If $\dim_{\mathbf{R}}(G/K) \geq 2kp'$, then $\mathcal{R}^\ell_{\mu,k}(g,\cdot)$ belongs to $L^p(G/K, \mathcal{E}nd(\Lambda^\ell\mathfrak{s}))$.

(2) Assume μ is a complex number such that: $Im(\mu) \neq 0$ or $Re(\mu) < \lambda_\ell(G/K)$. Then, outside any ball centered at the origin with finite radius, we have

(i) $\mathcal{R}^\ell_\mu(\cdot, e)$ belongs to $C^\infty(G/K, \mathcal{E}nd(\Lambda^\ell\mathfrak{s}))$,

(ii) there exist a positive real number C_μ and a continuous function ϕ_μ, both depending on μ, such that for all $g \in G$ satisfying $0 < \| g \| < 1$:

$$\| \mathcal{R}^\ell_\mu(g,e) \|_{End(\Lambda^\ell\mathfrak{s})} \leq \frac{C_\mu}{\| g \|^{\dim_{\mathbf{R}}(G/K)-2}} + \phi_\mu(g), \quad \text{if } \dim_{\mathbf{R}}(G/K) \geq 3$$
$$\| \mathcal{R}^\ell_\mu(g,e) \|_{End(\Lambda^\ell\mathfrak{s})} \leq C_\mu \log(\| g \|) + \phi_\mu(g), \quad \text{if } \dim_{\mathbf{R}}(G/K) = 2.$$

Proof. Throughout the proof A_j will denote a positive real number. Let us start with the large time behavior. We have

$$\Big\| \int_1^{+\infty} t^{k-1} e^{\mu t} p^\ell_t(g) dt \Big\|_{\mathrm{End}(\Lambda^\ell\mathfrak{s})} \leq \int_1^{+\infty} t^{k-1} e^{\mu t} \| p^\ell_t(g) \|_{\mathrm{End}(\Lambda^\ell\mathfrak{s})}\, dt$$
$$\leq A_1 \int_1^{+\infty} t^{k-1} e^{(\mu+\alpha_\ell)t} h_t(g) dt \ \text{ by (2.38)}.$$

Next, one has

$$\| h_t \|_{L^1(G/K)} = 1$$
$$\| h_t \|_{L^\infty(G/K)} \leq A_2 t^{-\frac{1}{2}\mathrm{rk}_{\mathbf{R}}(G) - |\Sigma^{++}|} e^{-\|\rho_{\mathfrak{a}_0}\|^2 t} \quad \text{by (2.37)}.$$

Writing

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{\infty}$$

for $\theta \in [0,1]$, we deduce, by interpolation, that

$$\| h_t \|_{L^p(G/K)} \leq \| h_t \|^\theta_{L^\infty(G/K)} = \| h_t \|^{\frac{1}{p'}}_{L^\infty(G/K)} \leq A_3 t^{-\frac{\mathrm{rk}_{\mathbf{R}}(G)+2|\Sigma^{++}|}{2p'}} e^{-\frac{\|\rho_{\mathfrak{a}_0}\|^2}{p'} t}.$$

Therefore

$$\|\int_1^{+\infty} t^{k-1}e^{\mu t}p_t^\ell dt\,\|_{L^p(G/K,\mathcal{E}nd(\Lambda^\ell\mathfrak{s}))}\leq A_3\int_1^{+\infty} t^{k-1-\frac{\mathrm{rk}_{\mathbf{R}}(G)+2|\Sigma^{++}|}{2p'}}e^{(\mu+\alpha_\ell-\frac{\|\rho_{\mathfrak{a}_0}\|^2}{p'})t}dt$$

is finite if $\mu+\alpha_\ell-\dfrac{\|\rho_{\mathfrak{a}_0}\|^2}{p'}<0$ or if $\mu+\alpha_\ell-\dfrac{\|\rho_{\mathfrak{a}_0}\|^2}{p'}=0$ and $\mathrm{rk}_{\mathbf{R}}(G)+2\mid\Sigma^{++}\mid>2kp'$. Observe that the function

$$g\mapsto\int_1^{+\infty} t^{k-1}e^{(\mu+\alpha_\ell)t}h_t(g)dt$$

is bounded and continuous provided $\mu+\alpha_\ell-\|\rho_{\mathfrak{a}_0}\|^2<0$, which is the case if $\mu+\alpha_\ell-\dfrac{\|\rho_{\mathfrak{a}_0}\|^2}{p'}<0$.

Similarly, we have

$$\begin{aligned}\|\int_0^1 t^{k-1}e^{\mu t}p_t^\ell(g)dt\,\|_{\mathrm{End}(\Lambda^\ell\mathfrak{s})} &\leq \int_0^1 t^{k-1}e^{\mu t}\,\|\,p_t^\ell(g)\,\|_{\mathrm{End}(\Lambda^\ell\mathfrak{s})}\,dt\\ &\leq A_4\int_0^1 t^{k-1}e^{(\mu+\alpha_\ell)t}h_t(g)dt \ \text{ by (2.38)}\\ &\leq A_5\int_0^1 t^{k-1-\frac12\dim_{\mathbf{R}}(G/K)}e^{-\frac{\|g\|^2}{4t}}dt \ \text{ by (2.36)}\end{aligned}\tag{5.3}$$

and, by interpolation,

$$\|\int_0^1 t^{k-1}e^{\mu t}p_t^\ell dt\,\|_{L^p(G/K,\mathcal{E}nd(\Lambda^\ell\mathfrak{s}))}\leq A_6\int_0^1 t^{k-1-\frac{1}{2p'}\dim_{\mathbf{R}}(G/K)}dt$$

which is finite if $\dim_{\mathbf{R}}(G/K)<2kp'$. This proves (1)(i) and (1)(ii).

We now turn to (2). We shall use the previous assertion. Fix a real number μ_0 such that $\mu_0+\alpha_\ell<0$ and $\mathcal{R}^\ell_{\mu_0}(\cdot,e)$ is smooth outside any ball centered at the origin with finite radius. Iterating the basic relation

$$R^\ell_\mu=R^\ell_{\mu_0}+(\mu-\mu_0)R^\ell_{\mu_0}\circ R^\ell_\mu,$$

we deduce that for all positive integer N

$$R^\ell_\mu=(\mu-\mu_0)^N R^\ell_{\mu_0,N}\circ R^\ell_\mu+\sum_{j=1}^N(\mu-\mu_0)^j R^\ell_{\mu_0,j}.\tag{5.4}$$

Let $\{X_i\}$ be a basis of $\mathfrak{g}_0$. Choose N even and sufficiently large so that both $\mathcal{R}^\ell_{\mu_0,\frac N2}(\cdot,e)$ and $L(X_j)^q\mathcal{R}^\ell_{\mu_0,\frac N2}(\cdot,e)$ are L^2 on any ball centered at the origin with finite radius. Then, since R^ℓ_μ is a bounded operator on $L^2(G/K,\Lambda^\ell\mathfrak{s})$, we see that the convolution product $\mathcal{R}^\ell_{\mu_0,\frac N2}(\cdot,e)\star\mathcal{R}^\ell_\mu(\cdot,e)$ is square integrable on any ball centered at the origin with finite radius, so that

$$L(X_j)^q\big(\mathcal{R}^\ell_{\mu_0,N}(\cdot,e)\star\mathcal{R}^\ell_\mu(\cdot,e)\big)=\big(L(X_j)^q\mathcal{R}^\ell_{\mu_0,\frac N2}(\cdot,e)\big)\star\big(\mathcal{R}^\ell_{\mu_0,\frac N2}\star\mathcal{R}^\ell_\mu(\cdot,e)\big)$$

is continuous on such neighborhoods. The identity (5.4) implies that $L(X_j)^q\mathcal{R}^\ell_\mu(\cdot,e)$ is continuous which proves (2)(i). Finally, from (5.3), we see that

$$\begin{aligned}
&|| \int_0^1 t^{k-1}e^{\mu t}p_t^\ell(g)dt ||_{\mathrm{End}(\Lambda^\ell\mathfrak{s})}\\
&\leq A_6\int_0^1 t^{k-1-\frac{1}{2p'}\dim_{\mathbf{R}}(G/K)}e^{-\frac{||g||^2}{4p't}}dt\\
&\leq A_7(p)\,||\,g\,||^{2k-1-\dim_{\mathbf{R}}(G/K)} \quad \text{if } \dim_{\mathbf{R}}(G/K)\geq 2k+1
\end{aligned}$$

for some positive real number $A_8(p)$ depending on p (after the change of variable $u=\frac{||g||^2}{4p't}$). Now (2)(ii) follows by taking $k=1$, with the obvious modification if $\dim_{\mathbf{R}}(G/K)=2$. □

Proposition 5.5. *(1) Assume that k and p are positive integers with $k>\dfrac{1}{4}\dim_{\mathbf{R}}(G/K)$, and μ is a real number satisfying $\mu+\alpha_\ell<0$. Let g be an element of G. Then, outside any ball centered at $\dot g$ with finite radius, $\widetilde{\mathcal{R}}^\ell_{\mu,k}(\dot g,\cdot)$ belongs to $L^p(\Gamma\backslash G/K,\mathcal{E}nd(\Lambda^\ell\mathfrak{s}))$ and, there exists a positive real number $A(g)$ depending on g such that*

$$|| \widetilde{\mathcal{R}}^\ell_{\mu,k}(\dot g,\cdot) ||_{L^p(\Gamma\backslash G/K,\mathcal{E}nd(\Lambda^\ell\mathfrak{s}))}\leq A(g)\Big(\inf\Big\{1;\frac{\delta(\dot g)}{2}\Big\}\Big)^{k-1-(\frac{p-1}{2p})(\dim_{\mathbf{R}}(G/K)-1)}.$$

where $\delta(\dot g)$ denotes the injectivity radius of $\Gamma\backslash G/K$ at $\dot g$.

(2) Assume $q\geq 1$ and μ is a complex number satisfying $Im(\mu)\neq 0$ or $Re(\mu)<\beta_\ell(\Gamma\backslash G/K)$. Then

(i) $\widetilde{\mathcal{R}}^\ell_{\mu,q}(\dot g,\cdot)$ is well defined, and
(ii) $\widetilde{\mathcal{R}}^\ell_{\mu,q}(\dot g,\cdot)$ belongs to $L^2(\Gamma\backslash G/K,\mathcal{E}nd(\Lambda^\ell\mathfrak{s}))$.

Proof. Throughout the proof A_j will denote a positive real number. First we have:

$$\begin{aligned}
&|| \widetilde{\mathcal{R}}^\ell_{\mu,k}(\dot g_1,\cdot) ||_{L^p(\Gamma\backslash G/K,\mathcal{E}nd(\Lambda^\ell\mathfrak{s}))}\\
&\leq \int_0^{+\infty} t^{k-1}e^{\mu t}\,||\,\widetilde{p}^\ell_t(\dot g_1,\cdot)\,||_{L^p(\Gamma\backslash G/K,\mathcal{E}nd(\Lambda^\ell\mathfrak{s}))}\,dt\\
&\leq A_1\int_0^{+\infty} t^{k-1}e^{(\mu+\alpha_\ell)t}\,||\,\widetilde{h}_t(\dot g_1,\cdot)\,||_{L^p(\Gamma\backslash G)}\,dt \text{ by (2.38).}
\end{aligned}$$

Next, by the semigroup property of the heat kernel, we have

$$\begin{aligned}
\|\, \widetilde{h}_t(\dot g_1, \cdot)\, \|_{L^1(\Gamma\backslash G/K)} &= \int_{\Gamma\backslash G} \widetilde{h}_t(\dot g_1, \dot g_2) d\nu(\dot g_2) \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} h_t(g_1^{-1}\gamma g_2) d\nu(\dot g_2) \\
&= \int_G h_t(g_1, g_2) dg_2 \\
&= 1 \\
\|\, \widetilde{h}_t(\dot g_1, \cdot)\, \|_{L^\infty(\Gamma\backslash G/K)} &= \sup_{\dot g_2} \widetilde{h}_t(\dot g_1, \dot g_2) \\
&= \widetilde{h}_t(\dot g_1, \dot g_1) \\
&= \|\, \widetilde{h}_{\frac{t}{2}}(\dot g_1, \cdot)\, \|^2_{L^2(\Gamma\backslash G/K)} \\
&\le \|\, \widetilde{H}_{\frac{t}{2}-\frac{1}{2}}\, \|^2_{L^2\to L^2} \|\, \widetilde{h}_{\frac{1}{2}}(\dot g_1, \cdot)\, \|^2_{L^2(\Gamma\backslash G/K)} \quad \text{for } t > 1.
\end{aligned}$$

Writing $p \ge 1$ as

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{\infty} \ \text{ for } \ \theta \in [0, 1[,$$

we deduce that

$$\begin{aligned}
\|\, \widetilde{h}_t(\dot g_1, \cdot)\, \|_{L^p(\Gamma\backslash G/K)} &\le \widetilde{h}^\theta_t(\dot g_1, \dot g_1) \\
&\le e^{-\theta\beta_0(\Gamma\backslash G/K)t} \widetilde{h}^\theta_1(\dot g_1, \dot g_1) \quad \text{if } t > 1
\end{aligned}$$

so that

$$\|\, \widetilde{\mathcal{R}}^k_\mu(\dot g_1, \cdot)\, \|_{L^p(\Gamma\backslash G/K, \mathcal{E}nd(\Lambda^\ell\mathfrak{s}))} \le A_1 \int_0^{+\infty} t^{k-1} e^{(\mu+\alpha_\ell)t} \widetilde{h}^\theta_t(\dot g_1, \dot g_1) dt.$$

Hence we have

$$\begin{aligned}
&\int_0^{+\infty} t^{k-1} e^{(\mu+\alpha_\ell-\theta\beta_0(\Gamma\backslash G/K))t} \widetilde{h}^\theta_1(\dot g_1, \dot g_1) dt \\
&\le A_2 \int_0^{\inf\{1; \frac{\delta(\dot g_1)}{2}\}} t^{k-1-\frac{\theta}{2}(\dim_{\mathbf{R}}(G/K)+1)} dt \\
&\qquad \text{(by Theorem 6 of [15])} \\
&\le A_3 \Big(\inf \Big\{ 1; \frac{\delta(\dot g_1)}{2} \Big\} \Big)^{k-1-(\frac{p-1}{2p})(\dim_{\mathbf{R}}(G/K)+1)} \\
&\qquad \Big(\text{since } k \ge \frac{1}{4} \dim_{\mathbf{R}}(G/K) + 1\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\inf\{1; \frac{\delta(\dot g_1)}{2}\}}^{+\infty} t^{k-1} e^{(\mu+\alpha_\ell)t} \widetilde{h}^\theta_t(\dot g_1, \dot g_1) dt \\
&\le A_4 \int_{\inf\{1; \frac{\delta(\dot g_1)}{2}\}}^{+\infty} t^{k-1} e^{(\mu+\alpha_\ell)t} \|\, \widetilde{H}_{\frac{t}{2}-\frac{1}{2}}\, \|^{2\theta}_{L^2\to L^2} \|\, \widetilde{h}_{\frac{1}{2}}(\dot g_1, \cdot)\, \|^{2\theta}_{L^2(\Gamma\backslash G/K)}\, dt \\
&\le A_5(g_1) \quad \text{a positive real number depending on } \ g_1
\end{aligned}$$

which proves (1). For (2), we shall first consider the case where $q = 1$. If μ_0 is a real number satisfying $\mu_0 + \alpha_\ell < 0$, then

$$\begin{aligned} || \int_{\Gamma\backslash G} \widetilde{\mathcal{R}}^\ell_{\mu_0}(\dot g_1, \dot g_2) f(\dot g_2) d\dot g_2 \,||_{\mathrm{End}(\Lambda^\ell \mathfrak{s})} &= || \int_{\Gamma\backslash G} \widetilde{\mathcal{R}}^\ell_{\mu_0,a}(\dot g_1, \dot g_2)(\widetilde{\Delta}_{\ell,\dot g_2} - \mu)^{a-1} f(\dot g_2) d\dot g_2 \,|| \\ &\leq C(f) \,||\, \widetilde{\mathcal{R}}^\ell_{\mu_0,a}(\dot g_1, \cdot) \,||_{L^2(\Gamma\backslash G/K, \Lambda^\ell \mathfrak{s})} \end{aligned}$$

for some positive constant depending on f. Choosing an integer $a \geq 1 + \frac14 \dim_{\mathbf{R}}(G/K)$, we deduce from (1) that $\widetilde{\mathcal{R}}^\ell_{\mu_0}(\dot g_1, \cdot)$ is well defined outside any ball centered at $\dot g$ with finite radius. The same argument shows that $\widetilde{\mathcal{R}}^\ell_{\mu_0,k}$ is well defined on such neighborhoods for all $k \geq 1$. Next similarly to (5.4), we have

$$\widetilde{R}^\ell_\mu = (\mu - \mu_0)^N \widetilde{R}^\ell_{\mu_0,N} \circ \widetilde{R}^\ell_\mu + \sum_{j=1}^N (\mu - \mu_0)^j \widetilde{R}^\ell_{\mu_0,j}. \tag{5.6}$$

On the other hand, by the semigroup property of the heat kernel, one has

$$\begin{aligned} \widetilde{\mathcal{R}}^\ell_\mu(\dot g_2, \cdot) \widetilde{\mathcal{R}}^\ell_{\mu_0,N}(\dot g_1, \cdot) &= \int_{\Gamma\backslash G} \widetilde{\mathcal{R}}^\ell_{\mu_0,N}(\dot g_1, \dot g_3) \circ \widetilde{\mathcal{R}}^\ell_\mu(\dot g_3, \dot g_2) d\dot g_3 \\ &= \int_{\Gamma\backslash G} \widetilde{\mathcal{R}}^\ell_{\mu_0,\frac N2}(\dot g_1, \dot g_4) \int_{\Gamma\backslash G} \widetilde{\mathcal{R}}^\ell_\mu(\dot g_3, \dot g_2) \circ \widetilde{\mathcal{R}}^\ell_{\mu_0,\frac N2}(\dot g_4, \dot g_3) d\dot g_3 d\dot g_4. \end{aligned}$$

Fix $\dot g_4$ and choose an even integer N sufficiently large such that $\dot g_3 \mapsto \widetilde{\mathcal{R}}^\ell_{\mu_0,\frac N2}(\dot g_4, \dot g_3)$ belongs to $L^2(\Gamma\backslash G/K, \mathcal{E}nd(\Lambda^\ell \mathfrak{s}))$ (by (1)). Then

$$\int_{\Gamma\backslash G} \widetilde{\mathcal{R}}^\ell_\mu(\dot g_3, \dot g_2) \circ \widetilde{\mathcal{R}}^\ell_{\mu_0,\frac N2}(\dot g_4, \dot g_3) d\dot g_3 = \widetilde{\mathcal{R}}^\ell_\mu(\dot g_2, \cdot) \widetilde{\mathcal{R}}^\ell_{\mu_0,\frac N2}(\dot g_4, \cdot) \in L^2(\Gamma\backslash G/K, \mathcal{E}nd(\Lambda^\ell \mathfrak{s})),$$

since $\widetilde{R}^\ell_\mu$ is a bounded operator on $L^2(\Gamma\backslash G/K, \Lambda^\ell \mathfrak{s})$, and (5.6) implies that $\widetilde{\mathcal{R}}^\ell_\mu(\dot g_1, \cdot)$ is square integrable outside any ball centered at $\dot g$ with finite radius. Actually the same argument shows that $\dot g_2 \mapsto \widetilde{\Delta}^a_{\ell,\dot g_2}(\widetilde{\mathcal{R}}^1_\mu \circ \widetilde{\mathcal{R}}^N_{\mu_0})(\dot g_1, \dot g_2)$ belongs to $L^2(\Gamma\backslash G/K, \mathcal{E}nd(\Lambda^\ell \mathfrak{s}))$ for all integer $a \geq 1$, so that $\widetilde{\mathcal{R}}^\ell_\mu \circ \widetilde{\mathcal{R}}^\ell_{\mu_0,N}$ is C^∞ outside any ball centered at $\dot g$ with finite radius. Applying (5.6) we deduce that $\widetilde{\mathcal{R}}^\ell_\mu$ is C^∞ on such neighborhoods. This proves (2)(i)(ii) for $q = 1$. The case where $q \geq 2$ follows by induction from the following formula

$$\widetilde{R}^\ell_{\mu,q} = \sum_{j=0}^q C^j_q (\mu - \mu_0)^{Nj} \widetilde{R}^\ell_{\mu_0,Nj} \circ \widetilde{R}^\ell_{\mu,j} \circ \Big[\sum_{r=1}^N (\mu - \mu_0)^r \widetilde{R}^\ell_{\mu_0,r} \Big]^{q-j}. \tag{5.7}$$

□

REMARK 5.8. *If Γ is of finite covolume in G, we may assume that the injectivity radius is small enough so that Min$\Big\{1; \dfrac{\delta(\dot g_1)}{2}\Big\} = \dfrac{\delta(\dot g_1)}{2}$. In this case, the assertion (1) of the previous proposition can be restated as follows. $\widetilde{\mathcal{R}}^\ell_{\mu,k}$ is L^p outside the diagonal of $\Gamma\backslash G/K$ for all integers $k > \dfrac14 \dim_{\mathbf{R}}(G/K)$ and $p \geq 1$, and real numbers $\mu < -\alpha_\ell$. The assertion (2) may be restated accordingly.*

THEOREM 5.9. *Let G be a non compact connected semisimple real Lie group with finite center, K a maximal compact subgroup of G and Γ a torsion free discrete subgroup of G with finite covolume. Fix an element g in G. Then for all complex number μ with positive imaginary part, there exists a positive number ϵ such that $\widetilde{\mathcal{R}}^{\ell}_{\mu,k}(\dot{g},\cdot) \in L^{2+\epsilon}(\Gamma\backslash G/K, \mathcal{E}nd(\Lambda^{\ell}\mathfrak{s}))$, for all integer $k > \frac{1}{4}\dim_{\mathbf{R}}(G/K)$.*

Proof. Throughout the proof C_j will denote a positive real number.

Step 1: Let us decompose the Hilbert space $L^2(\Gamma\backslash G/K, \Lambda^{\ell}\mathfrak{s})$ as follows

$$L^2(\Gamma\backslash G/K, \Lambda^{\ell}\mathfrak{s}) = \operatorname{Ker}(\widetilde{\Delta}_{\ell}) \oplus \operatorname{Ker}(\widetilde{\Delta}_{\ell})^{\perp}$$

where $\operatorname{Ker}(\widetilde{\Delta}_{\ell})$ denotes the space of square integrable harmonic ℓ-forms and $\operatorname{Ker}(\widetilde{\Delta}_{\ell})^{\perp}$ its orthogonal complement. By a result of A. Borel and H. Garland [9], $\operatorname{Ker}(\widetilde{\Delta}_{\ell})$ is finite dimensional for all ℓ when Γ is of finite covolume. So we may write the orthogonal projection $T_{\ell} : L^2(\Gamma\backslash G/K, \Lambda^{\ell}\mathfrak{s}) \to \operatorname{Ker}(\widetilde{\Delta}_{\ell})$ on $\operatorname{Ker}(\widetilde{\Delta}_{\ell})$ as

$$T_{\ell} = \sum_j \langle \varphi_j\,, \cdot\rangle \varphi_j$$

where $\{\varphi_j\}$ is some orthonormal basis of $\operatorname{Ker}(\widetilde{\Delta}_{\ell})$.

Step 2: By Lemma 2 of [23], there exists a positive number ϵ_0 such that

$$\varphi_j \in L^{2+\epsilon}(\Gamma\backslash G/K, \Lambda^{\ell}\mathfrak{s}) \ \ \forall \epsilon \in [0, \epsilon_0[.$$

It turns out that T_{ℓ} is a bounded operator on $L^{2+\epsilon}(\Gamma\backslash G/K, \Lambda^{\ell}\mathfrak{s})$ for all $\epsilon \in [0, \epsilon_0[$. Indeed let $f \in L^{2+\epsilon}(\Gamma\backslash G/K, \Lambda^{\ell}\mathfrak{s})$ and write ϵ' for the conjugate $\frac{2+\epsilon}{1+\epsilon}$ of $2+\epsilon$. Since Γ has finite covolume in G, the Hölder inequality implies that

$$\varphi_j \in L^{q}(\Gamma\backslash G/K, \Lambda^{\ell}\mathfrak{s}) \ \ \forall q \in [1,2].$$

In particular, $\varphi_j \in L^{\epsilon'}(\Gamma\backslash G/K, \Lambda^{\ell}\mathfrak{s})$ and $\langle \varphi_j\,, f\rangle \in L^1(\Gamma\backslash G/K)$. Observe that T_{ℓ} is also a bounded operator on $L^{\epsilon'}(\Gamma\backslash G/K, \Lambda^{\ell}\mathfrak{s})$ by selfadjointness.

Step 3: Let w be a complex number with positive imaginary part. Fix a real number $c \in]0,1[$ and let B_c be the ball of radius c centered at the origin. We have

$$\int_{||g_0||>B_c} ||\ \mathcal{R}^{\ell}_{w}(g,g_0)\ ||_{\operatorname{End}(\Lambda^{\ell}\mathfrak{s})}\, dg_0$$

$$\leq b_c \int_{||a^+(g_0)||>B_c} \Phi_0(a^+(g_0)) e^{-(1-c)\tau_{w,\ell}(G/K)||a^+(g_0)||} dg_0 \text{ by (4.2)}$$

$$\leq C_1 \int_{||a^+(g_0)||>B_c} \Big(\Pi_{\alpha\in\Sigma^{++}}\big(1+\alpha(a^+(g_0))\big)\Big) e^{-\rho_{\mathfrak{a}_0}(a^+(g_0))} e^{-(1-c)\tau_{w,\ell}(G/K)||a^+(g_0)||} dg_0$$

by (2.19)

$$\leq C_2 \int_{\mathfrak{a}_0^+} \Big(\Pi_{\alpha\in\Sigma^{++}}\big(1+\alpha(X)\big)\Big) e^{\rho_{\mathfrak{a}_0}(X)-(1-c)\tau_{w,\ell}(G/K)||X||} dX$$

by integration formula (Prop. 5.28 of [20]).

The latter integral is finite if

$$\rho_{\mathfrak{a}_0}(X) < (1-c)\tau_{w,\ell}(G/K)\ ||\ X\ ||, \ \ \forall X \in \mathfrak{a}_0^+$$

$$\Leftrightarrow \tau_{w,\ell}(G/K) > \frac{1}{1-c}\rho_{\mathfrak{a}_0}(Y), \ \ \forall Y \in \mathfrak{a}_0^+,\ ||\ Y\ || = 1$$

$$\Leftrightarrow \lambda_{\ell}(G/K) - \operatorname{Re}(w) + \sqrt{\big(\lambda_{\ell}(G/K) - \operatorname{Re}(w)\big)^2 + \operatorname{Im}(w)^2} > \frac{2}{(1-c)^2}\rho^2_{max}$$

where ρ_{max} is the positive real number defined by

$$\rho_{max} = \sup\Big\{\rho_{\mathfrak{a}_0}(Y) \mid Y \in \mathfrak{a}_0^+,\ || Y ||= 1\Big\}.$$

Moreover, Proposition 5.2 (2)(ii) shows that the integral

$$\int_{||g_0||\leq B_c} || \mathcal{R}_w^\ell(g,g_0) ||_{\mathrm{End}(\Lambda^\ell\mathfrak{s})}\, dg_0$$

is finite for all complex number w with positive imaginary part. On the other hand, observe that for all $\phi \in L^2(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$, one has

$$\begin{aligned} || (\widetilde{R}_w^\ell\phi)(\dot g) ||_{\Lambda^\ell\mathfrak{s}} &= || \int_{\Gamma\backslash G} \widetilde{\mathcal{R}}_w^\ell(\dot g,\dot g_0)\phi(\dot g_0)d\dot g_0 ||_{\Lambda^\ell\mathfrak{s}} \\ &= || \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma}\mathcal{R}_w^\ell(g^{-1}\gamma g_0)\phi(\dot g_0)d\dot g_0 ||_{\Lambda^\ell\mathfrak{s}} \\ &\leq || \phi ||_{L^1(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})}\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma} || \mathcal{R}_w^\ell(g^{-1}\gamma g_0) ||_{\mathrm{End}(\Lambda^\ell\mathfrak{s})}\, d\dot g_0 \\ &= || \phi ||_{L^1(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})}\int_G || \mathcal{R}_w^\ell(g,g_0) ||_{\mathrm{End}(\Lambda^\ell\mathfrak{s})}\, dg_0. \end{aligned}$$

Thus $\widetilde{R}_w^\ell$ is a bounded operator on $L^1(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$ if

$$\mathrm{Im}(w)^2 > \Big(\frac{2}{(1-c)^2}\rho_{max}^2 - \big(\lambda_\ell(G/K) - \mathrm{Re}(w)\big)\Big)^2 - \Big(\lambda_\ell(G/K) - \mathrm{Re}(w)\Big)^2.$$

Recall that, by definition, $\widetilde{R}_w^\ell$ is a bounded operator on $L^2(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$ if $\mathrm{Im}(w) > 0$.

Step 4: Let e_0 be the smallest non zero eigenvalue of $\widetilde{\Delta}_\ell$ and z a complex number. Then $\widetilde{R}_{z+e_0}^\ell \circ \big(\mathbb{1} - T_\ell\big)$ is a bounded operator $L^2(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$ if $\mathrm{Im}(z) > 0$, and, from the previous step, $\widetilde{R}_{z+e_0}^\ell$ is bounded on $L^1(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$ if

$$\mathrm{Im}(z)^2 > \Big(\frac{2}{(1-c)^2}\rho_{max}^2 - \big(\lambda_\ell(G/K) - e_0 - \mathrm{Re}(z)\big)\Big)^2 - \Big(\lambda_\ell(G/K) - e_0 - \mathrm{Re}(z)\Big)^2.$$

Therefore, for all $f \in L^{\epsilon'}(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$, one has

$$\begin{aligned} || \widetilde{R}_{z+e_0}^\ell \circ \big(\mathbb{1} - T_\ell\big)f ||_{L^{\epsilon'}(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})} &\leq C_3\, || \big(\mathbb{1} - T_\ell\big)f ||_{L^{\epsilon'}(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})} \\ &\leq || f ||_{L^{\epsilon'}(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})} \end{aligned}$$

i.e $\widetilde{R}_{z+e_0}^\ell \circ \big(\mathbb{1} - T_\ell\big)$ is a bounded operator $L^{\epsilon'}(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$. Now, using the Stein interpolation theorem (Theorem V.4.1 in [35]), we deduce that $\widetilde{R}_{z+e_0}^\ell \circ \big(\mathbb{1} - T_\ell\big)$ is a bounded operator on $L^p(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$ for $\epsilon' \leq p \leq 2$ and $\mathrm{Im}(z)$ satisfying

$$\mathrm{Im}(z) > \theta\sqrt{\Big(\frac{2}{(1-c)^2}\rho_{max}^2 - \big(\lambda_\ell(G/K) - e_0 - \mathrm{Re}(z)\big)\Big)^2 - \Big(\lambda_\ell(G/K) - e_0 - \mathrm{Re}(z)\Big)^2}$$

where $\dfrac{1}{p} = \dfrac{1-\theta}{\epsilon'} + \dfrac{\theta}{2}$ with $\theta \in [0,1]$ and $c \in]0,1[$. Finally, observing that

$$\Big(\big(\widetilde{\Delta}_\ell - e_0 - z\big)^{-1} \circ T_\ell\Big)(f) = -\frac{1}{z+e_0}T_\ell(f)\ \ \forall f \in L^p(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$$

and writing

$$\big(\widetilde{\Delta}_\ell - e_0 - z\big)^{-1} = \big(\widetilde{\Delta}_\ell - e_0 - z\big)^{-1} \circ T_\ell + \big(\widetilde{\Delta}_\ell - e_0 - z\big)^{-1} \circ \big(\mathbb{1} - T_\ell\big),$$

we see that $\widetilde{R}^\ell_{z+e_0}$ is a bounded operator on $L^p(\Gamma\backslash G/K, \Lambda^\ell \mathfrak{s})$. In particular, $\widetilde{R}^\ell_{z+e_0}$ is bounded on $L^p(\Gamma\backslash G/K, \Lambda^\ell \mathfrak{s})$.

Step 5: Let μ_0 be a real number satisfying $\mu_0 + \alpha_\ell < 0$ and μ a complex number with $\mathrm{Im}(\mu) > 0$. Recall that, from formulas (5.4) and (5.7), one has

$$\widetilde{\mathcal{R}}^\ell_{\mu,k} = \sum_{j=0}^{k} C_k^j (\mu - \mu_0)^{Nj} \widetilde{\mathcal{R}}^\ell_{\mu_0,Nj} \circ \widetilde{\mathcal{R}}^\ell_{\mu,j} \circ \Big[\sum_{r=1}^{N} (\mu - \mu_0)^r \widetilde{\mathcal{R}}^\ell_{\mu_0,r}\Big]^{k-j}$$

where the generic term is of the form $\widetilde{\mathcal{R}}^\ell_{\mu_0,Nj} \circ \widetilde{\mathcal{R}}^\ell_{\mu,j} \circ \widetilde{\mathcal{R}}^\ell_{\mu_0,r(k-j)}$. Then it is enough to consider the case where $j = r = 1$. By the semigroup property, we have

$$\widetilde{\mathcal{R}}^\ell_{\mu}(\dot{g}_1, \cdot)\widetilde{\mathcal{R}}^\ell_{\mu_0,k+N-1}(\dot{g}_2, \cdot) = \int_{\Gamma\backslash G} \widetilde{\mathcal{R}}^\ell_{\mu}(\dot{g}_1, \dot{g}_3) \circ \widetilde{\mathcal{R}}^\ell_{\mu_0,k+N-1}(\dot{g}_3, \dot{g}_2) d\dot{g}_3.$$

But, by proposition 5.5(1), we know that $\widetilde{\mathcal{R}}^\ell_{\mu_0,k}(\dot{g}_1, \cdot)$ is L^q outside any ball centered at $\dot{g}_1$ with finite radius, for all $q \geq 2$. Therefore $\widetilde{\mathcal{R}}^\ell_{\mu,k}(\dot{g}_1, \cdot)$ is L^p on such neighborhoods if $\mathrm{Im}(\mu)$ satisfies the condition

$$\mathrm{Im}(\mu) > \theta \sqrt{\Big(\frac{2}{(1-c)^2}\rho^2_{max} - \big(\lambda_\ell(G/K) - \mathrm{Re}(\mu)\big)\Big)^2 - \Big(\lambda_\ell(G/K) - \mathrm{Re}(\mu)\Big)^2}.$$

Finally, in the case where

$$0 < \mathrm{Im}(\mu) \leq \theta \sqrt{\Big(\frac{2}{(1-c)^2}\rho^2_{max} - \big(\lambda_\ell(G/K) - \mathrm{Re}(\mu)\big)\Big)^2 - \Big(\lambda_\ell(G/K) - \mathrm{Re}(\mu)\Big)^2}$$

we apply the previous result, replacing θ by θ' with

$$0 < \theta' < \mathrm{Im}(\mu) \Big[\Big(\frac{2}{(1-c)^2}\rho^2_{max} - \big(\lambda_\ell(G/K) - \mathrm{Re}(\mu)\big)\Big)^2 - \Big(\lambda_\ell(G/K) - \mathrm{Re}(\mu)\Big)^2\Big]^{-\frac{1}{2}}$$

and $\dfrac{1}{p} = \dfrac{1-\theta'}{\epsilon'} + \dfrac{\theta'}{2}$. □

Remark 5.10. *Using the same argument as above, one can show that the resolvent R^ℓ_μ of Δ_ℓ is a bounded operator on $L^q(G/K, \Lambda^\ell \mathfrak{s})$ for $q \geq 2$ and*

$$Im(\mu) > \theta \sqrt{\Big(\frac{2}{(1-c)^2}\rho^2_{max} - \big(\lambda_\ell(G/K) - Re(\mu)\big)\Big)^2 - \Big(\lambda_\ell(G/K) - Re(\mu)\Big)^2}.$$

where $q = \dfrac{2}{1-\theta}$ with $\theta \in [0, 1[$ and $c \in]0, 1[$.

6. Lower bounds for the bottom of the spectrum of $\widetilde{\Delta}_\ell$ and L^2-cohomology of $\Gamma\backslash G/K$.

THEOREM 6.1. *Let G be a non compact connected semisimple real Lie group with finite center, K a maximal compact subgroup of G and Γ a torsion free discrete subgroup of G. We assume that Γ is of infinite covolume in G and that the bottom of the spectrum of Δ_ℓ does not vanish. Then we have*

(i) if $\delta(\Gamma) \leq \rho_{min}$ then $\beta_\ell(\Gamma\backslash G/K) \geq \lambda_\ell(G/K)$,

(ii) if $\rho_{min} \leq \delta(\Gamma) \leq \| \rho_{\mathfrak{a}_0} \| + \sqrt{\lambda_\ell(G/K)}$ then $\beta_\ell(\Gamma\backslash G/K) \geq \lambda_\ell(G/K) - \big(\delta(\Gamma) - \rho_{min}\big)^2$, and

(iii) if $| \delta(\Gamma) - \rho_{min} | \leq \| \rho_{\mathfrak{a}_0} \| < \delta(\Gamma)$ and $\lambda_\ell(G/K) \geq \big(\delta(\Gamma) - \rho_{min}\big)^2$ then $\beta_\ell(\Gamma\backslash G/K) \geq \lambda_\ell(G/K) - \big(\delta(\Gamma) - \rho_{min}\big)^2$.

Proof. Throughout the proof the symbols B_j will denote positive real numbers. Let θ be a smooth function on G defined by $\theta(g) = 1$ for $\| g \| \leq 1$ and $\theta(g) = 0$ for $\| g \|$ sufficiently large. For a real number $\mu < \lambda_\ell(G/K)$, decompose the kernel $\mathcal{R}^\ell_\mu$ of the resolvent $(\Delta_\ell - \mu)^{-1}$ as follows

$$\mathcal{R}^\ell_\mu = \mathcal{R}^{\ell,1}_\mu + \mathcal{R}^{\ell,2}_\mu$$

where $\mathcal{R}^{\ell,1}_\mu = \theta\mathcal{R}^\ell_\mu$ and $\mathcal{R}^{\ell,2}_\mu = (1-\theta)\mathcal{R}^\ell_\mu$. Accordingly the kernel $\widetilde{\mathcal{R}}^\ell_\mu$ of the resolvent $(\widetilde{\Delta}_\ell - \mu)^{-1}$ decomposes as

$$\widetilde{\mathcal{R}}^\ell_\mu = \widetilde{\mathcal{R}}^{\ell,1}_\mu + \widetilde{\mathcal{R}}^{\ell,2}_\mu. \tag{6.2}$$

Now we have, using (2.39)

$$\begin{aligned}\int_{\Gamma\backslash G} \| \widetilde{\mathcal{R}}^{\ell,1}_\mu(\dot{g}_1, \dot{g}_2) \|_{\mathrm{End}(\Lambda^\ell\mathfrak{s})} \, d\dot{g}_2 &= \int_{\Gamma\backslash G} \| \sum_{\gamma\in\Gamma} \mathcal{R}^{\ell,1}_\mu(g_1^{-1}\gamma g_2) \|_{\mathrm{End}(\Lambda^\ell\mathfrak{s})} \, d\dot{g}_2 \\ &= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \| \mathcal{R}^{\ell,1}_\mu(g_1^{-1}\gamma g_2) \|_{\mathrm{End}(\Lambda^\ell\mathfrak{s})} \, d\dot{g}_2 \\ &= \int_G \| \mathcal{R}^{\ell,1}_\mu(g_1, g_2) \|_{\mathrm{End}(\Lambda^\ell\mathfrak{s})} \, dg_2.\end{aligned}$$

Since $\mathcal{R}^{\ell,1}_\mu$ is integrable on the unit ball of G, by Proposition 5.2 (2)(ii), we deduce that, for all ϕ in $L^2(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$

$$\| \widetilde{R}^{\ell,1}_\mu \phi \|_{L^2(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})} \leq B_1 \| \phi \|_{L^2(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})} \, . \tag{6.3}$$

Next choose a real number μ_ϵ such that

$$\| \rho_{\mathfrak{a}_0} \|^2 - (1-\epsilon)^2(\lambda_\ell(G/K) - \mu) \leq \mu_\epsilon \leq \| \rho_{\mathfrak{a}_0} \|^2$$

for some $\epsilon \in]0, 1[$. Then, combining Theorem 4.1 and (5.1), there exists a positive constant C_{μ_ϵ}, depending on ϵ, such that

$$\mathcal{R}^{\ell,2}_\mu(g, \cdot) \leq C_{\mu_\epsilon}\mathcal{S}_{\mu_\epsilon}(g, \cdot) \ \text{ for } \ \| g \| \ \text{ sufficiently large.}$$

In particular we have

$$\begin{aligned}\widetilde{\mathcal{R}}^{\ell,2}_\mu(\dot{g}_1, \dot{g}_2) &= \sum_{\gamma\in\Gamma} \mathcal{R}^{\ell,2}_\mu(g_1^{-1}\gamma g_2) \\ &\leq C_{\mu_\epsilon} \sum_{\gamma\in\Gamma} \mathcal{S}_{\mu_\epsilon}(g_1^{-1}\gamma g_2) \\ &= C_{\mu_\epsilon}\widetilde{\mathcal{S}}_{\mu_\epsilon}(\dot{g}_1, \dot{g}_2)\end{aligned}$$

so that

$$\| (\widetilde{R}_\mu^{\ell,2}\phi)(\dot{g}_1) \|_{\Lambda^\ell\mathfrak{s}} = \| \int_{\Gamma\backslash G} \widetilde{\mathcal{R}}_\mu^{\ell,2}(\dot{g}_1,\dot{g}_2)\phi(\dot{g}_2)d\nu(\dot{g}_2) \|_{\Lambda^\ell\mathfrak{s}}$$
$$\leq C_{\mu_\epsilon} \int_{\Gamma\backslash G} \widetilde{\mathcal{S}}_{\mu_\epsilon}(\dot{g}_1,\dot{g}_2) \| \phi(\dot{g}_2) \|_{\Lambda^\ell\mathfrak{s}} \, d\nu(\dot{g}_2).$$

Now, by the assertion (i) of Leuzinger's theorem (see Section 2.11), we know that $\widetilde{S}_{\mu_\epsilon}$ is bounded on $L^2(\Gamma\backslash G/K)$, so that

$$\| \widetilde{R}_\mu^{\ell,2}\phi \|_{L^2(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})} \leq B_2 \| \phi \|_{L^2(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})} . \tag{6.4}$$

Finally, we deduce, from (6.2), (6.3) and (6.4), that R_μ^ℓ is bounded on $L^2(\Gamma\backslash G/K, \Lambda^\ell\mathfrak{s})$

$$\| R_\mu^\ell\phi \|_{L^2(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})} \leq B_3 \| \phi \|_{L^2(\Gamma\backslash G/K,\Lambda^\ell\mathfrak{s})} .$$

Therefore $\beta_\ell(\Gamma\backslash G/K) \geq \lambda_\ell(G/K)$, which proves (i). The proof of (ii) and (iii) is very similar.

For (ii) we choose $\mu < \lambda_\ell(G/K) - \big(\delta(\Gamma) - \rho_{min}\big)^2$ and

$$\| \rho_{\mathfrak{a}_0} \|^2 - (1-\epsilon)^2(\lambda_\ell(G/K) - \mu) \leq \mu_\epsilon \leq \| \rho_{\mathfrak{a}_0} \|^2 - \big(\delta(\Gamma) - \rho_{min}\big)^2.$$

For (iii) we choose $\mu < \lambda_\ell(G/K) - \big(\delta(\Gamma) - \rho_{min}\big)^2$ and

$$\| \rho_{\mathfrak{a}_0} \|^2 - (1-\epsilon)^2(\lambda_\ell(G/K) - \mu) \leq \mu_\epsilon \leq \sup\Big\{0; \| \rho_{\mathfrak{a}_0} \|^2 - \big(\delta(\Gamma) - \rho_{min}\big)^2\Big\}.$$

□

COROLLARY 6.5. *Under the assumptions of the previous theorem, the (reduced or unreduced) L^2-cohomology group of degree ℓ of $\Gamma\backslash G/K$ vanishes in the following cases:*

(i) $\delta(\Gamma) \leq \rho_{min}$,
(ii) $\rho_{min} \leq \delta(\Gamma) \leq \| \rho_{\mathfrak{a}_0} \| + \sqrt{\lambda_\ell(G/K)}$ *and* $\sqrt{\lambda_\ell(G/K)} > \delta(\Gamma) - \rho_{min}$, *and*
(iii) $| \delta(\Gamma) - \rho_{min} | \leq \| \rho_{\mathfrak{a}_0} \| < \delta(\Gamma)$ *and* $\sqrt{\lambda_\ell(G/K)} > | \delta(\Gamma) - \rho_{min} |$.

In particular, in these cases, the kernel of $\widetilde{\Delta}_\ell$ is reduced to $\{0\}$.

Proof. We deduce from the previous theorem that, in each case (i)-(ii)-(iii), any square integrable closed ℓ-form on $\Gamma\backslash G/K$ is exact. In other words, the unreduced L^2-cohomology group $H^{(\ell)}(\Gamma\backslash G/K)$ of degree ℓ is trivial, and therefore the reduced L^2-cohomology group $\overline{H}^{(\ell)}(\Gamma\backslash G/K)$ vanishes as well. □

Finally, to sum up, we can say that using algebraic and analytic tools from representation theory of semisimple Lie groups, we obtained estimates for large time behavior of the heat kernel for dfferential forms on symmetric spaces of the type G/K, where G is a non compact connected semisimple Lie group with finite center and K a maximal compact subgroup of G (Theorem 3.1). Then, combining these estimates with some techniques from the theory of special functions, we deduced estimates for the resolvent of the form Laplacian on G/K (Theorem 4.1). As a byproduct, we obtained $L^{2+\epsilon}$-estimates for the resolvent of the form Laplacian $\widetilde{\Delta}_\ell$ on locally symmetric spaces $\Gamma\backslash G/K$ when Γ is a torsion free discrete subgroup of G with finite covolume

(Theorem 5.9). The latter estimates play an important role in the theory of "Eisenstein transforms" and the Langlands' decomposition of $L^2(\Gamma\backslash G/K)$ (see Theorems 4.2 and 4.7 in [30]). As an application of these $L^{2+\epsilon}$-estimates, we derived lower bounds for the bottom of the spectrum of $\widetilde{\Delta}_\ell$ when Γ has infinite covolume (Theorem 6.1) and a vanishing criterion for L^2-cohomology of $\Gamma\backslash G/K$ (Corollary 6.5). We also mention that, after our results were announced in [25], G. Carron posted a preprint on his webpage in which he proves, using techniques rather different from ours, analogous estimates for the Green kernel and the heat kernel for Laplacian-type operators on symmetric spaces [12].

Appendix A. On the computation of $\lambda_\ell(G/K)$.

A.1. The complex case. Let G be a connected complex semisimple Lie group and K a compact real form of G. If $\mathfrak{b}_0$ is a maximal abelian subspace in the Lie algebra $\mathfrak{k}_0$ of K then the complexification of $\mathfrak{b}_0+\sqrt{-1}\mathfrak{b}_0$ is a Cartan subalgebra of $\mathfrak{g}$. Write $\Delta=\Delta(\mathfrak{g},\mathfrak{h})$ for the set of $\mathfrak{g}$-roots relative to $\mathfrak{h}$. If we define the following real number

$$C_\ell(G)=\sup\{||< Q >_{\mathfrak{b}_0}||^2 \mid \ Q\subset\Delta,\ \sup\{0;\ell-\mathrm{rk}_{\mathbf{C}}(G)\}\leq\mid Q\mid\leq\ell\}$$

where $< Q >_{\mathfrak{b}_0}$ denotes the restriction to $\mathfrak{b}_0$ of the sum of elements in the subset Q and $\mid Q\mid$ is the number of elements in Q, then we have

$$\lambda_\ell(G/K)=\frac{1}{12}\dim_{\mathbf{R}}(G/K)-C_\ell(G).$$

A.2. The real case. We consider the Hermitian spaces $G/K = SO_e(2,n)/SO(2)\times SO(n)$ with $n\geq 2$. Recall from Section 2, if $\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}$ is the principal series representation associated with a cuspidal parabolic subgroup $P_i=M_iA_iN_i$ of G, δ_i a discrete series representation of M_i and ν_i a linear form on $\mathfrak{a}_{i,0}$, then the Casimir operator of G acts on the (smooth vectors of the) $\pi_{P_i,\delta_i,\sqrt{-1}\nu_i}$ as the scalar operator $\omega_{P_i,\delta_i,\sqrt{-1}\nu_i}\mathrm{Id}$ where $\omega_{P_i,\delta_i,\sqrt{-1}\nu_i}$ is the real number $||\ \mathrm{char}(\delta_i)\ ||^2-||\ \nu_i\ ||^2-||\ \rho_i\ ||^2$ defined by (2.4). In particular, to compute the numbers $\lambda_\ell(G/K)$, defined by (2.31), we may assume that $\nu_i=0$. If λ is the minimal $M_i\cap K$-type of δ_i then the infinitesimal character $\mathrm{char}(\delta_i)$ of δ_i is given by (see p. 310 of [20]):

$$\mathrm{char}(\delta_i)=\lambda-\rho(\mathfrak{m}_i)+2\rho(\mathfrak{m}_i\cap\mathfrak{k})$$

where $\rho(\mathfrak{m}_i)$ (resp. $\rho(\mathfrak{m}_i\cap\mathfrak{k})$) is the half sum of positive roots of $\mathfrak{m}_i$ (resp. $\mathfrak{m}_i\cap\mathfrak{k}$) relative to $\mathfrak{t}_i$. Here $\mathfrak{t}_i$ is a Cartan subalgebra of $\mathfrak{m}$ such that $\mathfrak{h}_i=\mathfrak{t}_i\oplus\mathfrak{a}_i$ is a Cartan subalgebra of $\mathfrak{g}$. Under the Cayley transform, $\mathfrak{h}_i$ becomes a compact Cartan subalgebra of $\mathfrak{g}$, and the roots in Δ_i transform accordingly (see p. 417 of [20]). Note that when M_i is compact then δ_i is just a highest weight representation with highest weight λ and infinitesimal character $\lambda+\rho(\mathfrak{m}_i)$.

1) $\underline{G/K=SO_e(2,2n)/SO(2)\times SO(2n),\ n\geq 1.}$
   By Hodge isomorphism $\lambda_\ell(SO_e(2,2n)/SO(2)\times SO(2n)) = \lambda_{4n-\ell}(SO_e(2,2n)/SO(2)\times SO(2n))$ for $0\leq\ell\leq 4n$, so we may restrict our attention to $0\leq\ell\leq 2n$.
   Up to a conjugacy, we need only to consider the minimal parabolic with $M_i = M = M\cap K\simeq SO(2n-2)$. Using branching laws for $SO(2n)\to SO(2n-1)$ and $SO(2n-1)\to SO(2n-2)$ (Theorems 8.1.3 and 8.1.4 of [17]) along with

the explicit decomposition of the isotropy representation for the Hermitian groups given in [31], one finds that

$n=1$: $\lambda_\ell(SO_e(2,2n)/SO(2)\times SO(2n)) = \dfrac{1}{4}$ for $\ell = 0,1,2$.

$n=2$: $\lambda_0(SO_e(2,2n)/SO(2)\times SO(2n)) = \dfrac{5}{8}$,

$\lambda_1(SO_e(2,2n)/SO(2)\times SO(2n)) = \dfrac{1}{2}$,

$\lambda_\ell(SO_e(2,2n)/SO(2)\times SO(2n)) = \dfrac{1}{8}$ for $\ell = 2,3,4$.

$n\geq 3$: $\lambda_0(SO_e(2,2n)/SO(2)\times SO(2n)) = \dfrac{2n^2-2n+1}{4n}$,

$\lambda_1(SO_e(2,2n)/SO(2)\times SO(2n)) = \dfrac{2n^2-4n+4}{4n}$,

$\lambda_2(SO_e(2,2n)/SO(2)\times SO(2n)) = \dfrac{2n^2-6n+5}{4n}$,

$\lambda_\ell(SO_e(2,2n)/SO(2)\times SO(2n)) = \dfrac{1}{4n}\Big(2n^2+\dfrac{1}{2}+\dfrac{\ell(\ell-4n)}{2}\Big)$ for $\ell\geq 3$ and ℓ odd,

$\lambda_\ell(SO_e(2,2n)/SO(2)\times SO(2n)) = \dfrac{1}{4n}\Big(2n^2+2n+1+\dfrac{\ell(\ell-4n-2)}{2}\Big)$ for $\ell\geq 4$ and ℓ even.

2) $\underline{G/K = SO_e(2,2n+1)/SO(2)\times SO(2n+1),\ n\geq 1.}$
Again $\lambda_\ell(SO_e(2,2n+1)/SO(2)\times SO(2n+1)) = \lambda_{4n+2-\ell}(SO_e(2,2n+1)/SO(2)\times SO(2n+1))$ for $0\leq \ell\leq 4n+2$, by Hodge isomorphism, so we may restrict our attention to $0\leq\ell\leq 2n+1$.
Now we need to consider two parabolic subgroups P_1 and P_2 with $M_1 = M_1\cap K \simeq SO(2n-1)$ and $M_2\simeq SO_e(1,2n)$, $M_2\cap K\simeq SO(2n)$. In particular, using branching laws for $SO(2n+1)\to SO(2n)$ and $SO(2n)\to SO(2n-1)$ (Theorems 8.1.3 and 8.1.4 of [17]), one finds that

$n=1$: $\lambda_0(SO_e(2,2n+1)/SO(2)\times SO(2n+1)) = \dfrac{3}{8}$,

$\lambda_\ell(SO_e(2,2n+1)/SO(2)\times SO(2n+1)) = \dfrac{1}{24}$ for $\ell = 1,2,3$.

$n\geq 2$: $\lambda_0(SO_e(2,2n+1)/SO(2)\times SO(2n+1)) = \dfrac{8n^2+1}{16n+8}$,

$\lambda_1(SO_e(2,2n+1)/SO(2)\times SO(2n+1)) = \dfrac{8n^2-8n+9}{16n+8}$,

$\lambda_2(SO_e(2,2n+1)/SO(2)\times SO(2n+1)) = \dfrac{8n^2-16n+9}{16n+8}$,

$\lambda_\ell(SO_e(2,2n+1)/SO(2)\times SO(2n+1)) = \dfrac{1}{4n+2}\Big(2n^2+\dfrac{1}{4}+\dfrac{\ell(\ell-4n)}{2}\Big)$ for $\ell\leq 2n$, ℓ even,

$$\lambda_\ell(SO_e(2,2n+1)/SO(2)\times SO(2n+1)) = \frac{1}{4n+2}\Big(2n^2+\frac{7}{4}+\frac{\ell(\ell-4n)}{2}\Big)$$
for $\ell < 2n-1$, ℓ odd,

$$\lambda_\ell(SO_e(2,2n+1)/SO(2)\times SO(2n+1)) = \frac{1}{16n+8} \text{ for } \ell = 2n-1, 2n+1.$$

REFERENCES

[1] P. Albuquerque, *Patterson-Sullivan theory in higher rank symmetric spaces*, Geom. Funct. Anal., 9 (1999), pp. 1–28.

[2] G. Alexopoulos et N. Lohoué, *On the large time behavior of heat kernels on Lie groups*, Duke Math. J., 120 (2003), pp. 311–351.

[3] J.-P. Anker and L. Ji, *Heat kernel and Green function estimates on noncompact symmetric spaces*, Geom. Funct. Anal., 9 (1999), pp. 1035–1091.

[4] I. G. Avramidi, *Heat Kernel and Quantum Gravity*, Springer Verlag, 2000.

[5] E. van den Ban and S. Souaifi, *A comparison of Paley-Wiener theorems for real reductive groups*, preprint.

[6] D. Barbasch and H. Moscovici, *L^2-index and the Selberg trace formula*, J. Funct. Analysis, 53 (1983), pp. 151–201.

[7] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Spinger Verlag, 1991.

[8] A. Borel, *The L^2-cohomology of negatively curved Riemannian symmetric spaces*, Ann. Acad. Scient. Fenn. Series A. I. Math., 10 (1985), pp. 95–105.

[9] A. Borel and H. Garland, *Laplacian and the discrete spectrum of an arithmetic group*, Amer. J. Math., 105 (1983), pp. 309–335.

[10] A. Borel and N. R. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Annals of mathematics studies, Princeton University Press, 1980.

[11] R. Camporesi, *The Helgason-Fourier transform for homogeneous vector bundles over Riemannian symmetric spaces*, Pacific J. Math., 179 (1997), pp. 263–300.

[12] G. Carron, *Estimée du noyau de Green et de la chaleur sur les espaces symétriques,* preprint 2007. Available at http://www.math.sciences.univ-nantes.fr/ carron/publication.html

[13] G. Carron and E. Pedon, *On the differential form spectrum of hyperbolic manifods*, Ann. Sc. Norm. Super. Pisa Cl. Sci., 3 (2004), pp. 705–747.

[14] T. Coulhon, L. Saloff-Coste and N. T. Varopoulos, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, Vol. 100, Cambridge University Press, 1992.

[15] S. Y. Cheng, P. Li, Peter and S. T. Yau, *On the upper estimate of the heat kernel of a complete Riemannian manifold*, Amer. J. Math., 103 (1981), pp. 1021–1063.

[16] P. Delorme, *Sur le théorème de Paley-Wiener d'Arthur*, Ann. of Math., 162 (2005), pp. 987–1029.

[17] R. Goodman and N. R. Wallach, *Representations and invariants of the classical groups*, Encyclopedia of Mathematics and its Applications, Vol. 68, Cambridge University Press, 1998.

[18] Harish-Chandra, *Harmonic analysis on semisimple Lie groups*, Bull. AMS, 76 (1970), pp. 529–551.

[19] S. Helgason, *Geometric analysis on symmetric spaces,* Mathematical Surveys and Monographs, Vol. 39, AMS, 1994.

[20] A. W. Knapp, *Representation theory of semisimple groups: An overview based on examples*, Princeton University Press, Vol. 36, 1986.

[21] E. Leuzinger, *Critical exponents of discrete groups and L^2-spectrum*, Proc. of the AMS, 132 (2003), pp. 919–927.

[22] N. Lohoué, *Estimation des projecteurs de de Rham-Hodge de certaines variétés Riemanniennes non compactes*, Math. Nachr., 279 (2006), pp. 272–298.

[23] N. Lohoué, *Stabilité de la cohomologie L^p au voisinage de 2 pour les espaces localement symétriques*, Ann. Global Anal. Geom., 16 (1998), pp. 543–571.

[24] N. Lohoué, *Inégalités de Sobolev pour les sous-laplaciens de certains groupes modulaires*, Geom. Funct. Anal., 2 (1992), pp. 394–420.

[25] N. Lohoué et S. Mehdi, *Estimées du noyau de la chaleur pour les formes différentielles sur les espaces symétriques et L^2-cohomologie des espaces localement symétriques*, C. R. Math. Acad. Sci. Paris., 345 (2007), pp. 119–122.

[26] N. Lohoué and S. Mehdi, *The Novikov-Shubin invariants of locally symmetric spaces*, J. Math. Pures et App., 79 (2000), pp. 111–140.
[27] J. Lott, *The zero-in-spectrum question*, Ens. Math., 42 (1996), pp. 341–376.
[28] W. Lück, *L^2-invariants of regular coverings of compact manifolds and CW-complexes*, Handbook of geometric topology, pp. 735–817, Editors: Davermann, R.J. and Sher, R.B., Elsevier, 2002.
[29] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Die Grundlehren der mathematischen Wissenschaften, Band 52, Springer-Verlag, 1966.
[30] R. Miatello and N. R. Wallach, *The resolvent of the Laplacian on locally symmetric spaces*, J. Differential Geom., 36 (1992), pp. 663–698.
[31] S. Murakami, *Cohomology groups of vector-valued forms on symmetric spaces*, Lectures, University of Chicago, 1966.
[32] S. Mustapha, *Gaussian estimates for heat kernels on Lie groups*, Math. Proc. Cambridge Philos. Soc., 128 (2000), pp. 45–64.
[33] M. Olbrich, *L^2-invariants of locally symmetric spaces*, Doc. Math., 7 (2002), pp. 219–237.
[34] P. Ostellari, *Global behavior of the heat kernel associated with certain sub-Laplacian on semisimple Lie groups*, J. Funct. Analysis, 199 (2003), pp. 521–534.
[35] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on euclidian spaces*, Princeton Math. Series No. 32, Princeton University Press, 1971.
[36] R. Strichartz, *Analysis of the Laplacian on complete Riemannian manifolds*, J. Funct. Analysis, 52 (1983), pp. 48–79.
[37] D. V. Vassilevich, *Heat kernel expansion: user's manual*, Phys. Rept., 388 (2003), pp. 279–360.
[38] D. A. Vogan and G. J. Zuckerman, *Unitary representations with non-zero cohomology*, Comp. Math., 53 (1984), pp. 51–90.
[39] N. R. Wallach, *Cyclic vectors and irreducibility for principal series representations*, Trans. of the AMS, 158 (1971), pp. 389–396.
[40] N. R. Wallach, *Harmonic analysis on homogeneous spaces*, Pure and Appl. Math. series, Vol 19, Marcel Dekker, 1973.
[41] D. P. Želobenko, *Compact Lie groups and their representations*, Translations Math. Monographs, Vol. 40, AMS, 1973.