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ON THE K-THEORY OF TORIC STACK BUNDLES*

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Abstract. Simplicial toric stack bundles are smooth Deligne-Mumford stacks over smooth varieties with fibre a toric Deligne-Mumford stack. We compute the Grothendieck K-theory of simplicial toric stack bundles and study the Chern character homomorphism.

Key words. K-theory, toric stack bundle, finite abelian gerbe.

AMS subject classifications. 14A20, 14C35

1. Introduction. Simplicial toric stack bundles, as defined in [7], are bundles over a smooth base variety B with fibers toric Deligne-Mumford stacks in the sense of [3]. In this paper we compute the Grothendieck K-theory of simplicial toric stack bundles. To state the results we briefly recall the (slightly generalized) construction of toric stacks following [7]. A stacky fan[1] is a triple $\mathbf{\Sigma} := (N, \Sigma, \beta)$, where N is a finitely generated abelian group of rank d with N_{tor} the subgroup of torsion elements, Σ is a simplicial fan in the lattice $\overline{N} = N/N_{tor} \subset N_{\mathbb{Q}}$, and $\beta : \mathbb{Z}^m \to N$ is a map determined by integral vectors $b_1, \ldots, b_n, b_{n+1}, \ldots, b_m \in N$ $(m \geq n)$ satisfying the condition that for $1 \leq i \leq n$ the image $\overline{b}_i \in \overline{N}$ under the projection $N \to \overline{N}$ generates the ray $\rho_i \in \Sigma$. We call $\{b_{n+1}, \cdots, b_m\}$ the extra data in $\mathbf{\Sigma}$. The stacky fan $\mathbf{\Sigma}$ yields an exact sequence,

$$1 \longrightarrow \mu \longrightarrow G \xrightarrow{\alpha} (\mathbb{C}^*)^m \longrightarrow T \longrightarrow 1 \tag{1}$$

where $T = (\mathbb{C}^*)^d$. We associated to $\mathbf{\Sigma}$ a *toric Deligne-Mumford stack* $\mathcal{X}(\mathbf{\Sigma}) := [Z/G]$, where $Z = (\mathbb{C}^n \setminus \mathbb{V}(J_\Sigma)) \times (\mathbb{C}^*)^{m-n}$, the ideal J_Σ is the irrelevant ideal of the fan Σ, and G acts on Z via the homomorphism $\alpha : G \to (\mathbb{C}^*)^m$ above. Removing the extra data $\{b_{n+1}, \cdots, b_m\}$ from the map β yields $\beta_{min} : \mathbb{Z}^n \to N$ given by the integral vectors $\{b_1, \cdots, b_n\}$. The triple $\mathbf{\Sigma_{min}} := (N, \Sigma, \beta_{min})$ is the stacky fan in the sense of [3], and defines a toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma_{min}}) := [Z_{min}/G_{min}]$, where $Z_{min} = (\mathbb{C}^n \setminus \mathbb{V}(J_\Sigma))$ and G_{min} acts on Z_{min} through the homomorphism $\alpha_{min} : G_{min} \to (\mathbb{C}^*)^n$ determined by the stacky fan $\mathbf{\Sigma_{min}}$. It is known [7] that $\mathcal{X}(\mathbf{\Sigma_{min}}) \simeq \mathcal{X}(\mathbf{\Sigma})$.

For a principal $(\mathbb{C}^*)^m$-bundle $P \to B$, let ${}^P\mathcal{X}(\mathbf{\Sigma})$ be the quotient stack $[(P \times_{(\mathbb{C}^*)^m} Z)/G]$, where G acts on B trivially and on $(\mathbb{C}^*)^m$ via the map α above. ${}^P\mathcal{X}(\mathbf{\Sigma})$ is a toric stack bundle over B with fibre the toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$. The extra data $\{b_{n+1}, \cdots, b_m\}$ in Σ do not influence the structure of the toric Deligne-Mumford stack, but do influence the structure of the toric stack bundle ${}^P\mathcal{X}(\Sigma)$, see [7].

Let R be the character ring of the group G_{min}. Every character $\chi \in R$ gives a line bundle $\mathcal{L}_\chi$ over ${}^P\mathcal{X}(\mathbf{\Sigma})$. Let x_i be the standard generator of $\mathbb{Z}^n$. Then x_i gives a standard character χ_i of G_{min}. Write $\mathcal{L}_i := \mathcal{L}_{\chi_i}$. Then x_i represent the class $[\mathcal{L}_i]$ in the K-theory. For $\theta \in M := N^\star$, $\xi_\theta \to B$ is the line bundle coming from the principal

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[1] In [7] this is called an *extended* stacky fan.

T-bundle $E \to B$ by "extending" the structure group via $\chi^\theta : T \to \mathbb{C}^*$, where $E \to B$ is induced from P via the map $(\mathbb{C}^*)^m \to T$ in (1). Let $\{v_1, \cdots, v_d\}$ be a basis of $\overline{N} = \mathbb{Z}^d$ with $v_i \in \rho_i$. Choose a basis $\{u_1, \cdots, u_d\}$ of M dual to $\{v_1, \cdots, v_d\}$ and write $\xi_i = \xi_{u_i}$.

Let $K(B)$ be the K-theory ring of the smooth variety B with rational coefficients $\mathbb{Q}$ and $C({}^P\mathbf{\Sigma})$ the ideal in the ring $K(B) \otimes R$ generated by the elements

$$\prod_{1\leq j\leq n} x_j^{\langle\theta,b_j\rangle} - \prod_{1\leq i\leq d} (\xi_i^\vee)^{\langle\theta,v_i\rangle}, \quad \theta \in M, \tag{2}$$

where $\xi_i^\vee$ is the dual of the line bundle ξ_i. Let $I_{\mathbf{\Sigma}}$ be the ideal generated by

$$\prod_{i\in I}(1 - x_i), \quad I \subseteq [1, \cdots, n] \text{ such that } \{\rho_i | i \in I\} \text{ do not form a cone in } \Sigma. \tag{3}$$

Theorem 1.1. *If $K_0({}^P\mathcal{X}(\mathbf{\Sigma}))$ is the Grothendieck K-theory ring of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma})$. Then there is an isomorphism*

$$\varphi : \frac{K(B) \otimes R}{I_{\mathbf{\Sigma}} + C({}^P\mathbf{\Sigma})} \longrightarrow K_0({}^P\mathcal{X}(\mathbf{\Sigma})), \quad \chi \mapsto [\mathcal{L}_\chi].$$

By Theorem 1.1, the extra data $\{b_{n+1}, \cdots, b_m\}$ in $\mathbf{\Sigma}$ do not affect the K-theory of ${}^P\mathcal{X}(\mathbf{\Sigma})$.

In the reduced case, i.e. the abelian group N is torsion-free, the stack $\mathcal{X}(\mathbf{\Sigma})$ is an orbifold. Then every character of G can be lifted to a character of $(\mathbb{C}^*)^n$. We have the corollary:

Corollary 1.2. *For a reduced toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ there is an isomorphism*

$$\varphi : \frac{K(B)[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]}{I_{\mathbf{\Sigma}} + C({}^P\mathbf{\Sigma})} \longrightarrow K_0({}^P\mathcal{X}(\mathbf{\Sigma})), \quad x_i \mapsto [\mathcal{L}_i].$$

Our proof of the main theorem is based on computations of the K-theory rings of toric Deligne-Mumford stacks [4], and of toric bundles [9].

This paper is organized as follows. The result of Chen-Ruan orbifold cohomology ring of toric stack bundles is reviewed in Section 2. In Section 3 we compute the K-theory ring of toric stack bundles, and in Section 4 we show that there is a Chern character isomorphism from the K-theory of the toric stack bundle to the Chen-Ruan cohomology ring. In Section 5 we study an interesting example, the K-theory ring of finite abelian gerbes over smooth varieties.

In this paper we work algebraically over $\mathbb{C}$. Cohomology and K-theory are taken with $\mathbb{Q}$ coefficients. By an orbifold we mean a smooth Deligne-Mumford stack with trivial generic stabilizer. We refer to [3] for the construction of Gale dual $\beta^\vee : \mathbb{Z}^m \to DG(\beta)$ from $\beta : \mathbb{Z}^m \to N$. We write $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. $M := N^\star$ denotes the dual of N and $N \to \overline{N}$ is the natural map modulo torsion. For the cones in Σ, we assume that the rays $\rho_1, \cdots, \rho_d$ span a top dimensional cone $\sigma \in \Sigma$, and $\rho_{d+1}, \cdots, \rho_n$ are the other rays.

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2. Chen-Raun Cohomology of Toric Stack Bundles. In this section we briefly review the result of Chen-Ruan orbifold cohomology of toric stack bundles. We refer to [3], [7] for the detailed construction of toric Deligne-Mumford stacks and toric stack bundles.

Let $\mathbf{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan. For each top dimensional cone σ in Σ, denote by $Box(\sigma)$ the set of elements $v \in N$ such that $\overline{v} = \sum_{\rho_i \subseteq \sigma} a_i \overline{b}_i$ for some $0 \leq a_i < 1$. Elements in $Box(\sigma)$ are in one-to-one correspondence with elements in the finite group $N(\sigma) = N/N_\sigma$, where $N(\sigma)$ is a local group of the stack $\mathcal{X}(\mathbf{\Sigma})$. Let $Box(\mathbf{\Sigma})$ be the union of $Box(\sigma)$ for all d-dimensional cones $\sigma \in \Sigma$. For $v_1, \ldots, v_n \in N$, let $\sigma(\overline{v}_1, \ldots, \overline{v}_n)$ be the minimal cone in Σ containing $\overline{v}_1, \ldots, \overline{v}_n$.

Let ${}^P\mathcal{X}(\mathbf{\Sigma}) \to B$ be a toric stack bundle. The inertia stack of ${}^P\mathcal{X}(\mathbf{\Sigma})$ is

$$\mathcal{I}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right) = \coprod_{v \in Box(\mathbf{\Sigma})} {}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v})), \tag{4}$$

where ${}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))$ is a closed substack of ${}^P\mathcal{X}(\mathbf{\Sigma})$ corresponding to the cone $\sigma(\overline{v})$.

We recall the Chen-Ruan orbifold cohomology ring for ${}^P\mathcal{X}(\mathbf{\Sigma})$. For $\theta \in M$, let $\chi^\theta : (\mathbb{C}^*)^m \to \mathbb{C}^*$ be the map induced by $\theta \circ \beta : \mathbb{Z}^m \to \mathbb{Z}$. Let $\xi_\theta \to B$ be the line bundle $P \times_{\chi^\theta} \mathbb{C}$. We introduce the deformed ring $H^*(B)[N]^{\mathbf{\Sigma}} = H^*(B) \otimes \mathbb{Q}[N]^{\mathbf{\Sigma}}$, where $\mathbb{Q}[N]^{\mathbf{\Sigma}} := \bigoplus_{c \in N} \mathbb{Q} \cdot y^c$, y is a formal variable, and $H^*(B)$ is the cohomology ring of B. The multiplication of $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ is defined as in [3], Section 1. Let $\mathcal{I}({}^P\mathbf{\Sigma})$ be the ideal in $H^*(B)[N]^{\mathbf{\Sigma}}$ generated by the following elements:

$$c_1(\xi_\theta) + \sum_{i=1}^n \theta(b_i) y^{b_i}, \quad \theta \in M, \tag{5}$$

and $H^*_{CR}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right)$ the Chen-Ruan cohomology ring of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma})$. From [7], we have an isomorphism of $\mathbb{Q}$-graded rings:

$$H^*_{CR}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right) \cong \frac{H^*(B)[N]^{\mathbf{\Sigma}}}{\mathcal{I}({}^P\mathbf{\Sigma})}.$$

From the definition of Chen-Ruan cohomology and (4), we have

$$H^*_{CR}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right) = \bigoplus_{v \in Box(\mathbf{\Sigma})} H^*\left({}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))\right). \tag{6}$$

The closed substack ${}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))$ is also a toric stack bundle over B with fibre the toric stack $\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))$ associated to the quotient stacky fan $\mathbf{\Sigma}/\sigma(\overline{v})$. Let $link(\sigma(\overline{v})) = \{\rho_1, \cdots, \rho_l\}$ be the rays in the quotient fan $\Sigma/\sigma(\overline{v})$. Let $I_{\mathbf{\Sigma}/\sigma(\overline{v})}$ be the ideal of $H^*(B)[y^{\widetilde{b}_1}, \cdots, y^{\widetilde{b}_l}]$ generated by

$$\{y^{\widetilde{b}_{i_1}} \cdots y^{\widetilde{b}_{i_k}} | \rho_{i_1}, \cdots, \rho_{i_k} \text{ do not span a cone in } \Sigma/\sigma(\overline{v})\}.$$

Then the cohomology ring of ${}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))$ is isomorphic to the Stanley-Reisner ring of the quotient fan over the cohomology ring $H^*(B)$ of the base B:

$$H^*({}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))) \cong \frac{H^*(B)[y^{\widetilde{b}_1}, \cdots, y^{\widetilde{b}_l}]}{I_{\mathbf{\Sigma}/\sigma(\overline{v})} + \mathcal{I}({}^P\mathbf{\Sigma}/\sigma(\overline{v}))}. \tag{7}$$

The ring structure on $H^*_{CR}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right)$ requires the use of certain *obstruction bundle*. Since we will not discuss ring structures in this paper, we omit the details.

Remark 2.1. *As pointed out in [4], if the toric stack $\mathcal{X}(\mathbf{\Sigma})$ is reduced (i.e. N is free), then $H^*_{CR}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right)$ is an Artinian module over the cohomology ring $H^*(B)$ of the base. If N has torsion, then $H^*_{CR}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right)$ is not Artinian in general since it has degree zero elements.*

3. The K-Theory of Toric Stack Bundles. In this section we study the Grothendieck ring of toric stack bundles and prove the main theorem.

3.1. The K-theory of toric Deligne-Mumford stacks. We recall the result of [4]. Let $\mathbf{\Sigma}$ be a stacky fan and $\mathcal{X}(\mathbf{\Sigma})$ the corresponding toric Deligne-Mumford stack. For each ray ρ_i in the fan Σ, define the line bundle L_i over $\mathcal{X}(\mathbf{\Sigma})$ to be the quotient of the trivial line bundle $Z\times\mathbb{C}$ over Z under the action of G on $\mathbb{C}$ through i-th component of α in (1). Let x_i represent the class $[L_i]$ in the Grothendieck K-theory ring.

Let R be the character ring of the group G_{min} and $Cir(\mathbf{\Sigma})$ the ideal in $K(B)\otimes R$ generated by

$$\prod_{1\leq j\leq n} x_j^{\langle\theta,v_j\rangle}-1,\quad \theta\in M, \tag{8}$$

where Cir means the circuit ideal of the fan Σ in K-theory. Let $I_{\mathbf{\Sigma}}$ be the ideal in (3). According to [4], the Grothendieck K-theory ring $K_0(\mathcal{X}(\mathbf{\Sigma}))$ of $\mathcal{X}(\mathbf{\Sigma})$ can be described as follows.

Theorem 3.1 ([4]). *There is an isomorphism*

$$\phi:\frac{R}{I_{\mathbf{\Sigma}}+Cir(\mathbf{\Sigma})}\longrightarrow K_0(\mathcal{X}(\mathbf{\Sigma})),\quad \chi\mapsto[L_\chi].$$

Let $\mathbf{\Sigma_{min}}$ be the minimal stacky fan associated to $\mathbf{\Sigma}$. There is an underlying *reduced* stacky fan $\mathbf{\Sigma_{red}}=(\overline{N},\Sigma,\overline{\beta})$, where $\overline{N}=N/N_{tor}$, $\overline{\beta}:\mathbb{Z}^n\to\overline{N}$ is the natural projection given by the vectors $\{\overline{b}_1,\cdots,\overline{b}_n\}\subseteq\overline{N}$. Consider the following diagram

$$\begin{array}{ccc}\mathbb{Z}^n & \xrightarrow{\beta} & N\\ {\scriptstyle id}\downarrow & & \downarrow\\ \mathbb{Z}^n & \xrightarrow{\overline{\beta}} & \overline{N}.\end{array}$$

Taking Gale dual and $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$-functor yields

$$\begin{array}{ccccccccccc}1 & \longrightarrow & \mu & \longrightarrow & G & \xrightarrow{\alpha} & (\mathbb{C}^*)^n & \longrightarrow & T & \longrightarrow & 1\\ & & \downarrow & & \downarrow{\scriptstyle \alpha(\varphi)} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & 1 & \longrightarrow & \overline{G} & \xrightarrow{\overline{\alpha}} & (\mathbb{C}^*)^n & \longrightarrow & T & \longrightarrow & 1.\end{array} \tag{9}$$

The stack $\mathcal{X}(\mathbf{\Sigma_{red}})$ is a toric orbifold. By construction $\mathcal{X}(\mathbf{\Sigma_{red}})=[(\mathbb{C}^n\setminus\mathbb{V}(J_\Sigma))/\overline{G}]$, where $\overline{G}=\mathrm{Hom}_{\mathbb{Z}}(DG(\overline{\beta}),\mathbb{C}^*)$ and $DG(\overline{\beta})$ is the Gale dual $\overline{\beta}^\vee:\mathbb{Z}^n\to DG(\overline{\beta})$ of the map $\overline{\beta}$. We can see from (9) that every character of $\overline{G}$ can be represented as a character of $(\mathbb{C}^*)^n$. So we have:

Theorem 3.2. *For the reduced toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma_{red}})$ there is an isomorphism*

$$\phi:\frac{\mathbb{Z}[x_1,x_1^{-1},\cdots,x_n,x_n^{-1}]}{I_{\mathbf{\Sigma}}+Cir(\mathbf{\Sigma})}\longrightarrow K_0(\mathcal{X}(\mathbf{\Sigma_{red}})),\quad x_i\mapsto[L_i].$$

3.2. Proof of main theorem. We begin with some lemmas.

LEMMA 3.3. *Let U be a smooth scheme. Let $[M/G]$ be a quotient stack, where M admits a cellular decomposition (in the sense of [10]) which is G-equivariant. Let G act on U trivially. Then the map*

$$K_0(U) \otimes K_G(M) \to K_G(U \times M)$$

is surjective.

Proof. This is an G-equivariant version of [10], Expose 0, Proposition 2.13. This can be proven by adopting the arguments in [10], together with Lemmas 3.4 and 3.5 below. □

LEMMA 3.4. *Let X be a smooth scheme with trivial G-action and G acts on $\mathbb{A}^1$. Let $p : X \times \mathbb{A}^1 \to X$ be the projection. Then the pull-back $p^* : K_G(X) \to K_G(X \times \mathbb{A}^1)$ is surjective.*

Proof. Let V be a G-equivariant vector bundle over $X \times \mathbb{A}^1$. Then by the non-equivariant version of this Lemma (see [10], Expose 0, Proposition 2.9), there is a vector bundle V' over X such that $V = p^*(V')$. Since G acts trivially on X, it is easy to see that the G-action on V actually yields a G-action on V', making p^* G-equivariant. □

LEMMA 3.5. *Let X be a smooth G-scheme and $Y \subset X$ a smooth closed subscheme preserved by G-action. Suppose that the quotient $[X/G]$ is a noetherian Deligne-Mumford stack. Set $U := X \setminus Y$. Then the following natural sequence is exact,*

$$K_G(Y) \to K_G(X) \to K_G(U) \to 0.$$

Proof. The exactness in the middle is a general fact, see e.g. [11], Section 3.1. The surjectivity of the restriction map $K_G(X) \to K_G(U)$ follows from the following argument (we interpret G-equivariant sheaves as sheaves on the quotient stacks).

Let $\mathcal{F}$ be a coherent sheaf on $[U/G]$. Define a *quasi-coherent* sheaf $\bar{\mathcal{F}}$ on $[X/G]$ as follows. For an open subset $V \subset [X/G]$ define $\bar{\mathcal{F}}(V) := \mathcal{F}(V \cap [U/G])$. By construction $\bar{\mathcal{F}}|_{[U/G]} = \mathcal{F}$, which is coherent. Then by [8], Corollaire 15.5, there exists a coherent sheaf $\mathcal{F}'$ on $[X/G]$ such that $\mathcal{F}'|_{[U/G]} = \mathcal{F}$. □

Proof of Theorem 1.1 Let $\mathbf{\Sigma}$ be a stacky fan, and $\mathcal{X}(\mathbf{\Sigma})$ the associated toric stack. Let $P \to B$ be a principal $(\mathbb{C}^*)^m$-bundle over the smooth variety B and $\pi : {}^P\mathcal{X}(\mathbf{\Sigma}) \to B$ the toric stack bundle. Each ray ρ_i in the fan Σ gives a line bundle L_i over $\mathcal{X}(\mathbf{\Sigma})$. Twisting by the principal bundle P gives the line bundle $\mathcal{L}_i$ over ${}^P\mathcal{X}(\mathbf{\Sigma})$.

As in [3] and [7] we have a codimension one closed substack $\mathcal{X}(\mathbf{\Sigma}/\rho_j) \subset \mathcal{X}(\mathbf{\Sigma})$. There is a canonical section s_j of the line bundle L_j whose zero locus is $\mathcal{X}(\mathbf{\Sigma}/\rho_j)$.

Suppose that $\rho_{j_1}, \cdots, \rho_{j_r}$ do not span a cone in Σ. The section $s = (s_{j_1}, \cdots, s_{j_r})$ of $L_{j_1} \oplus \cdots \oplus L_{j_r}$ is nowhere vanishing and extends to a nowhere vanishing section

$$P(s) : \ {}^P\mathcal{X}(\mathbf{\Sigma}) \longrightarrow \mathcal{L}_{j_1} \oplus \cdots \oplus \mathcal{L}_{j_r}$$

after twisting by the principle $(\mathbb{C}^*)^m$-bundle P. Hence by Remark 4.4 in [9],

$$\prod_{1 \le p \le r} (1 - \mathcal{L}_{j_p}) = 0. \tag{10}$$

For any $\theta \in M$, the P-equivariant isomorphism of bundles $\prod_{1\leq j\leq n} L_j^{\langle \theta, b_j\rangle} \cong L_\theta$ over $\mathcal{X}(\mathbf{\Sigma})$ yields an isomorphism of bundles $\prod_{1\leq j\leq n}\mathcal{L}_j^{\langle\theta,b_j\rangle}\cong \mathcal{L}_\theta$ over ${}^P\mathcal{X}(\mathbf{\Sigma})$. Since $\mathcal{L}_\theta = \prod_{1\leq i\leq d}\xi_i^{-\langle\theta,v_i\rangle}$, we obtain

$$\prod_{1\leq j\leq n}\mathcal{L}_j^{\langle\theta,b_j\rangle}\cong \xi_\theta^\vee, \quad \text{where } \xi_\theta = \prod_{1\leq i\leq d}\xi_i^{\langle\theta,v_i\rangle}. \tag{11}$$

Consider the following map

$$\varphi : \frac{K(B)\otimes R}{I_{\mathbf{\Sigma}} + C({}^P\mathbf{\Sigma})} \longrightarrow K_0({}^P\mathcal{X}(\mathbf{\Sigma})), \quad b\otimes\chi\mapsto [\pi^* b\otimes \mathcal{L}_\chi], \quad b\in K(B), \chi\in R.$$

We prove that φ is surjective by induction on the dimension of B. It is obvious when B is a point. Let $U\subset B$ be a Zariski open subset over which the bundle is trivial, i.e. $\pi^{-1}(U)\cong U\times\mathcal{X}(\mathbf{\Sigma})$. Set $Z = B\setminus U$. Consider the following diagram:

$$\begin{array}{ccccccc} K_0(Z)\otimes K(\mathcal{X}(\mathbf{\Sigma})) & \longrightarrow & K_0(B)\otimes K(\mathcal{X}(\mathbf{\Sigma})) & \longrightarrow & K_0(U)\otimes K(\mathcal{X}(\mathbf{\Sigma})) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ K_0(\pi^{-1}Z) & \longrightarrow & K_0({}^P\mathcal{X}(\mathbf{\Sigma})) & \longrightarrow & K_0(\pi^{-1}U) & \longrightarrow & 0. \end{array} \tag{12}$$

It is clear that the top row is exact. By Lemma 3.5, the bottom row is also exact. By Lemma 3.3, the vertical map on the right of (12) is surjective. The vertical map on the left of (12) is surjective by induction, since $dim(Z_k) < dim(B)$ for each irreducible component Z_k of Z. Thus the map φ is surjective.

Now we prove that φ is injective. Let $\sum_{i=1}^m b_i[F_i]\in K(B)\otimes R$ such that

$$\varphi\left(\sum_{i=1}^m b_i[F_i]\right) = \sum_{i=1}^m \pi^* b_i\otimes[\mathcal{F}_i] = 0,$$

where $\mathcal{F}_i$ is the twist of F_i by the $(\mathbb{C}^*)^m$-bundle P. The sheaf $\mathcal{F}_i$ is generated by $\mathcal{L}_j$'s corresponding to rays and the torsion line bundles corresponding to torsion subgroup in G_{min}. From the relations in (10) and (11), it is easy to see that if one of $b_i\neq 0$, then $\sum_{i=1}^m \pi^* b_i\otimes[\mathcal{F}_i]\neq 0$. So φ is injective, hence is an isomorphism. The concludes the proof of Theorem 1.1. □

4. Combinatorial Chern Character. In this section we give a combinatorial description of the Chern character homomorphism from the K-theory to Chen-Ruan cohomology. For simplicity, we assume that the toric Deligne-Mumofrd stack $\mathcal{X}(\mathbf{\Sigma})$ is reduced. By Corollary 1.2,

$$K_0({}^P\mathcal{X}(\mathbf{\Sigma}),\mathbb{C}) := K_0({}^P\mathcal{X}(\mathbf{\Sigma}))\otimes_{\mathbb{Q}}\mathbb{C}\simeq \frac{K(B)\otimes R}{I_{\mathbf{\Sigma}}+C({}^P\mathbf{\Sigma})}\otimes\mathbb{C}, \tag{13}$$

where $R\cong\mathbb{C}[x_1,x_1^{-1},\cdots,x_n,x_n^{-1}]$. Let $\widetilde{R}$ denote the right-hand side of (13). Again let $[\xi_i]\in K(B,\mathbb{C}) := K(B)\otimes_{\mathbb{Z}}\mathbb{C}$ represent the class of ξ_i in the K-theory of B. The following Lemma generalizes [4], Lemma 5.1.

Lemma 4.1. *The maximal ideals of $\widetilde{R}$ as $K(B,\mathbb{C})$-algebras are in bijective correspondence with elements of $Box(\mathbf{\Sigma})$. A box element $v=\sum_{\rho_i\subset\sigma} a_i\overline{b}_i$ corresponds to the n-tuple $(y_1,\cdots,y_n)\in K(B,\mathbb{C})^n$ such that*

$$y_i = \begin{cases} e^{2\pi i a_i}\sqrt[r_i]{\xi_i^\vee} & \textit{if } \rho_i\subset\sigma, \\ 1 & \textit{otherwise}, \end{cases}$$

where $\xi_i \in K(B,\mathbb{C})$ *and* r_i *is the order of* $e^{2\pi i a_i}$.

Proof. The maximal ideals of $\widetilde{R}$ viewed as $K(B,\mathbb{C})$-algebras correspond to points $(y_1,\cdots,y_n)$ in $K(B,\mathbb{C})^n$ such that

$$\prod_{1\leq j\leq n} y_j^{\langle\theta,b_j\rangle} - \prod_{1\leq i\leq d}(\xi_i^\vee)^{\langle\theta,v_i\rangle}=0 \tag{14}$$

and

$$\prod_{i\in I}(1-x_i)=0$$

for θ and I in (2) and (3).

Suppose that the $K(B,\mathbb{C})$-point $(y_1,\cdots,y_n)$ satisfies the above condition. Since $\prod_{i\in I}(1-x_i)=0$, there is some cone $\sigma\in\Sigma$ such that $y_i=1$ for ρ_i outside the cone σ. Assume that σ is generated by rays $\rho_1,\cdots,\rho_k$.

Consider the relation (14). Since this relation holds for any $\theta\in M$, and $y_i=1$ for ρ_i outside the cone σ, we can take $\theta: N_\sigma\to\mathbb{Z}$, where N_σ is the intersection of N with the rational span of $\rho_1,\cdots,\rho_k$. Then we can choose θ such that $\theta(v_i)=1$, and $\theta(v_j)=0$ for $j\neq i$. The value y_i is an r_i-th root of ξ_i for some integer r_i. So $y_i = e^{2\pi i a_i}\sqrt[r_i]{\xi_i^\vee}$. The relation now reads $\prod_{1\leq i\leq k} e^{2\pi i a_i\langle\theta,b_i\rangle}=1$, and then $\sum_i\langle\theta,b_i\rangle a_i\in\mathbb{Z}$ for all θ. This is equivalent to $v=\sum_{\rho_i\subset\sigma}a_i\overline{b}_i\in N$. So the maximal ideals are in one-to-one correspondence to the box elements $Box(\mathbf{\Sigma})$. □

In the reduced case the ring $\widetilde{R}$ is an Artinian module over $K(B,\mathbb{C})$. The localization $\widetilde{R}_v$ can be taken as a submodule of $\widetilde{R}$, which is simple. According to [12], we have

$$\widetilde{R} := \frac{K(B)\otimes R}{I_{\mathbf{\Sigma}}+C({}^P\mathbf{\Sigma})}\otimes\mathbb{C} = \bigoplus_{v\in Box(\mathbf{\Sigma})}\widetilde{R}_v. \tag{15}$$

PROPOSITION 4.2. *Let* $v\in Box(\mathbf{\Sigma})$ *and* $\sigma(\overline{v})$ *the minimal cone in* Σ *containing* $\overline{v}$. *Then the* $K(B,\mathbb{C})$*-algebra* $\widetilde{R}_v$ *is isomorphic to the cohomology of the closed substack* ${}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))$ *of* ${}^P\mathcal{X}(\mathbf{\Sigma})$.

Proof. Let $\sigma(\overline{v})$ be generated by the rays $\rho_1,\cdots,\rho_k$, and let $\overline{v}=\sum_{1\leq i\leq k}a_i\overline{b}_i$ with $a_i\in(0,1)$. For the rest of rays $\rho_{k+1},\cdots,\rho_n$, we may assume that $\rho_{k+1},\cdots,\rho_l$ are contained in some cone σ' containing σ, and $\rho_{l+1},\cdots,\rho_n$ are not.

Now localizing gives the $K(B,\mathbb{C})$-algebra $\widetilde{R}_v$. Then x_i-1 is nilpotent for $i>k$, and $x_i-e^{2\pi i a_i}\sqrt[r_i]{\xi_i^\vee}$ is nilpotent for $1\leq i\leq k$. Similar to Lemma 5.2 of [4], let

$$z_i=\begin{cases}\log(x_i), & i>k,\\ \log(x_i e^{-2\pi i a_i}(\sqrt[r_i]{\xi_i^\vee})^{-1}), & 1\leq i\leq k.\end{cases}$$

Now we work over the quotient ring $\widetilde{R}_1$ of $\widetilde{R}$ by a sufficiently high power of the maximal ideal. Using the same method as in [4], we see that $z_j=0$ in $\widetilde{R}_v$ for $j>l$. And the relations $\prod_{i\in I}(x_i-1)=0$ are translated to $\prod_{i\in I_{\Sigma/\sigma}}z_i=0$, where $I_{\Sigma/\sigma}$ represents the subset of $\{k+1,\cdots,l\}$ such that $\{\rho_i|i\in I_{\Sigma/\sigma}\}$ are not contained in any cone of Σ/σ. (Note that $\{\rho_{k+1},\cdots,\rho_l\}$ are the link set of σ). So the relations $\prod_{i\in I}(x_i-1)=0$ determine the relations $\prod_{i\in I_{\Sigma/\sigma}}z_i=0$ in the quotient fan Σ/σ.

Let $ch: K(B,\mathbb{C}) \to H^*(B,\mathbb{C})$ be the Chern character isomorphism from the K-theory of B to the cohomology. Then $ch(\xi_i) = e^{c_1(\xi_i)}$. Consider the linear relations

$$\prod_{1\leq j\leq n} x_j^{\langle\theta,b_j\rangle} - \prod_{1\leq i\leq d}(\xi_i^{\vee})^{\langle\theta,v_i\rangle} = 0, \quad \theta\in M.$$

Replacing the relations by z_i we get

$$\prod_{i=1}^{k} e^{2\pi i a_i\langle\theta,b_i\rangle}\xi_i^{\langle\theta,v_i\rangle}\prod_{i=1}^{k+l} e^{z_i\langle\theta,b_j\rangle} - \prod_{1\leq i\leq d}(\xi_i^{\vee})^{\langle\theta,v_i\rangle} = 0. \tag{16}$$

Let $N_{\sigma(v)}$ be the sublattice generated by $\sigma(v)$, and $N(\sigma(v)) = N/N_{\sigma(v)}$. Let $\overline{N}(\sigma(v))$ be the free part of $N(\sigma(v))$, and $M(\sigma(v)) := N(\sigma(v))^{\star}$. Consider the following diagram:

$$\begin{array}{ccc} N & \xrightarrow{\pi} & N(\sigma(v)) \\ \theta\downarrow & \swarrow_{\widetilde{\theta}} & \\ \mathbb{Z} & & \end{array} \tag{17}$$

where π is the natural morphism. For any $\widetilde{\theta}\in M(\sigma(v))$, there is an element $\theta\in M$ induced from diagram (17). Since $\xi_\theta = \prod_{1\leq i\leq d}\xi_i^{\langle\theta,v_i\rangle}$, and $e^{c_1(\xi_\theta)} = ch(\xi_\theta)$, passing to the quotient fan $\Sigma/\sigma(v)$ in the lattice $\overline{N}(\sigma(v))$ the equation (16) becomes

$$e^{\sum_{i=k+1}^{k+l} z_i\langle\theta,b_i\rangle} - e^{c_1(\xi_{\widetilde{\theta}}^{\vee})} = 0.$$

So these relations yield

$$\sum_{i=k+1}^{k+l} z_i\langle\theta,b_i\rangle + c_1(\xi_{\widetilde{\theta}}) = 0$$

which are exactly the linear relations in the cohomology ring of toric stack bundles. Since $x_1,\cdots,x_k$ can be represented as linear combinations of $z_{k+1},\cdots,z_l$, the algebra $\widetilde{R}_v$ is isomorphic to the ring $H^*(B)[z_{k+1},\cdots,z_l]$ with relations

$$\prod_{i\in I_{\Sigma/\sigma}} z_i = 0, \quad \text{and} \quad \sum_{i=k+1}^{k+l} z_i\langle\theta,b_i\rangle + c_1(\xi_{\widetilde{\theta}}) = 0.$$

So compared to the result in (7), $\widetilde{R}_v$ is isomorphic to $H^*({}^P\mathcal{X}(\mathbf{\Sigma}/\sigma),\mathbb{C})$. □

The decomposition (6) together with Theorem 1.1, Lemma 4.1, and Proposition 4.2 then yields

THEOREM 4.3. *Assume that $\mathcal{X}(\mathbf{\Sigma})$ is semi-projective. Then the Chern character map*

$$ch: K_0({}^P\mathcal{X}(\mathbf{\Sigma}),\mathbb{C}) \longrightarrow H^*_{CR}({}^P\mathcal{X}(\mathbf{\Sigma}),\mathbb{C}), \quad \mathcal{L}\mapsto ch(\mathcal{L}),$$

is a module isomorphism.

REMARK 4.4. *In [6], a "stringy" product on the K-theory of smooth Deligne-Mumford stacks is defined. The definition is similar to the Chen-Ruan cup product* *on the orbifold cohomology, which involves obstruction bundles. It is proven in [6] that the Chern character homomorphism yields a ring isomorphism. In our setting we can explicitly show this for toric stack bundles. Since this is a special case, we omit the details.*

5. Example: Finite Abelian Gerbes. In [7], the degenerate case of toric stack bundles, namely finite abelian gerbes over smooth varieties, were studied. In this section we compute their K-theory.

We first recall the construction of these gerbes. Let $N = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{n_s}}$ be a finite abelian group, where $p_1, \cdots, p_s$ are prime numbers and $n_1, \cdots, n_s \geq 1$. Let $\beta : \mathbb{Z} \to N$ be given by the vector $(1, 1, \cdots, 1)$. $N_{\mathbb{Q}} = 0$ implies that $\Sigma = 0$, then $\mathbf{\Sigma} = (N, \Sigma, \beta)$ is a stacky fan. Let $n := lcm(p_1^{n_1}, \cdots, p_s^{n_s}) := p_{i_1}^{n_{i_1}} \cdots p_{i_t}^{n_{i_t}}$, where $p_{i_1}, \cdots, p_{i_t}$ are the distinct prime numbers which have the highest powers $n_{i_1}, \cdots, n_{i_t}$. Note that the vector $(1, 1, \cdots, 1)$ generates an order n cyclic subgroup of N. We calculate the Gale dual $\beta^\vee : \mathbb{Z} \to \mathbb{Z} \oplus \bigoplus_{i \notin \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i}$, where $DG(\beta) = \mathbb{Z} \oplus \bigoplus_{i \notin \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i}$. We have the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{\beta} N \longrightarrow \bigoplus_{i \notin \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \longrightarrow 0,$$

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{(\beta)^\vee} \mathbb{Z} \oplus \bigoplus_{i \notin \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \longrightarrow \mathbb{Z}_n \oplus \bigoplus_{i \notin \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \longrightarrow 0.$$

So we obtain

$$1 \longrightarrow \mu \longrightarrow \mathbb{C}^* \times \prod_{i \notin \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i} \xrightarrow{\alpha} \mathbb{C}^* \longrightarrow 1, \tag{18}$$

where the map α in (18) is given by the matrix $[n, 0, \cdots, 0]^t$ and $\mu = \mu_n \times \prod_{i \notin \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i} \cong N$. The toric Deligne-Mumford stack associated with the data is the classifying stack of the group μ,

$$\mathcal{X}(\mathbf{\Sigma}) = [\mathbb{C}^* / \mathbb{C}^* \times \prod_{i \notin \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i}] = \mathcal{B}\mu.$$

Let L be a line bundle over a smooth variety B and L^* the principal $\mathbb{C}^*$-bundle induced from L by removing the zero section. From our twist we have

$$\mathcal{X} := {}^{L^*}\mathcal{X}(\mathbf{\Sigma}) = L^* \times_{\mathbb{C}^*} [\mathbb{C}^* / \mathbb{C}^* \times \prod_{i \notin \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i}] = [L^* / \mathbb{C}^* \times \prod_{i \notin \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i}],$$

which is a μ-gerbe over B. The stack $\mathcal{X}$ is the product of a μ_n-gerbe coming from the line bundle L and a trivial $\prod_{i \notin \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i}$-gerbe over B. Its Chen-Ruan cohomology was computed in [7].

PROPOSITION 5.1 ([7]). *The Chen-Ruan cohomology ring of $\mathcal{X}$ is:*

$$H^*_{CR}(\mathcal{X}, \mathbb{Q}) \cong H^*(B, \mathbb{Q}) \otimes H^*_{CR}(\mathcal{B}\mu, \mathbb{Q}) \simeq H^*(B, \mathbb{Q})[t_1, \cdots, t_s]/(t_1^{p_1^{n_1}} - 1, \cdots, t_s^{p_s^{n_s}} - 1).$$

For the stacky fan $\mathbf{\Sigma} = (N, 0, \beta)$, the minimal stacky fan is given by $\mathbf{\Sigma_{min}} = (N, 0, \beta_{min})$, where $\beta_{min} = 0 : 0 \to N$ is the zero map. So the Gale dual is still β_{min}, and $G_{min} = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*) \cong \mu$. The characters of $\mu \simeq N$ are given by all the maps $\chi : \mu \to \mathbb{C}^*$. Since $N = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{n_s}}$, let $\chi_1, \cdots, \chi_s$ be the base generators of the characters of N such that $\chi_1^{p_1^{n_1}}, \cdots, \chi_1^{p_s^{n_s}}$ are trivial. Every character χ_i determines a line bundle $\mathcal{L}_i$ over $\mathcal{X}$ such that $\mathcal{L}_1^{p_i^{n_i}}$ is trivial. Then Theorem 1.1 implies

PROPOSITION 5.2. *The K-theory ring of the finite abelian gerbe $\mathcal{X}$ is:*

$$K_0(\mathcal{X}) \simeq K(B)[\mathcal{L}_1, \cdots, \mathcal{L}_s]/(\mathcal{L}_1^{p_1^{n_1}}, \cdots, \mathcal{L}_s^{p_s^{n_s}}).$$

It is easy to see from Proposition 5.2 that the K-theory ring of the finite abelian gerbes is independent to the triviality and nontriviality of the gerbes.

Propositions 5.1 and 5.2 yields a Chern character isomorphism $K_0(\mathcal{X}, \mathbb{C}) \simeq H^*_{CR}(\mathcal{X}, \mathbb{C})$ for $\mathcal{X}$.

REMARK 5.3. *Suppose that we have two finite abelian μ-gerbes over B, one is trivial and the other is nontrivial. We see that the K-theory ring and the Chen-Ruan cohomology ring cannot distinguish these two different stacks. However* quantum *cohomology rings of different gerbes are different in general [1], [2].*

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