ASIAN J. MATH.
Vol. 13, No. 2, pp. 251–270, June 2009

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FOUR-MANIFOLDS WITH 1/4-PINCHED FLAG CURVATURES*

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Abstract. The Ricci flow on a compact four-manifold preserves the condition of pointwise 1/4-pinching of flag curvatures. Any compact Riemannian four-manifold with 1/4-pinched flag curvatures is either isometric to $\mathbb{CP}^2$ or diffeomorphic to a space-form.

Key words. Ricci flow, Sphere theorem.

AMS subject classifications. Primary 53C44, 53C20; Secondary 35K55, 58J35, 53C21

1. Introduction. It is well known that various conditions of positive curvature imply topological restrictions on a Riemannian manifold. A famous example is the 1/4-pinching theorem of Klingenberg, Berger and Rauch, which states that a simply-connected manifold with globally 1/4-pinched sectional curvatures is necessarily homeomorphic to a sphere. An intriguing aspect of this result is that the result gives only topological equivalence and not diffeomorphism equivalence, leaving open the possibility that the manifold could in fact be an exotic sphere. The question of whether such exotic spheres can exist is made more interesting by examples of exotic spheres with non-negative sectional curvatures, or even strictly positive sectional curvatures almost everywhere (though there are no examples known which are close to having 1/4-pinched sectional curvatures).

In this paper our aim is to provide an alternative condition, that of pointwise 1/4-pinched flag curvatures, which is preserved by the Ricci flow and which avoids all but the trivial singularity asymptotic to a shrinking spaceform. The required analysis follows the model introduced by Hamilton in his groundbreaking work on compact three-manifolds with positive Ricci curvature. The condition of pointwise 1/4-pinched flag curvatures is weaker than 1/4-pinched sectional curvatures since only a subset of sectional curvatures are compared.

Let (M, g) be a compact Riemannian 4-manifold, with curvature tensor R. The condition we consider is as follows: We suppose that M has positive sectional curvatures and that for every $x \in M$ and every orthonormal basis $\{e_1, \ldots, e_4\}$ for T_xM, we have

$$R(e_2, e_1, e_2, e_1) \geq \lambda R(e_3, e_1, e_3, e_1). \tag{1}$$

To put this in a more geometric way, for each e_1 in T_xM there is an associated bilinear form R_{e_1} on the orthogonal subspace, the flag curvature in direction e_1, defined by $R_{e_1}(v, v) = R(e_1, v, e_1, v)$. The condition (1) says precisely that the ratio of any two eigenvalues of R_{e_1} is bounded below by λ. That is, each of the flag curvatures of M is λ-pinched.

THEOREM 1. *Let M be a compact four-manifold, and g_0 a Riemannian metric on M with λ-pinched flag curvatures, where $\lambda \geq 1/4$. Then either $\lambda = 1/4$ and (M, g_0) is*

*Received November 11, 2008; accepted for publication January 30, 2009. Research partially supported by Discovery Grant DP0344221 of the Australian Research Council.

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isometric to a multiple of the Fubini-Study metric on $\mathbb{CP}^2$, or (M, g_0) is diffeomorphic to a spherical space form.

In outline, the proof follows approximately the stages of the proof used by Hamilton in [H1] and [H2]: After introducing some notation and acquainting the reader with some preliminary results in Section 2, we show that λ-pinching of flag curvatures is preserved by the Ricci flow (in Section 3), and also show that if the pinching ratio is initially equal to 1/4 then it becomes strictly greater than 1/4 unless the initial metric is isometric to the standard one on $\mathbb{CP}^2$ modulo scaling. It is perhaps important that the methods we use in this analysis do not rely heavily on the special algebraic structure of the curvature operator in four dimensions, and so seem likely to generalise in some way to higher dimensions. In Section 3, we show that lower bounds on the flag curvature pinching are also preserved. In Section 5 we deduce that the pinching ratio approaches one in regions of high curvature. Unlike in earlier works [H1], [H2], [Hu] we reduce this to the computation showing that quarter pinched flag curvature is preserved and so eliminate some of the detailed computations required in those works. In Section 6 we prove the convergence result, mostly drawing on existing results and methods. In Section 7 we show that for weak pinching, the manifold is isometric to $\mathbb{CP}^2$ or diffeomorphic to a spherical space form. We note that H. Chen has proved that four-manifolds with 2-positive curvature operator, and so in particular four-manifolds with 1/4-pinched sectional curvature, deform to constant curvature under Ricci flow [C]. We also note as this paper was being written up, the authors learnt that the quarter pinching diffeomorphism sphere theorem in all dimensions was proven in the preprint [BS1] by Brendle and Schoen. The key step there is to prove that positive isotropic curvature is preserved by the Ricci flow. This was also proved by the second author in his PhD dissertation [N]. It is not clear whether 1/4-pinched flag curvature for four-manifolds implies the conditions in [C] or [BS1] or more recent work of Brendle [B][1].

2. Notation and preliminary results.

2.1. Short time existence. The following theorem was originally proved in [H1] using the Nash-Moser implicit function theorem. However a considerably simpler proof was discovered by DeTurck [D]. Subsequently, other simpler proofs were also discovered [CK].

THEOREM 2. *[CK]*Theorem 3.13 Let $(M, g_{ij}(0))$ be a connected compact Riemannian manifold, then there exists $\epsilon > 0$ such that*

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \quad g_{ij}(0,x) = g_{ij}(x)|_{t=0}$$

has a unique solution for $x \in M$ and $t \in [0, \epsilon)$.

2.2. Existence until curvature blowup.

THEOREM 3. *[H3, Theorem 7.1] [CK]*Theorem 7.1 There exist constants C_k for $k \geq 1$ such that if for $0 \leq T < \frac{1}{K}$*

$$\sup_M |\operatorname{Rm}| \leq K$$

[1]Added in proof: Recently Ni-Wilking has shown that 1/4-pinched flag curvature implies positive isotropic curvature on $M \times \mathbb{R}^2$.

then the covariant derivatives of the curvature are bounded:

$$|\nabla \operatorname{Rm}| \leq \frac{C_1 K^2}{t^{1/2}},$$

for each $x \in M$ *and* $0 < t \leq K$, *and for each* $k \in \mathbb{N}$ *we also have*

$$|\nabla^k \operatorname{Rm}(x,t)| \leq \frac{C_k K^{1+k}}{t^{k/2}}.$$

A consequence of the global derivative estimates is the following:

COROLLARY 4. *[H3]*Theorem 8.1, [CK]*Corollary 7.2 If* $g_{ij}(0)$ *is a smooth metric on a compact manifold* M, *then there is a unique solution* $g_{ij}(t)$ *to the Ricci flow which exists on a maximal time interval* $0 \leq t \leq T \leq \infty$. *Furthermore if* $T \leq \infty$ *then*

$$\lim_{t \to T^-} \left(\sup_{x \in M^n} |\operatorname{Rm}(x,t)| \right) = \infty.$$

2.3. Hamilton's maximum principle. In the following, we state an advanced maximum principle for tensors which we will use in this paper. We note that there is a more general maximum principle for time dependents sets with an avoidance set. We refer to [CL]. The original tensor maximum principle was proved by Hamilton [H1]*Theorem 9.1. A version of the advanced maximum principle was proved in [H2]*Lemmas 4.1, 8.1.

Let M^n be a closed manifold with a one-parameter family of metrics $g(t) \mid t \in [0,T)$ and associated Levi-Civita connection. Let $\pi : \xi \to M$ be a vector bundle over M, with a connection $\bar{\nabla}$, and let $\hat{\nabla}$ be the connection induced on $\xi \otimes T^*M$ by ∇ and $\bar{\nabla}$, so that

$$\bar{\nabla}(t) : C^\infty(\xi) \to C^\infty(\xi \otimes T^*M), \hat{\nabla}(t) : C^\infty(\xi \otimes T^*M) \to C^\infty(\xi \otimes T^*M \otimes T^*M).$$

We define the Laplacian with respect to g and $\bar{\nabla}$ by

$$\hat{\triangle}\varphi = tr_{g(t)} \hat{\nabla}\bar{\nabla}\varphi, \text{where } \varphi \in C^\infty(\xi).$$

Let $F(\cdot, x, t) : \xi_x \to \xi_x$ be a continuous map such that $F(\cdot,\cdot,t) : \xi \to \xi$ is a fibre preserving map for all $t \in [0,T)$.

PROPOSITION 5. *[CK]*Theorem 4.8 Let* $z(t), t \in [0,T]$ *be a solution of the non-linear system of partial differential equations*

$$\frac{\partial}{\partial t} z = \hat{\triangle} z + \Phi(z)$$

such that $z(x,0) \in \mathcal{C}$ *for all* $x \in M$ *where* $\mathcal{C}$ *satisfies*

1. $\mathcal{C}$ *is invariant under parallel translation by* $\nabla(t)$ *for all* $t \in [0,T]$,
2. $\mathcal{C}_x \equiv \mathcal{C} \cap \pi^{-1}(x)$ *is a closed convex subset of* $\pi^{-1}(x)$ *for all* $x \in M^n$.

If every solution of the ODE

$$\frac{d}{dt} Z = \Phi(Z), \quad Z(0) \in \mathcal{C}_x$$

defined in the fibre $\pi^{-1}(x)$ *remains in* $\mathcal{C}_x$ *for each* $x \in M$ *and* $t \geq 0$, *then* $z(x,t) \in \mathcal{C}$ *for all* $x \in M$ *and* $t \geq 0$.

2.4. Evolution of curvature, finite time existence. By applying the evolution equation for the scalar curvature, we can show finite time existence.

LEMMA 6. *If* $R \geq \rho$ *at* $t = 0$ *then* $T \leq \frac{3}{2\rho}$.

3. Preserving flag pinching. In this section we analyze the reaction ODE for the evolution of curvature in the Ricci flow, and show that the condition of λ-pinched flag curvatures is preserved for $\lambda \geq 1/4$. The argument we employ should be applicable to the analysis of many other situations involving Ricci flow and related equations. In a subsequent paper we will apply similar ideas to the preservation of positive curvature on isotropic 2-planes in higher dimensions.

We consider a function defined on the frame bundle $\mathbb{O}M = \{(x, e_1, \ldots, e_4) : x \in M,\ e_i \in T_xM,\ g_x(e_i, e_j) = \delta_{ij}\}$ by

$$Z(x, e_1, \ldots, e_4) = \frac{1}{\lambda} R_x(e_2, e_1, e_2, e_1) - \lambda R_x(e_3, e_1, e_3, e_1). \tag{2}$$

Our goal is to prove that positivity of Z is preserved if $\lambda \geq \frac{1}{2}$.

Denote by Ω_x the set of algebraic curvature operators at x, i.e. the set of symmetric bilinear forms R defined on the space of antisymmetric $(0,2)$-tensors and satisfying the Bianchi identity. We observe that the set of curvature operators $\{Z \geq 0\} \subset \Omega_x$ is an $O(4)$-invariant, convex set for each $x \in M$, and is invariant under parallel transport, and so by Proposition 5 the Ricci flow preserves the positivity of Z if the reaction ODE does. To analyse this we consider any curvature operator Ω_x for which $Z \geq 0$ for all frames in $\mathbb{O}_x$, and suppose there exists some frame $e_1, \ldots, e_4$ for which $Z = 0$. We will use the first and second order conditions for minimality within $\mathbb{O}_x$ to deduce inequalities for components of the curvature, which in turn will imply that the reaction ODE points into $\{Z \geq 0\}$.

Let $\Gamma(s)$ denote a curve in $O(n)$ such that $\Gamma(0) = Id$. Then locally, $\Gamma(s) = \exp_{Id}(\gamma(s))$ where $\gamma(s)$ is curve in $\mathfrak{so}(n)$. Hence $\frac{d\Gamma}{ds} = d\exp_{Id}(\gamma'(s))$. We compute the first and second derivatives of Z along the curves in $\mathbb{O}_x$ defined by

$$\begin{aligned} \frac{d}{ds} e_i(s) &= \Lambda_{ij} e_j(s); \\ e_i(0) &= e_i, \end{aligned}$$

where Λ is an arbitrary antisymmetric 2-tensor. Since $Z = 0$ we have

$$\frac{1}{\lambda} R_{2121} = \lambda R_{3131}.$$

The first order condition then gives

$$\begin{aligned} \frac{d}{ds} Z &= \frac{2}{\lambda} \Lambda_{2k} R_{k121} + \frac{2}{\lambda} \Lambda_{1k} R_{2k21} - 2\lambda \Lambda_{3k} R_{k131} - 2\lambda \Lambda_{1k} R_{3k31} \\ &= -2\lambda \Lambda_{12} R_{3231} + \frac{2}{\lambda} \Lambda_{13} R_{2321} + 2\Lambda_{14} \left(\frac{1}{\lambda} R_{2421} - \lambda R_{3431} \right) \\ &\quad + 2\Lambda_{23} \left(\frac{1}{\lambda} R_{3121} + 2\lambda R_{2131} \right) + \frac{2}{\lambda} \Lambda_{24} R_{4121} - 2\lambda \Lambda_{34} R_{4131}. \end{aligned}$$

The various choices of Λ then imply the following identities:

$$
\begin{aligned}
R_{1323} &= 0;\\
R_{1223} &= 0;\\
R_{1214} &= 0;\\
R_{1314} &= 0;\\
R_{1213} &= 0;\\
\frac{1}{\lambda}R_{1224} &= \lambda R_{1334}.
\end{aligned}
\tag{3}
$$

For convenience we denote the quantities on each side of the last identity by F. We also define $S = R_{1324} = R_{1423} + R_{1234}$ and $T = R_{1423} - R_{1234}$, so that $R_{1423} = (T+S)/2$ and $R_{1234} = (S-T)/2$.

Next we consider the second order conditions: The second derivative along the curve above is given by

$$
\begin{aligned}
\frac{1}{2}\frac{d^2}{ds^2}Z = {} & \frac{1}{\lambda}\Lambda_{2k}\Lambda_{kl}R_{l121} + \frac{1}{\lambda}\Lambda_{2k}\Lambda_{1l}R_{kl21} + \frac{1}{\lambda}\Lambda_{2k}\Lambda_{2l}R_{k1l1} + \frac{1}{\lambda}\Lambda_{2k}\Lambda_{1l}R_{k12l}\\
& + \frac{1}{\lambda}\Lambda_{1k}\Lambda_{2l}R_{lk21} + \frac{1}{\lambda}\Lambda_{1k}\Lambda_{kl}R_{2l21} + \frac{1}{\lambda}\Lambda_{1k}\Lambda_{2l}R_{2kl1} + \frac{1}{\lambda}\Lambda_{1k}\Lambda_{1l}R_{2k2l}\\
& - \lambda\Lambda_{3k}\Lambda_{kl}R_{l131} - \lambda\Lambda_{3k}\Lambda_{1l}R_{kl31} - \lambda\Lambda_{3k}\Lambda_{3l}R_{k1l1} - \lambda\Lambda_{3k}\Lambda_{1l}R_{k13l}\\
& - \lambda\Lambda_{1k}\Lambda_{3l}R_{lk31} - \lambda\Lambda_{1k}\Lambda_{kl}R_{3l31} - \lambda\Lambda_{1k}\Lambda_{3l}R_{3kl1} - \lambda\Lambda_{1k}\Lambda_{1l}R_{3k3l}\\
= {} & \Lambda\mathbb{M}\Lambda^t
\end{aligned}
$$

where $\Lambda = \begin{bmatrix} \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & \Lambda_{23} & \Lambda_{24} & \Lambda_{34} \end{bmatrix}$, and $\mathbb{M}$ is the 6×6 matrix given by

$$
\begin{bmatrix}
\frac{1}{\lambda}R_{1212}-\lambda R_{2323} & 0 & \lambda R_{2334} & 0 & F & -\frac{\lambda}{2}(T+3S)\\
0 & \frac{1}{\lambda}R_{2323}-\lambda R_{1313} & \frac{1}{\lambda}R_{2324} & 0 & -\frac{T}{\lambda} & -F\\
\lambda R_{2334} & \frac{1}{\lambda}R_{2324} & \frac{1}{\lambda}R_{2424}-\lambda R_{3434} & \frac{1+\lambda^2}{2\lambda}(3S-T) & -\frac{1}{\lambda}R_{1424} & \lambda R_{1434}\\
0 & 0 & \frac{1+\lambda^2}{2\lambda}(3S-T) & \frac{1}{\lambda}R_{1313}-\lambda R_{1212} & 0 & 0\\
F & -\frac{T}{\lambda} & -\frac{1}{\lambda}R_{1424} & 0 & \frac{1}{\lambda}R_{1414}-\lambda R_{1313} & 0\\
-\frac{\lambda}{2}(T+3S) & -F & \lambda R_{1434} & 0 & 0 & \frac{1}{\lambda}R_{1212}-\lambda R_{1414}
\end{bmatrix}
$$

It follows that this matrix is positive semidefinite. Note that we have used the identities derived above in deriving this form.

Next we compute the evolution of Z under the reaction ODE for evolution of curvature under the Ricci flow: Using the formula derived by Richard Hamilton [H1]

$$
\frac{1}{2}\frac{d}{dt}R_{ijkl} = R_{ipjq}R_{kplq} - R_{ipjq}R_{lpkq} + R_{ipkq}R_{jplq} - R_{iplq}R_{jpkq} \tag{4}
$$

and noting the identities (3), we arrive at the following equation for the rate of change of Z at its minimum point:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}Z &= \frac{1}{\lambda}\left(R_{1p2q}^2 + R_{1p1q}R_{2p2q} - 2R_{1p2q}R_{2p1q}\right)\\ &\quad - \lambda\left(R_{1p3q}^2 + R_{1p1q}R_{3p3q} - 2R_{1p3q}R_{3p1q}\right)\\ &= \frac{1}{\lambda}\big(R_{1212}^2 + R_{1224}^2 + R_{1324}^2 + R_{1423}^2 + R_{1424}^2 + R_{1313}R_{2323}\\ &\qquad + R_{1414}R_{2424} - 4R_{1324}R_{2314} - 2R_{1424}^2\big)\\ &\quad - \lambda\big(R_{1313}^2 + R_{1334}^2 + R_{1234}^2 + R_{1432}^2 + R_{1434}^2 + R_{1212}R_{2323}\\ &\qquad + R_{1414}R_{3434} - 4R_{1234}R_{3214} - 2R_{1434}^2\big)\\ &= \frac{1}{\lambda}\left(R_{1212}^2 + R_{1313}R_{2323} + R_{1414}R_{2424} - R_{1424}^2\right)\\ &\quad - \lambda\left(R_{1313}^2 + R_{1212}R_{2323} + R_{1414}R_{3434} - R_{1434}^2\right)\\ &\quad + \left(\lambda - \frac{1}{\lambda}\right)F^2 + \frac{1+2\lambda^2}{4\lambda}T^2 - \frac{3(1+2\lambda^2)}{4\lambda}S^2 - \frac{3}{2\lambda}TS.\end{aligned}$$

We wish to show that the right-hand side is non-negative at the minimum point, by making use of the non-negativity of the matrix $\mathbb{M}$. The components of $\mathbb{M}$ are linear in the components of the curvature tensor, while the quantity we wish to estimate is quadratic, so we proceed by using the non-negativity of the matrix $\mathbb{M}\otimes\mathbb{M}$ induced by $\mathbb{M}$ on the space of 2-tensors on the space of 2-planes, which is defined by

$$(\mathbb{M}\otimes\mathbb{M})_{\alpha\beta,\gamma\delta} = \mathbb{M}_{\alpha\gamma}\mathbb{M}_{\beta\delta}$$

where the indices range over an orthonormal basis for Λ^2. Diagonal elements of this matrix are products of diagonal elements of $\mathbb{M}$. We observe first that certain of the terms arising in $\frac{dZ}{dt}$ can be expressed naturally in terms of diagonal elements of $\mathbb{M}$:

$$\begin{aligned}&\frac{1}{\lambda}\left(R_{1212}^2 + R_{1313}R_{2323} + R_{1414}R_{2424}\right) - \lambda\left(R_{1313}^2 + R_{1212}R_{2323} + R_{1414}R_{3434}\right)\\ &= R_{1212}\left(\frac{1}{\lambda}R_{1212} - \lambda R_{2323}\right) + R_{1313}\left(\frac{1}{\lambda}R_{2323} - \lambda R_{1313}\right) + R_{1414}\left(\frac{1}{\lambda}R_{2424} - \lambda R_{3434}\right)\\ &= R_{1212}\mathbb{M}_{1212} + R_{1313}\mathbb{M}_{1313} + R_{1414}\mathbb{M}_{1414}.\end{aligned}$$

In order to write this in terms of $\mathbb{M}\otimes\mathbb{M}$, we observe that the coefficients R_{1212}, R_{1313} and R_{1414} can be written as diagonal elements of $\mathbb{M}$, in various different ways: In particular, for any α, β, γ we can write

$$\begin{aligned}&\frac{1}{\lambda}\left(R_{1212}^2 + R_{1313}R_{2323} + R_{1414}R_{2424}\right) - \lambda\left(R_{1313}^2 + R_{1212}R_{2323} + R_{1414}R_{3434}\right)\\ &\quad = \frac{\lambda^3\cos^2\alpha}{1-\lambda^2}\mathbb{M}_{1212}\mathbb{M}_{2424} + \frac{\lambda\cos^2\alpha}{1-\lambda^2}\mathbb{M}_{1212}\mathbb{M}_{3434} + \frac{\lambda^3\sin^2\alpha}{1-\lambda^4}\mathbb{M}_{1212}\mathbb{M}_{2323}\\ &\qquad + \frac{\lambda\cos^2\beta}{1-\lambda^2}\mathbb{M}_{1313}\mathbb{M}_{2424} + \frac{\cos^2\beta}{\lambda(1-\lambda^2)}\mathbb{M}_{1313}\mathbb{M}_{3434} + \frac{\lambda\sin^2\beta}{1-\lambda^4}\mathbb{M}_{1313}\mathbb{M}_{2323}\\ &\qquad + \frac{\lambda(1-\lambda^2\sin^2\gamma)}{1-\lambda^2}\mathbb{M}_{1414}\mathbb{M}_{2424} + \frac{\lambda\cos^2\gamma}{1-\lambda^2}\mathbb{M}_{1414}\mathbb{M}_{3434} + \frac{\lambda^3\sin^2\gamma}{1-\lambda^4}\mathbb{M}_{1414}\mathbb{M}_{2323}.\end{aligned}$$

We will apply $\mathbb{M}\otimes\mathbb{M}$ to particular vectors chosen to produce useful off-diagonal terms. First define the operation $\hat\wedge$ to be the wedge product on the space Λ^2, so that

$$(e_i\wedge e_j)\hat\wedge(e_k\wedge e_l) = \frac{1}{\sqrt{2}}\left[(e_i\wedge e_j)\otimes(e_k\wedge e_l) - (e_k\wedge e_l)\otimes(e_i\wedge e_j)\right].$$

The vectors of interest are then

$$V_1^{\pm} = \sqrt{\frac{\lambda^3}{1-\lambda^2}}\cos\alpha(e_1\wedge e_2)\hat{\wedge}(e_2\wedge e_4) \pm \sqrt{\frac{1}{\lambda(1-\lambda^2)}}\cos\beta(e_1\wedge e_3)\hat{\wedge}(e_3\wedge e_4);$$
$$V_2^{\pm} = \sqrt{\frac{\lambda}{1-\lambda^2}}\cos\alpha(e_1\wedge e_2)\hat{\wedge}(e_3\wedge e_4) \pm \sqrt{\frac{\lambda}{1-\lambda^2}}\cos\beta(e_1\wedge e_3)\hat{\wedge}(e_2\wedge e_4);$$
$$V_3^{\pm} = \sqrt{\frac{\lambda^3}{1-\lambda^4}}\sin\alpha(e_1\wedge e_2)\hat{\wedge}(e_2\wedge e_3) \pm \sqrt{\frac{\lambda}{1-\lambda^2}}\cos\gamma(e_1\wedge e_4)\hat{\wedge}(e_3\wedge e_4);$$
$$V_4^{\pm} = \sqrt{\frac{\lambda}{1-\lambda^4}}\sin\beta(e_1\wedge e_3)\hat{\wedge}(e_2\wedge e_3) \pm \sqrt{\frac{\lambda(1-\lambda^2\sin^2\gamma)}{1-\lambda^2}}(e_1\wedge e_4)\hat{\wedge}(e_2\wedge e_4);$$
$$V_5 = \sqrt{\frac{\lambda^3}{1-\lambda^4}}\sin\gamma(e_1\wedge e_4)\hat{\wedge}(e_2\wedge e_3).$$

The inequality $\varepsilon_1\mathbb{M}\otimes\mathbb{M}(V_1^+,V_1^+)+(1-\varepsilon_1)\mathbb{M}\otimes\mathbb{M}(V_1^-,V_1^-)\geq 0$ can be written as

$$\frac{\lambda^3\cos^2\alpha}{1-\lambda^2}\mathbb{M}_{1212}\mathbb{M}_{2424} + \frac{\cos^2\beta}{\lambda(1-\lambda^2)}\mathbb{M}_{1313}\mathbb{M}_{3434} \geq \left(\frac{\lambda^3\cos^2\alpha}{1-\lambda^2} + \frac{\cos^2\beta}{\lambda(1-\lambda^2)}\right)F^2 + (1-2\varepsilon_1)\frac{\lambda\cos\alpha\cos\beta}{1-\lambda^2}T(T+3S). \tag{5}$$

The inequality $\mathbb{M}\otimes\mathbb{M}(V_2^+,V_2^+)\geq 0$ yields

$$\frac{\lambda\cos^2\alpha}{1-\lambda^2}\mathbb{M}_{1212}\mathbb{M}_{3434} + \frac{\lambda\cos^2\beta}{1-\lambda^2}\mathbb{M}_{1313}\mathbb{M}_{2424} \geq \frac{\lambda^3\cos^2\alpha}{4(1-\lambda^2)}(T+3S)^2 + \frac{\cos^2\beta}{\lambda(1-\lambda^2)}T^2 + 2\frac{\lambda\cos\alpha\cos\beta}{1-\lambda^2}F^2. \tag{6}$$

From $\mathbb{M}\otimes\mathbb{M}(V_3^{\pm},V_3^{\pm})\geq 0$ we obtain

$$\frac{\lambda^3}{1-\lambda^4}\sin^2\alpha\mathbb{M}_{1212}\mathbb{M}_{2323} + \frac{\lambda}{1-\lambda^2}\cos^2\gamma\mathbb{M}_{1414}\mathbb{M}_{3434} \geq \frac{\lambda^3}{1-\lambda^2}\cos^2\gamma R_{1434}^2 \pm \frac{\lambda^2\sqrt{1+\lambda^2}}{2(1-\lambda^2)}\sin\alpha\cos\gamma\left(9S^2-T^2\right). \tag{7}$$

From $\varepsilon_4\mathbb{M}\otimes\mathbb{M}(V_4^+,V_4^+)+(1-\varepsilon_4)\mathbb{M}\otimes\mathbb{M}(V_4^-,V_4^-)\geq 0$ we obtain

$$\frac{\lambda}{1-\lambda^4}\sin^2\beta\mathbb{M}_{1313}\mathbb{M}_{2323} + \frac{\lambda(1-\lambda^2\sin^2\gamma)}{1-\lambda^2}\mathbb{M}_{1414}\mathbb{M}_{2424} \geq \frac{1-\lambda^2\sin^2\gamma}{\lambda(1-\lambda^2)}R_{1424}^2 + (1-2\varepsilon_4)\frac{\sqrt{1-\lambda^2\sin^2\gamma}\sqrt{1+\lambda^2}}{\lambda(1-\lambda^2)}\sin\beta T(T-3S). \tag{8}$$

Finally, from $\mathbb{M}\otimes\mathbb{M}(V_5,V_5)\geq 0$ we have

$$\frac{\lambda^3}{1-\lambda^4}\sin^2\gamma\mathbb{M}_{1414}\mathbb{M}_{2323} \geq \frac{\lambda(1+\lambda^2)}{4(1-\lambda^2)}\sin^2\gamma(3S-T)^2. \tag{9}$$

These identities give the following for the time evolution of Z:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}Z \geq & \left(-\frac{1}{\lambda}+\frac{1-\lambda^2\sin^2\gamma}{\lambda(1-\lambda^2)}\right)R_{1424}^2+\left(\lambda+\frac{\lambda^3\cos^2\gamma}{1-\lambda^2}\right)R_{1434}^2\\ &+\left(\lambda-\frac{1}{\lambda}+\frac{\lambda^3\cos^2\alpha}{1-\lambda^2}+\frac{\cos^2\beta}{\lambda(1-\lambda^2)}+\frac{2\lambda\cos\alpha\cos\beta}{1-\lambda^2}\right)F^2\\ &+\frac{1+4\lambda^2}{4\lambda}T^2-\frac{3(1+2\lambda^2)}{4\lambda}S^2-\frac{3}{2\lambda}TS+(1-2\varepsilon_1)\frac{\lambda\cos\alpha\cos\beta}{1-\lambda^2}T(T+3S)\\ &+\frac{\lambda^3\cos^2\alpha}{4(1-\lambda^2)}(T+3S)^2+\frac{\cos^2\beta}{\lambda(1-\lambda^2)}T^2\pm\frac{\lambda^2\sqrt{1+\lambda^2}\sin\alpha\cos\gamma}{2(1-\lambda^2)}\left(9S^2-T^2\right)\\ &+(1-2\varepsilon_4)\frac{\sqrt{1-\lambda^2\sin^2\gamma}\sqrt{1+\lambda^2}}{\lambda(1-\lambda^2)}\sin\beta\, T(T-3S)\\ &+\frac{\lambda(1+\lambda^2)}{4(1-\lambda^2)}\sin^2\gamma(3S-T)^2.\end{aligned}$$

The coefficient of R_{1434}^2 is positive, while that of R_{1424}^2 is non-negative (positive if $\sin\gamma<1$). The coefficient of F^2 can be rearranged to give

$$\frac{\left(\lambda^2\cos\alpha+\cos\beta\right)^2}{\lambda(1-\lambda^2)}-\frac{1-\lambda^2}{\lambda},$$

which is non-negative provided

$$\left|\lambda^2\cos\alpha+\cos\beta\right|\geq 1-\lambda^2. \tag{10}$$

We observe that the curvature operator of $\mathbb{CP}^2$ evolves by simply scaling under the Ricci flow reaction ODE, and has 1/4-pinched flag curvatures. That is, in this case we have $Z=0$ for $\lambda=\frac{1}{2}$, and $\frac{dZ}{dt}=0$. For this case we have $T=0$ but $S\neq 0$, while $F=0$. It follows that there can be no way to produce a positive coefficient for S^2 in $\frac{dZ}{dt}$. Clearly, however, to preserve 1/4-pinching we must be able to achieve at least a non-negative coefficient for S^2. We consider this coefficient in more detail: It can be written as follows, choosing the inequality for V_3^+ rather than that for V_3^-:

$$\frac{3(1+2\lambda^2)(4\lambda^2-1)}{4\lambda(1-\lambda^2)}-\frac{9\lambda}{4(1-\lambda^2)}\left(\lambda\sin\alpha-\sqrt{1+\lambda^2}\cos\gamma\right)^2.$$

In particular, for $\lambda=\frac{1}{2}$ this is non-positive, with equality precisely when

$$\cos\gamma=\frac{\lambda}{\sqrt{1+\lambda^2}}\sin\alpha. \tag{11}$$

We make this choice (for all $\lambda\geq\frac{1}{2}$) to achieve a positive coefficient for S^2. Next we consider the coefficient of ST: In the case $\lambda=\frac{1}{2}$ this must be made to vanish to obtain a sign on dZ/dt, and we opt to produce this in all cases. This coefficient is given by

$$-\frac{3}{2\lambda(1-\lambda^2)}+\frac{3}{\lambda(1-\lambda^2)}\left((1-2\varepsilon_1)\lambda^2\cos\alpha\cos\beta-(1-2\varepsilon_4)\sqrt{1-\lambda^4\cos^2\alpha}\sin\beta\right).$$

This can be made to vanish in many different ways, for example by choosing $\cos\alpha=1$, $\cos\beta=\frac{1}{2}$ (note that the previous requirement is then satisfied since $\lambda\geq\frac{1}{2}$), and $\varepsilon_1=0$

and $\varepsilon_4 = \frac{1}{2}\left(1+\sqrt{\frac{1-\lambda^2}{1+\lambda^2}}\right)$ (the latter lies in the range $\left(\frac{1}{2},\frac{1}{2}\left[1+\sqrt{\frac{3}{5}}\right]\right)$ and so is an admissible choice). It remains to check the coefficient of T^2, which is then

$$\frac{\lambda(4-\lambda^2)}{2(1-\lambda^2)} > 0.$$

4. Lower bound. In this section we show that if at time $t = 0, Z_\lambda \geq \bar{m}R_{1212}$ then this inequality is preserved. By compactness of the frame bundle, if (M, g_{ij}) is strictly quarter pinched such an estimate always exists. As in the above case we consider the case where Z_λ attains its minimum that is $Z_\lambda = \bar{m}R_{1212}$ or $(\frac{1}{\lambda}-\bar{m})R_{1212} - \lambda R_{1313} = 0$. Let us consider the following quantity,

$$Z_{\lambda,\bar{m}} = \frac{1}{\lambda_{\bar{m}}}R_{1212} - \lambda R_{1313},$$

where $\lambda_{\bar{m}} = \frac{\lambda}{1-\bar{m}\lambda}$ and let $\mathcal{Z}_{\bar{m}}$ be the set of algebraic curvature tensors such that $Z_{\lambda,m} \geq 0$. This a closed convex subset of Ω_x. The inequality follows from the computation in the previous section, note that

$$R_{1212} - \lambda_m \lambda R_{1313} \geq 0. \tag{12}$$

Define $\mu_m = \sqrt{\lambda_m \lambda}$, then (12) becomes

$$\frac{1}{\mu_m}R_{1212} - \mu_m R_{1313} \geq 0.$$

Hence this inequality holds as long as $\mu_m \geq 1/4$.

5. Improving flag pinching. In this section, we will show that in regions of high curvature, the ratio sectional curvatures approaches unity. Let us consider the following quantity, let

$$Q_{\epsilon,\lambda} = \left(\frac{R_{1313}-R_{1212}}{R_{1212}}\right)R^\epsilon = \left(\frac{Z_\lambda}{\lambda R_{1212}} + \frac{1}{\lambda^2} - 1\right)R^\epsilon.$$

Note that $Q_{\lambda,\epsilon}$ has no explicit dependence on λ, we use this notation merely to remind us that λ will be used in the estimate. Furthermore let us also consider, $\mathcal{C}_x$, a convex closed subset of the bundle of algebraic curvature tensors,

$$\mathcal{C} = \{\mathbb{P} \mid \exists K \in \mathbb{R}, \epsilon \in (0,1] \mid \mathbb{P}_{1212} - \mathbb{P}_{1313} \leq K \operatorname{tr}(\mathbb{P})^\epsilon \mathbb{P}_{1212}\} \subset \Omega_x. \tag{13}$$

We will show that the Ricci flow preserves the set $\mathcal{C}_x$. If we consider a frame maximum of $Q_{\epsilon,\lambda}$, this is equivalent to a frame minimum of $\frac{Z_\lambda}{R_{1212}}$ and the reaction ODE is

$$\frac{d}{dt}Q_{\lambda,\epsilon} = R^\epsilon\left(-\frac{d}{dt}\frac{Z_\lambda}{\lambda R_{1212}} - 2\epsilon\frac{|Rc|^2}{R}\left(-\frac{Z_\lambda}{\lambda R_{1212}} + \frac{1}{\lambda^2} - 1\right)\right). \tag{14}$$

THEOREM 7. *Assume that there exists a $m \geq \bar{m}$ such that $\min_{O\in\mathbb{O}_x}\frac{Z_\lambda}{R_{1212}} = m$, then at a minimal frame $O = \{e_1, e_2, \ldots, e_4\}$, we have that*

$$\frac{d}{dt}\left(\frac{Z_\lambda}{R_{1212}}\right) \geq \delta\left(2\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}} - \delta\right)\frac{1}{R_{1212}}\left(\frac{R_{2424}}{\lambda_m} - \lambda R_{3434}\right)\left(\frac{R_{1414}}{\lambda_m} - \lambda R_{1313}\right). \tag{15}$$

where

$$\delta \leq \min\left\{2\sqrt{\frac{\lambda_{\bar{m}}}{1-\lambda\lambda_{\bar{m}}}}, \frac{3}{4}\frac{\sqrt{\lambda}(4-\lambda\lambda_{\bar{m}})}{\sqrt{1-\lambda\lambda_{\bar{m}}}}\right\}$$

Consider the following equations,

$$\frac{d}{dt}\left(\frac{Z_\lambda}{R_{1212}}\right) = \frac{1}{R_{1212}}\frac{dZ_\lambda}{dt} - \frac{Z_\lambda}{R_{1212}^2}\frac{dR_{1212}}{dt} = \frac{1}{R_{1212}}\frac{d}{dt}\left(\left(\frac{1}{\lambda} - m\right)R_{1212} - \lambda R_{1313}\right).$$

Furthermore note that if we have $\frac{d}{ds}e_i(s) = \Lambda_{ij}e_j(s)$, then at a minimum of Z_λ,

$$\frac{d}{ds}\left(\frac{Z_\lambda}{R_{1212}}\right) = \frac{1}{R_{1212}}\frac{d}{ds}Z_{\lambda,m} = 0,$$

and

$$\frac{d^2}{ds^2}\left(\frac{Z_\lambda}{R_{1212}}\right) = \frac{1}{R_{1212}^2}\frac{d^2}{ds^2}Z_{\lambda,m}.$$

Hence it suffices to prove the following proposition,

PROPOSITION 8. *Assume that there exists a $m \geq \bar{m}$ such that $\min_{O\in\mathbb{O}_x}\frac{Z_\lambda}{R_{1212}} = m$, then at a minimal frame $O = \{e_1, e_2, \ldots, e_4\}$, we have that*

$$\frac{d}{dt}Z_{\lambda,m} \geq \delta\left(2\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}} - \delta\right)\left(\frac{R_{2424}}{\lambda_m} - \lambda R_{3434}\right)\left(\frac{R_{1414}}{\lambda_m} - \lambda R_{1313}\right) \tag{16}$$

where

$$\delta \leq \min\left\{2\sqrt{\frac{\lambda_{\bar{m}}}{1-\lambda\lambda_{\bar{m}}}}, \frac{3}{4}\frac{\sqrt{\lambda}(4-\lambda\lambda_{\bar{m}})}{\sqrt{1-\lambda\lambda_{\bar{m}}}}\right\}$$

Proof. We replace the vector $V_2^{\pm}$, with the vector,

$$V_{2,m,\delta}^{\pm} = \sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}}\cos\alpha(e_1\wedge e_2)\hat{\wedge}(e_3\wedge e_4) \pm \left(\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}} - \delta\right)\cos\beta(e_1\wedge e_3)\hat{\wedge}(e_2\wedge e_4)$$

As a consequence, the inequality (6) becomes,

$$\begin{aligned}\frac{\lambda_m\cos^2\alpha}{1-\lambda\lambda_m}\mathbb{M}_{1313}\mathbb{M}_{2424} + \frac{\lambda_m\cos^2\beta}{1-\lambda\lambda_m}\mathbb{M}_{1313}\mathbb{M}_{2424} &\geq \delta\left(2\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}} - \delta\right)\mathbb{M}_{1212}\mathbb{M}_{3434} \\ &+ \frac{\lambda_m\lambda^2\cos^2\alpha}{4(1-\lambda\lambda_m)}(T+3S)^2 + \frac{1}{\lambda_m^2}\left(\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}} - \delta\right)^2\cos^2\beta T^2 \\ &+ 2\left(\frac{\lambda_m}{1-\lambda\lambda_m} - \delta\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}}\right)\cos\alpha\cos\beta F^2.\end{aligned}$$

To ensure that we have a non-negative coefficient on $\mathbb{M}_{1313}\mathbb{M}_{2424}$ on the right hand side we require that $\delta < 2\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}}$. Choosing $\cos\alpha{=}1, \cos\beta{=}\frac{1}{2}$, the coefficient of F_m^2 then becomes,

$$-\frac{1-\lambda\lambda_m}{\lambda_m} + \frac{(\lambda\lambda_m + \frac{1}{2})^2}{\lambda(1-\lambda\lambda_m)} - \delta\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}},$$

which is then non-negative if

$$\delta < \frac{3}{4}\frac{4\lambda\lambda_m - 1}{\lambda\sqrt{\lambda_m(1-\lambda\lambda_m)}}.$$

Finally we must check the coefficient of the T^2 term,

$$\frac{\lambda(4-\lambda\lambda_m)}{2(1-\lambda\lambda_m)} + \frac{1}{4\lambda_m^2}\left(\delta^2 - 2\delta\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}}\right) = \frac{1}{4\lambda_m^2}\left[\delta^2 - 2\delta\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}} + \frac{2\lambda\lambda_m(4-\lambda\lambda_m)}{1-\lambda\lambda_m}\right]. \tag{17}$$

We compute the discriminant of this polynomial,

$$\text{disc} = \frac{4\lambda_m}{1-\lambda\lambda_m}\left[1 + 2\lambda\lambda_m - 8\lambda^2\lambda_m^2\right],$$

The zeroes of this polynomial are given by $\lambda\lambda_m = 2 \pm \sqrt{\frac{7}{2}}$ and as $2 + \sqrt{\frac{7}{2}} \geq 1 \geq \lambda\lambda_m \geq \frac{1}{4} \geq 2 - \sqrt{\frac{7}{2}}$, disc < 0, and the polynomial (17) is positive for $\delta \geq 0$. Hence if we choose

$$\delta < \min\left\{2\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}}, \frac{3}{4}\frac{4\lambda\lambda_m - 1}{\lambda\sqrt{\lambda_m(1-\lambda\lambda_m)}}\right\}.$$

then

$$\frac{d}{dt}Z_\lambda \geq \delta\left(2\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}} - \delta\right)(\frac{1}{\lambda_m}R_{2424} - \lambda R_{3434})(\frac{1}{\lambda_m}R_{1414} - \lambda R_{1313}). \tag{18}$$

It remains to show that we can choose δ independent of m, this follows from the following two inequalities

$$\sqrt{\frac{\lambda_{\bar{m}}}{1-\lambda\lambda_{\bar{m}}}} \leq \sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}},$$
$$\frac{3}{4}\frac{\sqrt{\lambda}(4-\lambda\lambda_{\bar{m}})}{\sqrt{1-\lambda\lambda_{\bar{m}}}} \leq \frac{3}{4}\frac{4\lambda\lambda_m - 1}{\lambda\sqrt{\lambda_m}\sqrt{1-\lambda\lambda_m}}.$$

□

Finally, let us note a trivial consequence of λ pinching

Lemma 9. *Let (M, g) be a Riemannian manifold such that $\frac{1}{\lambda}R_{ijij} - \lambda R_{klkl} \geq 0$ for all frames $\mathcal{O} = \{e_i, e_j, e_k, e_l\}$, then for any sectional curvature R_{mnmn}*

$$1 \leq \frac{R}{R_{mnmn}} \leq \frac{n(n-1)}{\lambda^4}.$$

Returning to the equation (14), applying the estimate (15) we get

$$\begin{aligned}\frac{d}{dt}Q_{\lambda,\epsilon} \leq R^{\epsilon}\Bigg[&-\delta\left(2\sqrt{\frac{\lambda_m}{1-\lambda\lambda_m}}-\delta\right)\frac{1}{R_{1212}}\left(\frac{R_{2424}}{\lambda_m}-\lambda R_{3434}\right)\left(\frac{R_{1414}}{\lambda_m}-\lambda R_{1313}\right)\\ &+2\epsilon\frac{|Rc|^2}{R}\left(-m+\frac{1}{\lambda^2}-1\right)\Bigg]\\ \leq R^{\epsilon}&\left[-\delta\left(2\sqrt{\frac{\lambda_{\bar{m}}}{1-\lambda\lambda_{\bar{m}}}}-\delta\right)\frac{m^2R_{2424}R_{1414}}{R_{1212}}+2\epsilon R\left(-\bar{m}+\frac{1}{\lambda^2}-1\right)\right]\\ \leq R^{\epsilon}&\left[-\delta\left(2\sqrt{\frac{\lambda_{\bar{m}}}{1-\lambda\lambda_{\bar{m}}}}-\delta\right)\bar{m}^2R_{2424}+2\epsilon\frac{n(n-1)}{\lambda^4}R_{2424}\left(-\bar{m}+\frac{1}{\lambda^2}-1\right)\right]\end{aligned}$$

Hence if we choose

$$\epsilon \leq \left(\frac{\delta\lambda^4\left(2\sqrt{\frac{\lambda_{\bar{m}}}{1-\lambda\lambda_{\bar{m}}}}-\delta\right)}{2n(n-1)\left(-\bar{m}+\frac{1}{\lambda^2}-1\right)}\right)$$

we get that

$$\frac{d}{dt}Q_{\lambda,\epsilon} \leq 0$$

at a maximal frame $O \in \mathbb{O}_x$ which shows that $Q(t) \in \mathcal{C}$. This implies that we have an estimate of the form, $Q_{\lambda,\epsilon}(t) \leq C(n,\lambda,\bar{m},Q_{\lambda,\epsilon}(0))$.

6. Convergence. Using Hamilton's compactness theorem, we can prove that M is diffeomorphic to either $\mathbb{S}^4$ or $\mathbb{RP}^4$. In this case we show that after rescaling a sequence of metrics, $g(t_i) \to g$, where g is a spherical space form.

We gather the necessary theorems from [P] and [CCG+].

Theorem 10 (No local Collapsing). *Let $g_{ij}(t), t \in [0,T]$ be a smooth solution to the Ricci Flow on a closed manifold M^n. If $T < \infty$, then for any $\rho > 0$ there exists $\kappa = \kappa(g_{ij}(0),T,\rho)$ such that $g(t)$ is κ-collapsed below the scale ρ for all $t \in [0,T)$.*

This is equivalent to the following local injectivity radius

Theorem 11 (Local Injectivity Radius). *Let $(M^n, g_{ij}(t))$ be a solution to the Ricci flow on a closed manifold. Then for every constant C, there exists a constant $a > 0$ depending only on $C, g(0)$ and T such that if (x_i,t_i) is a sequence of points and times such that*

$$|Rm[g(t_i)]| \leq CK_i,$$

in $B_{g_{t_i}}(x_i,(CK_i)^{-\frac{1}{2}})$ then

$$\mathrm{inj}_{g(t_i)}(x_i) \geq \frac{a}{\sqrt{K_i}}.$$

Theorem 12. *There exists a subsequence which converges in C^∞ to a complete Riemannian manifold $(M^n_\infty, g_\infty, \mathcal{O}_\infty)$ with $|\mathrm{Rm}[g_\infty]| \leq C$ on M^n_∞, $\mathrm{inj}(g_\infty,\mathcal{O}_\infty) \geq c$ and $|\nabla^k \mathrm{Rm}[g_\infty] \leq C_k$ on M^n_∞.*

Theorem 13 (Compactness theorem-local). *Let $(M_i^n, g_i(t), \mathcal{O}_i)_{i\in\mathbb{N}}, t \in [O,T]$ with $T > 0$ be a sequence of complete pointed solutions to the Ricci flow. Let $p_0 > 4$ be an integer and $s_0 > 0$. Suppose that we have:*

1. *the uniform derivative of curvature bounds*

$$\sup_{\mathbb{B}_{g_i(0)(\mathcal{O}_i, s_0)\times[0,T]}} |\nabla^q \operatorname{Rm}[g_i(t)]| \leq C_{q,s_0} < \infty$$

2. *an injectivity radius bound*

$$\operatorname{inj}_{g_i(0)}(\mathcal{O}_i) \geq \delta > 0 \forall i \in \mathbb{N}.$$

Then there exists $c(n) < \infty$ and a subsequence of

$$\left\{ \left(\mathbb{B}_{g_i(0)} \left(\mathcal{O}_i, e^{-c(n)TC_{0,s_0}} \right), g_i(t), \mathcal{O}_i \right) \right\}_{i\in\mathcal{N}}, t \in [0,T]$$

which converges to an evolving pointed Riemannian manifold, $\{\mathbb{B}_\infty^n, g_\infty(t), \mathcal{O}_i\}, t \in [0,T]$ in the $C^{p_0-2}(g_\infty(0))$-topology and $g_\infty(t)$ is a solution of the Ricci flow. Furthermore, if we assume the global bounds

$$\sup_{\mathbb{M}_i^n\times[0,T]} |\nabla^q \operatorname{Rm}[g_i(t)]| \leq C_q < \infty$$

for all $0 \leq q \leq p_0$ then there exists a subsequence of $\{(M_i^n, g_i(t), \mathcal{O}_i\}_{i\in\mathcal{N}}, t \in [0,T]$ which converges to an evolving complete Riemannian manifold $\{(M_\infty^n, g_\infty(t), \mathcal{O}_\infty\}, t \in [0,T]$ in the $C^{p_0-2}(g_\infty(0))$ topology and $g_\infty(t)$ is a solution of the Ricci flow.

The proof of the diffeomorphism results then follows standard convergence arguments, which we include for the sake of completeness. The argument for positive Ricci curvature in dimension 3 is given in [CCG+]*Section 4.2, we modify the argument where necessary.

Theorem 14. *Let (M^4, g_{ij}) be a closed 4-manifold with strictly quarter pinched flag curvature, then M^4 is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^2$.*

Proof. By the above theorems , we know that there exists a sequence of points and times, (x_i, t_i) such that $t_i \to T$ where

$$K_i = \sup_{x\in M^4} |\operatorname{Rm}[x, t_i]|$$

and the rescaled solutions $g_i(t) = K_i g(t_i + \frac{t_i}{K_i})$ converge in C^∞ on compact subsets to a complete ancient solution, $(M, g_\infty), t \in (-\infty, \omega) \mid \omega > 0$ with quarter pinched flag curvature and $|\operatorname{Rm}(x_\infty, 0)| = 1$. This implies that $R[g_\infty(x_\infty, 0)] \geq 0$. By the maximum principle we then have $R[g_\infty(t)] \geq 0$ and $R[g_\infty]$ and hence we have a positive lower bound $R[g_\infty(t)] \geq c > 0$ in $\mathbb{B}_{g_\infty}(x_\infty, \rho)$. We recall the flag difference pinching result,

$$\left(\frac{R_{1313}}{R_{1213}} - 1 \right) \leq C(n, \lambda, \bar{m}) R^\epsilon.$$

Now $g_i(0) = K_i g(t_i)$ converges to $g_\infty(0)$ in C^∞ on compact sets. Hence for sufficiently large i, we have

$$R[g_i](x_\infty, 0) \geq \frac{1}{2} \inf_{x\in\mathbb{B}_{g_\infty(0)}(x_\infty,\rho)} R[g_\infty](x) \geq \frac{c}{2},$$

for $x \in \mathbb{B}_{g_i(0)}(x_i, \rho - 1)$. This implies that

$$R[g(t_i)] = K_i R[g_i(0)] \geq \frac{c}{2} K_i$$

in $\mathbb{B}_{g_i(0)}(x_i, \rho - 1)$. Hence we have that

$$\begin{aligned}\left(\frac{R_{1313}}{R_{1212}} - 1\right)[g_i(0)] &= \left(\frac{R_{1313}}{R_{1212}} - 1\right)[g(t_i] \\ &\leq CR[g(t_i)]^{-\epsilon} \leq C\left(\frac{c}{2}K_i\right)^{-\epsilon}\end{aligned}$$

in $\mathbb{B}_{g_i(0)}(x_i, \rho - 1)$. Since $g_i(0) \to g_{\infty(0)}$ we see that

$$\left(\frac{R_{1313}}{R_{1212}} - 1\right)[g_\infty(0)] \leq 2C\left(\frac{c}{2}K_i\right)^{-\epsilon},$$

in $\mathbb{B}_{g_\infty(0)}(x_\infty, \rho - 2)$ for all i sufficiently large. Since $\lim_{i\to\infty} K_i = \infty$, we have that $R_{ijij} - R_{klkl} \leq f(\lambda)(R_{1313} - R_{1212}) \to 0$ in $\mathbb{B}_{g_\infty(0)}(x_\infty, \rho - 2)$. As ρ is arbitrary we conclude that (M^4, g_∞) has pointwise constant positive sectional curvature on M^4 for the metric $g_\infty(0)$. Hence by Schur's lemma, the metric $g_\infty(0)$ has constant sectional curvature and by the Bonnet-Myer's theorem, M^4_∞ is compact and furthermore as M^4_∞ is compact and admits a metric which is the limit of metrics on M^4 we conclude that M^4_∞ is diffeomorphic to M^4. This proves that that M^4 with quarter pinched flag curvature admits a C^∞ metric which has constant positive sectional curvature. ∎

7. Weak quarter pinched flag curvature. In this section, we consider the case of weakly quarter pinched flag curvature. To this end we use a degenerate maximum principle first introduced by Bony. We use a form introduced by Brendle and Schoen [BS2].

Theorem 15 (Bony's Maximum Principle, [BS2]). *Let M be a compact manifold and let $\mathbb{E}$ be a vector bundle over M with a fixed bundle metric h. Let $\mathbb{P}$ be the bundle whose fiber over $p \in M$ consists of all orthonormal k-frames $\{e_1, \ldots, e_k\} \subset E_p$. Let $g(t), t \in [0, \delta]$ be a smooth family of Riemannian metrics on M and let $D(t), t \in [0, \delta]$ be a smooth family of connections on E that are compatible with the metric h. Assume that u is a non-negative smooth function on $\mathbb{P} \times (0, \delta)$ satisfying*

$$\frac{\partial}{\partial t} u \geq \mathcal{L}u + \alpha\left\{0, \inf_{\xi \in V, |\xi|=1} D^2 u(\xi, \xi)\right\} - \alpha \sup_{\xi \in V, |\xi|=1} Du(\xi) - \alpha u,$$

where $\mathcal{L}$ is the horizontal Laplacian on P, V denotes the vertical subspace and α is a positive constant. Then the set $F = \{u = 0, \subset P \times (0, \delta)$ is invariant under parallel translation.

The evolution equation for the term Z_λ without simplifications, is given by

$$\begin{aligned}\frac{dZ}{dt} = &\frac{1}{\lambda}\left(R_{1212}^2 + R_{1313}R_{2323} + R_{1414}R_{2424} - R_{1424}^2\right) \\ &- \lambda\left(R_{1313}^2 + R_{1212}R_{2323} + R_{1414}R_{3434} - R_{1434}^2\right) && (19) \\ &+ \lambda R_{1242}^2 - \frac{1}{\lambda}R_{1343}^2 + \frac{1+2\lambda^2}{4\lambda}T^2 - \frac{3(1+2\lambda^2)}{4\lambda}S^2 - \frac{3}{2\lambda}TS && (20) \\ &+ \frac{2}{\lambda}R_{1413}R_{2342} - 2\lambda R_{1412}R_{2343} + (\lambda - \frac{1}{\lambda})R_{1323}^2 + (\frac{1}{\lambda} - \lambda)R_{1242}^2 && (21) \\ &+ \frac{1}{\lambda}R_{1412}^2 - \lambda R_{1413}^2 + (\frac{1}{\lambda} - \lambda)R_{1213}^2. && (22)\end{aligned}$$

We compute the first and second derivatives of the flag pinching in the frame bundle,

$$
\begin{aligned}
\frac{1}{2}\frac{\partial Z_\lambda}{\partial \Lambda_{12}} &= -\lambda R_{1323},\\
\frac{1}{2}\frac{\partial Z_\lambda}{\partial \Lambda_{13}} &= \frac{1}{\lambda} R_{1232},\\
\frac{1}{2}\frac{\partial Z_\lambda}{\partial \Lambda_{14}} &= (\frac{1}{\lambda} R_{1242} - \lambda R_{1343}) = F,\\
\frac{1}{2}\frac{\partial Z_\lambda}{\partial \Lambda_{23}} &= \frac{(1+\lambda^2)}{\lambda} R_{1213},\\
\frac{1}{2}\frac{\partial Z_\lambda}{\partial \Lambda_{24}} &= \frac{1}{\lambda} R_{1214},\\
\frac{1}{2}\frac{\partial Z_\lambda}{\partial \Lambda_{34}} &= -\lambda R_{1314},
\end{aligned}
$$

where we let $F = \frac{1}{\lambda}R_{1242} - \lambda R_{1343}$ and

$$
\begin{bmatrix}
\lambda R_{1313}-\lambda R_{2323} & \frac{\lambda}{2(1+\lambda^2)}\frac{\partial Z_\lambda}{\partial \Lambda_{23}} & \lambda R_{2334} & \frac{-\lambda^2}{2}\frac{\partial Z_\lambda}{\partial \Lambda_{13}} & -\lambda R_{1343} & -\frac{\lambda}{2}(T+3S)\\
* & \frac{1}{\lambda}R_{2323}-\frac{1}{\lambda}R_{1212} & \frac{1}{\lambda}R_{2324} & \frac{1}{2\lambda^2}\frac{\partial Z_\lambda}{\partial \Lambda_{12}} & -\frac{T}{\lambda} & -\frac{1}{\lambda}R_{1242}\\
\lambda R_{2334} & \frac{1}{\lambda}R_{2324} & \frac{1}{\lambda}R_{2424}-\lambda R_{3434}-Z_\lambda & \frac{1+\lambda^2}{2\lambda}(3S-T) & -\frac{1}{\lambda}R_{1424} & \lambda R_{1434}\\
* & * & \frac{1+\lambda^2}{2\lambda}(3S-T) & \frac{1}{\lambda}R_{1313}-\lambda R_{1212}-Z_\lambda & -\frac{1+\lambda^2}{2\lambda^2}\frac{\partial Z_\lambda}{\partial \Lambda_{34}} & \frac{1+\lambda^2}{2}\frac{\partial Z_\lambda}{\partial \Lambda_{24}}\\
F & -\frac{T}{\lambda} & -\frac{1}{\lambda}R_{1424} & * & \frac{1}{\lambda}R_{1414}-\frac{1}{\lambda}R_{1212} & \frac{-\lambda^2}{2}\frac{\partial Z_\lambda}{\partial \Lambda_{13}}\\
-\frac{\lambda}{2}(T+3S) & -F & \lambda R_{1434} & * & * & \lambda R_{1313}-\lambda R_{1414}
\end{bmatrix}
$$

The terms (19) in the nonlinearity in the evolution equation have the form,

$$
\begin{aligned}
&\frac{1}{\lambda}\left(R_{1212}^2 + R_{1313}R_{2323} + R_{1414}R_{2424}\right) - \lambda\left(R_{1313}^2 + R_{1212}R_{2323} + R_{1414}R_{3434}\right)\\
&= R_{1212}\left(\frac{1}{\lambda}R_{1212} - \lambda R_{2323}\right) + R_{1313}\left(\frac{1}{\lambda}R_{2323} - \lambda R_{1313}\right) + R_{1414}\left(\frac{1}{\lambda}R_{2424} - \lambda R_{3434}\right)\\
&= R_{1212}\mathbb{M}_{1212} + R_{1313}\mathbb{M}_{1313} + R_{1414}\mathbb{M}_{1414} + (R_{1212} + R_{1313} + R_{1414})Z_\lambda
\end{aligned}
$$

As in the previous section we write this in terms of $\mathbb{M}\otimes\mathbb{M}$, and we observe that the coefficients R_{1212}, R_{1313} and R_{1414} can be written as diagonal elements of $\mathbb{M}$, in various different ways: In particular, for any α, β, γ we can write

$$
\begin{aligned}
&\frac{1}{\lambda}\left(R_{1212}^2 + R_{1313}R_{2323} + R_{1414}R_{2424}\right) - \lambda\left(R_{1313}^2 + R_{1212}R_{2323} + R_{1414}R_{3434}\right)\\
&\quad= \frac{\lambda^3\cos^2\alpha}{1-\lambda^2}\mathbb{M}_{1212}\mathbb{M}_{2424} + \frac{\lambda\cos^2\alpha}{1-\lambda^2}\mathbb{M}_{1212}\mathbb{M}_{3434} + \frac{\lambda^3\sin^2\alpha}{1-\lambda^4}\mathbb{M}_{1212}\mathbb{M}_{2323}\\
&\qquad+ \frac{\lambda\cos^2\beta}{1-\lambda^2}\mathbb{M}_{1313}\mathbb{M}_{2424} + \frac{\cos^2\beta}{\lambda(1-\lambda^2)}\mathbb{M}_{1313}\mathbb{M}_{3434} + \frac{\lambda\sin^2\beta}{1-\lambda^4}\mathbb{M}_{1313}\mathbb{M}_{2323}\\
&\qquad+ \frac{\lambda(1-\lambda^2\sin^2\gamma)}{1-\lambda^2}\mathbb{M}_{1414}\mathbb{M}_{2424} + \frac{\lambda\cos^2\gamma}{1-\lambda^2}\mathbb{M}_{1414}\mathbb{M}_{3434} + \frac{\lambda^3\sin^2\gamma}{1-\lambda^4}\mathbb{M}_{1414}\mathbb{M}_{2323}\\
&\qquad+ (\mathbb{M}_{1212} + \mathbb{M}_{1313} + \mathbb{M}_{1414})Z_\lambda + (R_{1212} + R_{1313} + R_{1414})Z_\lambda.
\end{aligned}
$$

As in the previous section this gives us a collection of diagonal elements of $\mathbb{M}\otimes\mathbb{M}$, that we can use to control the off diagonal elements. Using the same algebraic construction as the previous section we choose the same vectors,

$$
\begin{aligned}
V_1^{\pm} &= \sqrt{\frac{\lambda^3}{1-\lambda^2}}\cos\alpha(e_1\wedge e_2)\hat{\wedge}(e_2\wedge e_4) \pm \sqrt{\frac{1}{\lambda(1-\lambda^2)}}\cos\beta(e_1\wedge e_3)\hat{\wedge}(e_3\wedge e_4);\\
V_2^{\pm} &= \sqrt{\frac{\lambda}{1-\lambda^2}}\cos\alpha(e_1\wedge e_2)\hat{\wedge}(e_3\wedge e_4) \pm \sqrt{\frac{\lambda}{1-\lambda^2}}\cos\beta(e_1\wedge e_3)\hat{\wedge}(e_2\wedge e_4);\\
V_3^{\pm} &= \sqrt{\frac{\lambda^3}{1-\lambda^4}}\sin\alpha(e_1\wedge e_2)\hat{\wedge}(e_2\wedge e_3) \pm \sqrt{\frac{\lambda}{1-\lambda^2}}\cos\gamma(e_1\wedge e_4)\hat{\wedge}(e_3\wedge e_4);\\
V_4^{\pm} &= \sqrt{\frac{\lambda}{1-\lambda^4}}\sin\beta(e_1\wedge e_3)\hat{\wedge}(e_2\wedge e_3) \pm \sqrt{\frac{\lambda(1-\lambda^2\sin^2\gamma)}{1-\lambda^2}}(e_1\wedge e_4)\hat{\wedge}(e_2\wedge e_4);\\
V_5 &= \sqrt{\frac{\lambda^3}{1-\lambda^4}}\sin\gamma(e_1\wedge e_4)\hat{\wedge}(e_2\wedge e_3).
\end{aligned}
$$

The term $\varepsilon_1\mathbb{M}\otimes\mathbb{M}(V_1^+,V_1^+)+(1-\varepsilon_1)\mathbb{M}\otimes\mathbb{M}(V_1^-,V_1^-)$ can be written as

$$
\begin{aligned}
&\frac{\lambda^3\cos^2\alpha}{1-\lambda^2}\mathbb{M}_{1212}\mathbb{M}_{2424}+\frac{\cos^2\beta}{\lambda(1-\lambda^2)}\mathbb{M}_{1313}\mathbb{M}_{3434}\\
&=\left(\frac{\lambda^3\cos^2\alpha}{1-\lambda^2}R_{1343}^2+\frac{\cos^2\beta}{\lambda(1-\lambda^2)}R_{1242}^2\right)+(1-2\varepsilon_1)\frac{\lambda\cos\alpha\cos\beta}{1-\lambda^2}T(T+3S) \qquad (23)\\
&\quad+(1-2\epsilon_1)\frac{\lambda^4\cos\alpha\cos\beta}{2(1-\lambda^4)}\frac{\partial Z_\lambda}{\partial\Lambda_{23}}\frac{\partial Z_\lambda}{\partial\Lambda_{24}} \qquad (24)\\
&\quad+\varepsilon_1\mathbb{M}\otimes\mathbb{M}(V_1^+,V_1^+)+(1-\varepsilon_1)\mathbb{M}\otimes\mathbb{M}(V_1^-,V_1^-). \qquad (25)
\end{aligned}
$$

The term $\mathbb{M}\otimes\mathbb{M}(V_2^+,V_2^+)$ yields

$$
\begin{aligned}
\frac{\lambda\cos^2\alpha}{1-\lambda^2}\mathbb{M}_{1212}\mathbb{M}_{3434}+\frac{\lambda\cos^2\beta}{1-\lambda^2}\mathbb{M}_{1313}\mathbb{M}_{2424} &= \frac{\lambda^3\cos^2\alpha}{4(1-\lambda^2)}(T+3S)^2+\frac{\cos^2\beta}{\lambda(1-\lambda^2)}T^2\\
&\quad+2\frac{\lambda\cos\alpha\cos\beta}{1-\lambda^2}R_{1343}R_{1242} \qquad (26)\\
&\quad+\frac{\lambda^4\cos\alpha\cos\beta}{2(1-\lambda^2)^2}\frac{\partial Z_\lambda}{\partial\Lambda_{23}}\frac{\partial Z_\lambda}{\partial\Lambda_{24}} \qquad (27)\\
&\quad+\mathbb{M}\otimes\mathbb{M}(V_2^+,V_2^+). \qquad (28)
\end{aligned}
$$

From $\mathbb{M}\otimes\mathbb{M}(V_3^{\pm},V_3^{\pm})$ we obtain

$$
\begin{aligned}
\frac{\lambda^3}{1-\lambda^4}\sin^2\alpha\mathbb{M}_{1212}\mathbb{M}_{2323}+\frac{\lambda}{1-\lambda^2}\cos^2\gamma\mathbb{M}_{1414}\mathbb{M}_{3434} &= \frac{\lambda^3}{1-\lambda^2}\cos^2\gamma R_{1434}^2\\
&\quad\pm\frac{\lambda^2\sqrt{1+\lambda^2}}{2(1-\lambda^2)}\sin\alpha\cos\gamma\left(9S^2-T^2\right) \qquad (29)\\
&\quad\pm\frac{\sqrt{1+\lambda^2}}{\lambda(1-\lambda^2)}\sin\alpha\cos\gamma R_{2334}\frac{\partial Z_\lambda}{\partial\Lambda_{34}} \qquad (30)\\
&\quad+\mathbb{M}\otimes\mathbb{M}(V_3^{\pm},V_3^{\pm}). \qquad (31)
\end{aligned}
$$

From $\varepsilon_4\mathbb{M}\otimes\mathbb{M}(V_4^+,V_4^+)+(1-\varepsilon_4)\mathbb{M}\otimes\mathbb{M}(V_4^-,V_4^-)$ we obtain

$$\frac{\lambda}{1-\lambda^4}\sin^2\beta\mathbb{M}_{1313}\mathbb{M}_{2323}+\frac{\lambda(1-\lambda^2\sin^2\gamma)}{1-\lambda^2}\mathbb{M}_{1414}\mathbb{M}_{2424}=\frac{1-\lambda^2\sin^2\gamma}{\lambda(1-\lambda^2)}R_{1424}^2$$

$$+(1-2\varepsilon_4)\frac{\sqrt{1-\lambda^2\sin^2\gamma}\sqrt{1+\lambda^2}}{\lambda(1-\lambda^2)}\sin\beta T(T-3S) \tag{32}$$

$$+(1-2\epsilon_4)\frac{\sqrt{1-\lambda^2\sin^2\gamma}(1+\lambda^2)}{\lambda^3}R_{2324}\frac{\partial Z_\lambda}{\partial\Lambda_{34}} \tag{33}$$

$$+\varepsilon_4\mathbb{M}\otimes\mathbb{M}(V_4^+,V_4^+)+(1-\varepsilon_4)\mathbb{M}\otimes\mathbb{M}(V_4^-,V_4^-) \tag{34}$$

Finally, from $\mathbb{M}\otimes\mathbb{M}(V_5,V_5)$ we have

$$\frac{\lambda^3}{1-\lambda^4}\sin^2\gamma\mathbb{M}_{1414}\mathbb{M}_{2323}=\frac{\lambda(1+\lambda^2)}{4(1-\lambda^2)}\sin^2\gamma(3S-T)^2 \tag{35}$$

$$+\mathbb{M}\otimes\mathbb{M}(V_5,V_5). \tag{36}$$

Proposition 16. *At each point* $\{(e_1,e_2,e_3,e_4),t\}\in\mathbb{P}\times(0,\delta)$ *we have*

$$\frac{1}{\lambda}Q(R)_{1212}-\lambda Q(R)_{1313}\geq\alpha\min\left\{0,\inf_{\xi\in V,|\xi|=1}D^2Z_\lambda(\xi,\xi)\right\}-\alpha\sup_{\xi\in V,|\xi|=1}DZ_\lambda(\xi)-\alpha Z_\lambda. \tag{37}$$

Proof. As in the original proof and use the fact that before the singular time we have that the curvature is bounded. Then the inequality above is simple to verify. Applying Bony's maximum principle, (Theorem 15), we obtain the desired result. Consider the terms (21) and (22), then these clearly satisfy

$$\frac{2}{\lambda}R_{1413}R_{2342}-2\lambda R_{1412}R_{2343}+(\lambda-\frac{1}{\lambda})R_{1323}^2+$$
$$(\frac{1}{\lambda}-\lambda)R_{1242}^2+\frac{1}{\lambda}R_{1412}^2-\lambda R_{1413}^2+(\frac{1}{\lambda}-\lambda)R_{1213}^2\geq-\alpha\sup_{\xi\in V,|\xi|=1}DZ_\lambda(\xi). \tag{38}$$

Furthermore, the terms (19) may be used to control the terms (20) in the same way as the previous section. Hence the terms in this equation become,

$$\begin{aligned}
&\left(-\frac{1}{\lambda}+\frac{1-\lambda^2\sin^2\gamma}{\lambda(1-\lambda^2)}\right)R_{1424}^2+\left(\lambda+\frac{\lambda^3\cos^2\gamma}{1-\lambda^2}\right)R_{1434}^2\\
&\left(\frac{1}{\lambda}+\frac{\cos^2\beta}{\lambda^3(1-\lambda^2)}\right)R_{1242}^2+\left(\frac{\lambda^5\cos^2\alpha}{1-\lambda^2}-\lambda\right)R_{1343}^2+\frac{2\lambda\cos\alpha\cos\beta}{1-\lambda^2}R_{1343}R_{1242}\\
&+\frac{1+4\lambda^2}{4\lambda}T^2-\frac{3(1+2\lambda^2)}{4\lambda}S^2-\frac{3}{2\lambda}TS+(1-2\varepsilon_1)\frac{\lambda\cos\alpha\cos\beta}{1-\lambda^2}T(T+3S)\\
&+\frac{\lambda^3\cos^2\alpha}{4(1-\lambda^2)}(T+3S)^2+\frac{\cos^2\beta}{\lambda(1-\lambda^2)}T^2\pm\frac{\lambda^2\sqrt{1+\lambda^2}\sin\alpha\cos\gamma}{2(1-\lambda^2)}\left(9S^2-T^2\right)\\
&+(1-2\varepsilon_4)\frac{\sqrt{1-\lambda^2\sin^2\gamma}\sqrt{1+\lambda^2}}{\lambda(1-\lambda^2)}\sin\beta\, T(T-3S)\\
&+\frac{\lambda(1+\lambda^2)}{4(1-\lambda^2)}\sin^2\gamma(3S-T)^2\\
&+\varepsilon_1\mathbb{M}\otimes\mathbb{M}(V_1^+,V_1^+)+(1-\varepsilon_1)\mathbb{M}\otimes\mathbb{M}(V_1^-,V_1^-)+\mathbb{M}\otimes\mathbb{M}(V_2^+,V_2^+)\\
&+\mathbb{M}\otimes\mathbb{M}(V_3^\pm,V_3^\pm)+\varepsilon_4\mathbb{M}\otimes\mathbb{M}(V_4^+,V_4^+)+(1-\varepsilon_4)\mathbb{M}\otimes\mathbb{M}(V_4^-,V_4^-)\\
&+\mathbb{M}\otimes\mathbb{M}(V_5,V_5)\\
&+(1-2\epsilon_1)\frac{\lambda^4\cos\alpha\cos\beta}{2(1-\lambda^4)}\frac{\partial Z_\lambda}{\partial\Lambda_{23}}\frac{\partial Z_\lambda}{\partial\Lambda_{24}}+\frac{\lambda^4\cos\alpha\cos\beta}{2(1-\lambda^2)^2}\frac{\partial Z_\lambda}{\partial\Lambda_{23}}\frac{\partial Z_\lambda}{\partial\Lambda_{24}}\\
&\pm\frac{\sqrt{1+\lambda^2}}{\lambda(1-\lambda^2)}\sin\alpha\cos\gamma R_{2334}\frac{\partial Z_\lambda}{\partial\Lambda_{34}}+(1-2\epsilon_4)\frac{\sqrt{1-\lambda^2\sin^2\gamma}(1+\lambda^2)}{\lambda^3}R_{2324}\frac{\partial Z_\lambda}{\partial\Lambda_{34}}.
\end{aligned}$$

We make exactly the same choices of $\cos\alpha = 1, \cos\beta = \frac{1}{2}$ and $\cos\gamma = \frac{\lambda}{\sqrt{1+\lambda^2}}\sin\alpha, \epsilon_1 = 1, \epsilon_4 = \frac{1}{2}\left(1+\sqrt{\frac{1-\lambda^2}{1+\lambda^2}}\right)$ then the coefficient of the terms S^2 and ST are zero and the coefficients of R_{1424}^2, R_{1434}^2 and T^2 are positive. Finally, by writing $R_{1242} = \lambda F + \lambda^2 R_{1343}$, we may estimate the following term,

$$\begin{aligned}
&\left(\frac{1}{\lambda}+\frac{\cos^2\beta}{\lambda^3(1-\lambda^2)}\right)R_{1242}^2+\left(\frac{\lambda^5\cos^2\alpha}{1-\lambda^2}-\lambda\right)R_{1343}^2+\frac{2\lambda\cos\alpha\cos\beta}{1-\lambda^2}R_{1343}R_{1242}\geq\\
&\lambda^2R_{1343}^2\left(\frac{\left(\lambda^2\cos\alpha+\cos\beta\right)^2}{\lambda(1-\lambda^2)}-\frac{1-\lambda^2}{\lambda}\right)-\alpha\sup_{\xi\in V,|\xi|=1}DZ_\lambda(\xi).
\end{aligned}$$

By the above choices, the term in the brackets is positive. Therefore, collecting all the above estimates we have shown that (37) is true. ◻

Let $F = \{(e_1, e_2, e_3) \subset T_pM\}$ be the set of orthonormal 3-frames such that $F_\lambda(e_1, e_2, e_3) = 0$.

Corollary 17. *The set F is invariant under parallel translation.*

Proof. The corollary follows by Proposition 16 and Theorem 15. ◻

Theorem 18. *Let (M, g_0) have weakly $1/4$-pinched flag curvature. Further, let us suppose that there exists a real number $\tau \in (O, T)$ such that*

$$Hol^{\,0}(\mathbb{M}, g(\tau)) = SO(n).$$

then (M, g_0) is diffeomorphic to a spherical space form.

Proof. As in the paper of [BS2], the theorem follows from Theorem 14 if

$$F_\lambda(e_1, e_2, e_3) = \frac{1}{\lambda} R_{1212} - \lambda R_{1313} > 0$$

for all orthonormal three-frames $\{e_1, e_2, e_3\}$. Hence let us fix a point $p \in M$ and number $\lambda \in [\frac{1}{4}, 1]$ and suppose that $\{e_1, e_2, e_3\} \subset T_p M$ is an orthonormal three frame such that

$$\frac{1}{\lambda} R_{1212} - \lambda R_{1313} = 0.$$

Now the manifold M is not flat, hence we can find a $q \in M$ and an orthonormal two frame $\{v_1, v_2\} \subset T_q M$ such that

$$R_q(v_1, v_2, v_1, v_2) > 0.$$

Now we assume that the holonomy is $O(n)$, this implies that there exists a piecewise smooth path $\gamma(t), \gamma(0) = p, \gamma(1) = q$ such that under parallel transport along γ we have $v_1 = P_\gamma(e_1)$ and $v_2 = P_\gamma(e_2), v_3 = P_\gamma(e_3)$. Then by Corollary 17, we have that

$$\frac{1}{\lambda} R(v_1, v_2, v_1, v_2) - \lambda R(v_1, v_3, v_1, v_3) = 0.$$

By a similar argument we have that

$$\frac{1}{\lambda} R(v_1, v_3, v_1, v_3) - \lambda R(v_1, v_2, v_1, v_2) = 0.$$

This implies that $R(v_1, v_2, v_1, v_2) = 0$ which is a contradiction. □

PROPOSITION 19 (cf [BS2]*Proposition 11). *Assume that (M, g_0) is locally irreducible. Then one of the following statements holds:*

1. *(M, g_0) is diffeomorphic to a spherical space form,*
2. *The universal cover if (M, g_0) is Kähler manifold biholomorphic to $\mathbb{CP}^2$.*

Proof. The argument is similar to the argument in [BS2]. A (M, g_0) is not locally symmetric and has $\frac{1}{4}$ pinched flag curvature then (M, g_0) is not flat and is locally irreducible. Then by Theorem 18, there are two possibilities,

1. The Ricci flow exists up to time T, and the manifold is diffeomorphic to a spherical space form. Hence we are done.
2. The universal cover of (M, g_0) is a Kähler manifold. As (M, g_0) has quarter pinched flag curvature so does the universal cover $(\tilde{M}, g_0)$. This implies that $(\tilde{M}, g_0)$ has quarter pinched sectional curvature as follows, we only need to check that

$$\frac{1}{\lambda} R(v, w, v, w) - \lambda R(Jv, Jw, Jv, Jw) \geq 0. \quad (39)$$

But (M, g_0) is a Kähler manifold and the metric and the curvature operator are J invariant, that is

$$g(JX, JY) = g(X, Y), \quad R(X, Y)JZ = J(R(X, Y)Z).$$

In particular, using the symmetries of the curvature operator, we have that

$$R(Jv, Jw, Jv, Jw) = R(v, w, v, w).$$

Hence (39) holds and $(\tilde{M}, g_0)$ has quarter pinched sectional curvature and hence is a Kähler manifold of constant holomorphic sectional curvature [KN]. This implies that the universal cover is isometric to $\mathbb{CP}^2$ up to scaling and that (M, g_0) is locally symmetric, which is a contradiction.

$\square$

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