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CONFIGURATION OF THE CENTRAL STREAMS IN THE MODULI OF ABELIAN VARIETIES*

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Dedicated to Professor Iku Nakamura on his 60th birthday

Abstract. We study the structure of the moduli space of principally polarized abelian varieties in characteristic p. In this paper we determine the configuration of the central streams in the moduli space. As a corollary of our proof we obtain a new proof of the dimension formula of the central streams.

Key words. Newton polygon stratification, Ekedahl-Oort stratification, central streams.

AMS subject classifications. Primary 14K10; Secondary 11G10, 14L05, 14L20

1. Introduction. In this paper we study some combinatorial interrelation between *symmetric Newton polygons* and *elementary sequences*, where symmetric Newton polygons are combinatorial data classifying isogeny classes of quasi-polarized p-divisible groups, and elementary sequences are combinatorial data classifying polarized truncated Barsotti-Tate groups of level one (we shall give a brief review in §2.2 and §2.3). From this we deduce some geometrically meaningful results on the structure of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties over fields of characteristic $p > 0$.

In [17] Oort defined *central leaves* and *isogeny leaves* in the open Newton polygon stratum W^0_ξ for a symmetric Newton polygon ξ, and showed that a central leaf and an isogeny leaf give an "almost" product structure on each irreducible component of W^0_ξ ([17], (5.3)). Clearly this result tells us that it is important to investigate these two leaves in detail in order to know the structure of $\mathcal{A}_g$.

For each symmetric Newton polygon ξ, there is a special central leaf $\mathcal{Z}_\xi$ in $\mathcal{A}_g$ which is called the *central stream* (cf. §2.4). By definition, the p-divisible group of any geometric fiber of $\mathcal{Z}_\xi$ is minimal of Newton polygon ξ.

Our main theorem is Theorem 3.1. It would not be appropriate to state the theorem here, because some technical notations are necessary. What is important is that the theorem produces the following significant corollaries. Firstly we can determine the configuration of the central streams $\{\mathcal{Z}_\xi\}$ in $\mathcal{A}_g$ (Corollary 3.2). Secondly we obtain the dimension formula of the central streams (Corollary 3.4), which has been obtained by Oort and Chai-Oort (see [20]). Finally we give a contribution (Corollary 3.6) to Oort's conjecture (Conjecture 3.5) on intersections of Newton polygon strata and Ekedahl-Oort strata.

Let us explain the points of our proof. By Oort's theory [18] on minimal p-divisible groups, the central stream $\mathcal{Z}_\xi$ is nothing but the Ekedahl-Oort stratum S_{φ_ξ} for a certain elementary sequence φ_ξ (cf. §2.4). Thus our problem deals with the configuration of a certain class of Ekedahl-Oort strata. We emphasize here that there are two difficulties to solve this problem. One is that we can compute φ_ξ explicitly for each given example, but do not yet have a general formula. The other one is that we need some complicated combinatorics to show $S_{\varphi'} \subset \overline{S_\varphi}$, denoted by $\varphi' \subset \varphi$, for

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elementary sequences φ' and φ. For the former, we show some beautiful inductive formulas of φ_ξ's instead of an explicit general formula. For the latter, we use a sufficient condition for $\varphi' \subset \varphi$, which we can check more easily. From these partial answers, we can show $\varphi_\zeta \subset \varphi_\xi$ for any symmetric Newton polygons ζ and ξ with $\zeta \prec \xi$.

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NOTATIONS. We fix once for all a rational prime p. All base fields and all base schemes will be in characteristic p. We write $\mathbb{N} = \mathbb{Z}_{>0}$ the set of natural numbers. For non-negative integers m, n we denote by $\gcd(m, n)$ the greatest common divisor where for convenience we set $\gcd(m, 0) = \gcd(0, m) = m$ for $\forall m \in \mathbb{Z}_{\geq 0}$. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the biggest integer $\leq x$ and $\lceil x \rceil$ the smallest integer $\geq x$. For $a, b \in \mathbb{R}$ with $a \leq b$, we denoted by $[a, b]$ the set $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ and by $[a, b)$ the set $\{x \in \mathbb{R} \mid a \leq x < b\}$.

2. Stratifications. In this section, we start with reviewing the definition of the Newton polygon stratification, the Ekedahl-Oort stratification and the central streams, and some facts that we shall use later on.

2.1. Dieudonné theory. Let K be a perfect field of characteristic p and $W(K)$ the ring of Witt vectors with coordinates in K. Let A_K be the p-adic completion of the associative ring

$$W(K)[\mathcal{F}, \mathcal{V}]/(\mathcal{F}x - x^\rho \mathcal{F}, \mathcal{V}x^\rho - x\mathcal{V}, \mathcal{F}\mathcal{V} - p, \mathcal{V}\mathcal{F} - p, \forall x \in W(K))$$

with the Frobenius automorphism ρ of $W(K)$. A *Dieudonné module over* $W(K)$ is a left A_K-module which is finitely generated as a $W(K)$-module. There is a canonical categorical equivalence $\mathbb{D}$ (the covariant Dieudonné functor) from the category of p-torsion finite commutative group schemes (resp. p-divisible groups) over K to the category of Dieudonné modules over $W(K)$ which are of finite length (resp. free as $W(K)$-modules). We have $\mathbb{D}(F) = \mathcal{V}$ and $\mathbb{D}(V) = \mathcal{F}$ for the Frobenius F and the Verschiebung V on finite commutative group schemes (resp. p-divisible groups).

2.2. The NP-stratification. A pair (m, n) of non-negative integers with $\gcd(m, n) = 1$ is called *a segment*. For a segment (m, n), we define a p-divisible group $G_{m,n}$ over $\mathbb{F}_p$ by $\mathbb{D}(G_{m,n}) = A_{\mathbb{F}_p}/A_{\mathbb{F}_p}(\mathcal{F}^m - \mathcal{V}^n)$. The slope of $G_{m,n}$ (or $\varrho = (m, n)$) is is defined to be $\lambda(\varrho) := n/(m + n)$. (**Caution:** this slope is called the V-slope (or the $\mathcal{F}$-slope); in some papers the F-slope (= the $\mathcal{V}$-slope) is used, where the slope of $G_{m,n}$ is defined to be $m/(m + n)$.) A *Newton polygon* is a formal sum $\varrho_1 + \cdots + \varrho_t$ of segments. Arranging ρ_i's so that $\lambda(\varrho_1) \leq \lambda(\varrho_2) \leq \cdots \leq \lambda(\varrho_t)$, we regard the Newton polygon as the line graph passing through $P_0, \ldots, P_t$ in this order, where we put $P_j := (\sum_{i=1}^{j}(m_i + n_i), \sum_{i=1}^{j} n_i)$ for $0 \leq j \leq t$. The point P_j is called the *j-th breaking point* for $0 < j < t$. (**Caution:** for $0 < j < t$ we call P_j a breaking point even if $\lambda(\varrho_i) = \lambda(\varrho_{i+1})$; we shall call P_j a *true* breaking point if $\lambda(\varrho_i) \neq \lambda(\varrho_{i+1})$.) For two Newton polygons ξ, ζ with the same end point, we say $\zeta \prec \xi$ if every point of ζ is not below ξ.

By the Dieudonné-Manin classification ([11] and [2]), for any p-divisible group $\mathcal{G}$ over a field K of characteristic p, there is an isogeny over an algebraically closed field containing K from $\mathcal{G}$ to the direct sum of G_{m_i,n_i} for some finite set of segments $\varrho_i = (m_i, n_i)$. Thus we have a Newton polygon $\varrho_1 + \cdots + \varrho_t$, which is denoted by $\mathrm{NP}(\mathcal{G})$. For an abelian variety X, we have its Newton polygon $\mathrm{NP}(X) := \mathrm{NP}(X[p^\infty])$. By [14], Theorem 19.1, the p-divisible group $X^t[p^\infty]$ of the dual abelian variety X^t is canonically isomorphic to the Serre dual of $X[p^\infty]$; this implies that the Newton polygon $\mathrm{NP}(X)$ is symmetric, i.e., $\lambda(\varrho_i) + \lambda(\varrho_{t+1-i}) = 1$ for all $0 \le i \le t$ (cf. [2], Chapter V, 3). Also see [11], Chapter VI, 3 for an abelian variety over a finite field.

For a symmetric Newton polygon ξ of height $2g$, we define its *NP-stratum* by

$$W_\xi = \{(X,\mu) \in \mathcal{A}_g \mid \mathrm{NP}(X) \prec \xi\},$$

which is a closed subset of $\mathcal{A}_g$ by Grothendieck and Katz ([9], Theorem 2.3.1 on p. 143); we consider it as a closed subscheme of $\mathcal{A}_g$ by giving it the reduced induced scheme structure. We define the *open NP-stratum* by

$$W_\xi^0 = \{(X,\mu) \in \mathcal{A}_g \mid \mathrm{NP}(X) = \xi\},$$

which is a locally closed subset of $\mathcal{A}_g$ (with reduced induced scheme structure).

2.3. The EO-stratification. The main reference for the EO-stratification is [15]. See [3], [4], [12], [13] and [21] for a beautiful formulation in terms of the Weyl group. To use Weyl groups is starting to become more mainstream, but in this paper we follow the terminology in [15], because we can then more easily get information about Ekedahl-Oort strata (cf. Theorem 2.5 (2) and (3) with Definition 2.4 (1)).

Let K be a field of characteristic p. A finite commutative group scheme G over K is said to be a *truncated Barsotti-Tate group of level one* (BT_1) over K if it is annihilated by p and $\mathrm{Im}(V : G^{(p)} \to G) = \mathrm{Ker}(F : G \to G^{(p)})$. A *final sequence* of length d is a map $\psi : \{0,1,\dots,d\} \to \{0,1,\dots,d\}$ satisfying $\psi(0) = 0$ and $\psi(i-1) \le \psi(i) \le \psi(i-1)+1$ for $1 \le i \le d$. We frequently write $\psi = (\psi(1),\dots,\psi(d))$.

Let G be a BT_1 over K. For any subgroup scheme G' of G over $\overline{K}$ and for any word w of V, F^{-1}, we define $w \cdot G'$ inductively by $V \cdot G' := VG'^{(p)}$ and $F^{-1} \cdot G' := F^{-1}(G'^{(p)} \cap FG)$. Then there exists a unique final sequence ψ of a certain length d such that for any word w of V, F^{-1} we have $\psi(\mathrm{length}(w \cdot G)) = \mathrm{length}(Vw \cdot G)$, see [15], (2.4). Thus we have a canonical map

$$\mathrm{FS} : \{\mathrm{BT}_1 \text{ of length } d \text{ over } K\}/K\text{-isom.} \longrightarrow \{\text{final seq. of length } d\}.$$

The following theorem was first obtained by Kraft [10]:

THEOREM 2.1. *If K is algebraically closed, then* FS *is bijective.*

Let K be a perfect field. A *polarized* BT_1 over K is a pair $(G, \langle\ ,\ \rangle)$, where G is a BT_1 over K and $\langle\ ,\ \rangle$ is a non-degenerate alternating pairing on $\mathbb{D}(G)$ satisfying $\langle \mathcal{F}x, y\rangle = \langle x, \mathcal{V}y\rangle^\rho$ for all $x, y \in \mathbb{D}(G)$, see [15], (9.2). A *symmetric final sequence* of length $2g$ is a final sequence of length $2g$ satisfying $\psi(2g-i) = g + \psi(i) - i$. An *elementary sequence* of length g is the sequence obtained by restricting a symmetric final sequence of length $2g$ to $\{0,\dots,g\}$. (Abstractly, an elementary sequence of length g is nothing but a final sequence of length g.) Note any symmetric final sequence is uniquely determined by its elementary sequence.

For a polarized BT_1 $(G, \langle\ ,\ \rangle)$, its final sequence $\mathrm{FS}(G)$ is symmetric (cf. [15], (5.4)). Hence we have a canonical map

$$\mathrm{ES} : \{\text{pol. } \mathrm{BT}_1 \text{ of length } 2g \text{ over } K\}/K\text{-isom.} \longrightarrow \{\text{elementary seq. of length } g\}.$$

We recall [15], (9.4) (see [13], (5.4) for a formulation in terms of the Weyl group):

THEOREM 2.2. *If K is algebraically closed, then* ES *is bijective.*

For each elementary sequence φ of length g, the EO-stratum S_φ is defined to be the subset of $\mathcal{A}_g$ consisting of points $y \in \mathcal{A}_g$ where y comes over some field from a principally polarized abelian variety X_y such that $\mathrm{ES}(X_y[p]) = \varphi$, see [15], (5.11). As shown in [15], (3.2), S_φ is locally closed in $\mathcal{A}_g$; we consider it as a locally closed subscheme by giving it the reduced induced scheme structure.

Let us recall the inverse maps of FS and ES respectively. For this the notion of final types is useful:

DEFINITION 2.3.
(1) A *final type* of length d is a pair (B, δ), where B is a totally ordered finite set with $\sharp B = d$ and δ is a map $B \to \{0, 1\}$. We often write $\delta = (\delta(b_1), \dots, \delta(b_d))$, where $B = \{b_1 < \cdots < b_d\}$.
(2) Let (B, δ) be a final type of length $2g$ with $B = \{b_1 < \cdots < b_{2g}\}$. Let $\vee : B \to B$ be the map sending b_i to $b_i^\vee := b_{2g+1-i}$. We say (B, δ) to be *symmetric* if $\delta(b) + \delta(b^\vee) = 1$ for all $b \in B$.

For any (symmetric) final sequence ψ of length d, we define a (symmetric) final type (B, δ) by $B = \{b_1 < \ldots < b_d\}$ and

$$\delta(b_i) = 1 - \psi(i) + \psi(i-1). \tag{2.3.1}$$

Clearly this correspondence gives a bijection from the set of (symmetric) final sequences to the set of (symmetric) final types.

Let ψ be the final sequence of length d and (B, δ) the associated final type. Write $B = \{b_1 < \cdots < b_d\}$. In order to see the inverse map of FS, it suffices to describe the Dieudonné module $\mathbb{D}(G)$ of a BT_1 G with $\mathrm{FS}(G) = \psi$. It is known (see the proof of [15], (9.4)) that $\mathbb{D}(G)$ is isomorphic to the Dieudonné module N_ψ which is a K-vector space with a basis indexed by B, simply say

$$N_\psi = \bigoplus_{i=1}^{d} K b_i \tag{2.3.2}$$

with the $\mathcal{F}$ and $\mathcal{V}$-operations defined by

$$\mathcal{F}(b_i) := \begin{cases} \pi(b_i) & \text{if} \quad \delta(b_i) = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathcal{V}(\pi(b_i)) := \begin{cases} -(-1)^{\delta(\pi(b_i))} b_i & \text{if} \quad \delta(b_i) = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.3.3}$$

where π is the bijection

$$\pi_\delta : \quad \{b_1, \dots, b_d\} \longrightarrow \{b_1, \dots, b_d\} \tag{2.3.4}$$

defined by sending b_i to $b_{\psi(i)}$ if $\delta(b_i) = 0$ and to $b_{\psi(d)+i-\psi(i)}$ if $\delta(b_i) = 1$. If G is a polarized BT_1, then $\psi = \mathrm{FS}(G)$ is a symmetric final sequence. It was shown in

[15], (9.4) that the polarized Dieudonné module $\mathbb{D}(G)$ is isomorphic to N_ψ with the polarization $\langle\ ,\ \rangle$ defined by $\langle b_i, b_j^\vee \rangle = (-1)^{\delta(b_i)}$ if $i = j$ and zero otherwise.

Recall that there are two natural partial orderings on the set of elementary sequences of length g.

Definition 2.4. Let φ and φ' be two elementary sequences of length g.

(1) We say $\varphi' \leq_{BC} \varphi$ if $\varphi'(i) \leq \varphi(i)$ for all $i = 1, \ldots, g$. This order is called the Bruhat-Chevalley order.
(2) We say $\varphi' \subset \varphi$ if $S_{\varphi'}$ is contained in the Zariski closure $\overline{S_\varphi}$ of S_φ in $\mathcal{A}_g$.

We shall use results of [15]:

Theorem 2.5.

(1) *S_φ is not empty and is quasi-affine for every φ.*
(2) *Any irreducible component of S_φ has dimension $|\varphi| := \sum_{i=1}^g \varphi(i)$.*
(3) *$\varphi' \leq_{BC} \varphi$ implies $\varphi' \subset \varphi$.*
(4) *$\varphi' \subset \varphi$ is equivalent to $S_{\varphi'} \cap \overline{S_\varphi} \neq \emptyset$.*

Recall [3], Theorem 11.5 (with [1], (4.8), Step 2, also see [5], §4):

Theorem 2.6. *S_φ is irreducible if $S_\varphi \not\subset W_\sigma$.*

For two polarized BT_1's G and G', the direct sum $G \oplus G'$ becomes a polarized BT_1 canonically. Let φ and φ' be elementary sequences of G and G' respectively. We denote by $\varphi \oplus \varphi'$ the elementary sequence of $G \oplus G'$. Clearly $S_\varphi \times S_{\varphi'} \subset S_{\varphi\oplus\varphi'}$ holds.

Definition 2.7. Let φ and φ' be two elementary sequences. We say $\varphi' \leq \varphi$ if there exist elementary sequences $\varphi_0, \ldots, \varphi_l$ for some l with $0 \leq l < \infty$ such that

(1) $\varphi' = \varphi_0$ and $\varphi = \varphi_l$,
(2) for each i ($0 \leq i < l$), there are elementary sequences α, β and γ (depending on i) such that $\varphi_i = \alpha \oplus \gamma$ and $\varphi_{i+1} = \beta \oplus \gamma$ with $\alpha <_{BC} \beta$.

Lemma 2.8.

(1) $\varphi' \leq_{BC} \varphi \Longrightarrow \varphi' \leq \varphi$.
(2) $\varphi' \leq \varphi \Longrightarrow \varphi' \subset \varphi$.

Proof. (1) follows from the definition. (2) It suffices to show that $\varphi_1 \subset \varphi_2$ for $\varphi_1 = \alpha \oplus \gamma$ and $\varphi_2 = \beta \oplus \gamma$ with $\alpha <_{BC} \beta$. Clearly we have $S_\alpha \times S_\gamma \subset \overline{S_\beta} \times S_\gamma \subset \overline{S_\beta \times S_\gamma} \subset \overline{S_{\varphi_2}}$. Here we used Theorem 2.5 (3) to see the first inclusion. Since $S_\alpha \times S_\gamma$ is not empty (Theorem 2.5 (1)) and is contained in S_{φ_1}, we have $S_{\varphi_1} \cap \overline{S_{\varphi_2}} \neq \emptyset$. Then $\varphi_1 \subset \varphi_2$ follows from Theorem 2.5 (4). ◻

Remark 2.9.

(1) $\varphi' \leq \varphi$ does not imply $\varphi' \leq_{BC} \varphi$. Indeed put $\varphi = (0,0,1,2,2)$ and $\varphi' = (0,1,1,1,1)$. Since $\varphi = (0,1,1) \oplus (0,1)$ and $\varphi' = (0,1,1) \oplus (0,0)$, we have $\varphi' \leq \varphi$. However obviously $\varphi' \not\leq_{BC} \varphi$.
(2) $\varphi' \subset \varphi$ does not imply $\varphi' \leq \varphi$. See [3], Example 9.5 (iii): for $\varphi = (0,0,1,2,3,3)$ and $\varphi' = (0,1,1,2,2,2)$ we have $\varphi' \subset \varphi$ and $\varphi' \not\leq \varphi$.
(3) Quite recently in [22] Wedhorn determined when $\varphi' \subset \varphi$ in terms of Weyl groups. In this paper we do not use his result.

2.4. Central streams. For a segment (m,n), we define a p-divisible group $H_{m,n}$ over $\mathbb{F}_p$ (cf. [8], 5.3) by

$$\mathbb{D}(H_{m,n}) = \bigoplus_{i=0}^{m+n-1} \mathbb{Z}_p x_i \tag{2.4.1}$$

with $\mathcal{F},\mathcal{V}$ operations: $\mathcal{F}x_i = x_{i+n}$ and $\mathcal{V}x_i = x_{i+m}$, where x_i $(i \in \mathbb{Z}_{\geq m+n})$ are defined as satisfying $x_{i+m+n} = px_i$ for $i \in \mathbb{Z}_{\geq 0}$. For a Newton polygon $\xi = \sum_{i=1}^t (m_i, n_i)$, we write $H(\xi) = \bigoplus_{i=1}^t H_{m_i,n_i}$, which is called the *minimal p-divisible group* defined by ξ. Note the Newton polygon of $H(\xi)$ equals ξ.

For any symmetric Newton polygon ξ, we set

$$\mathcal{Z}_\xi = \{x = (A_x, \mu_x) \in \mathcal{A}_g \mid A_x[p^\infty] \otimes \Omega \simeq H(\xi) \otimes \Omega, \text{ for some } \Omega = \overline{\Omega} \supset k(x)\},$$

which is a closed subset of W_ξ^0 by [17], (3.3); we consider it as a closed subscheme of W_ξ^0 by giving it the reduced induced scheme structure. We call $\mathcal{Z}_\xi$ the *central stream* defined by ξ, see [17], (3.10). By [17], (3.7), there exists a principal quasi-polarization μ on $H(\xi)$, which is unique up to isomorphism of $H(\xi)$. We set $\varphi_\xi := \mathrm{ES}(H(\xi)[p], \mu[p])$. Then Oort's theory [18], (1.2) on minimal p-divisible groups shows that the central stream $\mathcal{Z}_\xi$ coincides with the EO-stratum S_{φ_ξ}. By Theorem 2.6, $\mathcal{Z}_\xi$ is irreducible if ξ is not supersingular σ. Let χ be the ordinary Newton polygon; then we have $\varphi_\chi = (1,\ldots,g)$; hence $\dim \mathcal{Z}_\chi = |\varphi_\chi| = g(g+1)/2$ by Theorem 2.5 (2). For the supersingular case σ, we have $\varphi_\sigma = (0,\ldots,0)$; hence the dimension of any irreducible component of $\mathcal{Z}_\sigma$ is $|\varphi_\sigma| = 0$.

3. Main theorem. For two symmetric Newton polygons ξ and ζ of height $2g$ with $\zeta \prec \xi$, we set

$$c(\xi;\zeta) = 2\sum_{1\leq i\leq g} (\zeta(i) - \xi(i))$$

and $c(\xi) = c(\xi;\sigma)$. This is an easy way to define the value $c(\xi)$, but is not sufficient for doing computations, see [20], (5.3) for various alternative ways to compute $c(\xi)$.

As our main result in this paper, we shall show:

THEOREM 3.1. *Let ξ and ζ be two symmetric Newton polygons with $\zeta \prec \xi$. Then there exists a series $\varphi_0, \ldots, \varphi_{c(\xi;\zeta)}$ of elementary sequences of length g such that*

$$\varphi_\zeta = \varphi_0 < \varphi_1 < \cdots < \varphi_{c(\xi;\zeta)} = \varphi_\xi.$$

It is not too much to say that this theorem is for the three corollaries below. The corollaries are more meaningful than this theorem itself. Let $\overline{\mathcal{Z}_\xi}$ denote the Zariski closure of $\mathcal{Z}_\xi$ in $\mathcal{A}_g$. Then we have

COROLLARY 3.2. *$\mathcal{Z}_\zeta \subset \overline{\mathcal{Z}_\xi}$ if and only if $\zeta \prec \xi$.*

Proof. Assume $\zeta \prec \xi$. Theorem 3.1 says in particular $\varphi_\zeta \leq \varphi_\xi$. Then we have $\varphi_\zeta \subset \varphi_\xi$ by Lemma 2.8 (2). The "only if"-part follows from Grothendieck-Katz ([9], Theorem 2.3.1). ∎

REMARK 3.3. Corollary 3.2 was expected in [17], (6.10).

We give a new proof of the dimension formula of $\mathcal{Z}_\xi$, which was first obtained in [20].

COROLLARY 3.4. *We have* $\dim(\mathcal{Z}_\xi) = c(\xi)$.

Proof. Let χ be the ordinary Newton polygon. We know $\dim(\mathcal{Z}_\chi) = g(g+1)/2 = c(\chi)$ and $\dim(\mathcal{Z}_\sigma) = 0 = c(\sigma)$. By Theorem 3.1 we have $\dim(\mathcal{Z}_\xi) - \dim(\mathcal{Z}_\zeta) \geq c(\xi;\zeta)$ for any symmetric Newton polygons $\zeta \prec \xi$. Applying this to $\sigma \prec \xi$ and $\xi \prec \chi$, we have $c(\sigma) + c(\xi;\sigma) \leq \dim(\mathcal{Z}_\xi) \leq c(\chi) - c(\chi;\xi)$. Since $c(\sigma) + c(\xi;\sigma) = c(\xi) = c(\chi) - c(\chi;\xi)$ by definition, we obtain $\dim(\mathcal{Z}_\xi) = c(\xi)$. □

In [17], (6.9) Oort conjectured

CONJECTURE 3.5. *If* $W^0_\xi \cap S_\varphi \neq \emptyset$, *then* $\mathcal{Z}_\xi \subset \overline{S_\varphi}$.

For an elementary sequence φ, let ξ_φ be the Newton polygon of a generic point of S_φ. (This definition is independent of the choice of the generic point. Indeed by Theorem 2.6, S_φ is irreducible if it is not contained in W_σ, and otherwise every generic point of S_φ has the supersingular Newton polygon σ.)

Now clearly $W^0_{\xi_\varphi} \cap S_\varphi \neq \emptyset$ holds; hence Conjecture 3.5 implies $\mathcal{Z}_{\xi_\varphi} \subset \overline{S_\varphi}$. Let us show that the inverse holds:

COROLLARY 3.6. $\mathcal{Z}_{\xi_\varphi} \subset \overline{S_\varphi}$ *implies Conjecture 3.5.*

Proof. Assume $\mathcal{Z}_{\xi_\varphi} \subset \overline{S_\varphi}$. If $W^0_\xi \cap S_\varphi \neq \emptyset$, we have $\xi \prec \xi_\varphi$ by Grothendieck and Katz ([9], Theorem 2.3.1); then Corollary 3.2 implies $\mathcal{Z}_\xi \subset \overline{\mathcal{Z}_{\xi_\varphi}}$, which is contained in $\overline{S_\varphi}$ by the assumption. □

REMARK 3.7. In [7] we shall prove $\mathcal{Z}_{\xi_\varphi} \subset \overline{S_\varphi}$.

4. Direct sums of BT_1**'s.** As written in §1, we need to investigate the final sequence ψ_ξ of $H(\xi)[p] = \bigoplus_i H_{m_i,n_i}[p]$, where $\xi = \sum_i (m_i, n_i)$. Although we can not give a general formula of ψ_ξ, it is possible to compute ψ_ξ for each example. In this section, we explain a way to determine the type of the direct sum of BT_1's in term of final types, and show some properties of ψ_ξ used later on.

4.1. A basic fact on final types. Let (B,δ) be a final type. The purpose of this subsection is to prove

PROPOSITION 4.1. *Let* π *be an automorphism of* B *such that* $\pi(b') > \pi(b) \Leftrightarrow \delta(b') > \delta(b)$ *for any* $b, b' \in B$ *with* $b' < b$. *Then* π *coincides with* π_δ *defined in (2.3.4).*

We need a lemma:

LEMMA 4.2. *Let* π *be as in the proposition above. Let* b *and* b' *be elements of* B.
(1) *If* $\delta(b') < \delta(b)$, *then* $\pi(b') < \pi(b)$.
(2) *If* $\delta(b) = \delta(b')$, *then* $b' < b \Leftrightarrow \pi(b') < \pi(b)$.

Proof. (1) Suppose $\delta(b') < \delta(b)$. Obviously we have $b \neq b'$. If $b' > b$, then $\pi(b') < \pi(b)$ holds. Thus we may assume $b' < b$. Since $\delta(b') \leq \delta(b)$, we have $\pi(b') \leq \pi(b)$. Since π is an automorphism and $b \neq b'$, we have $\pi(b') < \pi(b)$.

(2) Suppose $\delta(b) = \delta(b')$. First we prove $b' < b \Rightarrow \pi(b') < \pi(b)$. Assume $b' < b$. Since $\delta(b) \leq \delta(b')$, we get $\pi(b') \leq \pi(b)$; by $b' \neq b$ we have $\pi(b') \neq \pi(b)$; hence $\pi(b') < \pi(b)$. Exchanging the roles of b and b', we have $b' > b \Rightarrow \pi(b') > \pi(b)$. Also obviously $b' = b \Rightarrow \pi(b') = \pi(b)$; thus we have $b' \geq b \Rightarrow \pi(b') \geq \pi(b)$; this means $b' < b \Leftarrow \pi(b') < \pi(b)$. □

Proof. [Proof of Proposition 4.1] Let $B = \{b_1 < \cdots < b_d\}$ and set $B_- = \{b \in B \mid \delta(b) = 0\}$ and $B_+ = \{b \in B \mid \delta(b) = 1\}$. Put $d_0 = \sharp B_-$. Let f_- and f_+ be the order preserving bijections

$$\begin{aligned} f_-: \quad & B_- \longrightarrow \{b_1, \dots, b_{d_0}\}, \\ f_+: \quad & B_+ \longrightarrow \{b_{d_0+1}, \dots, b_d\}. \end{aligned} \tag{4.1.1}$$

Then Lemma 4.2 shows that π has to satisfy

$$\pi(b) = \begin{cases} f_-(b) & \text{if} \quad b \in B_-, \\ f_+(b) & \text{if} \quad b \in B_+. \end{cases} \tag{4.1.2}$$

Note π is uniquely determined by (4.1.2). Hence we obtain $\pi_\delta = \pi$, since π_δ in (2.3.4) also satisfies the condition that $\pi_\delta(b') > \pi_\delta(b) \Leftrightarrow \delta(b') > \delta(b)$ for any $b, b' \in B$ with $b' < b$. □

4.2. Direct decompositions of final types. We investigate direct decompositions of final types.

Let $\mathcal{B} = (B, \delta)$ be a final type. Let C be a subset of B and set $\varepsilon = \delta|_C$. We call $\mathcal{C} = (C, \varepsilon)$ a *final subtype* of $\mathcal{B}$ if $\pi(C) = C$, in which case we have $\pi|_C = \pi_\varepsilon$ by Proposition 4.1.

Definition 4.3. We call $\mathcal{B}$ *indecomposable*, if $\mathcal{B}$ has no proper final subtype, or equivalently B consists of one π-cycle. (Note that $\mathcal{B}$ is indecomposable if and only if the BT_1 associated to $\mathcal{B}$ is indecomposable, see [19], (1.5).)

Let $\mathcal{B} = (B, \delta)$ be a final type and $\mathcal{C} = (C, \epsilon)$ a final subtype of $\mathcal{B}$. Put $C' = B \setminus C$. Then clearly $\pi(C') = C'$ holds; hence we have a final subtype $\mathcal{C}' = (C', \delta|_{C'})$ of $\mathcal{B}$. In this case we write $\mathcal{B}$ as $\mathcal{C} \oplus \mathcal{C}'$. Let G, H and H' be the BT_1's associated to $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{C}'$ respectively. Then we have $G \simeq H \oplus H'$ by (2.3.2) and (2.3.3).

Let I be the set of isomorphism classes of indecomposable final types. A *multiplicity function* on I is a map $e : I \to \mathbb{Z}_{\geq 0}$ sending almost all to zero. For every final type $\mathcal{B} = (B, \delta)$, there is a unique multiplicity function e on I such that $\mathcal{B} = \bigoplus_{\mathcal{C} \in I} \mathcal{C}^{\oplus e(\mathcal{C})}$.

4.3. Computation of direct sums of final types. Now we present a way to compute direct sums of final types. The goal is to prove Corollary 4.8.

Before getting into the details, we give a remark. In [15], (2.2) we find an ordering on the set of words of $\mathcal{F}$ and $\mathcal{V}^{-1}$ (cyclic infinite words will be used below), which is closely related to our computation. If we associate such words to the 2-adic expansions of elements of $[0, 1] \subset \mathbb{R}$ by assigning 0 to $\mathcal{F}$ and 1 to $\mathcal{V}^{-1}$, then the ordering coincides with the usual ordering on $\mathbb{R}$. The following formulation is based on this fact.

Let D be the ordered subset of $[0, 1]$ consisting of $u \in [0, 1]$ with cyclic 2-adic expansion $u = \sum_{l=1}^{\infty} a_l 2^{-l}$ ($a_l = 0$ or 1), where "cyclic" means that for some $d \in \mathbb{N}$ we have $a_{l+d} = a_l$ for all $l \geq 1$; taking a minimal one among these d's, we write $u = \lceil a_1, \dots, a_d \rfloor$ and call d the *period* of u. Let U be the product $D \times \mathbb{N}$ with the lexicographic ordering. We define the partition map $\gamma : U \to \{0, 1\}$ by sending $(u, v) \in D \times \mathbb{N}$ to a_d, where $u = \lceil a_1, \dots, a_d \rfloor$. We define an automorphism τ of D by $\tau(\lceil a_1, \dots, a_d \rfloor) = \lceil a_d, a_1, \dots, a_{d-1} \rfloor$. This is extended to the automorphism of U defined by sending (u, v) to $(\tau(u), v)$, which is denoted by the same symbol τ.

Lemma 4.4. *Let B be a finite subset of U which is τ-stable. Put $\delta = \gamma|_B$. Then (B, δ) is a final type such that $\pi_\delta = \tau|_B$.*

Proof. Let $b = (u, v)$ and $b' = (u', v')$ be elements of B. Write the minimal cyclic expressions of u and u' as $\lceil a_1, \dots, a_d \rfloor$ and $\lceil a'_1, \dots, a'_{d'} \rfloor$ respectively. Assume

$b' < b$. By Proposition 4.1, it suffices to show $\tau(b') > \tau(b) \Leftrightarrow \gamma(b') > \gamma(b)$. Note $\tau(b) = (a_d + 2^{-1}u, v)$ and $\gamma(b) = a_d$; hence we have to show

$$(a'_{d'} + 2^{-1}u', v') > (a_d + 2^{-1}u, v) \Longleftrightarrow a'_{d'} > a_d. \tag{4.3.1}$$

From the assumption $b' < b$, we have (1) $u' < u$ or (2) $u' = u$ and $v' < v$. For (1), clearly $a'_{d'} + 2^{-1}u' > a_d + 2^{-1}u$ is equivalent $a'_{d'} > a_d$. For (2), clearly $a'_{d'} + 2^{-1}u' = a_d + 2^{-1}u$; then both of $v' > v$ and $a'_{d'} > a_d$ are false and therefore they are equivalent. $\square$

From now on, we shall show that any final type can be constructed as in Lemma 4.4. First let us check the indecomposable case. For an indecomposable final type $\mathcal{B} = (B, \delta)$ of length d with $\pi := \pi_\delta$, we define a map

$$\nu_{\mathcal{B}}: \quad B \longrightarrow D \tag{4.3.2}$$

by sending b to $\sum_{l=1}^{\infty} \delta(\pi^{-l}(b))2^{-l}$.

Lemma 4.5. *Let $\mathcal{B} = (B, \delta)$ be an indecomposable final type. If $b, b' \in B$ satisfy $\delta(\pi^i(b)) = \delta(\pi^i(b'))$ for all $i \in \mathbb{N}$, then we have $b = b'$. In other words $\nu_{\mathcal{B}}$ is injective.*

Proof. We assume $b' < b$ and derive a contradiction. Since $\mathcal{B}$ is indecomposable, there exists $l \in \mathbb{N}$ such that $b' = \pi^l(b)$. The conditions $\delta(\pi^j(b)) = \delta(\pi^j(b'))$ for all $j < i$ imply $\pi^i(b') < \pi^i(b)$ by using Lemma 4.2 (2) recursively. This means $\pi^{i+l}(b) < \pi^i(b)$ for all $i \in \mathbb{N}$, which contradicts $\sharp B < \infty$. $\square$

Lemma 4.6. *Let $\mathcal{B} = (B, \delta)$ be an indecomposable final type of length d and set $\nu = \nu_{\mathcal{B}}$. Then we have*

(1) $\delta = \gamma \circ \nu : \quad B \longrightarrow D \longrightarrow \{0, 1\}$*;*
(2) $\nu \circ \pi = \tau \circ \nu$*;*
(3) $\nu(b') < \nu(b)$ *if and only if* $b' < b$*.*

Proof. By Lemma 4.5, the period of $\nu(b)$ is equal to d. Hence (1) follows from the straightforward calculation: $(\gamma \circ \nu)(b) = \gamma(\lceil \delta(\pi^{-1}(b)), \ldots, \delta(\pi^{-d}(b)) \rfloor) = \delta(\pi^{-d}(b)) = \delta(b)$. (2) is obvious by definition. (3) By the injectivity of ν, it suffices to show $\nu(b') < \nu(b) \Rightarrow b' < b$. Note $\nu(b') < \nu(b)$ means that there is $l \in \mathbb{N}$ such that **(a)** $\delta(\pi^{-i}(b')) = \delta(\pi^{-i}(b))$ for all $1 \le i < l$ and **(b)** $\delta(\pi^{-l}(b')) < \delta(\pi^{-l}(b))$. **(b)** implies $\pi^{-l+1}(b') < \pi^{-l+1}(b)$ by Lemma 4.2 (1). Then **(a)** shows $b' < b$ by using Lemma 4.2 (2) recursively. $\square$

This lemma says that any indecomposable final type can be obtained as in Lemma 4.4, and also implies that there is a canonical bijection from the set of τ-orbits in D to the set I of isomorphism classes of indecomposable final types.

For a multiplicity function e on I, we consider the composition map

$$\tilde{e}: \quad D \longrightarrow D/\langle\tau\rangle \xrightarrow[\text{can}]{\sim} I \xrightarrow{e} \mathbb{Z}_{\ge 0} \tag{4.3.3}$$

and put

$$U^{\le e} = \{(x, y) \in U \mid y \le \tilde{e}(x)\}; \tag{4.3.4}$$

then $U^{\le e}$ is τ-stable and $U^{\le e}$ is a finite set; hence Lemma 4.4 defines a final type

$$\mathcal{U}^{\le e} = (U^{\le e}, \gamma|_{U^{\le e}}). \tag{4.3.5}$$

Clearly $\mathcal{U}^{\le e}$ is decomposed into the direct sum $\bigoplus_{\mathcal{C} \in I} \mathcal{C}^{\oplus e(\mathcal{C})}$. Thus we have

Proposition 4.7. *For any final type $\mathcal{B}$, there exists a unique multiplicity function e on I such that $\mathcal{B}$ is isomorphic to $\mathcal{U}^{\le e}$.*

This proposition tells us a concrete way to compute the direct sum of final types:

Corollary 4.8.

(1) *Let $\mathcal{B}$ be a final type of the form $\mathcal{C}^{\oplus e}$ for a certain indecomposable final type $\mathcal{C}$. Let $\mathcal{B} = (B, \delta)$ and $\mathcal{C} = (C, \epsilon)$. Write $(C^{(j)}, \epsilon^{(j)}) = (C, \epsilon)$ for $1 \le j \le e$ and $C^{(j)} = \{c_1^{(j)} < \cdots < c_d^{(j)}\}$. Then $\mathcal{B}$ can be given by*

$$B = \{c_1^{(1)} < \cdots < c_1^{(e)} < c_2^{(1)} < \cdots < c_2^{(e)} < \cdots\cdots < c_d^{(1)} < \cdots < c_d^{(e)}\}$$

and $\delta(c_i^{(j)}) = \epsilon^{(j)}(c_i^{(j)})$.

(2) *Let $\mathcal{C}^{(j)} = (C^{(j)}, \epsilon^{(j)})$ be final types. We write $C^{(j)} = \{c_1^{(j)} < \cdots < c_{d_j}^{(j)}\}$ and set $\nu^{(j)} = \nu_{\mathcal{C}^{(j)}}$. Assume any pair of distinct $\mathcal{C}^{(j)}$'s have no common indecomposable factor. Set $\mathcal{B} = (B, \delta) = \bigoplus_j \mathcal{C}^{(j)}$. Then $\mathcal{B}$ can be given as follows. First $B = \bigsqcup_j C^{(j)}$ as a set and the ordering on B is given by*

$$b < b' \Longleftrightarrow \begin{cases} b < b' & \text{for } b, b' \in C^{(j)}, \\ \nu^{(j)}(b) < \nu^{(j')}(b') & \text{for } b \in C^{(j)} \text{ and } b' \in C^{(j')} \text{ with } j \ne j', \end{cases}$$

and the partition map δ is given by $\delta(c_i^{(j)}) = \epsilon^{(j)}(c_i^{(j)})$.

Example 4.9. For the final sequences $\psi^{(1)} = (0, 1, 2)$ and $\psi^{(2)} = (0, 1)$, let us compute $\psi^{(1)} \oplus \psi^{(2)}$. Let $\mathcal{C}^{(i)} = (C^{(i)}, \epsilon^{(i)})$ be the final type of $\psi^{(i)}$. Write $C^{(1)} = \{c_1^{(1)} < c_2^{(1)} < c_3^{(1)}\}$ and $C^{(2)} = \{c_1^{(2)} < c_2^{(2)}\}$. By the rule (2.3.1) we have

$$\left(\epsilon^{(1)}(c_1^{(1)}), \epsilon^{(1)}(c_2^{(1)}), \epsilon^{(1)}(c_3^{(1)})\right) = (1, 0, 0),$$
$$\left(\epsilon^{(2)}(c_1^{(2)}), \epsilon^{(2)}(c_2^{(2)})\right) = (1, 0),$$

and by (2.3.4) or (4.1.2) the automorphisms $\pi_{\epsilon^{(i)}}$ of $C^{(i)}$ are given by

$$\begin{tikzcd} c_1^{(1)} \arrow[rr, bend right] & c_2^{(1)} \arrow[l, bend right] & c_3^{(1)} \arrow[l, bend right] & c_1^{(2)} \arrow[r, bend right] & c_2^{(2)} . \arrow[l, bend right] \end{tikzcd}$$

From the definition (4.3.2) of $\nu^{(i)} := \nu_{\mathcal{C}^{(i)}}$, we have

$$\begin{cases} \nu^{(1)}(c_1^{(1)}) = \lceil 0, 0, 1 \rfloor, \\ \nu^{(1)}(c_2^{(1)}) = \lceil 0, 1, 0 \rfloor, \\ \nu^{(1)}(c_3^{(1)}) = \lceil 1, 0, 0 \rfloor, \end{cases} \quad \begin{cases} \nu^{(2)}(c_1^{(2)}) = \lceil 0, 1 \rfloor, \\ \nu^{(2)}(c_2^{(2)}) = \lceil 1, 0 \rfloor. \end{cases}$$

Let $\mathcal{B} = (B, \delta)$ be the direct sum $\mathcal{C}^{(1)} \oplus \mathcal{C}^{(2)}$. Then by Corollary 4.8 (2), we have

$$B = \{c_1^{(1)} < c_2^{(1)} < c_1^{(2)} < c_3^{(1)} < c_2^{(2)}\}$$

with $\delta = (1, 0, 1, 0, 0)$. Thus we obtain $\psi^{(1)} \oplus \psi^{(2)} = (0, 1, 1, 2, 3)$ by (2.3.1). Note π_δ is given by

$$\begin{tikzcd} c_1^{(1)} \arrow[rrr, bend right] & c_2^{(1)} \arrow[l, bend right] & c_1^{(2)} \arrow[rr, bend right] & c_3^{(1)} \arrow[l, bend right] & c_2^{(2)} . \arrow[lll, bend right] \end{tikzcd}$$

Since we will see $\psi^{(1)} = \psi_{(2,1)}$ and $\psi^{(2)} = \psi_{(1,1)}$ in Lemma 4.13, we have $\psi_{(2,1)+(1,1)} = (0, 1, 1, 2, 3)$.

4.4. The first jumping element of a non-étale final type. We say that a final type (B,δ) is *not étale* (or *non-étale*) if $\delta \neq (1,\dots,1)$. Note that a BT_1 G is not étale (i.e., G is not isomorphic to a product of copies of $\mathbb{Z}/p\mathbb{Z}$) if and only if its final type is not étale.

Let us define the first jumping element of a non-étale final type. This is a very simple notion, but it plays important roles in this paper. In particular we shall see in Corollary 4.18 that the first jumping elements are beautifully arranged in the direct sum of minimal non-étale final types, which is a key step of our proof of the main theorem.

Definition 4.10. Assume $\mathcal{B} = (B,\delta)$ is not étale. Let ψ be the final sequence of $\mathcal{B}$.

(1) The *first jumping element* of $\mathcal{B}$ is the element $b \in B$ satisfying $\delta(b) = 0$ and $\delta(b') = 1$ for all $b' < b$.

(2) The *first jumping number* of ψ is the number J $(1 \leq J \leq d)$ satisfying $\psi(J-1) = 0$ and $\psi(J) = 1$:

$$\psi = (\underbrace{0,\dots,0,1}_{J},*,\dots,*).$$

Note b_J is the first jumping element of $\mathcal{B}$, where we write $B = \{b_1 < \dots < b_d\}$.

Let $\mathcal{C} = (C,\epsilon)$ be a final subtype of $\mathcal{B}$.

Definition 4.11. Assume $\mathcal{C}$ is not étale. The first jumping element of $\mathcal{C}$ considered as an element of B is called the *first jumping element of $\mathcal{C}$ in $\mathcal{B}$*.

We will use the obvious lemma:

Lemma 4.12. *Let $\mathcal{B} = \bigoplus_{i=1}^t \mathcal{B}^{(i)}$ be a decomposition of $\mathcal{B}$ into some final subtypes. Assume all $\mathcal{B}^{(i)}$ are not étale. Let $b_{J^{(i)}}$ be the first jumping element of $\mathcal{B}^{(i)}$ in $\mathcal{B}$. Then the first jumping element b_J of $\mathcal{B}$ is equal to* $\min\{b_{J^{(1)}},\dots,b_{J^{(t)}}\}$.

4.5. Minimal final types. Now we investigate the final sequence ψ_ξ of $H(\xi)[p] = \bigoplus_i H_{m_i,n_i}[p]$. Let us begin with studying the case $H_{m,n}[p]$, where (m,n) is a pair of non-negative integers. The following was shown in the proof of [5], Lemma 5.12:

Lemma 4.13. *We have* $\psi_{(m,n)} = (\underbrace{0,\dots,0}_{n},\underbrace{1,\dots,m}_{m})$.

Let $\mathcal{B}_{m,n} = (B_{m,n},\delta_{m,n})$ denote the final type of $\psi_{m,n}$. By Lemma 4.13 and (2.3.1) we have

$$\delta_{m,n} = (\underbrace{1,\dots,1}_{n},\underbrace{0,\dots,0}_{m}). \tag{4.5.1}$$

We call $\mathcal{B}_{m,n}$ *minimal of type* (m,n). By (4.1.2) we have:

Lemma 4.14. *Let (B,δ) be the minimal final type $\mathcal{B}_{m,n}$ with $B = \{b_1 < \dots < b_{m+n}\}$ and set $\pi = \pi_\delta$. Let ρ be the bijection from B to $\mathbb{Z}/(m+n)\mathbb{Z}$ defined by mapping b_i to the class of $i-1$. Then we have a commutative diagram*

$$\begin{array}{ccc} B & \xrightarrow{\rho} & \mathbb{Z}/(m+n)\mathbb{Z} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle -n} \\ B & \xrightarrow[\rho]{} & \mathbb{Z}/(m+n)\mathbb{Z}. \end{array}$$

This lemma gives us a simple proof of the following fact (cf. [19], (1.5) and (1.8)):

COROLLARY 4.15. *If* $\gcd(m,n)=1$*, then* $\mathcal{B}_{m,n}$ *is indecomposable.*

Proof. In the notation of Lemma 4.14, obviously B consists of one π-cycle. This means that $\mathcal{B}_{m,n}$ is indecomposable. ∎

By Corollary 4.8, we need only to have the formula of $\nu_{\mathcal{B}}$ for each $\mathcal{B}=\mathcal{B}_{m,n}$ with $\gcd(m,n)=1$, in order to calculate the final sequence ψ_ξ of $H(\xi)[p]$ for a concretely given ξ.

LEMMA 4.16. *Assume* $\gcd(m,n)=1$*. Let* $\mathcal{B}=\mathcal{B}_{m,n}$ *and write* $\mathcal{B}=(B,\delta)$ *with* $B=\{b_1<\cdots<b_{m+n}\}$*. Then for all* $1\le r\le m+n$ *we have*

$$\nu_{\mathcal{B}}(b_r)=\begin{cases}\displaystyle\sum_{l=1}^{\infty}2^{-\lfloor\{(m+n)l+n-r\}/n\rfloor} & \text{if}\quad n\neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. If $n=0$, we have $m=1$ by $\gcd(m,n)=1$; then $\mathcal{B}=\{b_1\}$, $\delta(b_1)=0$ and $\pi(b_1)=b_1$; hence $\nu_{\mathcal{B}}(b_1)=0$. Assume $n\neq 0$. By Lemma 4.14 we get $\pi^{-i}(b_r)=b_{r+ni}$ for all $i\in\mathbb{N}$. Hence $\delta(\pi^{-i}(b_r))=1$ if and only if for some $l\in\mathbb{N}$ we have $(m+n)l<r+ni\le(m+n)l+n$, namely $i=\lfloor\{(m+n)l+n-r\}/n\rfloor$. ∎

From now on we collect some partial results on the final sequence ψ_ξ, or equivalently on the direct sum of $\mathcal{B}_{m_i,n_i}$'s.

PROPOSITION 4.17. *Let* $(m_1,n_1),\dots,(m_t,n_t)$ *be pairs of integers* ≥ 0 *with* $\gcd(m_i,n_i)=1$ *and set* $\lambda_i=n_i/(m_i+n_i)$*. Assume* $\lambda_1<\cdots<\lambda_t$*. For* $1\le i\le t$*, let* $\mathcal{B}^{(i)}=\mathcal{B}_{m_i,n_i}$ *and write* $\mathcal{B}^{(i)}=(B^{(i)},\delta^{(i)})$ *with* $B^{(i)}=\{b_1^{(i)}<\cdots<b_{m_i+n_i}^{(i)}\}$*. Let* $\mathcal{B}=(B,\delta)$ *be their direct sum* $\bigoplus_i\mathcal{B}^{(i)}$*. Then for* $\varepsilon\in\{0,1\}$ *and for integers* α,β *satisfying* $1\le\alpha n_i+\beta m_i+\varepsilon\le m_i+n_i$ *for every* $1\le i\le t$*, we have*

$$b^{(1)}_{\alpha n_1+\beta m_1+\varepsilon}<\cdots<b^{(t)}_{\alpha n_t+\beta m_t+\varepsilon} \tag{4.5.2}$$

in $B=\coprod_i B^{(i)}$*.*

Proof. Put $\nu^{(i)}:=\nu_{\mathcal{B}^{(i)}}$. By Lemma 4.16, we have

$$\nu^{(i)}(b^{(i)}_{\alpha n_i+\beta m_i+\varepsilon})=\begin{cases}\displaystyle\sum_{l=1}^{\infty}2^{-\lfloor(l-\beta)/\lambda_i\rfloor-\alpha+\beta+1} & \text{for }\varepsilon=0,\\ \displaystyle\sum_{l=1}^{\infty}2^{-\lfloor(l-\beta)/\lambda_i+(n_i-1)/n_i\rfloor-\alpha+\beta} & \\ \qquad\qquad =\sum_{l=1}^{\infty}2^{-\lceil(l-\beta)/\lambda_i\rceil-\alpha+\beta} & \text{for }\varepsilon=1.\end{cases}$$

(Here these sums are regarded as 0 if $\lambda_i=0$.) Hence $\lambda_1<\cdots<\lambda_t$ implies $\nu^{(1)}(b^{(1)}_{\alpha n_1+\beta m_1+\varepsilon})<\cdots<\nu^{(t)}(b^{(t)}_{\alpha n_t+\beta m_t+\varepsilon})$. By Corollary 4.8 (2) we have the inequality (4.5.2). ∎

COROLLARY 4.18. *In the same notation as in Proposition 4.17, we assume* $m_t\ge 1$ *in addition. Then all* $\mathcal{B}^{(i)}$ *are not étale and we have* $b^{(1)}_{J_1}<\cdots<b^{(t)}_{J_t}$*, where* J_i *is the first jumping number* n_i+1 *of* ψ_{m_i,n_i}*.*

Proof. By $\lambda_i<\lambda_t<1$, we get $m_i\ge 1$ for all $1\le i\le t$. This means all $\mathcal{B}^{(i)}$ are not étale. Set $\alpha=1$, $\beta=0$ and $\varepsilon=1$. Then we have $1\le\alpha n_i+\beta m_i+\varepsilon\le m_i+n_i$. Applying Proposition 4.17 to this case, we have $b^{(1)}_{J_1}<\cdots<b^{(t)}_{J_t}$. ∎

LEMMA 4.19. *Let $\mathcal{B}^{(i)}$ $(i = 1, 2)$ be the minimal final types $\mathcal{B}_{m_i,n_i}$ with $\gcd(m_i, n_i) = 1$. Write $\mathcal{B}^{(i)} = (B^{(i)}, \delta^{(i)})$ with $B^{(i)} = \{b_1^{(i)} < \cdots < b_{m_i+n_i}^{(i)}\}$. Then in the direct sum $\mathcal{B}^{(1)} \oplus \mathcal{B}^{(2)}$, we have $b_{m_2}^{(2)} < b_{m_1+1}^{(1)}$.*

Proof. This follows from the computation of the first terms of $\nu_{\mathcal{B}^{(1)}}(b_{m_1+1}^{(1)})$ and $\nu_{\mathcal{B}^{(2)}}(b_{m_2}^{(2)})$. Indeed since $\lfloor\{(m_1+n_1)+n_1-m_1-1\}/n_1\rfloor = 1$ and $\lfloor\{(m_2+n_2)+n_2-m_2\}/n_2\rfloor = 2$, we have $\nu_{\mathcal{B}^{(2)}}(b_{m_2}^{(2)}) < \nu_{\mathcal{B}^{(1)}}(b_{m_1+1}^{(1)})$. $\square$

PROPOSITION 4.20. *Let $\mathcal{B}^{(i)}$ $(i = 1, 2)$ be the minimal final types $\mathcal{B}_{m_i,n_i}$ with $\gcd(m_i, n_i) = 1$. Write $\mathcal{B}^{(i)} = (B^{(i)}, \delta^{(i)})$ with $B^{(i)} = \{b_1^{(i)} < \cdots < b_{m_i+n_i}^{(i)}\}$. Set $\lambda_i = n_i/(m_i+n_i)$ and $h_i = m_i+n_i$. Assume $\lambda_1 \le 1/2 \le \lambda_2$. Then $(B, \delta) := \mathcal{B}^{(1)} \oplus \mathcal{B}^{(2)}$ can be given by*

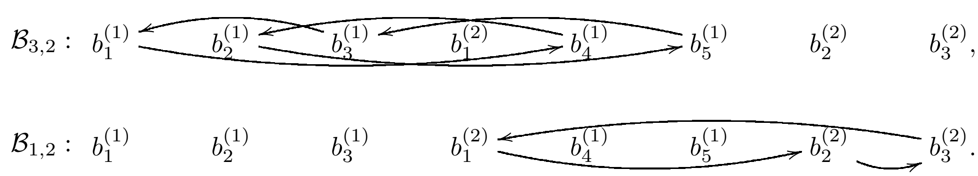
(1) *$B = \{b_1^{(1)} < \cdots < b_{m_1}^{(1)} < b_1^{(2)} < \cdots < b_{m_2}^{(2)} < b_{m_1+1}^{(1)} < \cdots < b_{h_1}^{(1)} < b_{m_2+1}^{(2)} < \cdots < b_{h_2}^{(2)}\}$,*

(2) $$\delta = (\underbrace{\overbrace{1,\dots,1}^{n_1},0,\dots,0}_{m_1},\underbrace{1,\dots,1}_{m_2},\underbrace{0,\dots,0}_{n_1},\underbrace{1,\dots,1,\overbrace{0,\dots,0}^{m_2}}_{n_2}).$$

Proof. (2) follows from (1) and (4.5.1). Let us prove (1). First we consider the case $\lambda_1 < 1/2 < \lambda_2$. We use an auxiliary final type $\mathcal{C} = (C, \epsilon)$ defined by $C = \{c_1 < c_2\}$, $\epsilon(c_1) = 1$ and $\epsilon(c_2) = 0$ (i.e., $\mathcal{C} \simeq \mathcal{B}_{1,1}$). By Proposition 4.17, we have $b_{m_1}^{(1)} < c_1$ $(\varepsilon = 0, \alpha = 0, \beta = 1)$ and $c_1 < b_1^{(2)}$ $(\varepsilon = 1, \alpha = 0, \beta = 0)$; hence $b_{m_1}^{(1)} < b_1^{(2)}$. Similarly $b_{h_1}^{(1)} < c_2$ $(\varepsilon = 0, \alpha = 1, \beta = 1)$ and $c_2 < b_{m_2+1}^{(2)}$ $(\varepsilon = 1, \alpha = 0, \beta = 1)$; hence $b_{h_1}^{(1)} < b_{m_2+1}^{(2)}$. We showed $b_{m_2}^{(2)} < b_{m_1+1}^{(1)}$ in Lemma 4.19.

By looking at the relation between $\mathcal{B}^{(1)}$ and $\mathcal{C}$ in the above argument, we obtain a proof of the case $\lambda_1 < 1/2 = \lambda_2$. Similarly we can prove the case $\lambda_1 = 1/2 < \lambda_2$. The case $\lambda_1 = 1/2 = \lambda_2$ is obvious by Corollary 4.8 (1). $\square$

EXAMPLE 4.21. The π-cycles of $\mathcal{B}_{3,2}$ and $\mathcal{B}_{1,2}$ in $\mathcal{B}_{3,2} \oplus \mathcal{B}_{1,2}$ are as follows:



By Proposition 4.20, we can determine how the elements of B_{m_1,n_1} and B_{m_2,n_2} are shuffled if $\lambda_1 \le 1/2 \le \lambda_2$. Thus clearly we have

COROLLARY 4.22. *Let $\eta = \sum_{i=1}^t (m_i, n_i)$ be a Newton polygon with $n_i/(m_i+n_i) \le 1/2$. Let ξ be the symmetric Newton polygon $s(1,1) + \sum_{i=1}^t (m_i, n_i) + (n_i, m_i)$ for $s \in \mathbb{Z}_{\ge 0}$. Set $m = \sum_i m_i$ and $n = \sum_i n_i$ and $g = m+n+s$. Put $\mathcal{C} = \bigoplus_{i=1}^t \mathcal{B}_{m_i,n_i}$ and $\mathcal{B} = \mathcal{C} \oplus \mathcal{B}_{s,s} \oplus \mathcal{C}^\vee$ and write $\mathcal{C} = (C, \epsilon)$ and $\mathcal{B} = (B, \delta)$ with $B = \{b_1 < \cdots < b_{2g}\}$.*

(1) *The decomposition $B = C \sqcup B_{s,s} \sqcup C^\vee$ is given by*

$$\begin{cases} C = \{b_1, \dots, b_m, b_{g+1}, \dots, b_{g+n}\}, \\ B_{s,s} = \{b_{m+1}, \dots, b_{m+s}, b_{g+n+1}, \dots, b_{g+n+s}\}, \\ C^\vee = \{b_{m+s+1}, \dots, b_g, b_{g+n+s+1}, \dots, b_{2g}\}, \end{cases}$$

(2) *The elementary sequence φ_ξ of $H(\xi)[p]$ is equal to*

$$(\psi_\eta(1), \ldots, \psi_\eta(m), \underbrace{m-n, \ldots, m-n}_{n+s}).$$

EXAMPLE 4.23. Consider the case $\eta = (m, n)$ and $\xi = (m, n) + s(1, 1) + (n, m)$. By Lemma 4.13 and Corollary 4.22 (2), we have

$$\varphi_\xi = (\underbrace{0, \ldots, 0}_{n}, \underbrace{1, 2, \ldots, m-n}_{m-n}, \underbrace{m-n, \ldots, m-n}_{n+s}). \tag{4.5.3}$$

5. Quasi-cycles in final types. In §4 we investigated ψ_ξ. Next we have to study the interrelation between ψ_ζ and ψ_ξ for $\zeta \prec \xi$. In §6 and §7 we shall explain "surgery", which is a basic tool to produce ψ_ζ from ψ_ξ. By surgery we mean a procedure to cut π-cycles into some pieces and join the pieces into new cycles, where each piece will be a quasi-cycle. In this section, after introducing the notion of quasi-cycles, we explain a way to construct a new final type from a quasi-cycle in §5.2, and we show some basic results on quasi-cycles in minimal final types in §5.3.

5.1. Definition of quasi-cycles. Let $\mathcal{B} = (B, \delta)$ be a final type and set $\pi = \pi_\delta$.

DEFINITION 5.1.
(1) A *π-path* of length l in $\mathcal{B}$ is an ordered subset Γ of B with $\sharp\Gamma = l + 1$ of the form $\{\pi^i(x) \mid 0 \leq i \leq l\}$ for some $x \in B$. We write Γ as Γ_{xy}, where $y = \pi^l(x)$.
(2) Let $\Gamma = \Gamma_{xy}$ be a π-path in $\mathcal{B}$. We call Γ a *quasi-cycle* in $\mathcal{B}$ if $x \neq y$ and there is no element z of Γ satisfying $x < z < y$ or $y < z < x$. We frequently write Γ as

$$\begin{tikzcd} x \arrow[r, bend right, "\Gamma"'] & y. \end{tikzcd} \tag{5.1.1}$$

EXAMPLE 5.2. Assume $\mathcal{B}$ is indecomposable. Then any adjacent elements $x < y$ of B define two quasi-cycles Γ_{yx} and Γ_{xy}. For example we consider $\mathcal{B}_{3,2} = (B_{3,2}, \delta_{3,2})$ with $B_{3,2} = \{b_1 < \cdots < b_5\}$ and put $x = b_2$ and $y = b_3$. The practical figures of (5.1.1) in this case are as follows:

$$\Gamma_{yx}: \quad \begin{tikzcd} b_1 \arrow[rrr, bend right] & x & y \arrow[ll, bend right] & b_4 \arrow[ll, bend right] & b_5, \end{tikzcd}$$

$$\Gamma_{xy}: \quad \begin{tikzcd} b_1 & x \arrow[rrr, bend right] & y & b_4 & b_5. \arrow[ll, bend right] \end{tikzcd}$$

5.2. The final type associated to a quasi-cycle. To a quasi-cycle $\Gamma = \Gamma_{xy}$ in $\mathcal{B}$, we associate a new final type $\mathcal{B}_\Gamma = (\tilde{\Gamma}, \tilde{\delta})$. We define $\tilde{\Gamma}$ to be the totally ordered set $\Gamma \setminus \{x\}$, and define $\tilde{\delta} : \tilde{\Gamma} \to \{0, 1\}$ by

$$\tilde{\delta}(c) = \begin{cases} \delta(x) & \text{if} \quad c = y, \\ \delta(c) & \text{otherwise} \end{cases}$$

for any $c \in \tilde{\Gamma}$. Put $\tilde{\pi} = \pi_{\tilde{\delta}}$.

Lemma 5.3. *We have $\tilde{\pi}(c) = \pi(c)$ for all $c \in \tilde{\Gamma} \setminus \{y\}$ and $\tilde{\pi}(y) = \pi(x)$.*

Proof. Let $\imath$ be the order preserving bijection from $\tilde{\Gamma}$ to $\Gamma \setminus \{y\}$, i.e., $\imath(c) = c$ for $c \neq y$ and $\imath(y) = x$. It is clear that

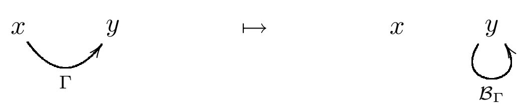
$$\tilde{\delta}(c) = \delta(\imath(c)) \quad \text{for any} \quad b \in \tilde{\Gamma}. \tag{5.2.1}$$

The lemma is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} \tilde{\Gamma} \arrow[r, "\tilde{\pi}"] \arrow[d, "\imath"'] & \tilde{\Gamma} \arrow[d, equal] \\ \Gamma \setminus \{y\} \arrow[r, "\pi"] & \Gamma \setminus \{x\}. \end{tikzcd} \tag{5.2.2}$$

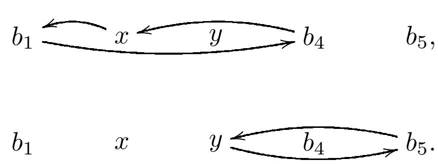
Proposition 4.1 says that $\tilde{\pi}$ is uniquely determined by the condition $\tilde{\delta}(c') > \tilde{\delta}(c) \Leftrightarrow \tilde{\pi}(c') > \tilde{\pi}(c)$ for any $c, c' \in \tilde{\Gamma}$ with $c' < c$. Similarly $\pi|_{\Gamma\setminus\{y\}}$ is uniquely determined by the condition $\delta(b') > \delta(b) \Leftrightarrow \pi(b') > \pi(b)$ for any $b, b' \in \Gamma \setminus \{y\}$ with $b' < b$. Hence (5.2.1) shows that the diagram is commutative. ∎

By the lemma above, we see that $\mathcal{B}_\Gamma$ is obtained from Γ only by sticking x on y; hence in §6 and §7 we shall express the association $\Gamma \mapsto \mathcal{B}_\Gamma$ as



since we will not touch other parts during surgery. Compare Example 5.2 and the following:

Example 5.4. In the notation in Example 5.2, the figures of $\mathcal{B}_{\Gamma_{yx}}$ and $\mathcal{B}_{\Gamma_{xy}}$ are as follows:

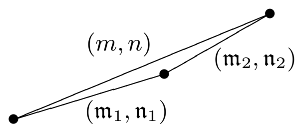


From this we have $\mathcal{B}_{\Gamma_{yx}} \simeq \mathcal{B}_{2,1}$ and $\mathcal{B}_{\Gamma_{xy}} \simeq \mathcal{B}_{1,1}$ (cf. Example 4.9). This is closely related to the inequality $(3, 2) \prec (2, 1) + (1, 1)$ of the Newton polygons(!), see Lemma 5.6 below.

5.3. Quasi-cycles in minimal final types. Let $\mathcal{C} = (C, \epsilon)$ be the minimal final type $\mathcal{B}_{m,n}$ with $\gcd(m, n) = 1$ and $m, n > 0$. Let us describe the final types $\mathcal{B}_{\Gamma_{yx}}$ and $\mathcal{B}_{\Gamma_{xy}}$ for any adjacent elements $x < y$ of C.

LEMMA 5.5. *Let m, n be positive integers with $\gcd(m, n) = 1$. The following conditions of non-negative integers $\mathfrak{m}_1, \mathfrak{n}_1, \mathfrak{m}_2, \mathfrak{n}_2$ with $m = \mathfrak{m}_1 + \mathfrak{m}_2$ and $n = \mathfrak{n}_1 + \mathfrak{n}_2$ are equivalent:*

(1) $\mathfrak{m}_1\mathfrak{n}_2 - \mathfrak{n}_1\mathfrak{m}_2 = 1$;
(2) $\mathfrak{m}_1 n \equiv 1 \pmod m$ *with* $0 < \mathfrak{m}_1 \le m$ *and* $\mathfrak{n}_1 m \equiv -1 \pmod n$ *with* $0 \le \mathfrak{n}_1 < n$. *Note such $\mathfrak{m}_1, \mathfrak{n}_1$ uniquely exist, and then $\mathfrak{m}_2, \mathfrak{n}_2$ also exist and are determined by $\mathfrak{m}_2 = m - \mathfrak{m}_1$ and $\mathfrak{n}_2 = n - \mathfrak{n}_1$;*
(3) *we have an inequality $(m, n) \prec (\mathfrak{m}_1, \mathfrak{n}_1) + (\mathfrak{m}_2, \mathfrak{n}_2)$ of Newton polygons:*



$\frac{\mathfrak{n}_1}{\mathfrak{m}_1+\mathfrak{n}_1} < \frac{\mathfrak{n}_2}{\mathfrak{m}_2+\mathfrak{n}_2}$

such that the area of the triangle is $1/2$.

Let $\mathfrak{m}_1, \mathfrak{n}_1, \mathfrak{m}_2$ and $\mathfrak{n}_2$ be the non-negative integers satisfying one of the conditions in Lemma 5.5.

LEMMA 5.6. *For any adjacent elements $x < y$ of C, we have $\mathcal{B}_{\Gamma_{yx}} \simeq \mathcal{B}_{\mathfrak{m}_1,\mathfrak{n}_1}$ and $\mathcal{B}_{\Gamma_{xy}} \simeq \mathcal{B}_{\mathfrak{m}_2,\mathfrak{n}_2}$.*

Proof. First note that a final type $\mathcal{B}' = (B', \delta')$ is minimal of a certain type (m', n') if and only if $\delta'(b') \ge \delta'(b'')$ for all $b' < b''$.

Since Γ_{yx} is an ordered subset of C, we have $\epsilon(c) \ge \epsilon(c')$ for any $c, c' \in \Gamma_{yx}$ with $c < c'$; hence $\mathcal{B}_{\Gamma_{yx}}$ is minimal of a certain type $(\mathfrak{m}'_1, \mathfrak{n}'_1)$. Similarly $\mathcal{B}_{\Gamma_{xy}}$ is minimal of a certain type $(\mathfrak{m}'_2, \mathfrak{n}'_2)$. Note $\mathfrak{m}'_1 + \mathfrak{m}'_2 = m$ and $\mathfrak{n}'_1 + \mathfrak{n}'_2 = n$. By Lemma 5.5, in order to prove $(\mathfrak{m}'_1, \mathfrak{n}'_1) = (\mathfrak{m}_1, \mathfrak{n}_1)$ and $(\mathfrak{m}'_2, \mathfrak{n}'_2) = (\mathfrak{m}_2, \mathfrak{n}_2)$, it suffices to show that $\mathfrak{m}'_1\mathfrak{n}'_2 - \mathfrak{m}'_2\mathfrak{n}'_1 = 1$ or equivalently $\mathfrak{m}'_1 n - m\mathfrak{n}'_1 = 1$. Write $C = \{c_1 < \cdots < c_{m+n}\}$. Note $\pi_\epsilon(c_i) = c_{i+m}$ if $1 \le i \le n$ and $\pi_\epsilon(c_i) = c_{i-n}$ if $n < i \le m + n$ by Lemma 4.14. Let $\Gamma_{yx} \setminus \{x\} = \{c_{j(1)} < \cdots < c_{j(\mathfrak{m}'_1+\mathfrak{n}'_1)}\}$. We have

$$\pi_\epsilon(c_{j(i)}) = \begin{cases} c_{j(i)+m} & \text{for} \quad 1 \le i \le \mathfrak{n}'_1, \\ c_{j(i)-n} & \text{for} \quad \mathfrak{n}'_1 < i \le \mathfrak{m}'_1 + \mathfrak{n}'_1. \end{cases}$$

Since $\pi(\Gamma_{yx} \setminus \{x\}) = \Gamma_{yx} \setminus \{y\}$ and $x < y$ are adjacent in C, we have $\mathfrak{m}'_1 n - m\mathfrak{n}'_1 = 1$. $\square$

Let $\mathcal{B} = (B, \delta)$ be a final type and set $\pi = \pi_\delta$. Assume that $\mathcal{B}$ contains $\mathcal{C} = \mathcal{B}_{m,n}$ as a final subtype. For later use we shall give a sufficient condition for that Γ_{yx} contains the first jumping element of $\mathcal{C}$ in $\mathcal{B}$. Write $\mathcal{B} = (B, \delta)$ with $B = \{b_1 < \cdots < b_d\}$. Let $b_{J(\mathcal{C})}$ be the first jumping element of $\mathcal{C}$ in $\mathcal{B}$.

LEMMA 5.7. *Let $x < y$ be adjacent elements of B with $x, y \in C$. Assume $\delta(x) = \delta(y)$ and $b_{J(\mathcal{C})+1} \notin C$. Then we have $b_{J(\mathcal{C})} \in \Gamma_{yx} \setminus \{y\}$.*

Proof. First note $y \ne b_{J(\mathcal{C})}$ by $\delta(x) = \delta(y)$, because for any adjacent elements $c' < c$ of C we have

$$\delta(c') \ne \delta(c) \quad \Longleftrightarrow \quad c = b_{J(\mathcal{C})}. \tag{5.3.1}$$

For any $c \in C$, let $v(c)$ denote the smallest non-negative integer i satisfying $\pi^i(c) = b_{J(\mathcal{C})}$. We assume $b_{J(\mathcal{C})} \notin \Gamma_{yx} \setminus \{y\}$ (i.e., $b_{J(\mathcal{C})} \in \Gamma_{xy} \setminus \{x\}$) and derive a contradiction. Let l be the smallest positive integer satisfying $\pi^l(y) = x$. Then we have $v(y) = l + v(x)$. Let us prove by induction:

CLAIM. $\pi^j(x)$ and $\pi^j(y)$ are adjacent in $\mathcal{B}$ with $\pi^j(x) < \pi^j(y)$ and $\delta(\pi^j(x)) = \delta(\pi^j(y))$ for all $0 \le j \le v(x)$.

This is obvious for $j = 0$. Assume Claim holds for $j - 1$. Then we have $\pi^j(x) < \pi^j(y)$ by Lemma 4.2 (2) and moreover $\pi^j(x)$ and $\pi^j(y)$ are adjacent by (4.1.2). Since $j \le v(x) < v(y)$, we get $\pi^j(y) \neq b_{J(\mathcal{C})}$ by the definition of $v(y)$; hence $\delta(\pi^j(x)) = \delta(\pi^j(y))$ by (5.3.1). Thus Claim holds for j.

Claim for $j = v(x)$ says that $b_{J(\mathcal{C})}$ and $\pi^{v(x)}(y)$ are adjacent in $\mathcal{B}$ with $b_{J(\mathcal{C})} < \pi^{v(x)}(y)$, namely $\pi^{v(x)}(y) = b_{J(\mathcal{C})+1}$. This contradicts $b_{J(\mathcal{C})+1} \notin C$. □

6. Surgeries - the unpolarized case. For a final type $\mathcal{B} = (B, \delta)$, we can construct a new final type by a "small" modification of δ, called a *twist*. In some cases, we find some beautiful relation between the old final type and the new one. Then we call those operations *surgeries*. By using surgeries, we derive some inductive formulas of ψ_ξ's (Corollaries 6.4, 6.7, 6.10 and 6.12).

6.1. Twists. Let $\mathcal{B} = (B, \delta)$ be a final type. Let κ be a permutation of B.

DEFINITION 6.1. The *twist* by κ of $\mathcal{B}$ is the new final type $(B, \delta \circ \kappa)$.

Put $B_- = \{b \in B \mid \delta(b) = 0\}$ and $B_+ = \{b \in B \mid \delta(b) = 1\}$.

LEMMA 6.2. *Assume that $\kappa^{-1}|_{B_-}$ and $\kappa^{-1}|_{B_+}$ preserve order. Then we have $\pi_{\delta\circ\kappa} = \pi_\delta \circ \kappa$.*

Proof. This follows from (4.1.2) and the assumption. □

6.2. Cutting - an indecomposable final type. Let $\mathcal{B} = (B, \delta)$ be an indecomposable final type. Suppose we are given adjacent elements $x < y$ of B such that

$$\delta(x) \neq \delta(y). \tag{6.2.1}$$

We have two quasi-cycles Γ_{yx} and Γ_{xy} in $\mathcal{B}$:

$$\begin{tikzcd} \mathcal{B}: & x \arrow[r, bend right, "\Gamma_{xy}"'] & y. \arrow[l, bend right, "\Gamma_{yx}"'] \end{tikzcd} \tag{6.2.2}$$

Let $\imath$ be the transposition (x, y). Let $\mathcal{B}' = (B, \delta')$ be the twist by $\imath$ of $\mathcal{B}$. By Lemma 6.2, we have $\pi_{\delta'} = \pi_\delta \circ \imath$. Set $\pi' = \pi_{\delta'}$.

LEMMA 6.3. *We have $\mathcal{B}' \simeq \mathcal{B}_{\Gamma_{yx}} \oplus \mathcal{B}_{\Gamma_{xy}}$.*

Proof. Clearly $\mathcal{B}'$ consists of two π'-cycles: one is obtained from Γ_{yx} by sticking y to x and the other is obtained from Γ_{xy} by sticking x to y, i.e., the π'-cycles in $\mathcal{B}'$ are written as:

$$\begin{tikzcd} \mathcal{B}': & x \arrow[loop above, "\mathcal{B}_{\Gamma_{yx}}"] & y \arrow[loop above, "\mathcal{B}_{\Gamma_{xy}}"] \end{tikzcd} \tag{6.2.3}$$

with $\delta'(x) = \delta(y)$ and $\delta'(y) = \delta(x)$. Thus we obtain $\mathcal{B}' \simeq \mathcal{B}_{\Gamma_{yx}} \oplus \mathcal{B}_{\Gamma_{xy}}$. □

We consider the case $\mathcal{B} = (B, \delta) = \mathcal{B}_{m,n}$. Write $B = \{b_1 < \cdots < b_{m+n}\}$. Then $(x, y) := (b_n, b_{n+1})$ satisfies (6.2.1) by (4.5.1). Let $\mathcal{B}' = (B, \delta')$ be the twist of $\mathcal{B}$ by (x, y) as above. Since δ is of the form (4.5.1), we have

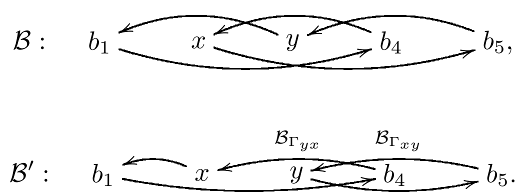
$$\delta' = (\underbrace{1, \ldots, 1}_{n-1}, 0, 1, \underbrace{0, \ldots, 0}_{m-1}). \tag{6.2.4}$$

Corollary 6.4. *Let* $(\mathfrak{m}_1, \mathfrak{n}_1)$ *and* $(\mathfrak{m}_2, \mathfrak{n}_2)$ *be as in Lemma 5.5. Then we have*

$$\psi_{(\mathfrak{m}_1,\mathfrak{n}_1)+(\mathfrak{m}_2,\mathfrak{n}_2)} = (\underbrace{0, \ldots, 0}_{n-1}, 1, \underbrace{1, \ldots, m}_{m}).$$

Proof. By Lemma 5.6, we have $\mathcal{B}_{\Gamma_{yx}} \simeq \mathcal{B}_{\mathfrak{m}_1,\mathfrak{n}_1}$ and $\mathcal{B}_{\Gamma_{xy}} \simeq \mathcal{B}_{\mathfrak{m}_2,\mathfrak{n}_2}$. Hence we obtain $\mathcal{B}' \simeq \mathcal{B}_{\mathfrak{m}_1,\mathfrak{n}_1} \oplus \mathcal{B}_{\mathfrak{m}_2,\mathfrak{n}_2}$. Then the corollary follows from (6.2.4) and (2.3.1). □

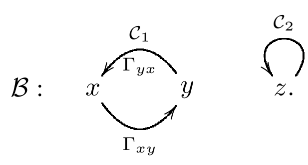
Example 6.5. Consider the case $(m, n) = (3, 2)$ and set $\mathcal{B} = \mathcal{B}_{3,2}$; then we have $(\mathfrak{m}_1, \mathfrak{n}_1) = (2, 1)$ and $(\mathfrak{m}_2, \mathfrak{n}_2) = (1, 1)$. Put $(x, y) = (b_2, b_3)$. Let $\mathcal{B}'$ be the final type twisted by (x, y). Then we have $\mathcal{B}' \simeq \mathcal{B}_{2,1} \oplus \mathcal{B}_{1,1}$. In this case we have $\psi_{(3,2)} = (0, 0, 1, 2, 3)$ and $\psi_{(2,1)+(1,1)} = (0, 1, 1, 2, 3)$ (cf. Example 4.9). The practical figures of (6.2.2) and (6.2.3) in this case are as follows:



6.3. Low cutting - a final type with two factors. Let $\mathcal{C}_1 = (C_1, \epsilon_1)$ and $\mathcal{C}_2 = (C_2, \epsilon_2)$ be two indecomposable final types. Let $\mathcal{B} = (B, \delta) = \mathcal{C}_1 \oplus \mathcal{C}_2$. Assume we are given adjacent elements $x < y < z$ of B such that

$$\begin{cases} (\delta(x), \delta(y), \delta(z)) = (0, 0, 1), \\ x, y \in C_1 \text{ and } z \in C_2. \end{cases} \tag{6.3.1}$$

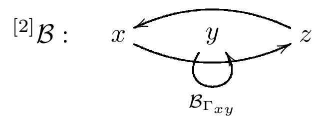
Then the π-cycles in $\mathcal{B}$ are written as



Let κ be the cyclic permutation (z, y, x). Let ${}^{[2]}\mathcal{B} = (B, {}^{[2]}\delta, {}^{[2]}\pi)$ be the twist by κ. By Lemma 6.2, we have

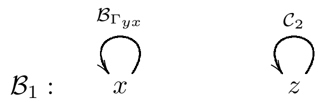
$${}^{[2]}\pi = \pi \circ \kappa. \tag{6.3.2}$$

Then ${}^{[2]}\pi$-cycles in ${}^{[2]}\mathcal{B}$ are written as



(6.3.3)

with $({}^{[2]}\delta(x), {}^{[2]}\delta(y), {}^{[2]}\delta(z)) = (1, 0, 0)$, where the complement of $\mathcal{B}_{\Gamma_{xy}}$ in ${}^{[2]}\mathcal{B}$ is isomorphic to the twist ${}^{[1]}\mathcal{B}_1$ by (x, z) of $\mathcal{B}_1 = (B_1, \delta_1)$ defined by

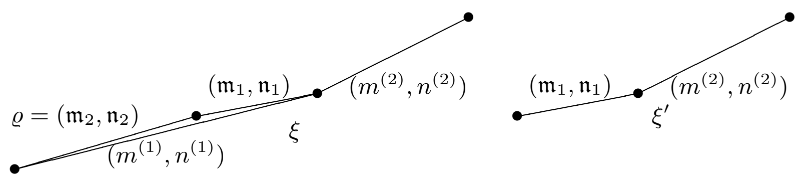


(6.3.4)

with $(\delta_1(x), \delta_1(z)) = (0, 1)$. Thus

Proposition 6.6. *We have* ${}^{[2]}\mathcal{B} \simeq {}^{[1]}\mathcal{B}_1 \oplus \mathcal{B}_{\Gamma_{xy}}$, *where* $\mathcal{B}_1 = \mathcal{B}_{\Gamma_{yx}} \oplus \mathcal{C}_2$.

Consider the case $\mathcal{C}_1 = \mathcal{B}_{m^{(1)},n^{(1)}}$ and $\mathcal{C}_2 = \mathcal{B}_{m^{(2)},n^{(2)}}$ with $0 < n^{(1)}/(m^{(1)} + n^{(1)}) < n^{(2)}/(m^{(2)} + n^{(2)})$. Assume that we are given a triple (x, y, z) satisfying (6.3.1). Let $(\mathfrak{m}_1, \mathfrak{n}_1)$ and $(\mathfrak{m}_2, \mathfrak{n}_2)$ be as in Lemma 5.5 for $(m, n) = (m^{(1)}, n^{(1)})$. Put $\xi = (m^{(1)}, n^{(1)}) + (m^{(2)}, n^{(2)})$ and $\xi' = (\mathfrak{m}_1, \mathfrak{n}_1) + (m^{(2)}, n^{(2)})$, and also $\varrho = (\mathfrak{m}_2, \mathfrak{n}_2)$.

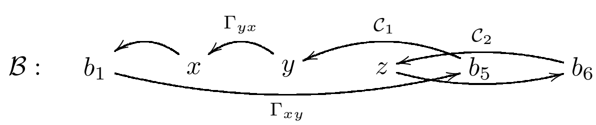


(6.3.5)

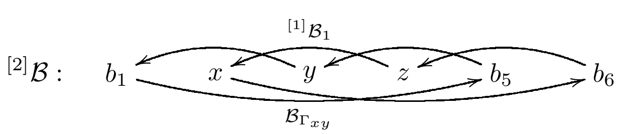
By Lemma 5.6, we have $\mathcal{B}_{\Gamma_{yx}} \simeq \mathcal{B}_{\mathfrak{m}_1,\mathfrak{n}_1}$ and $\mathcal{B}_{\Gamma_{xy}} \simeq \mathcal{B}_{\mathfrak{m}_2,\mathfrak{n}_2}$. Hence the final sequence of $\mathcal{B}_1$ (resp. $\mathcal{B}_{\Gamma_{xy}}$) is $\psi_{\xi'}$ (resp. ψ_ϱ). Let ${}^{[2]}\psi_\xi$ (resp. ${}^{[1]}\psi_{\xi'}$) denote the final sequence of ${}^{[2]}\mathcal{B}$ (resp. ${}^{[1]}\mathcal{B}_1$). Note ${}^{[2]}\psi_\xi$ and ${}^{[1]}\psi_{\xi'}$ depend on the choice of (x, y, z). In this case Proposition 6.6 is expressed as

Corollary 6.7. *We have* ${}^{[2]}\psi_\xi = {}^{[1]}\psi_{\xi'} \oplus \psi_\varrho$.

Example 6.8. Consider the case $\mathcal{C}_1 = \mathcal{B}_{3,1}$ and $\mathcal{C}_2 = \mathcal{B}_{1,1}$. In this case we have $\xi = (3, 1) + (1, 1)$ and $\xi' = (1, 0) + (1, 1)$ and also $\varrho = (2, 1)$. Note $\psi_\xi = (0, 1, 2, 2, 3, 4)$ and $\psi_{\xi'} = (1, 1, 2)$ and $\psi_\varrho = (0, 1, 2)$. Let $\mathcal{B} = (B, \delta)$ be the direct sum $\mathcal{C}_1 \oplus \mathcal{C}_2$ and write $B = \{b_1 < \ldots < b_6\}$. Set $(x, y, z) = (b_2, b_3, b_4)$. The π-cycles in $\mathcal{B}$ are written as



with $(\delta(b_1), \ldots, \delta(b_6)) = (1, 0, 0, 1, 0, 0)$. Thus (x, y, z) satisfies (6.3.1). Hence we have ${}^{[2]}\psi_\xi = {}^{[1]}\psi_{\xi'} \oplus \psi_\varrho$, where ${}^{[2]}\psi_\xi = (0, 0, 1, 2, 3, 4)$ and ${}^{[1]}\psi_{\xi'} = (0, 1, 2)$. Let us draw the practical figures of (6.3.3) and (6.3.4):



and

$$\begin{tikzcd} \mathcal{B}_1 : & x \arrow[loop above, "\mathcal{B}_{\Gamma_{yx}}"] & & z \arrow[r, bend right] & b_6 \arrow[l, bend right, "\mathcal{C}_2"'] . \end{tikzcd}$$

6.4. High cutting - a final type with two factors. Let $\mathcal{C}_1 = (C_1, \epsilon_1)$ and $\mathcal{C}_2 = (C_2, \epsilon_2)$ be two indecomposable final types. Let $\mathcal{B} = (B, \delta) = \mathcal{C}_1 \oplus \mathcal{C}_2$. Assume that we are given adjacent elements $x < y < z$ such that

$$\begin{cases} (\delta(x), \delta(y), \delta(z)) = (0, 1, 1), \\ x \in C_1 \text{ and } y, z \in C_2. \end{cases} \tag{6.4.1}$$

This case can be seen as the "dual" of (6.3.1). The π-cycles in $\mathcal{B}$ are written as

$$\begin{tikzcd} \mathcal{B} : & x \arrow[loop above, "\mathcal{C}_1"] & & y \arrow[r, bend right, "\Gamma_{yz}"'] & z \arrow[l, bend right, "\mathcal{C}_2", "\Gamma_{zy}"'] \end{tikzcd}$$

with $(\delta(x), \delta(y), \delta(z)) = (0, 1, 1)$. Let κ is the cyclic permutation (x, y, z). Let ${}^{[2]}\mathcal{B} = (B, {}^{[2]}\delta, {}^{[2]}\pi)$ be the twist by κ. By Lemma 6.2, we have

$${}^{[2]}\pi = \pi \circ \kappa. \tag{6.4.2}$$

Then ${}^{[2]}\pi$-cycles in ${}^{[2]}\mathcal{B}$ are written as

$$\begin{tikzcd} {}^{[2]}\mathcal{B} : & x \arrow[rr, bend right, "{}^{[1]}\mathcal{B}_1"'] & y \arrow[loop above, "\mathcal{B}_{\Gamma_{zy}}"] & z \arrow[ll, bend right] \end{tikzcd}$$

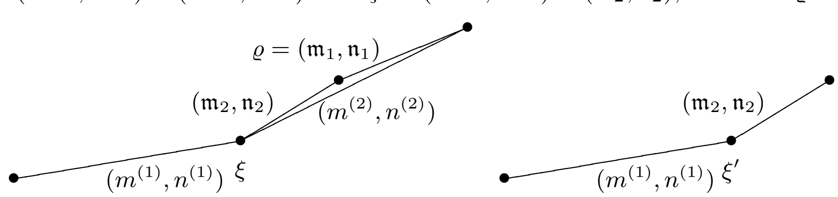
with $({}^{[2]}\delta(x), {}^{[2]}\delta(y), {}^{[2]}\delta(z)) = (1, 1, 0)$, where ${}^{[1]}\mathcal{B}_1$ is the twist by (x, z) of $\mathcal{B}_1 = (B_1, \delta_1)$

$$\begin{tikzcd} \mathcal{B}_1 : & x \arrow[loop above, "\mathcal{C}_1"] & & z \arrow[loop above, "\mathcal{B}_{\Gamma_{yz}}"] \end{tikzcd}$$

with $(\delta_1(x), \delta_1(z)) = (0, 1)$. Thus

PROPOSITION 6.9. *We have* ${}^{[2]}\mathcal{B} \simeq {}^{[1]}\mathcal{B}_1 \oplus \mathcal{B}_{\Gamma_{zy}}$, *where* $\mathcal{B}_1 = \mathcal{B}_{\Gamma_{yz}} \oplus \mathcal{C}_1$.

Consider the special case $\mathcal{C}_1 = \mathcal{B}_{m^{(1)}, n^{(1)}}$ and $\mathcal{C}_2 = \mathcal{B}_{m^{(2)}, n^{(2)}}$ with $n^{(1)}/(m^{(1)} + n^{(1)}) < n^{(2)}/(m^{(2)} + n^{(2)}) < 1$. Assume that we are given a triple (x, y, z) satisfying (6.4.1). Let $(\mathfrak{m}_1, \mathfrak{n}_1)$ and $(\mathfrak{m}_2, \mathfrak{n}_2)$ be as in Lemma 5.5 for $(m, n) = (m^{(2)}, n^{(2)})$. Put $\xi = (m^{(1)}, n^{(1)}) + (m^{(2)}, n^{(2)})$ and $\xi' = (m^{(1)}, n^{(1)}) + (\mathfrak{m}_2, \mathfrak{n}_2)$, and also $\varrho = (\mathfrak{m}_1, \mathfrak{n}_1)$.



(6.4.3)

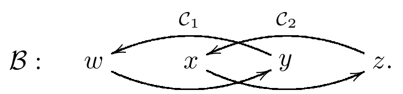
By Lemma 5.6, we have $\mathcal{B}_{\Gamma_{zy}} \simeq \mathcal{B}_{\mathfrak{m}_1,\mathfrak{n}_1}$ and $\mathcal{B}_{\Gamma_{yz}} \simeq \mathcal{B}_{\mathfrak{m}_2,\mathfrak{n}_2}$. Hence the final sequence of $\mathcal{B}_1$ (resp. $\mathcal{B}_{\Gamma_{zy}}$) is $\psi_{\xi'}$ (resp. ψ_ϱ). Let ${}^{[2]}\psi_\xi$ (resp. ${}^{[1]}\psi_{\xi'}$) denote the final sequence of ${}^{[2]}\mathcal{B}$ (resp. ${}^{[1]}\mathcal{B}_1$), which depends on the choice of (x, y, z). In this case Proposition 6.9 is expressed as

Corollary 6.10. *We have* ${}^{[2]}\psi_\xi = {}^{[1]}\psi_{\xi'} \oplus \psi_\varrho$.

6.5. High and low cutting - a final type with two factors. Let $\mathcal{C}_1 = (C_1, \epsilon_1)$ and $\mathcal{C}_2 = (C_2, \epsilon_2)$ be two indecomposable final types. Let $\mathcal{B} = (B, \delta) = \mathcal{C}_1 \oplus \mathcal{C}_2$. Assume that we are given adjacent elements $w < x < y < z$ such that

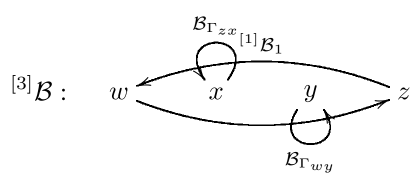
$$\begin{cases} (\delta(w), \delta(x), \delta(y), \delta(z)) = (0, 1, 0, 1), \\ w, y \in C_1 \text{ and } x, z \in C_2. \end{cases} \tag{6.5.1}$$

Then the figure of π-cycles in $\mathcal{B}$ is as follows:



(6.5.2)

Let κ be the permutation $(w, x, z, y) = (w, x)(y, z)(x, y)$. Let ${}^{[3]}\mathcal{B} = (B, {}^{[3]}\delta, {}^{[3]}\pi)$ be the twist by κ. By Lemma 6.2, we have ${}^{[3]}\pi = \pi \circ \kappa$. Then the ${}^{[3]}\pi$-cycles in ${}^{[3]}\mathcal{B}$ are written as

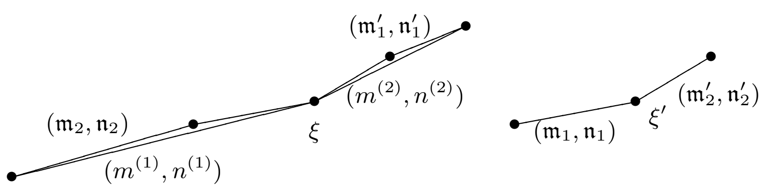


with $({}^{[3]}\delta(w), {}^{[3]}\delta(x), {}^{[3]}\delta(y), {}^{[3]}\delta(z)) = (1, 1, 0, 0)$, where ${}^{[1]}\mathcal{B}_1$ is the twist by (w, z) of $\mathcal{B}_1 = (B_1, \delta_1)$ defined by

with $(\delta_1(w), \delta_1(z)) = (0, 1)$. Thus

Proposition 6.11. *We have* ${}^{[3]}\mathcal{B} \simeq {}^{[1]}\mathcal{B}_1 \oplus \mathcal{B}_2$, *where* $\mathcal{B}_2 = \mathcal{B}_{\Gamma_{wy}} \oplus \mathcal{B}_{\Gamma_{zx}}$ *and* $\mathcal{B}_1 = \mathcal{B}_{\Gamma_{yw}} \oplus \mathcal{B}_{\Gamma_{xz}}$.

Consider the case $\mathcal{C}_1 = \mathcal{B}_{m^{(1)},n^{(1)}}$ and $\mathcal{C}_2 = \mathcal{B}_{m^{(2)},n^{(2)}}$ with $0 < n^{(1)}/(m^{(1)} + n^{(1)}) < n^{(2)}/(m^{(2)} + n^{(2)}) < 1$. Assume that we are given a quadruple (w, x, y, z) satisfying (6.5.1). Let $(\mathfrak{m}_1, \mathfrak{n}_1), (\mathfrak{m}_2, \mathfrak{n}_2)$ and $(\mathfrak{m}'_1, \mathfrak{n}'_1), (\mathfrak{m}'_2, \mathfrak{n}'_2)$ be as in Lemma 5.5 for $(m^{(1)}, n^{(1)})$ and $(m^{(2)}, n^{(2)})$ respectively. Put $\xi = (m^{(1)}, n^{(1)}) + (m^{(2)}, n^{(2)})$ and $\xi' = (\mathfrak{m}_1, \mathfrak{n}_1) + (\mathfrak{m}'_2, \mathfrak{n}'_2)$, and also $\varrho = (\mathfrak{m}_2, \mathfrak{n}_2) + (\mathfrak{m}'_1, \mathfrak{n}'_1)$.



(6.5.3)

By Lemma 5.6 we have $\mathcal{B}_{\Gamma_{yw}} \simeq \mathcal{B}_{\mathfrak{m}_1,\mathfrak{n}_1}$ and $\mathcal{B}_{\Gamma_{wy}} \simeq \mathcal{B}_{\mathfrak{m}_2,\mathfrak{n}_2}$ and also $\mathcal{B}_{\Gamma_{zx}} \simeq \mathcal{B}_{\mathfrak{m}'_1,\mathfrak{n}'_1}$ and $\mathcal{B}_{\Gamma_{xz}} \simeq \mathcal{B}_{\mathfrak{m}'_2,\mathfrak{n}'_2}$. Hence the final sequence of $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) is $\psi_{\xi'}$ (resp. ψ_{ϱ}). Let ${}^{[3]}\psi_\xi$ (resp. ${}^{[1]}\psi_{\xi'}$) denote the final sequence of ${}^{[3]}\mathcal{B}$ (resp. ${}^{[1]}\mathcal{B}_1$), which depends on the choice of (w, x, y, z). In this case Proposition 6.11 is expressed as

Corollary 6.12. *We have* ${}^{[3]}\psi_\xi = {}^{[1]}\psi_{\xi'} \oplus \psi_\varrho$.

7. Surgeries - the polarized case. We need to investigate surgeries of symmetric final types. In this case we have some inductive formulas of elementary sequences φ_ξ (Corollaries 7.5 and 7.8). In §8, by using these inductive formulas, we shall show $\varphi_\zeta \subset \varphi_\xi$ for $\zeta \prec \xi$.

7.1. Symmetric twists. Let $\mathcal{B} = (B, \delta)$ be a symmetric final type. Let κ be a permutation of B such that $\kappa(b^\vee) = \kappa(b)^\vee$ for all $b \in B$.

Definition 7.1. The *(symmetric) twist* of $\mathcal{B}$ by κ is the new symmetric final type $(B, \delta \circ \kappa)$.

By Lemma 6.2, we have

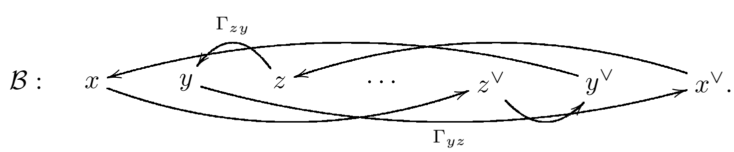
Lemma 7.2. *Assume $\kappa^{-1}|_{B_-}$ and $\kappa^{-1}|_{B_+}$ preserve order. Then we have $\pi_{\delta\circ\kappa} = \pi_\delta \circ \kappa$.*

7.2. Cutting - a symmetric final type with two asymmetric factors. Let $\mathcal{C} = (C, \epsilon)$ be an indecomposable final type. Let $\mathcal{B} = (B, \delta)$ be the symmetric final type $\mathcal{C} \oplus \mathcal{C}^\vee$. Write $B = \{b_1 < \cdots < b_{2g}\}$.

Assume that we are given adjacent elements $x < y < z$ of B with $z \leq b_g$ such that

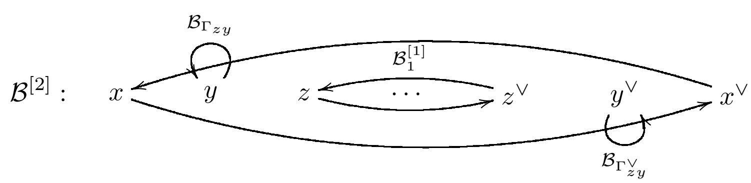
$$\begin{cases} (\delta(x), \delta(y), \delta(z)) = (0, 1, 1), \\ x \in C \text{ and } y, z \in C^\vee \text{ and } x^\vee \in \Gamma_{yz}. \end{cases} \tag{7.2.1}$$

Then the π-cycles in $\mathcal{B}$ are written as

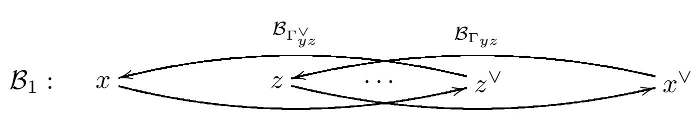


(7.2.2)

Let κ be the permutation $(x, y, z)(x^\vee, y^\vee, z^\vee)$. Let $\mathcal{B}^{[2]} = (B, \delta^{[2]}, \pi^{[2]})$ be the twist by κ. By Lemma 7.2, we have $\pi^{[2]} = \pi \circ \kappa$. Then the $\pi^{[2]}$-cycles in $\mathcal{B}^{[2]}$ are written as



with $(\delta^{[2]}(x), \delta^{[2]}(y), \delta^{[2]}(z)) = (1, 1, 0)$, where $\mathcal{B}_1^{[1]}$ is the twist by $(x, z)(x^\vee, z^\vee)$ of $\mathcal{B}_1 = (B_1, \delta_1)$ defined by



with $(\delta_1(x), \delta_1(z)) = (0,1)$. Thus

PROPOSITION 7.3. *We have* $\mathcal{B}^{[2]} \simeq \mathcal{B}_1^{[1]} \oplus \mathcal{B}_2$, *where* $\mathcal{B}_2 = \mathcal{B}_{\Gamma_{zy}} \oplus \mathcal{B}_{\Gamma_{zy}^\vee}$ *and* $\mathcal{B}_1 = \mathcal{B}_{\Gamma_{yz}} \oplus \mathcal{B}_{\Gamma_{yz}^\vee}$.

Next we shall investigate the case that we are given adjacent elements $z^\vee < y = b_g$ such that

$$\begin{cases} (\delta(z^\vee), \delta(y)) = (0,1), \\ z^\vee \in C \text{ and } y \in C^\vee. \end{cases} \tag{7.2.3}$$

We call this case the *confluent* case, because this case can be regarded as obtained by putting "$x = z^\vee$ and $y = b_g$" in (7.2.1).

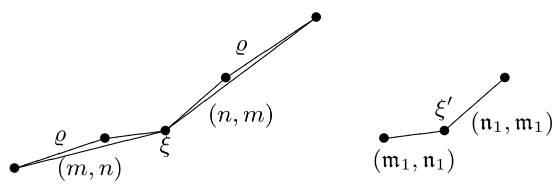
Then the π-cycles in $\mathcal{B}$ are written as

$$\begin{tikzcd} \mathcal{B}: & z^\vee \arrow[rr, bend right, "\Gamma_{zy}^\vee"'] & y \arrow[rr, bend right, "\Gamma_{yz}"'] & y^\vee \arrow[ll, bend right, "\Gamma_{yz}^\vee"'] & z. \arrow[ll, bend right, "\Gamma_{zy}"'] \end{tikzcd} \tag{7.2.4}$$

Let κ be the permutation $(z^\vee, y, z, y^\vee) = (z^\vee, y)(z, y^\vee)(y, y^\vee)$. Let $\mathcal{B}^{[2]} = (B, \delta^{[2]}, \pi^{[2]})$ be the twist by κ. By Lemma 7.2, we have $\pi^{[2]} = \pi \circ \kappa$. In the same way as in §6.5, we obtain

PROPOSITION 7.4. $\mathcal{B}^{[2]} \simeq \mathcal{B}_1^{[1]} \oplus \mathcal{B}_2$, *where* $\mathcal{B}_2 = \mathcal{B}_{\Gamma_{zy}} \oplus \mathcal{B}_{\Gamma_{zy}^\vee}$ *and* $\mathcal{B}_1^{[1]}$ *is the twist by* $(z^\vee, z)$ *of* $\mathcal{B}_1 = \mathcal{B}_{\Gamma_{yz}} \oplus \mathcal{B}_{\Gamma_{yz}^\vee}$.

Consider the case $\mathcal{C} = \mathcal{B}_{m,n}$ with $m > n > 0$. Let $\mathcal{B} = \mathcal{C} \oplus \mathcal{C}^\vee$. Let $(\mathfrak{m}_1, \mathfrak{n}_1)$ and $(\mathfrak{m}_2, \mathfrak{n}_2)$ be as in Lemma 5.5. Assume that we are given a triple (x, y, z) satisfying (7.2.1) or a pair $(z^\vee, y = b_g)$ satisfying (7.2.3). Set $\xi = (m,n) + (n,m)$ and $\xi' = (\mathfrak{m}_1, \mathfrak{n}_1) + (\mathfrak{n}_1, \mathfrak{m}_1)$ and also $\varrho = (\mathfrak{m}_2, \mathfrak{n}_2) + (\mathfrak{n}_2, \mathfrak{m}_2)$.



(7.2.5)

By Lemma 5.6, we have $\mathcal{B}_{\Gamma_{zy}^\vee} \oplus \mathcal{B}_{\Gamma_{zy}} \simeq \mathcal{B}_{\mathfrak{m}_2,\mathfrak{n}_2} \oplus \mathcal{B}_{\mathfrak{n}_2,\mathfrak{m}_2}$ and $\mathcal{B}_{\Gamma_{yz}^\vee} \oplus \mathcal{B}_{\Gamma_{yz}} \simeq \mathcal{B}_{\mathfrak{m}_1,\mathfrak{n}_1} \oplus \mathcal{B}_{\mathfrak{n}_1,\mathfrak{m}_1}$. Hence the final sequence of $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) is $\varphi_{\xi'}$ (resp. φ_ϱ). Let $\varphi_\xi^{[2]}$ (resp. $\varphi_{\xi'}^{[1]}$) denote the elementary sequence of $\mathcal{B}^{[2]}$ (resp. $\mathcal{B}_1^{[1]}$). Then in this case, Proposition 7.3 and Proposition 7.4 are expressed as

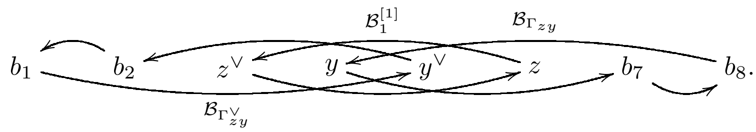
COROLLARY 7.5. *We have* $\varphi_\xi^{[2]} = \varphi_{\xi'}^{[1]} \oplus \varphi_\varrho$.

We give an example in the confluent case.

EXAMPLE 7.6. Consider the case $\mathcal{C} = \mathcal{B}_{3,1}$. Let $\xi = (3,1) + (1,3)$. In this case we have $\xi' = (1,0) + (0,1)$ and $\varrho = (2,1) + (1,2)$. Set $\mathcal{B} = \mathcal{C} \oplus \mathcal{C}^\vee$. Write $\mathcal{B} = (B, \delta)$ with $B = \{b_1 < \cdots < b_8\}$. The elementary sequence of $\mathcal{B}$ is $\varphi_\xi = (0,1,2,2)$ and we have $\delta = (1,0,0,1,0,1,1,0)$. Put $(z^\vee, y) := (b_3, b_4)$. Note that the π-cycles in $\mathcal{B}$ are written as

$$\begin{tikzcd} b_1 \arrow[rrrr, bend right] & b_2 \arrow[l, bend right] & z^\vee \arrow[l, bend right] & y \arrow[rr, bend right] & y^\vee \arrow[ll, bend right] & z \arrow[r, bend right] & b_7 \arrow[r, bend right] & b_8. \arrow[llll, bend right] \end{tikzcd}$$

Hence $(z^\vee, y)$ satisfies (7.2.3). Then $\mathcal{B}^{[2]}$ is given by



Confirm $\mathcal{B}_1 \simeq \mathcal{B}_{1,0} \oplus \mathcal{B}_{0,1}$ (with $\mathcal{B}_1^{[1]} \simeq \mathcal{B}_{1,1}$) and $\mathcal{B}_{\Gamma_{zy}^\vee} \oplus \mathcal{B}_{\Gamma_{zy}} \simeq \mathcal{B}_{2,1} \oplus \mathcal{B}_{1,2}$. Thus we have $\varphi_\xi^{[2]} = \varphi_{\xi'}^{[1]} \oplus \varphi_\varrho$, where $\varphi_\xi^{[2]} = (0,1,1,1)$ and $\varphi_{\xi'}^{[1]} = (0)$ and $\varphi_\varrho = (0,1,1)$.

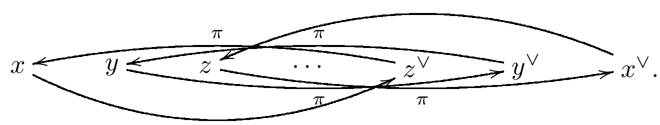
7.3. Reducing - a symmetric final type with certain three factors. Let $\mathcal{C}$ be an indecomposable final type. We consider the case $\mathcal{B} = (B, \delta) = \mathcal{C} \oplus \mathcal{B}_{1,1} \oplus \mathcal{C}^\vee$. Write $B = \{b_1 < \cdots < b_{2g}\}$. Assume that we are given adjacent elements $x < y < z$ with $z \le b_g$ such that

$$\begin{cases} (\delta(x), \delta(y), \delta(z)) = (0,1,1), \\ x \in C \text{ and } y \in B_{1,1} \text{ and } z \in C^\vee \end{cases} \tag{7.3.1}$$

or (confluent variant): adjacent elements $z^\vee < y = b_g$ such that

$$\begin{cases} (\delta(z^\vee), \delta(y)) = (0,1) \\ z^\vee \in C \text{ and } y \in B_{1,1}; \text{ we put } x = z^\vee. \end{cases} \tag{7.3.2}$$

Since $\pi(y^\vee) = y$ and $\delta(y^\vee) = 0 = \delta(z^\vee)$, we have $\pi(z^\vee) = x$ by (4.1.2); similarly we also have $\pi(z) = x^\vee$. Hence the figure of the π-cycles in $\mathcal{B}$ in the case (7.3.1) is as follows



As in §7.2, let κ be the permutation $(x, y, z)(x^\vee, y^\vee, z^\vee)$ for (7.3.1) and $(z^\vee, y, z, y^\vee)$ for (7.3.2). Let $\mathcal{B}^{[2]} = (B^{[2]}, \delta^{[2]})$ be the symmetric twist by κ. Set $\pi^{[2]} = \pi_{\delta^{[2]}}$. Then $\pi^{[2]} = \pi \circ \kappa$. Let $\mathcal{B}_1 = \mathcal{C} \oplus \mathcal{C}^\vee$ be the symmetric final subtype of $\mathcal{B}$; then $x < z$ is adjacent in $\mathcal{B}_1$; let $\mathcal{B}_1^{[1]} = (B_1^{[1]}, \delta_1^{[1]})$ be the symmetric twist by $(x, z)(x^\vee, z^\vee)$.

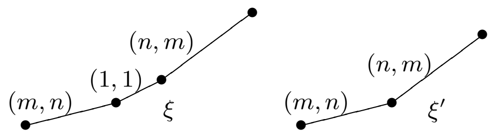
Proposition 7.7. *We have* $\mathcal{B}^{[2]} \simeq \mathcal{B}_1^{[1]} \oplus \mathcal{B}_{1,1}$.

Proof. Since $\pi^{[2]}(y^\vee) = \pi(z^\vee) = x$ and $\pi^{[2]}(x) = \pi(y) = y^\vee$, we have the $\pi^{[2]}$-cycle in $\mathcal{B}^{[2]}$:

$$y^\vee \xrightarrow{\pi^{[2]}} x \xrightarrow{\pi^{[2]}} y^\vee.$$

This cycle gives a factor in $\mathcal{B}^{[2]}$, which is isomorphic to $\mathcal{B}_{1,1}$. The remaining factor is $\mathcal{B}' = (B', \delta')$ defined by $B' = (B_1 \setminus x) \cup \{y\}$ and $\delta' = \delta^{[2]}|_{B'}$. Since $\delta'(y) = 1 = \delta_1^{[1]}(x)$, the map $\mathcal{B}'$ to $\mathcal{B}_1^{[1]}$ sending y to x and b to b for all $b \in B' \setminus \{y\}$ is an isomorphism as final types. $\square$

Consider the special case $\mathcal{C} = \mathcal{B}_{m,n}$. Let $\xi = (m,n) + (1,1) + (n,m)$ and $\xi' = (m,n) + (n,m)$.



(7.3.3)

Assume that we are given a triple (x,y,z) satisfying (7.3.1) or $(z^\vee, y = b_g)$ satisfying (7.3.2). Let $\varphi_\xi^{[2]}$ (resp. $\varphi_{\xi'}^{[1]}$) be the elementary sequence of $\mathcal{B}^{[2]}$ (resp. $\mathcal{B}_1^{[1]}$). In this case Proposition 7.7 is expressed as

Corollary 7.8. *We have* $\varphi_\xi^{[2]} = \varphi_{\xi'}^{[1]} \oplus \varphi_{(1,1)}$.

Example 7.9.

(1) Let $\zeta = 4(1,1)$ and $\xi = (2,1) + (1,1) + (1,2)$. Note $\xi' = (2,1) + (1,2)$ and $\varrho = (1,1)$. We have $\varphi_\xi = (0,1,1,1)$ and $\varphi_{\xi'} = (0,1,1)$. Note $(x,y,z) = (b_2, b_3, b_4)$ satisfies (7.3.1). Then one can check $\varphi_\xi^{[2]} = \varphi_{\xi'}^{[1]} \oplus \varphi_{(1,1)}$, i.e., $(0,0,0,1) = (0,0,1) \oplus (0)$.

(2) Let $\zeta = 2(1,1)$ and $\xi = (1,0) + (1,1) + (0,1)$. Note $\xi' = (1,0) + (0,1)$. We have $\varphi_\xi = (1,1)$ and $\varphi_{\xi'} = (1)$. Note $(z^\vee, y) = (b_1, b_2)$ satisfies (7.3.2). One can check $\varphi_\xi^{[2]} = \varphi_{\xi'}^{[1]} \oplus \varphi_{(1,1)}$, i.e., $(0,0) = (0) + (0)$.

8. Proof of the main theorem. Now we prove Theorem 3.1. In §8.1 we reduce our problem to three simple cases, and in §8.2 - §8.4 we give a proof in each case. The proof is done by showing some inductive formulas of φ_ξ's (Corollaries 8.6 and 8.10 and Propositions 8.20 - 8.23). For this we shall prove that the results in §6 and §7 are actually applicable (Lemmas 8.5, 8.9 and 8.18).

8.1. Reduction to the three cases. Let ζ and ξ be symmetric Newton polygons with $\zeta \prec \xi$. The proof of Theorem 3.1 is by induction on the number $c(\xi;\zeta)$ and the height of ξ. It suffices to show the case that

(S) (1) there is no symmetric Newton polygon η such that $\zeta \precneqq \eta \precneqq \xi$;
(2) $\zeta \cap \xi$ consists of finitely many points.

Indeed if there is a symmetric Newton polygon η such that $\zeta \precneqq \eta \precneqq \xi$, then by the induction hypothesis there exist two series of elementary sequences

$$\varphi_\zeta = \varphi_{1,0} < \cdots < \varphi_{1,c(\eta;\zeta)} = \varphi_\eta \quad \text{and} \quad \varphi_\eta = \varphi_{2,0} < \cdots < \varphi_{2,c(\xi;\eta)} = \varphi_\xi.$$

By $c(\xi;\zeta) = c(\xi;\eta) + c(\eta;\zeta)$, we get a required series $\varphi_{1,0} < \cdots < \varphi_{1,c(\eta;\zeta)} < \varphi_{2,1} < \cdots < \varphi_{2,c(\xi;\eta)}$. If $\zeta \cap \xi$ contains a segment, then there is a symmetric Newton polygon ϱ such that $\zeta = \zeta' + \varrho$ and $\xi = \xi' + \varrho$ with $\zeta' \prec \xi'$ and $c(\xi;\zeta) = c(\xi';\zeta')$. By the induction hypothesis, there is a series of elementary sequences $\varphi_{\zeta'} = \varphi'_0 < \cdots < \varphi'_{c(\xi';\zeta')} = \varphi_{\xi'}$. We set $\varphi_i = \varphi'_i \oplus \varphi_\varrho$ for $1 \le i \le c(\xi;\zeta) = c(\xi';\zeta')$. Then we have a required series $\varphi_\zeta = \varphi_0 < \cdots < \varphi_{c(\xi;\zeta)} = \varphi_\xi$.

From now on we assume that $\zeta \prec \xi$ satisfies **(S)**. Let $2g$ be the height of ξ.

Lemma 8.1. *One of the following three disjoint cases* **(A)**, **(A′)** *and* **(B)** *occurs.*

(A) $\xi(g) = \zeta(g) - 1/2 \in \mathbb{Z}$ *and the lower middle slope of* ξ *(the last slope over* $[0,g)$*) is less than* $1/2$.

(A′) $\zeta(g) = \xi(g) + 1/2 \in \mathbb{Z}$.

(B) $\xi(g) = \zeta(g) \in \mathbb{Z}$.

Proof. First note for any symmetric Newton polygon ϑ of height $2g$, we have $2\vartheta(g) \in \mathbb{Z}$. If there is an integer i with $\xi(g) < i < \zeta(g)$, then let η be the convex hull of ζ and the point (g, i); then we have $\zeta \precneqq \eta \precneqq \xi$, which contradicts the assumption. Hence we have $\zeta(g) - \xi(g) = 0, 1/2$ or $\zeta(g) - \xi(g) = 1$ with $\xi(g), \zeta(g) \in \mathbb{Z}$. If $\zeta(g) - \xi(g) = 1$ with $\xi(g), \zeta(g) \in \mathbb{Z}$, then we have the convex hull η of the points $(g-1, \xi(g))$, $(g+1, \zeta(g))$ and ζ, and the contradiction $\zeta \precneqq \eta \precneqq \xi$. The case $\zeta(g) - \xi(g) = 0$ is **(B)**. Now suppose $\zeta(g) - \xi(g) = 1/2$. If the lower middle slope of ξ is less than $1/2$, then $\xi(g)$ must be in $\mathbb{Z}$ and this is the case **(A)**. Otherwise we can assume that ξ does not have a breaking point at g and the slope of ξ is equal to $1/2$. If $\xi(g)$ were in $\mathbb{Z}$, then the convex hull η of ζ and the point $(g, \xi(g))$ satisfies $\zeta \precneqq \eta \precneqq \xi$, which is a contradiction. Hence this case is **(A′)**. ∎

8.2. Proof in the case (A). Assume $\zeta \prec \xi$ is of type **(A)**. First let us describe $\zeta \prec \xi$ concretely.

LEMMA 8.2. *We can write, for some $t \in \mathbb{Z}_{\geq 0}$,*

$$\begin{cases} \zeta = \zeta_0 + \zeta_1 + \cdots + \zeta_t, \\ \xi = (m, n) + (n, m) \end{cases}$$

with $\zeta_0 = (m_0, n_0) := (1, 1)$ and $\zeta_i = (m_i, n_i) + (n_i, m_i)$, where (m, n) and (m_i, n_i) $(1 \leq i \leq t)$ are segments such that

(1) *$\lambda_i = n_i/(m_i + n_i)$ satisfy*

$$\frac{n}{m+n} < \lambda_t \leq \cdots \leq \lambda_1 \leq \lambda_0 = 1/2;$$

(2) *$m = 1 + \sum_{i=1}^t m_i$ and $n = \sum_{i=1}^t n_i$.*

Note $g = m + n$. In this case we have $c(\xi; \zeta) = t + 1$.

Proof. We show that ξ has to be of the form above. Then ζ is automatically determined by the condition **(S)**, and it is straightforward to calculate the value of $c(\xi; \zeta)$ (cf. [17], (5.3)). The condition $\xi(g) \in \mathbb{Z}$ means that $Q = (g, \xi(g))$ is a breaking point of ξ. If ξ had another breaking point, then the convex hull η of ζ and the breaking points of ξ other than Q satisfies $\zeta \precneqq \eta \precneqq \xi$. This contradicts the assumption that $\zeta \prec \xi$ satisfies **(S)**. Hence $\xi = (m, n) + (n, m)$ for some non-negative integers m and n with $\gcd(m, n) = 1$. ∎

Set

$$\begin{aligned} \zeta' &:= \textstyle\sum_{i=0}^{t-1} \zeta_i, \\ \xi' &:= (m - m_t, n - n_t) + (n - n_t, m - m_t), \\ \varrho &:= \zeta_t. \end{aligned}$$

See the figure (7.2.5). Then $\zeta' \prec \xi'$ is lower dimensional of type **(A)**. Write $\xi' = (\mathfrak{m}_1, \mathfrak{n}_1) + (\mathfrak{n}_1, \mathfrak{m}_1)$ and $\varrho = (\mathfrak{m}_2, \mathfrak{n}_2) + (\mathfrak{n}_2, \mathfrak{m}_2)$. Note $\mathfrak{m}_1\mathfrak{n}_2 - \mathfrak{n}_1\mathfrak{m}_2 = 1$.

For a non-superspecial elementary sequence φ, i.e., $\varphi \neq (0, \ldots, 0)$, we define an elementary sequence $\varphi^{\langle 1 \rangle}$ as follows: let ψ be the final sequence stretched from φ and put

$$\mathfrak{r}(\varphi) = \max\{j \in \{1, \ldots, g\} \mid \psi(j-1) < \psi(j) = \psi(j+1)\}; \tag{8.2.1}$$

then we set

$$\varphi^{\langle 1\rangle}(i)=\begin{cases}\varphi(i)-1 & \text{if}\quad i=\mathfrak{r}(\varphi),\\ \varphi(i) & \text{otherwise}\end{cases}$$

for $i=0,\dots,g$ and also define $\varphi^{\langle k\rangle}$ as satisfying $\varphi^{\langle k\rangle}=(\varphi^{\langle k-1\rangle})^{\langle 1\rangle}$ for $2\le k\le|\varphi|$.

Let $\mathcal{C}=\mathcal{B}_{m,n}$ and set $\mathcal{B}=\mathcal{C}\oplus\mathcal{C}^\vee$. Write $\mathcal{B}=(B,\delta)$ and $\mathcal{C}=(C,\epsilon)$ with $C\subset B$. We set $\pi=\pi_\delta$ and write $B=\{b_1<\cdots<b_{2g}\}$. Then by Corollary 4.22 (1), we have

$$C=\{b_1,\dots,b_m,b_{g+1},\dots,b_{g+n}\} \tag{8.2.2}$$

and by (4.5.3) we have

$$\varphi_\xi=(\underbrace{0,\dots,0}_{n},\underbrace{1,\dots,m-n}_{m-n},\underbrace{m-n,\dots,m-n}_{n}). \tag{8.2.3}$$

Lemma 8.3. *We have $t=0$ if and only if $(m,n)=(1,0)$.*

Proof. Obvious by Lemma 8.2 (2). ∎

Lemma 8.4. *If $t=0$, then we have $\varphi_\zeta=(0)$ and $\varphi_\xi=(1)$. In particular we have $\varphi_\xi^{\langle 1\rangle}=\varphi_\zeta$.*

Proof. If $t=0$, then $\zeta=\zeta_0=(1,1)$; hence $\varphi_\zeta=(0)$. We have $\varphi_\xi=(1)$ by Lemma 8.3 and (8.2.3). ∎

Assume $t\ge 1$. Then we have $n\ge 1$ by Lemma 8.3, since $n=0$ implies $m=1$ by $\gcd(m,n)=1$. Hence we have $m+1=g-n+1\le g$. We put

$$\begin{cases}(x,y,z)=(b_m,b_{m+1},b_{m+2}) & \text{if}\quad m+1<g,\\ (z^\vee,y)=(b_m,b_{m+1}) & \text{if}\quad m+1=g.\end{cases}$$

Lemma 8.5.
(1) *If $m+1<g$, then (x,y,z) satisfies (7.2.1).*
(2) *If $m+1=g$, then $(z^\vee,y)$ satisfies (7.2.3).*

Proof. (1) Clearly $(x,y,z)=(b_m,b_{m+1},b_{m+2})$ satisfies $(\delta(x),\delta(y),\delta(z))=(0,1,1)$ by (2.3.1) and (8.2.3). Also we have $x\in C$ and $y,z\in C^\vee$ by (8.2.2). By (2.3.4) we have $\pi(y)=b_{g+(m+1)-(m-n)}=b_{2g+1-m}=x^\vee$; hence we obtain $x^\vee\in\Gamma_{yz}$. (2) Note $(z^\vee,y)=(b_m,b_{m+1})=(b_{g-1},b_g)$. By (2.3.1) and (8.2.3), we have $(\delta(z^\vee),\delta(y))=(0,1)$. It follows from (8.2.2) that $z^\vee\in C$ and $y\in C^\vee$. ∎

Corollary 8.6. *We have $\varphi_\xi^{\langle 2\rangle}=\varphi_{\xi'}^{\langle 1\rangle}\oplus\varphi_\varrho$.*

Proof. Let $\varphi_\xi^{[2]}$ and $\varphi_\xi^{[1]}$ be as in §7.2 for the triple (x,y,z) or the pair $(z^\vee,y)$ defined above. Then we have $\varphi_\xi^{[2]}=\varphi_\xi^{\langle 2\rangle}$ and $\varphi_{\xi'}^{[1]}=\varphi_{\xi'}^{\langle 1\rangle}$, since we have $\mathfrak{r}(\varphi_\xi)=m$ and $\mathfrak{r}(\varphi_\xi^{\langle 1\rangle})=m+1$. Hence the corollary is nothing but Corollary 7.5. ∎

Corollary 8.7. *There exists a series of elementary sequences $\varphi_0<\cdots<\varphi_{c(\xi;\zeta)}$ such that $\varphi_0=\varphi_\zeta$, $\varphi_{c(\xi;\zeta)-1}=\varphi_\xi^{\langle 1\rangle}$ and $\varphi_{c(\xi;\zeta)}=\varphi_\xi$. If $t\ge 1$, we can choose such a series satisfying $\varphi_{c(\xi;\zeta)-2}=\varphi_\xi^{\langle 2\rangle}$ in addition.*

Proof. For $t = 0$ this is nothing but Lemma 8.4. Assume $t \geq 1$. Put $c = c(\xi;\zeta)$ and $c' = c(\xi';\zeta')$. Note $c' = c - 1$. By the induction hypothesis, there are elementary sequences $\varphi'_0 < \cdots < \varphi'_{c'-1}$ such that $\varphi'_0 = \varphi_{\zeta'}$ and $\varphi'_{c'-1} = \varphi_{\xi'}^{\langle 1\rangle}$. We put $\varphi_i = \varphi'_i \oplus \varphi_\varrho$ for $i = 0, \ldots, c-2$ and set $\varphi_{c-1} = \varphi_\xi^{\langle 1\rangle}$ and $\varphi_c = \varphi_\xi$. Note $\varphi_0 = \varphi_{\zeta'} \oplus \varphi_\varrho = \varphi_\zeta$. It remains to show $\varphi_i < \varphi_{i+1}$ for all $0 \leq i < c$. This means

$$\begin{cases} \varphi'_i \oplus \varphi_\varrho < \varphi'_{i+1} \oplus \varphi_\varrho & \text{for } 0 \leq i \leq c-2, \\ \varphi_{\xi'}^{\langle 1\rangle} \oplus \varphi_\varrho < \varphi_\xi^{\langle 1\rangle} & \text{for } i = c-1, \\ \varphi_\xi^{\langle 1\rangle} < \varphi_\xi & \text{for } i = c. \end{cases}$$

The first and the third inequalities are obvious. The second inequality follows from Corollary 8.6. ∎

8.3. Proof in the case (A′). In this subsection we shall reduce the case **(A′)** to the case **(A)**. Let $\zeta \prec \xi$ be of type **(A′)**. The exact form of $\zeta \prec \xi$ is as follows.

Lemma 8.8. *We can write, for some $t \in \mathbb{Z}_{\geq 0}$,*

$$\begin{cases} \zeta = \zeta_0 + \zeta_1 + \cdots + \zeta_t, \\ \xi = (m,n) + (1,1) + (n,m) \end{cases}$$

with $\zeta_0 = (m_0, n_0) + (n_0, m_0) := (1,1) + (1,1)$ and $\zeta_i = (m_i, n_i) + (n_i, m_i)$, where (m,n) and (m_i, n_i) $(1 \leq i \leq t)$ are segments such that

(1) *$\lambda_i = n_i/(m_i + n_i)$ satisfy*

$$\frac{n}{m+n} < \lambda_t \leq \cdots \leq \lambda_1 \leq \lambda_0 = 1/2;$$

(2) *$m = 1 + \sum_{i=1}^t m_i$ and $n = \sum_{i=1}^t n_i$.*

Note $g = m + n + 1$. In this case we have $c(\xi;\zeta) = t + 2$.

Proof. It suffices to show that ξ has to be of the form above. Then ζ has to be of the above form by the condition **(S)**, and it is straightforward to compute the value of $c(\xi;\zeta)$ (cf.[20], (5.3)). Since $\xi(g)$ is not an integer, ξ contains a supersingular factor $(1,1)$. Thus ξ has the breaking points $P = (g-1, \xi(g-1))$ and $P^\vee = (g+1, \xi(g+1))$. If ξ had another breaking point, then the convex hull η of ζ and the breaking points other than $P, P^\vee$ satisfies $\zeta \precneqq \eta \precneqq \xi$. This is a contradiction. Hence $\xi = (m,n) + (1,1) + (n,m)$ for some non-negative integers m and n with $\gcd(m,n) = 1$. ∎

We define Newton polygons ξ' and ζ' by $\xi = \xi' + \varrho$ and $\zeta = \zeta' + \varrho$ with $\varrho = (1,1)$. See the figure (7.3.3). Then $\zeta' \prec \xi'$ is of type **(A)** with $c(\xi', \zeta') = t + 1$.

Set $\mathcal{C} = \mathcal{B}_{m,n}$ and $\mathcal{B} = \mathcal{C} \oplus \mathcal{B}_{1,1} \oplus \mathcal{C}^\vee$. Then $\mathcal{B}$ is the symmetric final type of φ_ξ. Write $\mathcal{B} = (B, \delta)$ with $B = \{b_1 < \cdots < b_{2g}\}$. Note $m + 1 = g - n \leq g$. Put

$$\begin{cases} (x,y,z) = (b_m, b_{m+1}, b_{m+2}) & \text{if} \quad m+1 < g, \\ (z^\vee, y) = (b_m, b_{m+1}) & \text{if} \quad m+1 = g. \end{cases}$$

Lemma 8.9.

(1) *If $m+1 < g$, then (x,y,z) satisfies (7.3.1).*

(2) *If $m+1 = g$, then $(z^\vee, y)$ satisfies (7.3.2).*

Proof. Write $\mathcal{C} = (C, \epsilon)$ and $\mathcal{B}_{1,1} = (B_2, \delta_2)$. By Corollary 4.22 (1), we have $C = \{b_1, \dots, b_m, b_{g+1}, \dots, b_{g+n}\}$ and $B_2 = \{b_{m+1}, b_{g+n+1}\}$ and $C^\vee = \{b_{m+2}, \dots, b_g, b_{g+n+2}, \dots, b_{2g}\}$. Also by (4.5.3) we have

$$\varphi_\xi = (\underbrace{0, \dots, 0}_{n}, 1, \dots, m-n, \underbrace{m-n, \dots, m-n}_{n+1}).$$

Hence if $m+1 < g$, then we have $(\delta(x), \delta(y), \delta(z)) = (0,1,1)$ by (2.3.1), and $x \in C$, $y \in B_2$ and $z \in C^\vee$. If $m+1 = g$, then we have $(\delta(z^\vee), \delta(y)) = (0,1)$ and also $z^\vee \in C$ and $y \in B_2$. ∎

Corollary 8.10. *We have* $\varphi_\xi^{\langle 2\rangle} = \varphi_{\xi'}^{\langle 1\rangle} \oplus \varphi_\varrho$.

Proof. Let $\varphi_\xi^{[2]}$ and $\varphi_\xi^{[1]}$ be as in §7.3 for the triple (x,y,z) or the pair $(z^\vee, y)$ defined above. Then we have $\varphi_\xi^{\langle 2\rangle} = \varphi_\xi^{[2]}$ and $\varphi_{\xi'}^{\langle 1\rangle} = \varphi_{\xi'}^{[1]}$, since $\mathfrak{r}(\varphi_\xi) = m$ and $\mathfrak{r}(\varphi_\xi^{\langle 1\rangle}) = m+1$. Hence the corollary is nothing but Corollary 7.8. ∎

See Example 7.9 for some examples. By the same argument as in Corollary 8.7, we obtain

Corollary 8.11. *There exist elementary sequences* $\varphi_0 < \cdots < \varphi_{c(\xi;\zeta)}$ *such that* $\varphi_0 = \varphi_\zeta$, $\varphi_{c(\xi;\zeta)-2} = \varphi_\xi^{\langle 2\rangle}$, $\varphi_{c(\xi;\zeta)-1} = \varphi_\xi^{\langle 1\rangle}$ *and* $\varphi_{c(\xi;\zeta)} = \varphi_\xi$.

8.4. Proof in the case (B). Assume $\zeta \prec \xi$ is of type **(B)**. We have the following description of $\zeta \prec \xi$.

Lemma 8.12. *We can write, for some* $r, s \in \mathbb{Z}_{\geq 0}$,

$$\begin{cases} \zeta = \displaystyle\sum_{i=-r}^{s} \zeta_i, \\ \xi = (m^{(1)}, n^{(1)}) + (m^{(2)}, n^{(2)}) + (n^{(2)}, m^{(2)}) + (n^{(1)}, m^{(1)}) \end{cases}$$

with $\zeta_i = (m_i, n_i) + (n_i, m_i)$, *where* $(m^{(i)}, n^{(i)})$ $(i = 1,2)$ *and* (m_i, n_i) $(-r \leq i \leq s)$ *are segments such that*

(1) $\lambda^{(i)} = n^{(i)}/(m^{(i)} + n^{(i)})$ *(*$i = 1,2$*) and* $\lambda_i = n_i/(m_i + n_i)$ *(*$-r \leq i \leq s$*) satisfy*

$$\lambda^{(1)} < \lambda_{-r} \leq \cdots \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_s < \lambda^{(2)};$$

(2) *we have* $m = \sum_{i=-r}^{s} m_i$ *and* $n = \sum_{i=-r}^{s} n_i$, *where* $m = m^{(1)} + m^{(2)}$ *and* $n = n^{(1)} + n^{(2)}$;

(3) *the first breaking point* $(m^{(1)} + n^{(1)}, n^{(1)})$ *of* ξ *is under* ζ_0. *(This condition determines* r *and* s.*)*

Note $g = m + n$. *In this case we have* $c(\xi;\zeta) = r + s + 1$.

Proof. It suffices to show that ξ has to be of the form above. Then the form of ζ is determined by the conditions **(S)** and $\xi(g) = \zeta(g)$, and it is straightforward to compute the value of $c(\xi;\zeta)$ (cf. [20], (5.3)). The condition $\xi(g) \in \mathbb{Z}$ means that $Q = (g, \xi(g))$ is a breaking point of ξ. If there were no other true breaking point, then $\zeta = \xi$ follows from the condition $\zeta(g) = \xi(g)$. This is a contradiction. Let P and $P^\vee$ be the first and the last true breaking points of ξ respectively. Note $P \neq Q \neq P^\vee$. If ξ had a breaking point other than $P, Q, P^\vee$, then the convex hull η of ζ and the breaking points of ξ other than $P, P^\vee$ satisfies $\zeta \precneqq \eta \precneqq \xi$. This is a contradiction. Thus ξ has to be as in the lemma. ∎

Put $\mathcal{C}_1 = (C_1, \epsilon_1) := \mathcal{B}_{m^{(1)},n^{(1)}}$ and $\mathcal{C}_2 = (C_2, \epsilon_2) := \mathcal{B}_{m^{(2)},n^{(2)}}$. We define $\mathcal{B} = \mathcal{C}_1 \oplus \mathcal{C}_2$ and $\overline{\mathcal{B}} = \mathcal{B} \oplus \mathcal{B}^\vee$. Write $\overline{\mathcal{B}} = (\overline{B}, \overline{\delta})$ with $\overline{B} = \{\overline{b}_1 < \cdots < \overline{b}_{2g}\}$. Note $\overline{\mathcal{B}}$ is the symmetric final type of φ_ξ. Recall $m = m^{(1)} + m^{(2)}$, $n = n^{(1)} + n^{(2)}$ and $g = m + n$ (Lemma 8.12).

We claim $n = n^{(1)} + n^{(2)} \geq 1$ and $m^{(i)} \geq 1$ for $i = 1, 2$. Indeed if $n = n^{(1)} + n^{(2)} = 0$, we have $n^{(i)} = 0$; then all slopes of ξ are 0 or 1; this contradicts $\lambda^{(1)} < \lambda^{(2)}$ (Lemma 8.12 (1)). Since $m^{(i)} \geq n^{(i)}$ and $\gcd(m^{(i)}, n^{(i)}) = 1$, we have $m^{(i)} \geq 1$. Note $m = m^{(1)} + m^{(2)} \geq 2$.

Lemma 8.13. *Let J be the first jumping number of φ_ξ (Definition 4.10). Then φ_ξ is of either of the following types:*

(I_ℓ) *φ_ξ is of the form $(0, \ldots, 0, \underbrace{1, 2, \ldots, \ell}_{\ell}, \ell, *, \ldots, *)$ for some ℓ with $2 \leq \ell \leq g - J$,*

(II_ℓ) *φ_ξ is of the form $(0, \ldots, 0, 1, \underbrace{1, \ldots, \ell}_{\ell}, \ell, *, \ldots, *)$ for some ℓ with $1 \leq \ell \leq g - J - 1$.*

In particular we have $|\varphi_\xi| \geq 3$.

Proof. By Corollary 4.22 (2) we have $J \leq n$. Thus we have $g - J \geq m \geq 2$. If $\varphi(J + 1) = 2$, then φ is of type (I). Otherwise φ is of type (II). □

For a non-superspecial elementary sequence φ, we define an elementary sequence ${}^{\langle 1\rangle}\varphi$ as follows: let ψ be the final sequence stretched from φ and let

$$\mathfrak{l}(\varphi) = \min\{j \in \{1, \ldots, g\} \mid 0 \neq \psi(j) = \psi(j+1)\}; \tag{8.4.1}$$

then we set

$${}^{\langle 1\rangle}\varphi(i) = \begin{cases} \varphi(i) - 1 & \text{if} \quad i = \mathfrak{l}(\varphi), \\ \varphi(i) & \text{otherwise} \end{cases}$$

for $i = 0, \ldots, g$ and define ${}^{\langle k\rangle}\varphi$ as satisfying ${}^{\langle k\rangle}\varphi = {}^{\langle 1\rangle}({}^{\langle k-1\rangle}\varphi)$ for $2 \leq k \leq |\varphi|$.

In this subsection we show a little stronger assertion than Theorem 3.1:

Proposition 8.14. *There exists a series of elementary sequences $\varphi_0 < \cdots < \varphi_{c(\xi;\zeta)}$ such that $\varphi_0 = \varphi_\zeta$, $\varphi_{c(\xi;\zeta)-1} = {}^{\langle 1\rangle}\varphi_\xi$ and $\varphi_{c(\xi;\zeta)} = \varphi_\xi$. If $(r, s) \neq (0, 0)$ we can choose such a series satisfying $\varphi_{c(\xi;\zeta)-2} = {}^{\langle 2\rangle}\varphi_\xi$ in addition. (Here (r, s) is as in Lemma 8.12.)*

The proof is by induction on height of ξ. Now we assume Proposition 8.14 holds for Newton polygons with lower height.

Firstly we consider the case of $r = s = 0$. In this case $\zeta = (m, n) + (n, m)$; hence by (4.5.3) we have

$$\varphi_\zeta = (\underbrace{0, \ldots, 0}_{n}, 1, \ldots, m - n, \underbrace{m - n, \ldots, m - n}_{n}).$$

Also ξ is written as

$$\xi = (\mathfrak{m}_1, \mathfrak{n}_1) + (\mathfrak{m}_2, \mathfrak{n}_2) + (\mathfrak{n}_2, \mathfrak{m}_2) + (\mathfrak{n}_1, \mathfrak{m}_1) \tag{8.4.2}$$

with $\mathfrak{m}_1 + \mathfrak{m}_2 = m$, $\mathfrak{n}_1 + \mathfrak{n}_2 = n$ and $\mathfrak{m}_1\mathfrak{n}_2 - \mathfrak{m}_2\mathfrak{n}_1 = 1$.

Lemma 8.15. *Let ξ be as in (8.4.2). Then we have*

$$\varphi_\xi = (\underbrace{0,\dots,0}_{n-1},1,\underbrace{1,\dots,m-n}_{m-n},\underbrace{m-n,\dots,m-n}_{n}). \tag{8.4.3}$$

In particular if $r = s = 0$, then we have ${}^{\langle 1\rangle}\varphi_\xi = \varphi_\zeta$.

Proof. This follows from Corollaries 6.4 and 4.22 (2). ☐

From now on we assume $(r, s) \neq (0, 0)$. Let J be the first jumping number of φ_ξ and ℓ as in Lemma 8.13. We put

$$v = \begin{cases} J+\ell & \text{for (I)}, \\ J+\ell+1 & \text{for (II)} \end{cases} \tag{8.4.4}$$

and set

$$\begin{cases} (x,y,z) = (\overline{b}_{v-2},\overline{b}_{v-1},\overline{b}_v) & \text{for (I) and } (\mathrm{II}_{\ell\neq 2}), \\ (w,x,y,z) = (\overline{b}_{v-3},\overline{b}_{v-2},\overline{b}_{v-1},\overline{b}_v) & \text{for } (\mathrm{II}_2). \end{cases} \tag{8.4.5}$$

Note $v \le g$ (see Lemma 8.13), i.e., we have $z \le \overline{b}_g$. By (2.3.1) we have

$$(\overline{\delta}(x),\overline{\delta}(y),\overline{\delta}(z)) = \begin{cases} (0,0,1) & \text{for (I) and } (\mathrm{II}_{\ell\geq 3}), \\ (0,1,1) & \text{for } (\mathrm{II}_1), \end{cases} \tag{8.4.6}$$

$$(\overline{\delta}(w),\overline{\delta}(x),\overline{\delta}(y),\overline{\delta}(z)) = (0,1,0,1) \quad \text{for } (\mathrm{II}_2). \tag{8.4.7}$$

Lemma 8.16. *We have $z \le \overline{b}_m$. (Therefore $w, x, y, z \in B$ by Corollary 4.22 (1)).*

Proof. Assume $z \in \{\overline{b}_{m+1},\dots,\overline{b}_g\}$ and derive a contradiction. By the definition (8.4.5) of z and Corollary 4.22 (2), we have $\ell = m - n$. Then by Corollary 4.22, the following has to hold:

$$\varphi_\xi = \begin{cases} (\underbrace{0,\dots,0}_{n},1,\dots,m-n,\underbrace{m-n,\dots,m-n}_{n}) & \text{for (I)}, \\ (\underbrace{0,\dots,0}_{n-1},1,\underbrace{1,\dots,m-n}_{m-n},\underbrace{m-n,\dots,m-n}_{n}) & \text{for (II)}. \end{cases}$$

Hence we have $\varphi_\xi = \varphi_{(m,n)+(n,m)}$ for (I); this is a contradiction. For (II), by (8.4.3) we have $\varphi_\xi = \varphi_{(\mathfrak{m}_1,\mathfrak{n}_1)+(\mathfrak{m}_2,\mathfrak{n}_2)+(\mathfrak{n}_2,\mathfrak{m}_2)+(\mathfrak{n}_1,\mathfrak{m}_1)}$, which contradicts $(r, s) \neq (0, 0)$. ☐

Lemma 8.17.

(1) *For the types* (I) *and* $(\mathrm{II}_{\ell\geq 3})$*, we have $x, y \in C_1$ and $z \in C_2$.*
(2) *For the type* (II_1)*, we have $x \in C_1$ and $y, z \in C_2$.*
(3) *For the type* (II_2)*, we have $w, y \in C_1$ and $x, z \in C_2$.*
(4) *For the type* $(\mathrm{II}_{\ell\geq 3})$ *we have $\overline{b}_{J+1} \in C_2$.*

Proof. By Corollary 4.18, we have $\overline{b}_J \in C_1$ for all cases. Since $\overline{\delta}(\overline{b}_J) = 0$, we have $\overline{\delta}(c) = 0$ for all $c \in C_1$ with $c > \overline{b}_J$. Thus for all $J < i \le m$ with $\overline{\delta}(\overline{b}_i) = 1$, we have $\overline{b}_i \in C_2$. Hence $z \in C_2$ holds; also for the type (II_1) we have $y \in C_2$; and for the type (II_2) we have $x \in C_2$; and for the type $(\mathrm{II}_{\ell\geq 3})$ we have $\overline{b}_{J+1} \in C_2$.

Since $z \in C_2$ and $\overline{\delta}(z) = 1$, we have $c \in C_1$ for all $c < z$ with $\overline{\delta}(c) = 0$. Hence we have $y \in C_1$ for the types (I), (II_2) and $(\mathrm{II}_{\ell\geq 3})$. ☐

By (8.4.6), (8.4.7) and Lemma 8.17 (1) - (3), we obtain

Lemma 8.18.

(1) *For the types* (I) *and* ($\mathrm{II}_{\ell\geq 3}$), (x,y,z) *satisfies (6.3.1) for* $\mathcal{B}=\mathcal{C}_1\oplus\mathcal{C}_2$.
(2) *For the type* (II_1), (x,y,z) *satisfies (6.4.1) for* $\mathcal{B}=\mathcal{C}_1\oplus\mathcal{C}_2$.
(3) *For the type* (II_2), (w,x,y,z) *satisfies (6.5.1) for* $\mathcal{B}=\mathcal{C}_1\oplus\mathcal{C}_2$.

Thus the results in §6 are applicable. In the case **(B)** we need an extra step to show that the surgeries actually produce φ_ζ from φ_ξ. More precisely we have to show $r\geq 1$ for (I) and ($\mathrm{II}_{\ell\leq 2}$), and $s\geq 1$ for ($\mathrm{II}_{\ell\geq 2}$), i.e., the first breaking point of ζ is above the first segment of ξ for (I) and ($\mathrm{II}_{\ell\leq 2}$), and the lower middle breaking point of ζ (=the last breaking point over $[0,g)$) is above the second segment of ξ for ($\mathrm{II}_{\ell\geq 2}$). This will be proved with a help from geometry: Grothendieck-Katz ([9], Theorem 2.3.1). From now on we shall use a line graph which may not be lower convex: for segments $\varrho_i=(m_i,n_i)$ $(i=1,\dots,t)$, putting $P_j:=(\sum_{i=1}^j(m_i+n_i),\sum_{i=1}^j n_i)$ for $0\leq j\leq t$, let $\mathcal{L}=\varrho_1\overrightarrow{+}\cdots\overrightarrow{+}\varrho_t$ denote the line graph passing through $P_0,\dots,P_t$ in this order. We call $\mathcal{L}$ *symmetric* if $\lambda(\varrho_i)+\lambda(\varrho_{t+1-i})=1$ for all $i=1,\dots,t$. Obviously we have

Lemma 8.19. *Let* $\mathcal{L}=\varrho_1\overrightarrow{+}\cdots\overrightarrow{+}\varrho_t$ *and* ζ *the Newton polygon* $\varrho_1+\cdots+\varrho_t$. *Then we have* $\mathcal{L}\prec\zeta$, *and moreover if* $\mathcal{L}\neq\zeta$, *then there exists a breaking point* P *of* ζ *which is below* $\mathcal{L}$, *i.e.,* $\mathcal{L}\precneqq P$.

Proposition 8.20. *For the type* (I) *we have* $r\geq 1$. *Put*

$$\begin{array}{rcl}\zeta' &:=& \sum_{i=-r+1}^{s}\zeta_i,\\ \xi' &:=& (m^{(1)}-m_{-r},n^{(1)}-n_{-r})+(m^{(2)},n^{(2)})+(n^{(2)},m^{(2)})\\ && +(n^{(1)}-n_{-r},m^{(1)}-m_{-r}),\\ \varrho &:=& (m_{-r},n_{-r})+(n_{-r},m_{-r}).\end{array}$$

(See the figure (6.3.5). Note $\zeta'\prec\xi'$ *is of type* **(B)**.*) Then we have* ${}^{\langle 2\rangle}\varphi_\xi={}^{\langle 1\rangle}\varphi_{\xi'}\oplus\varphi_\varrho$.

Proof. We define $\mathfrak{m}_1,\mathfrak{n}_1,\mathfrak{m}_2$ and $\mathfrak{n}_2$ by $\mathfrak{m}_1+\mathfrak{m}_2=m^{(1)}$, $\mathfrak{n}_1+\mathfrak{n}_2=n^{(1)}$ and $\mathfrak{m}_1\mathfrak{n}_2-\mathfrak{n}_1\mathfrak{m}_2=1$. We put

$$\begin{aligned}\varrho'&=(\mathfrak{m}_2,\mathfrak{n}_2)+(\mathfrak{n}_2,\mathfrak{m}_2),\\ \xi''&=(\mathfrak{m}_1,\mathfrak{n}_1)+(m^{(2)},n^{(2)})+(n^{(2)},m^{(2)})+(\mathfrak{n}_1,\mathfrak{m}_1).\end{aligned}$$

There exists a unique symmetric Newton polygon ζ'' with $\zeta''\prec\xi''$ satisfying **(S)** and $(\mathfrak{m}_1+\mathfrak{n}_1,\mathfrak{n}_1)\notin\zeta''$. Clearly $\zeta''\prec\xi''$ is of type **(B)**. Let $\mathcal{L}$ be the symmetric line graph

$$(\mathfrak{m}_2,\mathfrak{n}_2)\ \overrightarrow{+}\ \zeta''\ \overrightarrow{+}\ (\mathfrak{n}_2,\mathfrak{m}_2). \tag{8.4.8}$$

Note there is no lattice point Q with $\mathcal{L}\precneqq Q\precneqq\xi$ by the construction.

By Lemma 8.18 (1) and Corollary 6.7, we obtain

$${}^{\langle 2\rangle}\varphi_\xi={}^{\langle 1\rangle}\varphi_{\xi''}\oplus\varphi_{\varrho'}, \tag{8.4.9}$$

since we have $\mathfrak{l}(\varphi_\xi)=J+\ell-1$ and $\mathfrak{l}({}^{\langle 1\rangle}\varphi_\xi)=J+\ell-2$.

In order to show $r\geq 1$, $\zeta''=\zeta'$, $\xi''=\xi'$ and $\varrho'=\varrho$, it suffices to check $\mathcal{L}=\varrho'+\zeta''$. By the induction hypothesis, we have $\varphi_{\zeta''}\leq{}^{\langle 1\rangle}\varphi_{\xi''}$; hence by (8.4.9) we have

$$\varphi_{\zeta''}\oplus\varphi_{\varrho'}\leq{}^{\langle 1\rangle}\varphi_{\xi''}\oplus\varphi_{\varrho'}={}^{\langle 2\rangle}\varphi_\xi<\varphi_\xi;$$

thus we get $\varrho'+\zeta''\precneqq\xi$ by Grothendieck-Katz ([9], Theorem 2.3.1). If $\mathcal{L}\neq\varrho'+\zeta''$, then by Lemma 8.19 there exists a breaking point P of $\varrho'+\zeta''$ such that $\mathcal{L}\precneqq P\precneqq\xi$. This is a contradiction. $\square$

PROPOSITION 8.21. *For the type* (II$_1$) *we have* $s \geq 1$. *Put*

$$\begin{array}{lll} \zeta' & := & \sum_{i=-r}^{s-1} \zeta_i, \\ \xi' & := & (m^{(1)}, n^{(1)}) + (m^{(2)} - m_s, n^{(2)} - n_s) + (n^{(2)} - n_s, m^{(2)} - m_s) + (n^{(1)}, m^{(1)}), \\ \varrho & := & (m_s, n_s) + (n_s, m_s). \end{array}$$

(See the figure (6.4.3). Note $\zeta' \prec \xi'$ *is of type* **(B)***.) Then we have* ${}^{\langle 2\rangle}\varphi_\xi = {}^{\langle 1\rangle}\varphi_{\xi'} \oplus \varphi_\varrho$.

Proof. We define $\mathfrak{m}_1, \mathfrak{n}_1, \mathfrak{m}_2$ and $\mathfrak{n}_2$ by $\mathfrak{m}_1 + \mathfrak{m}_2 = m^{(2)}$, $\mathfrak{n}_1 + \mathfrak{n}_2 = n^{(2)}$ and $\mathfrak{m}_1\mathfrak{n}_2 - \mathfrak{n}_1\mathfrak{m}_2 = 1$. Put

$$\begin{aligned} \varrho' &= (\mathfrak{m}_1, \mathfrak{n}_1) + (\mathfrak{n}_1, \mathfrak{m}_1), \\ \xi'' &= (m^{(1)}, n^{(1)}) + (\mathfrak{m}_2, \mathfrak{n}_2) + (\mathfrak{n}_2, \mathfrak{m}_2) + (n^{(1)}, m^{(1)}). \end{aligned}$$

There exists a unique symmetric Newton polygon ζ'' such that $\zeta'' \prec \xi''$ satisfies **(S)** and $(m^{(1)} + n^{(1)}, n^{(1)}) \notin \zeta''$. Clearly $\zeta'' \prec \xi''$ is of type **(B)**. We write $\zeta'' = \zeta''_- \overrightarrow{+} \zeta''_+$ with $\mathrm{ht}(\zeta''_-) = \mathrm{ht}(\zeta''_+)$ and set

$$\mathcal{L} = \zeta''_- \overrightarrow{+} \varrho' \overrightarrow{+} \zeta''_+.$$

Note by the construction there is no lattice point Q with $\mathcal{L} \precneqq Q \precneqq \xi$.

By Lemma 8.18 (2) and Corollary 6.10, we have

$$ {}^{\langle 2\rangle}\varphi_\xi = {}^{\langle 1\rangle}\varphi_{\xi''} \oplus \varphi_{\varrho'}, \tag{8.4.10}$$

since we have $\mathfrak{l}(\varphi_\xi) = J$ and $\mathfrak{l}({}^{\langle 1\rangle}\varphi_\xi) = J + 1$.

In order to show $s \geq 1$, $\zeta'' = \zeta'$, $\xi'' = \xi'$ and $\varrho' = \varrho$, it suffices to check $\mathcal{L} = \zeta'' + \varrho'$. By the induction hypothesis we have $\varphi_{\zeta''} \leq {}^{\langle 1\rangle}\varphi_{\xi''}$; hence by (8.4.10), we have

$$\varphi_{\zeta''} \oplus \varphi_\varrho \leq {}^{\langle 1\rangle}\varphi_{\xi''} \oplus \varphi_{\varrho'} = {}^{\langle 2\rangle}\varphi_\xi < \varphi_\xi;$$

thus we get $\zeta'' + \varrho' \precneqq \xi$ by Grothendieck-Katz ([9], Theorem 2.3.1). If $\mathcal{L} \neq \zeta'' + \varrho'$, then by Lemma 8.19 there exists a breaking point P of $\zeta'' + \varrho'$ such that $\mathcal{L} \precneqq P \precneqq \xi$. This is a contradiction. ∎

PROPOSITION 8.22. *For the type* (II$_2$) *we have* $r, s \geq 1$. *Put*

$$\begin{array}{lll} \zeta' & := & \sum_{i=-r+1}^{s-1} \zeta_i, \\ \xi' & := & (m^{(1)} - m_{-r}, n^{(1)} - n_{-r}) + (m^{(2)} - m_s, n^{(2)} - n_s) \\ & & +(n^{(2)} - n_s, m^{(2)} - m_s) + (n^{(1)} - n_{-r}, m^{(1)} - m_{-r}), \\ \varrho & := & (m_{-r}, n_{-r}) + (m_s, n_s) + (n_s, m_s) + (n_{-r}, m_{-r}). \end{array}$$

(See the figure (6.5.3). Note $\zeta' \prec \xi'$ *is of type* **(B)***.) Then we have* ${}^{\langle 3\rangle}\varphi_\xi = {}^{\langle 1\rangle}\varphi_{\xi'} \oplus \varphi_\varrho$.

Proof. This can be shown in the similar way as in the cases (I) and (II$_1$). Use Corollary 6.12. ∎

For the type (II$_{\ell\geq 2}$) we define ${}^{\{1\}}\varphi_\xi$ by

$${}^{\{1\}}\varphi_\xi(i) := \begin{cases} \varphi_\xi(i) & \text{for} \quad i \neq J + \ell, \\ \varphi_\xi(i) - 1 & \text{for} \quad i = J + \ell \end{cases}$$

for $i = 1, \ldots, g$, and for type (II$_{\ell\geq 3}$) we define ${}^{\{2\}}\varphi_\xi$ by

$${}^{\{2\}}\varphi_\xi(i) := \begin{cases} \varphi_\xi(i) & \text{for} \quad i \neq J + \ell,\ J + \ell - 1, \\ \varphi_\xi(i) - 1 & \text{for} \quad i = J + \ell,\ J + \ell - 1 \end{cases}$$

for $i = 1, \ldots, g$. Note ${}^{\langle 2\rangle}\varphi_\xi = {}^{\langle 1\rangle}({}^{\{1\}}\varphi_\xi)$ for ($\mathrm{II}_{\ell\geq 2}$) and ${}^{\langle 3\rangle}\varphi_\xi = {}^{\langle 1\rangle}({}^{\{2\}}\varphi_\xi)$ for ($\mathrm{II}_{\ell\geq 3}$).

Proposition 8.23. *For the type* ($\mathrm{II}_{\ell\geq 3}$)*, we have* $r \geq 1$*. Put*

$$\begin{array}{lll} \zeta' & := & \sum_{i=-r+1}^{s} \zeta_i, \\ \xi' & := & (m^{(1)} - m_{-r}, n^{(1)} - n_{-r}) + (m^{(2)}, n^{(2)}) \\ & & +(n^{(2)}, m^{(2)}) + (n^{(1)} - n_{-r}, m^{(1)} - m_{-r}), \\ \varrho & := & (m_{-r}, n_{-r}) + (n_{-r}, m_{-r}). \end{array}$$

(See the figure (6.3.5). Note $\zeta' \prec \xi'$ *is of type* **(B)***.) Then we have* ${}^{\langle 3\rangle}\varphi_\xi = {}^{\langle 2\rangle}\varphi_{\xi'} \oplus \varphi_\varrho$*. Moreover we have* $\varphi_{\zeta'} \leq {}^{\langle 2\rangle}\varphi_{\xi'}$*.*

Proof. We define $\mathfrak{m}_1, \mathfrak{n}_1, \mathfrak{m}_2$ and $\mathfrak{n}_2$ by $\mathfrak{m}_1 + \mathfrak{m}_2 = m^{(1)}$, $\mathfrak{n}_1 + \mathfrak{n}_2 = n^{(1)}$ and $\mathfrak{m}_1\mathfrak{n}_2 - \mathfrak{n}_1\mathfrak{m}_2 = 1$. We put

$$\begin{aligned} \varrho' &= (\mathfrak{m}_2, \mathfrak{n}_2) + (\mathfrak{n}_2, \mathfrak{m}_2), \\ \xi'' &= (\mathfrak{m}_1, \mathfrak{n}_1) + (m^{(2)}, n^{(2)}) + (n^{(2)}, m^{(2)}) + (\mathfrak{n}_1, \mathfrak{m}_1). \end{aligned}$$

Let ζ'' be the symmetric Newton polygon such that $\zeta'' \prec \xi''$ satisfies **(S)** and $(\mathfrak{m}_1 + \mathfrak{n}_1, \mathfrak{n}_1) \notin \zeta''$. Clearly $\zeta'' \prec \xi''$ is of type **(B)**. Let $\mathcal{L}$ be as in (8.4.8). Note by the construction there is no lattice point Q with $\mathcal{L} \precneqq Q \precneqq \xi$.

Let ${}^{[2]}\mathcal{B}$, $\mathcal{B}_1 = (B_1, \delta_1)$ and $\mathcal{B}_{\Gamma_{yx}}$ be as in § 6.3. Clearly the final type of ${}^{\{2\}}\varphi_\xi$ is equal to ${}^{[2]}\mathcal{B} \oplus {}^{[2]}\mathcal{B}^\vee$, and the final type of $\varphi_{\xi''}$ is equal to $\mathcal{B}_1$. Moreover the decomposition $\mathcal{B}_1 = \mathcal{B}_{\Gamma_{yx}} \oplus \mathcal{C}_2$ is the decomposition "$\mathcal{B} = \mathcal{C}_1 \oplus \mathcal{C}_2$" for $\varphi_{\xi''}$. Recall $\overline{b}_{J+1} \in C_2$ (Lemma 8.17 (4)); then applying Lemma 5.7 to $x, y \in C_1 \subset B$, we see that $\overline{b}_J$ is an element of $\Gamma_{yx} \setminus \{y\}$; hence we have $\overline{b}_J \in B_1$. Clearly the four elements $\overline{b}_J, \overline{b}_{J+1}, x$ and z of B_1 satisfy $\overline{b}_J < \overline{b}_{J+1} < x < z$ and

$$(\delta_1(\overline{b}_J),\ \delta_1(\overline{b}_{J+1}),\ \delta_1(x),\ \delta_1(z)) = (0, 1, 0, 1). \tag{8.4.11}$$

Hence $\varphi_{\xi''}$ is of type ($\mathrm{II}_{\ell\geq 2}$). By Lemma 8.18 (1) and Corollary 6.7, we have

$${}^{\{2\}}\varphi_\xi = {}^{\{1\}}\varphi_{\xi''} \oplus \varphi_{\varrho'}. \tag{8.4.12}$$

Applying $\langle 1\rangle$ to the both sides of (8.4.12), we have

$${}^{\langle 3\rangle}\varphi_\xi = {}^{\langle 2\rangle}\varphi_{\xi''} \oplus \varphi_{\varrho'}. \tag{8.4.13}$$

Let (r'', s'') be the "(r, s)" for $\zeta'' \prec \xi''$. Then by (6.2.4) and (8.4.11) we have $(r'', s'') \neq (0, 0)$. Hence we have $\varphi_{\zeta''} \leq {}^{\langle 2\rangle}\varphi_{\xi''}$ by the induction hypothesis.

It remains to show $r \geq 1$, $\zeta'' = \zeta'$, $\xi'' = \xi'$ and $\varrho' = \varrho$. For this, it suffices to check $\mathcal{L} = \varrho' + \zeta''$. By (8.4.13), we have

$$\varphi_{\zeta''} \oplus \varphi_{\varrho'} \leq {}^{\langle 2\rangle}\varphi_{\xi''} \oplus \varphi_{\varrho'} = {}^{\langle 3\rangle}\varphi_\xi < \varphi_\xi;$$

thus we get $\varrho' + \zeta'' \precneqq \xi$ by Grothendieck-Katz ([9], Theorem 2.3.1). If $\mathcal{L} \neq \varrho' + \zeta''$, then by Lemma 8.19 there exists a breaking point P of $\varrho' + \zeta''$ such that $\mathcal{L} \precneqq P \precneqq \xi$. This is a contradiction. □

Proof of Proposition 8.14. For $r = s = 0$ this is nothing but Lemma 8.15. If $\zeta \prec \xi$ is of types (I) and ($\mathrm{II}_{\ell\leq 2}$), then the proof can be done by the same way as that of Corollary 8.7 thanks to Propositions 8.20, 8.21 and 8.22.

Assume $\zeta \prec \xi$ is of type ($\mathrm{II}_{\ell\geq 3}$). Let $\zeta' \prec \xi'$ be as in Proposition 8.23. Put $c = c(\xi; \zeta)$ and $c' = c(\xi'; \zeta')$. Note $c' = c - 1$. By the induction hypothesis, there are elementary sequences $\varphi'_0 < \cdots < \varphi'_{c'-2}$ such that $\varphi'_0 = \varphi_{\zeta'}$ and $\varphi'_{c'-2} = {}^{\langle 2\rangle}\varphi_{\xi'}$. We put $\varphi_i = \varphi'_i \oplus \varphi_\varrho$ for $i = 0, \ldots, c-3$ and set $\varphi_{c-2} = {}^{\langle 2\rangle}\varphi_\xi$, $\varphi_{c-1} = {}^{\langle 1\rangle}\varphi_\xi$ and $\varphi_c = \varphi_\xi$. Note $\varphi_0 = \varphi_{\zeta'} \oplus \varphi_\varrho = \varphi_\zeta$. It remains to show $\varphi_i < \varphi_{i+1}$ for all $0 \le i < c$. This means

$$\begin{cases} \varphi'_i \oplus \varphi_\varrho < \varphi'_{i+1} \oplus \varphi_\varrho & \text{for } 0 \le i \le c-3, \\ {}^{\langle 2\rangle}\varphi_{\xi'} \oplus \varphi_\varrho < {}^{\langle 2\rangle}\varphi_\xi & \text{for } i = c-2, \\ {}^{\langle 2\rangle}\varphi_\xi < {}^{\langle 1\rangle}\varphi_\xi & \text{for } i = c-1, \\ {}^{\langle 1\rangle}\varphi_\xi < \varphi_\xi & \text{for } i = c. \end{cases}$$

The first, the third and the fourth inequalities are obvious. The second inequality follows from Proposition 8.23: ${}^{\langle 3\rangle}\varphi_\xi = {}^{\langle 2\rangle}\varphi_{\xi'} \oplus \varphi_\varrho$. □

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