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A SIMPLE APPROACH TO THE STRUCTURE THEOREM FOR NEFVALUE MORPHISMS*

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Abstract. Let L be an ample line bundle on a smooth complex projective variety X of dimension n, let τ be the nefvalue of (X, L), and let $\phi : X \to W$ be the nefvalue morphism of (X, L). A simple approach to the complete structure theorem for nefvalue morphisms with $\tau > n - 2$ is developed.

Key words. Nefvalue, nefvalue morphism.

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Introduction. In this paper varieties are always assumed to be defined over the field $\mathbb{C}$ of complex numbers.

Let X be a smooth projective variety of dimension $n \geq 1$, and let L be an ample line bundle on X. Assume that the canonical bundle K_X of X is not nef. Then, as is well known, $\tau = \min\{t \in \mathbb{R} \mid K_X + tL \text{ is nef}\}$ is a positive rational number, and τ is called the *nefvalue* of (X, L). Keep in mind that τ is the unique rational number characterized by the condition that $K_X + \tau L$ is nef but not ample. Write $\tau = u/v$ for two coprime positive integers u, v. Then the complete linear system $|m(vK_X + uL)|$ for $m \gg 0$ defines a surjective morphism $\phi : X \to W$ onto a normal projective variety W with connected fibers such that $vK_X + uL = \phi^*A$ for some ample line bundle A on W, and ϕ is called the *nefvalue morphism* of (X, L).

Assume that $\tau > n - 2$. Then the structure of nefvalue morphisms is supplied, for example, in Chapter 7 of [BS]. The purpose of this paper is to complement the above structure theorem perfectly and to offer the complete structure theorem. The precise statement of our result is as follows:

THEOREM. *Let L be an ample line bundle on a smooth projective variety X of dimension $n \geq 1$, let τ be the nefvalue of (X, L), and let $\phi : X \to W$ be the nefvalue morphism of (X, L). Assume that $\tau > n - 2$. Then one of the following holds*:

(i) *$\tau = n + 1$, $\phi(X)$ is a point, and $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*

For $n \geq 2$,

(ii-1) *$\tau = n$, $\phi(X)$ is a point, X is a quadric hypersurface $\mathbb{Q}^n$ in $\mathbb{P}^{n+1}$, and $L = \mathcal{O}_{\mathbb{Q}^n}(1)$;*

(ii-2) *$\tau = n$, X is a $\mathbb{P}^{n-1}$-bundle over a smooth projective curve W, and $L_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ for any fiber $F = \mathbb{P}^{n-1}$ of ϕ;*

(ii-3) *$\tau = 3/2$, $\phi(X)$ is a point, and $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.*

For $n \geq 3$,

(iii-1) *$\tau = n - 1$, $\phi(X)$ is a point, and $K_X + (n - 1)L = \mathcal{O}_X$;*

(iii-2) *$\tau = n - 1$, W is a smooth projective curve, and any fiber F of ϕ is a quadric hypersurface in $\mathbb{P}^n$ with $L_F = \mathcal{O}_F(1)$;*

(iii-3) *$\tau = n - 1$, X is a $\mathbb{P}^{n-2}$-bundle over a smooth projective surface W, and $L_F = \mathcal{O}_{\mathbb{P}^{n-2}}(1)$ for any fiber $F = \mathbb{P}^{n-2}$ of ϕ;*

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(iii-4) $\tau = n-1$, ϕ *expresses* X *as the blow-up of a smooth projective variety* W *at a nonempty finite set* B *of points, and there exists an ample line bundle* H *on* W *such that* $L = \phi^*H \otimes \mathcal{O}_X(-\phi^{-1}(B))$ *and that* $K_W + (n-1)H$ *is ample*;

(iii-5) $\tau = 5/2$, $\phi(X)$ *is a point, and* $(X, L) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$;

(iii-6) $\tau = 4/3$, $\phi(X)$ *is a point, and* $(X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$;

(iii-7) $\tau = 3/2$, $\phi(X)$ *is a point, and* $(X, L) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$;

(iii-8) $\tau = 3/2$, X *is a* $\mathbb{P}^2$*-bundle over a smooth projective curve* W*, and* $L_F = \mathcal{O}_{\mathbb{P}^2}(2)$ *for any fiber* $F = \mathbb{P}^2$ *of* ϕ.

The core of this study is to investigate the case where $\tau = n-1$. At this point scrolls and quadric fibrations come into being. Their structure results are discussed, for example, in [BS, Theorem 14.1.1 and Theorem 14.2.1] by using families of unbreakable rational curves. On the other hand, the method developed here relies heavily on [I]. At least for the case $\tau = n-1$, our method seems to be simple and direct. The proof of the Theorem takes Section 1. Section 2 is devoted to some remarks on the Theorem.

We use the standard notation from algebraic geometry. The tensor products of line bundles are denoted additively. The pullback $i^*\mathcal{E}$ of a vector bundle $\mathcal{E}$ on X by an embedding $i : Y \hookrightarrow X$ is denoted by $\mathcal{E}_Y$. In particular, for a closed subvariety V of $\mathbb{P}^N$, $(\mathcal{O}_{\mathbb{P}^N}(1))_V$ is denoted by $\mathcal{O}_V(1)$. For a vector bundle $\mathcal{E}$ on a projective variety X, the tautological line bundle on the projective space bundle $\mathbb{P}_X(\mathcal{E})$ associated to $\mathcal{E}$ is denoted by $H(\mathcal{E})$. A vector bundle $\mathcal{E}$ on a projective variety X is said to be *ample* if $H(\mathcal{E})$ is ample. We denote by K_X the canonical bundle of a smooth variety X.

1. Proof of the Theorem. Before we proceed with the proof, we need the following

LEMMA. *Let* $\mathcal{E}$ *be an ample vector bundle of rank* r *on a smooth projective variety* X *of dimension* $n \geq 2$*. Assume that* $r \geq n$.

(i) *If* $K_X + \det\mathcal{E}$ *is not ample, then either* $K_X + \det\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(-1)$ *or* $(K_X + \det\mathcal{E})^n = 0$.

(ii) *Suppose that* $K_X + \det\mathcal{E}$ *is nef. If* $K_X + \det\mathcal{E}$ *is not ample and* $K_X + \det\mathcal{E} \neq \mathcal{O}_X$*, then there exists a vector bundle* $\mathcal{F}$ *of rank* n *on a smooth projective curve* C *such that* $X = \mathbb{P}_C(\mathcal{F})$*, and* $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus n}$ *for any fiber* F *of the bundle projection.*

Proof. If $K_X + \det\mathcal{E}$ is not ample, then it follows from [F, Theorem 20.1 and Theorem 20.8] that $(X, \mathcal{E})$ is one of the following:

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)})$;

(2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$;

(3) there exists a vector bundle $\mathcal{F}$ of rank n on a smooth projective curve C such that $X = \mathbb{P}_C(\mathcal{F})$, and $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus n}$ for any fiber F of the bundle projection;

(4) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-1)})$;

(5) $(\mathbb{P}^n, T_{\mathbb{P}^n})$, where $T_{\mathbb{P}^n}$ is the tangent bundle of $\mathbb{P}^n$;

(6) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n})$.

In cases (1), (4), (5) and (6) we obtain $K_X + \det\mathcal{E} = \mathcal{O}_X$. In case (2) we get $K_X + \det\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(-1)$. Suppose that $(X, \mathcal{E})$ is as in case (3). Then there exists a vector bundle $\mathcal{G}$ of rank n on C such that $\mathcal{E} = H(\mathcal{F}) \otimes \pi^*\mathcal{G}$, where $H(\mathcal{F})$ is the tautological line bundle on the projective space bundle $\mathbb{P}_C(\mathcal{F})$ associated to $\mathcal{F}$ and $\pi : X \to C$ is the bundle projection. We have $K_X = -nH(\mathcal{F}) + \pi^*(K_C + \det\mathcal{F})$ and $\det\mathcal{E} = nH(\mathcal{F}) + \pi^*(\det\mathcal{G})$, so that $K_X + \det\mathcal{E} = \pi^*(K_C + \det\mathcal{F} + \det\mathcal{G})$. Therefore $(K_X + \det\mathcal{E})^n = 0$, and (i) is proved. Moreover, (ii) also follows from the above argument. □

Let us prove the Theorem. The assertions (i), (ii-1) and (ii-2) follow from [BS, Proposition 7.2.2], and (ii-3) follows from [BS, Theorem 7.2.4]. Moreover, the assertions (iii-5), (iii-6) and (iii-7) follow from [BS, Theorem 7.3.4] except when $n = 3$, $\tau = 3/2$, W is a smooth projective curve, and $(F, L_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for a general fiber F of ϕ (see [BS, Theorem 7.3.4]). Set $H = K_X + 2L$. Then, since $2 > \tau$, H is ample, and $H_F = \mathcal{O}_{\mathbb{P}^2}(1)$. Since ϕ is flat, we obtain $H_G^2 = H_F^2 = 1$ for any fiber G of ϕ, so that G is irreducible and reduced. By the upper semicontinuity theorem we get $0 \le \Delta(G, H_G) \le \Delta(F, H_F) = 2 + 1 - h^0(F, H_F) = 0$. Hence $(G, H_G) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, and we conclude that X is a $\mathbb{P}^2$-bundle over W. Since $L_G = \mathcal{O}_{\mathbb{P}^2}(2)$, we are in (iii-8). Hence it suffices to consider the case $\tau = n - 1$ under the assumption that $n \ge 3$.

Assume that $\tau = n - 1$ with $n \ge 3$. Then $K_X + (n-1)L$ is nef but not ample. If $K_X + (n-1)L$ is big, then from the proof of [BS, Theorem 7.3.2] the nefvalue morphism $\phi : X \to W$ of (X, L) expresses X as the blow-up of a smooth projective variety W at a nonempty finite set B of points, and there exists an ample line bundle H on W such that $L = \phi^* H \otimes \mathcal{O}_X(-\phi^{-1}(B))$ and that $K_W + (n-1)H$ is ample. We are in (iii-4). Hence we can assume that $K_X + (n-1)L$ is not big. Then $\dim W < n$. Let F be a general fiber of ϕ. Then, since $K_X + (n-1)L = \phi^* A$ for some ample line bundle A on W, we have $K_F + (n-1)L_F = \mathcal{O}_F$, so that F is a Fano manifold with $\dim F \ge n - 2$. This implies that $\dim W \le 2$. If $\dim W = 0$, then $\phi(X)$ is a point, and $K_X + (n-1)L = \mathcal{O}_X$. We are in (iii-1). In what follows we suppose that either $\dim W = 1$ or $\dim W = 2$. If $\dim W = 1$, then W is smooth, and $(F, L_F) = (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1))$ because $K_F + (n-1)L_F = \mathcal{O}_F$. On the other hand, if $\dim W = 2$, then $(F, L_F) = (\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}^{n-2}}(1))$. In either event there exists a curve C on X such that $(K_X + (n-1)L)C = 0$. This directly indicates that there exists an extremal ray R of X such that $(K_X + (n-1)L)R = 0$. Let $\rho : X \to Y$ be the contraction of R. If R is not nef, then by the proof of [I, Lemma, (b)] there exists an effective divisor E on X such that $(E, L_E, (\mathcal{O}_X(E))_E) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1), \mathcal{O}_{\mathbb{P}^{n-1}}(-1))$, and ρ is nothing but the contraction of E. Thus there exists a line bundle M on Y such that $L = \rho^* M - \mathcal{O}_X(E)$, so that M is also ample by means of [F, Lemma 7.16]. Since $K_X = \rho^* K_Y + (n-1)\mathcal{O}_X(E)$, we have $K_X + (n-1)L = \rho^*(K_Y + (n-1)M)$. We note that $K_Y + (n-1)M$ is nef but not big because so is $K_X + (n-1)L$. By the assumption that $K_X + (n-1)L \ne \mathcal{O}_X$, we obtain $K_Y + (n-1)M \ne \mathcal{O}_Y$. Similarly there exists an extremal ray R of Y such that $(K_Y + (n-1)M)R = 0$. If R is not nef, then the same argument as above applies to (Y, M), and there exists a chain $(X, L) = (X_0, L_0) \xrightarrow{\rho_1} (X_1, L_1) \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_k} (X_k, L_k) = (Z, N)$ satisfying the following conditions:

(1) each X_i is a smooth projective variety with $\dim X_i = n$;
(2) each L_i is an ample line bundle on X_i;
(3) each ρ_i is a birational contraction as above;
(4) $K_{X_i} + (n-1)L_i$ is nef but not big, and $K_{X_i} + (n-1)L_i \ne \mathcal{O}_{X_i}$;
(5) every extremal ray of the final variety Z is nef.

In a similar way there exists an extremal ray R of Z such that $(K_Z + (n-1)N)R = 0$. Let $\rho : Z \to Y$ be the contraction of R. Then, since R is nef and $K_Z + (n-1)N \ne \mathcal{O}_Z$, we have $0 < \dim Y < n$. Moreover, if G denotes a general fiber of ρ, then $K_G + (n-1)N_G = \mathcal{O}_G$, so that $\dim G \ge n - 2$, which implies that either $\dim Y = 1$ or $\dim Y = 2$. By virtue of the proof of [I, Lemma, (c)] one of the following holds:

(a) Y is a smooth projective curve, and any fiber G of ρ is a quadric hypersurface in $\mathbb{P}^n$ with $N_G = \mathcal{O}_G(1)$;

(b) Z is a $\mathbb{P}^{n-2}$-bundle over a smooth projective surface Y, and $N_G = \mathcal{O}_{\mathbb{P}^{n-2}}(1)$ for any fiber $G = \mathbb{P}^{n-2}$ of ρ.

Let E be the exceptional divisor on X_{k-1} with respect to ρ_k, and set $z = \rho_k(E)$. Then in either event there exists a smooth rational curve l through z such that $Nl = 1$. Let $\widetilde{l}$ be the strict transform of l by ρ_k. Then, since $L_{k-1} = \rho_k^* N - \mathcal{O}_{X_{k-1}}(E)$, we have $L_{k-1}\widetilde{l} = Nl - \mathcal{O}_{X_{k-1}}(E)\widetilde{l} = 0$, which contradicts the ampleness of L_{k-1}. Consequently $k = 0$, i.e., $(X, L) = (Z, N)$, and there exists a surjective morphism $g : Y \to W$ with connected fibers such that $g \circ \rho = \phi$.

Assume first that (X, L) is as in case (a). Then, since $\dim Y = 1$, we have $\dim W = 1$, and we see that g is an isomorphism, i.e., $\phi = \rho$ and $Y \cong W$. We are in (iii-2).

Finally assume that (X, L) is as in case (b). Then there exists an ample vector bundle $\mathcal{E}$ of rank $n-1$ on Y such that $(X, L) = (\mathbb{P}_Y(\mathcal{E}), H(\mathcal{E}))$, where $H(\mathcal{E})$ is the tautological line bundle on the projective space bundle $\mathbb{P}_Y(\mathcal{E})$ associated to $\mathcal{E}$. Let us recall that $K_X + (n-1)L = \phi^* A$ for some ample line bundle A on W. On the other hand, we can write $K_X + (n-1)L = \rho^*(K_Y + \det \mathcal{E})$, so that $\rho^*(K_Y + \det \mathcal{E}) = \rho^* g^* A$. Thus $K_Y + \det \mathcal{E} = g^* A$. Now, since $\dim Y = 2$, we have either $\dim W = 2$ or $\dim W = 1$. If $\dim W = 2$, then g is birational, so that $K_Y + \det \mathcal{E}$ is nef and big. Note that $\operatorname{rank} \mathcal{E} \geq \dim Y$ because $n \geq 3$. Hence by (i) of the Lemma, $K_Y + \det \mathcal{E}$ itself is ample, so that g is finite. The Zariski main theorem tells us that g is an isomorphism, that is, $\phi = \rho$ and $(Y, K_Y + \det \mathcal{E}) \cong (W, A)$. We are in (iii-3). Next, for the case $\dim W = 1$, let F be a general fiber of ϕ again, and take a general fiber D of g. Then $F = \mathbb{P}_D(\mathcal{E}_D)$. Since $F = \mathbb{Q}^{n-1}$, we obtain $n = 3$ and $F = \mathbb{P}^1 \times \mathbb{P}^1$. Hence $D = \mathbb{P}^1$, which directly indicates that g is a $\mathbb{P}^1$-fibration. Furthermore, $K_Y + \det \mathcal{E} = g^* A$ is nef with $(K_Y + \det \mathcal{E})^2 = (g^* A)^2 = 0$. Moreover, it should be emphasized that $K_Y + \det \mathcal{E} \neq \mathcal{O}_Y$ because A is ample. Thus by (ii) of the Lemma, there exists a vector bundle $\mathcal{F}$ of rank two on a smooth projective curve C such that $Y = \mathbb{P}_C(\mathcal{F})$, and $\mathcal{E}_f = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ for any fiber f of the bundle projection $\pi : Y \to C$. In particular, Y is a geometrically ruled surface. Therefore $g = \pi$ and $W \cong C$ unless $Y = \mathbb{P}^1 \times \mathbb{P}^1$, $W = \mathbb{P}^1$, $C = \mathbb{P}^1$ and g is another ruling different from π. We claim that the latter does not occur. To see this, let D denote an arbitrary fiber of g. Then $D = \mathbb{P}^1$. Since $K_Y + \det \mathcal{E} = g^* A$, we have $(\det \mathcal{E})D = (-K_Y)D = 2$, which implies that $\mathcal{E}_D = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Combining this with the fact that $\mathcal{E}_f = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ for any fiber f of π gives $\det \mathcal{E} = \mathcal{O}(2, 2)$, so that $K_Y + \det \mathcal{E} = \mathcal{O}_Y$. This is a contradiction. Consequently $n = 3$, $g = \pi$ and $W \cong C$. What we want to emphasize is that every fiber F of ρ is a smooth quadric surface $\mathbb{P}_f(\mathcal{E}_f) = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}) = \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ with $L_F = \mathcal{O}_F(1)$. We are still in (iii-2), and this completes the proof of the Theorem.

2. Remarks. (2.1) In [BS, Remark 7.3.5], when (X, L) is as in (iii-8), the conclusion that $(F, L_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for any fiber F of ϕ is given under the assumption that $K_X + 2L$ is generated by its global sections. However, in order to reach this conclusion, as we have seen in the proof of the Theorem, it is enough to assume that L is simply ample.

(2.2) When $\tau > n-2$ and $n \geq 3$, if we assume that $\dim \phi(X) \geq 1$, then the nefvalue morphism is almost equal to the contraction of an extremal ray. There are two exceptions with the aid of the proof of the Theorem.

COROLLARY 1. *Let L be an ample line bundle on a smooth projective variety X of dimension $n \geq 3$, let τ be the nefvalue of (X, L), and let $\phi : X \to W$ be the nefvalue morphism of (X, L). Assume that $\tau > n - 2$ and that* $\dim W \geq 1$. *If ϕ is not the contraction of an extremal ray, then one of the following holds*:

(1) *$\tau = n - 1$, and X has at least two effective divisors E such that $(E, L_E, (\mathcal{O}_X(E))_E) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1), \mathcal{O}_{\mathbb{P}^{n-1}}(-1))$;*

(2) *$n = 3$, $\tau = 2$, W is a smooth projective curve, and any fiber F of ϕ is a smooth quadric surface $\mathbb{Q}^2$ in $\mathbb{P}^3$ with $L_F = \mathcal{O}_{\mathbb{Q}^2}(1)$.*

Furthermore, we obtain the following

COROLLARY 2. *Let L be an ample line bundle on a smooth projective variety X of dimension $n \geq 3$, let τ be the nefvalue of (X, L), and let $\phi : X \to W$ be the nefvalue morphism of (X, L). Assume that $\tau = n - 1$ and that* $\dim W \geq 1$. *Let ρ be the contraction of an extremal ray R with $(K_X + (n-1)L)R = 0$. Assume that $(X, L) = (\mathbb{P}_Y(\mathcal{E}), H(\mathcal{E}))$ for some ample vector bundle $\mathcal{E}$ of rank $n - 1$ on a smooth projective surface Y under ρ. Then $K_X + (n-1)L$ is the pullback of an ample line bundle on Y unless $n = 3$, W is a smooth projective curve, there exists a vector bundle $\mathcal{F}$ of rank two on W such that $Y = \mathbb{P}_W(\mathcal{F})$, and $\mathcal{E}_f = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ for any fiber f of the bundle projection.*

(2.3) Let L be an ample line bundle on a smooth projective variety X of dimension $n \geq 1$. Then the following follows from the Theorem:

(i) If $K_X + nL$ is not nef, then (X, L) is as in (i) of the Theorem.

(ii) Assume that $n \geq 2$ and that $K_X + nL$ is nef. If $K_X + (n-1)L$ is not nef, then (X, L) is as in (ii-1), (ii-2) or (ii-3).

(iii) Assume that $n \geq 3$ and that $K_X + (n-1)L$ is nef. If $K_X + (n-2)L$ is not nef, then (X, L) satisfies one of the conditions (iii-1)–(iii-8).

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