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A UNIVERSAL METRIC FOR THE CANONICAL BUNDLE OF A HOLOMORPHIC FAMILY OF PROJECTIVE ALGEBRAIC MANIFOLDS*

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Dedicated to M. Salah Baouendi on the occasion of his 70th birthday

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1. Introduction. In his celebrated work [S-98, S-02], Siu proved that the plurigenera of any algebraic manifold are invariant in families. More precisely, let $\pi : \mathscr{X} \to \mathbb{D}$ be a holomorphic submersion (i.e., $d\pi$ is nowhere zero) from a complex manifold $\mathscr{X}$ to the unit disk $\mathbb{D}$, and assume that every fiber $\mathscr{X}_t := \pi^{-1}(t)$ is a compact projective manifold. Then for every $m \in \mathbb{N}$, the function $P_m : \mathbb{D} \to \mathbb{N}$ defined by $P_m(t) := h^0(\mathscr{X}_t, mK_{\mathscr{X}_t})$ is constant.

Siu's approach to the problem begins with the observation that the function P_m is upper semi-continuous. Thus in order to prove that P_m is continuous (hence constant) it suffices to show that given a global holomorphic section s of $mK_{\mathscr{X}_0}$, there is a family of global holomorphic sections s_t of $\mathscr{X}_t$, for all t in a neighborhood of 0, that varies holomorphically with t and satisfies $s_0 = s$.

To prove such an extension theorem, Siu establishes a generalization of the Ohsawa-Takegoshi Extension Theorem to the setting of complex submanifolds of a Kahler manifold having codimension 1 and cut out by a single, bounded holomorphic function. This theorem, which we will discuss below, requires the existence of a singular Hermitian metric on the ambient manifold having non-negative curvature current, with respect to which the section to be extended is L^2. Thus in the presence of the extension theorem, the approach reduces to construction of such a metric.

The case where the fibers $\mathscr{X}_t$ of our holomorphic family are of general type was treated in [S-98]. In this setting, Siu produced a single singular Hermitian metric $e^{-\kappa}$ for K_X so that every m-canonical section is L^2 with respect to $e^{-(m-1)\kappa}$.

However, in the case where the fibers $\mathscr{X}_t$ of our holomorphic family are assumed only to be algebraic, and not necessarily of general type, Siu's proof in [S-02] does not construct a single metric as in the case of general type. Instead, Siu constructs for every section s of $mK_{\mathscr{X}_0}$ a singular Hermitian metric for $mK_{\mathscr{X}}$ of non-negative curvature so that s is L^2 with respect to this metric.

DEFINITION. Let $\mathscr{X} \to \Delta$ be a holomorphic family of complex manifolds and $\mathscr{X}_0$ the cental fiber of $\mathscr{X}$. A universal canonical metric for the pair $(\mathscr{X}, \mathscr{X}_0)$ is a singular Hermitian metric $e^{-\kappa}$ for the canonical bundle $K_{\mathscr{X}}$ of $\mathscr{X}$ such that for every global holomorphic section $s \in H^0(\mathscr{X}_0, mK_{\mathscr{X}_0})$,

$$\int_{\mathscr{X}_0} |s|^2 e^{-(m-1)\kappa} < +\infty.$$

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The goal of this paper is to prove that for any holomorphic family $\mathscr{X} \to \Delta$ of compact complex algebraic manifolds with central fiber $\mathscr{X}_0$, the pair $(\mathscr{X}, \mathscr{X}_0)$ has a universal canonical metric having non-negative curvature current. To this end, our main theorem is the following result.

THEOREM 1. *Let X be a complex manifold admitting a positive line bundle $A \to X$, and $Z \subset X$ a smooth compact complex submanifold of codimension 1. Assume there is a subvariety $V \subset X$ not containing Z such that $X - V$ is a Stein manifold. Let $T \in H^0(X, Z)$ be a holomorphic section of the line bundle associated to Z, thought of as a divisor. Let $E \to X$ be a holomorphic line bundle and denote by K_X the canonical bundle of X. Assume we are given singular metrics $e^{-\varphi_E}$ for E and $e^{-\varphi_Z}$ for the line bundle associated to Z.*

Suppose in addition that the above data satisfy the following assumptions.

(R) The metrics $e^{-\varphi_E}$ and $e^{-\varphi_Z}$ restrict to singular metrics on Z.

(B)

$$\sup_X |T|^2 e^{-\varphi_Z} < +\infty.$$

(G) For each $m > 0$, the line bundles $p(K_X + Z + E) + A$, $0 \le p \le m-1$, are globally generated, in the sense that a finite number of sections of $H^0(X, p(K_X + Z + E) + A)$ generate the sheaf $\mathcal{O}_X(p(K_X + Z + E) + A)$.

(P) $\sqrt{-1}\partial\bar\partial\varphi_E \ge 0$ and there exists a constant μ such that $\mu\sqrt{-1}\partial\bar\partial\varphi_E \ge \sqrt{-1}\partial\bar\partial\varphi_Z$.

(T) The singular metric $e^{-(\varphi_Z+\varphi_E)}|Z$ has trivial multiplier ideal:

$$\mathscr{I}(Z, e^{-(\varphi_Z+\varphi_E)}|Z) = \mathcal{O}_Z.$$

Then there is a metric $e^{-\kappa}$ for $K_X + Z + E$ with the following properties:

(C) $\sqrt{-1}\partial\bar\partial\kappa \ge 0$.

(L) For every $m > 0$ and every section $s \in H^0(Z, m(K_Z + E|Z))$, $|s|^2 e^{-((m-1)\kappa+\varphi_E+\varphi_Z)}$ is locally integrable.

(I) For every integer $m > 0$ and every section $s \in H^0(Z, m(K_Z + E))$,

$$\int_Z |s|^2 e^{-(m-1)\kappa+\varphi_E} < +\infty.$$

REMARKS.

(i) For the ambient manifold X, we have in mind the following two examples: either X is compact complex projective (in which case the variety V could be taken to be a hyperplane section of some embedding of X) or else X is a family of compact complex algebraic manifolds. In the former case, it is well-known [S-98] that the hypothesis (G) holds for any sufficiently ample A, while in the latter case, one might have to shrink X a little to obtain (G). Of course, there are many other examples of such X.

(ii) Note that in condition (L), the local functions $|s|^2 e^{-((m-1)\kappa+\varphi_E+\varphi_Z)}$ depend on the local trivializations of the line bundles in question. However, the local integrability condition is independent of these choices.

Together with a variant of the Ohsawa-Takegoshi Theorem (Theorem 4 below), Theorem 1 implies a generalization of Siu's extension theorem to the case where the normal bundle of the submanifold Z is not necessarily trivial. The first extension theorem of this type was established by Takayama [Ta-05, Theorem 4.1]under some additional hypotheses. The general case was done in [V-06], where Theorem 4 was also established. (In the case where Z is a fiber in a smooth familty, the result in [V-06] was also proved by Claudon in [C-06].) The argument here is related to that of [V-06], but the focus is on construction of the metric rather than on the extension theorem.

As a result of Theorem 1, we have the following corollary, which is our stated goal.

COROLLARY 2. *For every holomorphic family $\mathscr{X} \to \Delta$ of smooth projective varieties with central fiber $\mathscr{X}_0$, the pair $(\mathscr{X}, \mathscr{X}_0)$ has, perhaps after slightly shrinking the family, a universal canonical metric having non-negative curvature current.*

Proof. Let X be a family of compact projective manifolds $\pi : \mathscr{X} \to \mathbb{D}$, and $Z = \mathscr{X}_0$ the central fiber. Take $T = \pi$, $E = \mathcal{O}_{\mathscr{X}}$ and $\varphi_E \equiv 0$. Since $\mathscr{X}_0$ is cut out by a single holomorphic function, the line bundle associated to $\mathscr{X}_0$ is trivial. Take $\varphi_Z \equiv 0$. Then the hypotheses of Theorem 1 are satisfied, perhaps after shrinking the family, and we obtain a metric $e^{-\kappa}$ for $K_{\mathscr{X}}$ such that $\sqrt{-1}\partial\bar{\partial}\kappa \geq 0$ and $|s|^2 e^{-(m-1)\kappa}$ is integrable for every integer $m > 0$ and every section $s \in H^0(\mathscr{X}_0, mK_{\mathscr{X}_0})$. $\square$

REMARK. Note that in the setting of families, the constant μ is not needed, and the hypotheses (L) and (I) are the same.

REMARK. In his paper [Ts-02], Tsuji has claimed the existence of a metric with the properties stated in Corollary 2. As in our approach, Tsuji's proof makes use of an infinite process. It seems that convergence of Tsuji's process was not checked; in fact, it is demonstrated in [S-02] that Tsuji's process, as well as any reasonable modification of it, diverges.

PROPOSITION 3. *For each integer $m > 0$, fix a basis $s_1^{(m)}, ..., s_{N_m}^{(m)}$ of $H^0(X, m(K_Z + E|Z))$. Choose constants ε_m such that the metric*

$$\kappa_0 := \log \left(\sum_{m=1}^{\infty} \varepsilon_m \left(\sum_{\ell=1}^{N_m} |s_\ell^{(m)}|^2 \right)^{1/m} \right)$$

is convergent. Suppose $e^{-\varphi_E}$ is locally integrable. Then for each $m > 0$ and every $s \in H^0(X, m(K_Z + E|Z))$,

$$\int_Z |s|^2 e^{-((m-1)\kappa_0 + \varphi_E)} < +\infty.$$

Proof. Fix $s \in H^0(X, m(K_Z + E|Z))$, and let $\kappa_{0,m} = \log \left(\sum_{\ell=1}^{N_m} |s_\ell^{(m)}|^2 \right)^{1/m}$.

Note that $e^{-\kappa_0} \lesssim e^{-\kappa_{0,m}}$, and thus we have

$$\begin{aligned}
&\int_Z |s|^2 e^{-(m-1)\kappa_0 + \varphi_E} \\
&\lesssim \int_Z |s|^2 e^{-(m-1)\kappa_{0,m} + \varphi_E} \\
&= \int_Z |s|^{2/m} \left(\frac{|s|^2}{|s_1^{(m)}|^2 + \dots + |s_{N_m}^{(m)}|^2} \right)^{(m-1)/m} e^{\gamma_E - \varphi_E} e^{-\gamma_E} \\
&\lesssim \int_Z |s|^{2/m} e^{\gamma_E - \varphi_E} e^{-\gamma_E} \\
&\lesssim \left(\int_Z |s|^2 e^{\gamma_E - \varphi_E} e^{-m\gamma_E} \omega^{-(n-1)(m-1)} \right)^{1/m} \left(\int_Z e^{\gamma_E - \varphi_E} \omega^{n-1} \right)^{(m-1)/m},
\end{aligned}$$

where ω is a fixed Kähler form for Z and $e^{-\gamma_Z}$ is a smooth metric for $E|Z$. The last inequality is a consequence of Hölder's Inequality. Since $e^{-\varphi_E}$ is locally integrable, we are done. □

A calculation similar to the proof of Proposition 3 shows that $|s|^2 e^{-((m-1)\kappa_0 + \varphi_Z + \varphi_E)}$ is locally integrable on Z. Thus in view of Proposition 3, Theorem 1 follows if we construct a metric $e^{-\kappa}$ with non-negative curvature current such that $e^{-\kappa}|Z = e^{-\kappa_0}$. This is precisely what we do. We employ a technical simplification, due to Paun [P-05], of Siu's original idea of extending metrics using an Ohsawa-Takegoshi-type extension theorem for sections. Paun's simplification allows one to get rid of a rather difficult part of Siu's original proof; the use (and proof) of an effective version of global generation of multiplier ideal sheaves. As a consequence of Paun's methods, the present paper is also substantially shortened.

2. The Ohsawa-Takegoshi Extension theorem. Let Y be a Kähler manifold of complex dimension n. Assume there exists an analytic hypersurface $V \subset Y$ such that $Y - V$ is Stein. Examples of such manifolds are Stein manifolds (where V is empty) and projective algebraic manifolds (where one can take V to be the intersection of Y with a projective hyperplane in some projective space in which Y is embedded).

Fix a smooth hypersurface $Z \subset Y$ such that $Z \not\subset V$. In [V-06] we proved the following generalization of the Ohsawa-Takogoshi Extension Theorem.

Theorem 4. *Suppose given a holomorphic line bundle $H \to Y$ with a singular Hermitian metric $e^{-\psi}$, and a singular Hermitian metric $e^{-\varphi_Z}$ for the line bundle associated to the divisor Z, such that the following properties hold.*

(i) The restrictions $e^{-\psi}|Z$ and $e^{-\varphi_Z}|Z$ are singular metrics.

(ii) There is a global holomorphic section $T \in H^0(Y, Z)$ such that

$$Z = \{T = 0\} \quad \textit{and} \quad \sup_Y |T|^2 e^{-\varphi_Z} = 1.$$

(iii) $\sqrt{-1}\partial\bar\partial\psi \ge 0$ and there is an integer $\mu > 0$ such that $\mu\sqrt{-1}\partial\bar\partial\psi \ge \sqrt{-1}\partial\bar\partial\varphi_Z$. Then for every $s \in H^0(Z, K_Z + H)$ such that

$$\int_Z |s|^2 e^{-\psi} < +\infty \quad \textit{and} \quad s \wedge dT \in \mathscr{I}(e^{-(\varphi_Z + \psi)}|Z),$$

there exists a section $S \in H^0(Y, K_Y + Z + H)$ such that

$$S|Z = s \wedge dT \quad \textit{and} \quad \int_Y |S|^2 e^{-(\varphi_Z + \psi)} \le 40\pi\mu \int_Z |s|^2 e^{-\psi}.$$

REMARK. The list of L^2 extension theorems is by now rather long. For a large collection of such results and additional references, see [MV-05].

3. Inductive construction of certain sections by extension. In this section we use the method of Paun [P-05] mentioned in the introduction. Fix a holomorphic line bundle $A \to X$ such that the property (G) in Theorem 1 holds.
Let us fix bases

$$\{\tilde{\sigma}_j^{(m,0,p)}\ ;\ 1 \le j \le M_p\}$$

of $H^0(X, p(K_X + Z + E) + A)$. We let $\sigma_j^{(m,0,p)} \in H^0(Z, p(K_Z + E|Z) + A|Z)$ be such that

$$\tilde{\sigma}_j^{(m,0,p)}|Z = \sigma_j^{(m,0,p)} \wedge (dT)^{\otimes p}.$$

We also fix smooth metrics

$$e^{-\gamma_Z} \text{ and } e^{-\gamma_E} \quad \text{for } Z \to X, \text{ and } E \to X$$

respectively. Finally, let us fix bases

$$s_1^{(m)}, ..., s_{N_m}^{(m)} \text{ for } H^0(X, m(K_Z + E|Z)), \quad m = 1, 2, ...,$$

orthonormal with respect to the singular metric $(\omega^{-(n-1)}e^{-\gamma_E})^{m-1}e^{-\varphi_E}$ for $(m-1)K_Z + mE|Z$. (Since $e^{-\varphi_E}$ is locally integrable, every holomorphic section is integrable with respect to this metric.)

PROPOSITION 5. *For each $m = 1, 2, ...$ there exist a constant $C_m < +\infty$ and sections*

$$\tilde{\sigma}_{j,\ell}^{(m,k,p)} \in H^0(X, (km + p)(K_X + Z + E) + A)$$

where $p = 1, 2, ..., m-1$, $1 \le j \le M_p$, $1 \le \ell \le N_m$ and $k = 1, 2, ...$, with the following properties.

(a) $\tilde{\sigma}_{j,\ell}^{(m,k,p)}|Z = (s_\ell^{(m)})^{\otimes k} \otimes \sigma_j^{(m,0,p)} \wedge (dT)^{(km+p)}$

(b) If $k \ge 1$,

$$\int_X \frac{\sum_{j=1}^{M_0} |\tilde{\sigma}_{j,\ell}^{(m,k,0)}|^2 e^{-(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_{j,\ell}^{(m,k-1,m-1)}|^2} \le C_m.$$

(c) For $1 \le p \le m-1$,

$$\int_X \frac{\sum_{j=1}^{M_p} |\tilde{\sigma}_{j,\ell}^{(m,k,p)}|^2 e^{-(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{M_{p-1}} |\tilde{\sigma}_{j,\ell}^{(m,k,p-1)}|^2} \le C_m.$$

Proof. (Double induction on k and p.) Fix a constant $\widehat{C}_m$ such that the

$$\sup_X \frac{\sum_{j=1}^{M_0} |\tilde{\sigma}_j^{(m,0,0)}|^2 \omega^{n(m-1)} e^{(m-1)(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_j^{(m,0,m-1)}|^2} \le \widehat{C}_m$$

and

$$\sup_Z \frac{\sum_{j=1}^{M_0} |\sigma_j^{(m,0,0)}|^2 \omega^{(n-1)(m-1)} e^{(m-1)\gamma_E}}{\sum_{j=1}^{M_{m-1}} |\sigma_j^{(m,0,m-1)}|^2} \le \widehat{C}_m,$$

and for all $0 \le p \le m-2$,

$$\sup_X \frac{\sum_{j=1}^{N_{p+1}} |\tilde{\sigma}_j^{(m,0,p+1)}|^2 \omega^{-n} e^{-(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{M_p} |\tilde{\sigma}_j^{(m,0,p)}|^2} \le \widehat{C}_m,$$

and

$$\sup_Z \frac{\sum_{j=1}^{N_{p+1}} |\sigma_j^{(m,0,p+1)}|^2 \omega^{-(n-1)} e^{-\gamma_E}}{\sum_{j=1}^{M_p} |\sigma_j^{(m,0,p)}|^2} \le \widehat{C}_m.$$

($k = 0$) We set $\tilde{\sigma}_{j,\ell}^{(m,0,p)} := \tilde{\sigma}_j^{(m,0,p)}$ and simply observe that

$$\int_X \frac{\sum_{j=1}^{M_p} |\tilde{\sigma}_{j,\ell}^{(m,0,p)}|^2 e^{-(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{M_{p-1}} |\tilde{\sigma}_{j,\ell}^{(m,0,p-1)}|^2} \le \widehat{C}_m \int_X \omega^n.$$

($k \ge 1$) Assume the result has been proved for $k-1$.
<u>(($p = 0$))</u>: Consider the sections $(s_\ell^{(m)})^{\otimes k} \otimes \sigma_j^{(m,0,0)}$, and define the semi-positively curved metric

$$\psi_{k,\ell,0} := \log \sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_{j,\ell}^{(m,k-1,m-1)}|^2$$

for the line bundle $(mk-1)(K_X + Z + E) + A$. Observe that locally on Z,

$$\begin{aligned} |(s_\ell^{(m)} \wedge dT^m)^k \otimes \sigma_j^{(m,0,0)}|^2 e^{-(\varphi_Z+\psi_{k,\ell,0}+\varphi_E)} &= |s_\ell^{(m)} \wedge dT^m|^2 \frac{|\sigma_j^{(m,0,0)}|^2 e^{-(\varphi_Z+\varphi_E)}}{\sum_{j=1}^{M_{m-1}} |\sigma_j^{(m,0,m-1)}|^2} \\ &\lesssim |s_\ell^{(m)}|^2 e^{-(\varphi_Z+\varphi_E)}. \end{aligned}$$

Moreover, we have

$$\sqrt{-1}\partial\bar{\partial}(\psi_{k,\ell,0} + \varphi_E) \ge 0 \quad \text{and} \quad \mu\sqrt{-1}\partial\bar{\partial}(\psi_{k,\ell,0} + \varphi_E) \ge \sqrt{-1}\partial\bar{\partial}\varphi_Z.$$

Finally,

$$\begin{aligned} &\int_Z |(s_\ell^{(m)})^k \otimes \sigma_j^{(m,0,0)}|^2 e^{-(\psi_{k,\ell,0}+\varphi_E)} \\ &= \int_Z |s_\ell^{(m)}|^2 \frac{|\sigma_j^{(m,0,0)}|^2 e^{(m-1)\gamma_E} e^{-((m-1)\gamma_E+\varphi_E)}}{\sum_{j=1}^{M_{m-1}} |\sigma_j^{(m,0,m-1)}|^2} < +\infty. \end{aligned}$$

We may thus apply Theorem 4 to obtain sections

$$\tilde{\sigma}_{j,\ell}^{(m,k,0)} \in H^0(X, mk(K_X + Z + E) + A), \quad 1 \le j \le M_0,\ 1 \le \ell \le N_m,$$

such that

$$\tilde{\sigma}_{j,\ell}^{(m,k,0)}|Z = (s_\ell^{(m)})^{\otimes k} \otimes \sigma_{j,\ell}^{(m,0,0)} \wedge (dT)^{\otimes km}, \quad 1 \le j \le M_0,\ 1 \le \ell \le N_m,$$

and

$$\int_X |\tilde{\sigma}_{j,\ell}^{(m,k,0)}|^2 e^{-(\psi_{k,\ell,0}+\varphi_Z+\varphi_E)} \le 40\pi\mu \int_Z |s_\ell^{(m)}|^2 \frac{|\sigma_j^{(0)}|^2 e^{-(\varphi_E+\varphi_B)}}{\sum_{j=1}^{N_{m-1}} |\sigma_j^{(m-1)}|^2}.$$

Summing over j, we obtain

$$\begin{aligned}
&\int_X \frac{\sum_{j=1}^{M_0} |\tilde{\sigma}_{j,\ell}^{(m,k,0)}|^2 e^{-(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_{j,\ell}^{(m,k-1,m-1)}|^2} \\
&\le \sup_X e^{\varphi_Z+\varphi_E-\gamma_Z-\gamma_E} \int_X \frac{\sum_{j=1}^{M_0} |\tilde{\sigma}_{j,\ell}^{(m,k,0)}|^2 e^{-(\varphi_Z+\varphi_E)}}{\sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_{j,\ell}^{(m,k-1,m-1)}|^2} \\
&\le 40\pi \sup_X e^{\varphi_Z+\varphi_E-\gamma_Z-\gamma_E} \int_Z |s_\ell^{(m)}|^2 \frac{\sum_{j=1}^{M_0} |\sigma_j^{(m,0,0)}|^2 e^{-\varphi_E}}{\sum_{j=1}^{M_{m-1}} |\sigma_j^{(m,0,m-1)}|^2} e^{-\kappa} \\
&\le 40\pi \widehat{C}_m \sup_X e^{\varphi_Z+\varphi_E-\gamma_Z-\gamma_E} \int_Z |s_\ell^{(m)}|^2 \omega^{-(n-1)(m-1)} e^{-((m-1)\gamma_E+\varphi_E)} \\
&= 40\pi \widehat{C}_m \sup_X e^{\varphi_Z+\varphi_E-\gamma_Z-\gamma_E}.
\end{aligned}$$

$\underline{((1 \le p \le m-1))}$: Assume that we have obtained the sections $\tilde{\sigma}_{j,\ell}^{(m,k,p-1)}$, $1 \le j \le M_{p-1}$, $1 \le \ell \le N_m$. Consider the non-negatively curved singular metric

$$\psi_{k,\ell,p} := \log \sum_{j=1}^{M_{p-1}} |\tilde{\sigma}_{j,\ell}^{(m,k,p-1)}|^2$$

for $(km+p-1)(K_X+Z+E)+A$. We have

$$|(s_\ell^{(m)})^k \otimes \sigma_j^{(m,0,p)}|^2 e^{-(\varphi_Z+\psi_{k,\ell,p}+\varphi_E)} = \frac{|\sigma_j^{(m,0,p)}|^2 e^{-(\varphi_Z+\varphi_E)}}{\sum_{j=1}^{M_{p-1}} |\sigma_j^{(m,0,p-1)}|^2} \lesssim e^{-(\varphi_Z+\varphi_E)},$$

which is locally integrable on Z by the hypothesis (T). Next,

$$\begin{aligned}
\int_Z |(s_\ell^{(m)})^k \otimes \sigma_j^{(m,0,p)}|^2 e^{-(\psi_{k,\ell,p}+\varphi_E)} &= \int_Z \frac{|\sigma_j^{(m,0,p)}|^2 e^{-\varphi_E}}{\sum_{j=1}^{M_{p-1}} |\sigma_j^{(m,0,p-1)}|^2} \\
&\le C^\star \int_Z e^{\gamma_Z} \frac{|\sigma_j^{(m,0,p)}|^2 e^{-(\varphi_Z+\varphi_E)}}{\sum_{j=1}^{M_{p-1}} |\sigma_j^{(m,0,p-1)}|^2} < +\infty,
\end{aligned}$$

where

$$C^\star := \sup_Z e^{\varphi_Z-\gamma_Z}.$$

Moreover,

$$\sqrt{-1}\partial\bar{\partial}(\psi_{k,\ell,p}+\varphi_E) \ge 0 \quad \text{and} \quad \sqrt{-1}\partial\bar{\partial}(\psi_{k,\ell,p}+\varphi_E) \ge \sqrt{-1}\partial\bar{\partial}\varphi_Z.$$

By Theorem 4 there exist sections

$$\tilde{\sigma}_{j,\ell}^{(m,k,p)} \in H^0(X,(mk+p)(K_X+Z+E)+A), \quad 1 \le j \le M_0$$

such that

$$\tilde{\sigma}_{j,\ell}^{(m,k,p)}|Z = (s_\ell^{(m)})^{\otimes k} \otimes \sigma_{j,\ell}^{(m,0,p)} \wedge (dT)^{\otimes km+p}, \quad 1 \le j \le M_p,$$

and

$$\int_X |\tilde{\sigma}_{j,\ell}^{(m,k,p)}|^2 e^{-(\psi_{k,\ell,p}+\varphi_Z+\varphi_E)} \le 40\pi\mu \int_Z \frac{|\sigma_j^{(m,0,p)}|^2 e^{-\varphi_E}}{\sum_{j=1}^{M_{p-1}} |\sigma_j^{(m,0,p-1)}|^2}.$$

Summing over j, we obtain

$$\int_X \frac{\sum_{j=1}^{M_p} |\tilde{\sigma}_{j,\ell}^{(m,k,p)}|^2 e^{-(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{M_{p-1}} |\tilde{\sigma}_{j,\ell}^{(m,k,p-1)}|^2} \le 40\pi\mu \sup_X e^{\varphi_Z+\varphi_E-\gamma_Z-\gamma_E} \widehat{C}_m \int_Z e^{-\varphi_E}\omega^{n-1}.$$

Letting

$$C_m := 40\pi\mu\widehat{C}_m \max\left(\int_X \omega^n,\ \sup_X e^{\varphi_Z+\varphi_E+\varphi_B-\gamma_Z-\gamma_E},\ \sup_X e^{\varphi_Z+\varphi_E-\gamma_Z-\gamma_E} \int_Z e^{-\varphi_E}\omega^{n-1}\right)$$

completes the proof. □

4. Construction of the metric. This part of the proof follows the ideas of Siu set forth in [S-02].

4.1. A metric associated to $\mathbf{m(K_X+Z+E)}$. Fix a smooth metric $e^{-\psi}$ for $A \to X$. Consider the functions

$$\lambda_{\ell,N}^{(m)} := \log \sum_{j=1}^{M_p} |\tilde{\sigma}_{j,\ell}^{(m,k,p)}|^2 \omega^{-n(mk+p)} e^{-(km(\gamma_Z+\gamma_E)+\psi)},$$

where $N = mk+p$. Set

$$\lambda_N^{(m)} := \log \sum_{\ell=1}^{N_m} e^{\lambda_{\ell,N}^{(m)}}.$$

Lemma 6. *For any non-empty open subset $V \subset X$ and any smooth function $f : \overline{V} \to \mathbb{R}_+$,*

$$\frac{1}{\int_V f\omega^n} \int_V (\lambda_N^{(m)} - \lambda_{N-1}^{(m)}) f\omega^n \le \log\left(\frac{N_m C_m \sup_V f}{\int_V f\omega^n}\right).$$

Proof. Observe that by Proposition 5, there exists a constant C_m such that for any open subset $V \subset X$,

$$\int_V (e^{\lambda_{\ell,N}^{(m)}-\lambda_{\ell,N-1}^{(m)}}) f\omega^n \le C_m \sup_V f,$$

and thus

$$\int_V (e^{\lambda_N^{(m)}-\lambda_{N-1}^{(m)}})f\omega^n = \sum_{\ell=1}^{N_m}\int_V (e^{\lambda_{\ell,N}^{(m)}-\lambda_{\ell,N-1}^{(m)}})f\omega^n \le N_m C_m \sup_V f.$$

An application of (the concave version of) Jensen's inequality to the concave function log then gives

$$\frac{1}{\int_V f\omega^n}\int_V (\lambda_N^{(m)}-\lambda_{N-1}^{(m)})f\omega^n \le \log\left(\frac{N_m C_m \sup_V f}{\int_V f\omega^n}\right).$$

The proof is complete. □

Consider the function

$$\Lambda_k^{(m)} = \frac{1}{k}\lambda_{mk}^{(m)}.$$

Note that $\Lambda_k^{(m)}$ is locally the sum of a plurisubharmonic function and a smooth function. By applying Lemma 6 and using the telescoping property, we see that for any open set $V \subset X$ and any smooth function $f : \overline{V} \to \mathbb{R}_+$,

$$\frac{1}{\int_V f\omega^n}\int_V \Lambda_k^{(m)} f\omega^n \le m\log\left(\frac{N_m C_m \sup_V f}{\int_V f\omega^n}\right). \tag{1}$$

Proposition 7. *There exists a constant $C_o^{(m)}$ such that*

$$\Lambda_k^{(m)}(x) \le C_o^{(m)}, \quad x \in X.$$

Proof. Let us cover X by coordinate charts $V_1, ..., V_N$ such that for each j there is a biholomorphic map F_j from V_j to the ball $B(0,2)$ of radius 2 centered at the origin in $\mathbb{C}^n$, and such that if $U_j = F_j^{-1}(B(0,1))$, then $U_1, ..., U_N$ is also an open cover. Let $W_j = V_j \setminus F_j^{-1}(B(0,3/2))$.

Now, on each V_j, $\Lambda_k^{(m)}$ is the sum of a plurisubharmonic function and a smooth function. Say $\Lambda_k^{(m)} = h + g$ on V_j, where h is plurisubharmonic and g is smooth. Then for constant A_j we have

$$\begin{aligned}\sup_{U_j}\Lambda_k^{(m)} &\le \sup_{U_j} g + \sup_{U_j} h\\ &\le \sup_{U_j} g + A_j\int_{W_j} h\cdot F_{j*}dV\\ &\le \sup_{U_j} g - A_j\int_{W_j} g\cdot F_{j*}dV + A_j\int_{W_j}\Lambda_k^{(m)}\cdot F_{j*}dV\end{aligned}$$

Let

$$C_j^{(m)} := \sup_{U_j} g - A_j\int_{W_j} g\cdot F_{j*}dV$$

and define the smooth function f_j by

$$f_j\omega^n = F_{j*}dV.$$

Then by (1) applied with $V = W_j$ and $f = f_j$, we have

$$\sup_{U_j} \Lambda_k^{(m)} \le C_j^{(m)} + mA_j \log\left(\frac{N_m C_m \sup_{W_j} f_j}{\int_{W_j} f_j \omega^n}\right) \int_{W_j} f_j \omega^n.$$

Letting

$$C_o^{(m)} := \max_{1\le j\le N} \left\{ C_j^{(m)} + mA_j \log\left(\frac{N_m C_m \sup_{W_j} f_j}{\int_{W_j} f_j \omega^n}\right) \int_{W_j} f_j \omega^n \right\}$$

completes the proof. □

Since the upper regularization of the lim sup of a uniformly bounded sequence of plurisubharmonic functions is plurisubharmonic (see, e.g., [H-90, Theorem 1.6.2]), we essentially have the following corollary.

Corollary 8. *The function*

$$\Lambda^{(m)}(x) := \limsup_{y\to x} \limsup_{k\to\infty} \Lambda_k^{(m)}(y)$$

is locally the sum of a plurisubharmonic function and a smooth function.

Proof. One need only observe that the function Λ_k is obtained from a singular metric on the line bundle $m(K_X+Z+E)$ (this singular metric $e^{-\kappa_k^{(m)}}$ will be described shortly) by multiplying by a fixed smooth metric of the dual line bundle. □

Consider the singular Hermitian metric $e^{-\kappa^{(m)}}$ for $m(K_X + Z + E)$ defined by

$$e^{-\kappa^{(m)}} = e^{-\Lambda^{(m)}} \omega^{-nm} e^{-m(\gamma_Z+\gamma_E)}.$$

This singular metric is given by the formula

$$e^{-\kappa^{(m)}(x)} = \exp\left(-\limsup_{y\to x} \limsup_{k\to\infty} \kappa_k^{(m)}(y)\right),$$

where

$$e^{-\kappa_k^{(m)}} = e^{-\Lambda_k^{(m)}} \omega^{-nm} e^{-m(\gamma_Z+\gamma_E)}.$$

The curvature of $e^{-\kappa_k^{(m)}}$ is thus

$$\begin{aligned}\sqrt{-1}\partial\bar\partial\kappa_k^{(m)} &= \frac{\sqrt{-1}}{k}\partial\bar\partial \log \sum_{\ell=1}^{N_m}\sum_{j=1}^{N_0} |\tilde\sigma_{j,\ell}^{(m,k,0)}|^2 - \frac{1}{k}\sqrt{-1}\partial\bar\partial\psi \\ &\ge -\frac{1}{k}\sqrt{-1}\partial\bar\partial\psi.\end{aligned}$$

We claim next that the curvature of $e^{-\kappa}$ is non-negative. To see this, it suffices to work locally. Then we have that the functions

$$\kappa_k^{(m)} + \frac{1}{k}\psi$$

are plurisubharmonic. But

$$\limsup_{y\to x}\limsup_{k\to\infty} \kappa_k^{(m)} + \frac{1}{k}\psi = \limsup_{y\to x}\limsup_{k\to\infty}\kappa_k^{(m)} = \kappa^{(m)}.$$

It follows that $\kappa^{(m)}$ is plurisubharmonic, as desired.

4.2. The metric for $\mathbf{K_X + Z + E}$; Proof of Theorem 1. Let ε_m be constants, chosen so $\varepsilon_m \searrow 0$ sufficiently rapidly that the sum

$$e^{\kappa} := \sum_{m=1}^{\infty} \varepsilon_m e^{\frac{1}{m}\kappa^{(m)}} = \sum_{m=1}^{\infty} \exp(\tfrac{1}{m}\kappa^{(m)} + \log \varepsilon_m),$$

converges everywhere on X (to a metric for $-(K_X + Z + E)$). It is possible to find such constants since, by Proposition 7, each $\kappa^{(m)}$ is locally uniformly bounded from above. (The lower bound $e^{\kappa^{(m)}} \ge 0$ is trivial.) Moreover, by elementary properties of plurisubharmonic functions, κ is plurisubharmonic. Indeed, for any $r \in \mathbb{N}$, the function

$$\psi_r := \log \sum_{m=1}^{r} \exp(\tfrac{1}{m}\kappa^{(m)} + \log \varepsilon_m)$$

is plurisubharmonic, and $\psi_r \nearrow \kappa$. It follows that $\kappa = \sup_r \psi_r$ is plurisubharmonic. (Again, see [H-90, Theorem 1.6.2].)Thus $e^{-\kappa}$ is a singular Hermitian metric for $K_X + Z + E$ with non-negative curvature current.

Observe that, after identifying K_Z with $(K_X + Z)|Z$ by dividing by dT,

$$\kappa_k^{(m)}|Z = \log\left(\sum_{\ell=1}^{N_m} |s_\ell^{(m)}|^2\right) + \frac{1}{k}\log \sum_{j=1}^{M_0} |\sigma_j^{(m,0,0)}|^2.$$

Thus we obtain $e^{-\kappa^{(m)}}|Z = \left(\sum_{\ell=1}^{N_m} |s_\ell^{(m)}|^2\right)^{-1}$. It follows that

$$e^{-\kappa}|Z = \frac{1}{\sum_{m=1}^{\infty} \varepsilon_m \left(\sum_{\ell=1}^{N_m} |s_\ell^{(m)}|^2\right)^{2/m}}.$$

In view of the short discussion following the proof of Proposition 3, the metric $e^{-\kappa}$ satisfies the conclusions of Theorem 1. The proof of Theorem 1 is thus complete. □

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REFERENCES

[C-06] Claudon, B., *Invariance of multiples of the twisted canonical bundle*, preprint 2006.

[H-90] Hörmander, L., *An introduction to complex analysis in several variables*, Third edition. North-Holland Mathematical Library, 7. North-Holland Publishing Co., Amsterdam, 1990.

[MV-05] McNeal, J.; Varolin, D., *Analytic inversion of adjunction: L^2 extension theorems with gain*, preprint 2005. To appear in Ann. Inst. Fourier.

[P-05] Paun, M., *Siu's invariance of plurigenera: a one-tower proof*, preprint 2005.

[S-98] Siu, Y.-T., *Invariance of plurigenera*, Invent. Math., 134:3 (1998), pp. 661–673.

[S-02] Siu, Y.-T., *Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type,* Complex geometry. Collection of papers dedicated to Hans Grauert. Springer-Verlag, Berlin, 2002, pp. 223–277.

[Ta-05] Takayama, S, *Pluricanonical systems on algebraic varieties of general type*, to appear in Compositio Math.

[Ts-02] Tsuji, H., *Deformation invariance of plurigenera*, Nagoya Math. J., 166 (2002), pp. 117–134.

[V-06] Varolin, D., *A Takayama-type Extension Theorem,* preprint 2006.