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MEAN VALUE THEOREMS ON MANIFOLDS*

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Abstract. We derive several mean value formulae on manifolds, generalizing the classical one for harmonic functions on Euclidean spaces as well as the results of Schoen-Yau, Michael-Simon, etc, on curved Riemannian manifolds. For the heat equation a mean value theorem with respect to 'heat spheres' is proved for heat equation with respect to evolving Riemannian metrics via a space-time consideration. Some new monotonicity formulae are derived. As applications of the new local monotonicity formulae, some local regularity theorems concerning Ricci flow are proved.

Key words. Green's function, mean value theorem, heat spheres/balls, Ricci flow, local regularity theorem

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1. Introduction. The mean value theorem for harmonic functions plays an central role in the theory of harmonic functions. In this article we discuss its generalization on manifolds and show how such generalizations lead to various monotonicity formulae. The main focuses of this article are the corresponding results for the parabolic equations, on which there have been many works, including [Fu, Wa, FG, GL1, E1], and the application of the new monotonicity formula to the study of Ricci flow.

Let us start with the Watson's mean value formula [Wa] for the heat equation. Let U be a open subset of $\mathbb{R}^n$ (or a Riemannian manifold). Assume that $u(x,t)$ is a C^2 solution to the heat equation in a parabolic region $U_T = U \times (0,T)$. For any (x,t) define the 'heat ball' by

$$E(x,t;r) := \left\{ (y,s)\,|s \le t, \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(4\pi(t-s))^{\frac{n}{2}}} \ge r^{-n} \right\}.$$

Then

$$u(x,t) = \frac{1}{r^n}\int_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{4(t-s)^2}\,dy\,ds$$

for each $E(x,t;r) \subset U_T$. This result plays an important role in establishing the Wiener's criterion for heat equation on $\mathbb{R}^n$ [EG1].

In [FG], the above result was generalized to linear parabolic equations of divergence form. In fact a mean value theorem in terms of 'heat spheres' $\partial E(x,t;r)$ was also derived. This was later applied in establishing the Wiener's criterion for linear parabolic equations of divergence form on $\mathbb{R}^n$ [GL1].

More recently, Ecker [E1] made the remarkable discovery that the similar mean value property (in terms of 'heat balls') can be established for the heat equation coupled with the (nonlinear) mean curvature flow equation. In particular, he discovered a new local monotonicity formulae for the mean curvature flow through this consideration.

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Since Perelman's celebrated paper [P], it becomes more evident that new monotonicity formulae (besides the maximum principle) also play important roles in the study of Ricci flow (various monotonicity quantities, such as isoperimetric constants, entropy like quantities, etc, were used in Hamilton's work before). In a recent joint work [EKNT], the authors proved a mean value theorem (in terms of 'heat balls') for the heat equation coupled with rather arbitrary deformation equation on the Riemannian metrics. This in particular gives a local monotonicity formula for Ricci flow, thanks to the new 'reduced distance' ℓ-function discovered by Perelman, which serves a localization of Perelman's 'reduced volume' monotonicity.

In view of the work of [FG], it is natural to look for mean value theorem in terms of 'heat spheres' under this general setting. This is one of the results of this paper. The new mean value theorem in terms of 'heat spheres' also leads new local monotonicity formulae. The new mean value theorem and monotonicity formula (in terms of 'heat spheres') imply the previous results in terms of 'heat balls' via the integration, as expected. The proof of the result is through a space-time consideration, namely via the study of the geometry of $\widetilde{U} = U \times (0,T)$ with the metric $\tilde{g}(x,t) = g_{ij}(x,t)dx^i dx^j + dt^2$, where $g(x,t)$ are evolving (by some parabolic equation, such as Ricci flow) metrics on $\widetilde{U}$. This space-time geometry is considered un-natural for the parabolic equations since it is not compatible with parabolic scaling. (More involved space-time consideration was taken earlier by Chow and Chu in [CC].) Nevertheless, it worked well here for our purpose of proving the mean value theorem and the monotonicity formulae.

Another purpose of this paper is to derive general mean value theorem for harmonic functions on Riemannian manifolds. Along this line there exist several known results before, including the one due to Schoen and Yau [SY] on manifolds with non-negative Ricci curvature, the one of Michael and Simon [MS] on minimal sub-manifolds of Euclidean spaces, as well as the one on Cartan-Hadamard manifolds [GW]. Our derivation here unifies them all, despite of its simplicity. This also serves the relative easier model for the more involved parabolic case. Hence we treat it in Section 2 before the parabolic case in Section 3.

As an application of the new monotonicity formula of [EKNT], we formulate and prove a local regularity result for the Ricci flow. The result and its proof, which are presented in Section 4, are motivated by [E1], [W1] and [P]. This result gives pointwise curvature estimates under assumption on integral quantities and hopefully it can shed some lights on suitable formulation of weak solutions for Ricci flow. One can refer to [Y2] for results in similar spirits.

It is interesting (also somewhat mysterious) that the quantity used by Li and Yau in their fundamental paper [LY] on the gradient estimates of positive solutions to the heat equation, also appears in the local monotonicity formulae derived in this article. Recall that Li and Yau proved that if M is a complete Riemannian manifold with non-negative Ricci curvature, for any positive solution $u(x,\tau)$ to the heat equation $\frac{\partial}{\partial \tau} - \Delta$,

$$\mathcal{Q}(u) := |\nabla \log u|^2 - (\log u)_\tau \le \frac{n}{2\tau}.$$

Here we show two monotonicity formulae for Ricci flow, which all involve the expression $\mathcal{Q}$. To state the result we need to introduce the notion of the 'reduced distance' ℓ-function of Perelman. Let $\ell^{(x_0,t_0)}(y,\tau)$ be the 'reduced distance' centered at (x_0,t_0), with $\tau = t_0 - t$. (See [P], as well as Section 4 of current paper, for a definition.) Define the 'sub-heat kernel' $\hat{K}(y,\tau;x_0,t_0) = \frac{e^{-\ell(y,\tau)}}{(4\pi\tau)^{\frac{n}{2}}}$. Let $\hat{E}(x_0,t_0;r)$ be the 'heat balls' defined in terms of $\hat{K}$ instead. Also define the 'reduced volume' $\theta(\tau) := \int_M \hat{K}\, d\mu(y)$. Then we have the following result.

THEOREM 1.1. *Let $(\widetilde{U}, g(t))$ be a solution to Ricc flow. Define*

$$\hat{J}(r) := \frac{1}{r^n} \int_{\partial \hat{E}(x_0,t_0;r)} \frac{\mathcal{Q}(\hat{K})}{\sqrt{|\nabla \log \hat{K}|^2 + |(\log \hat{K})_\tau|^2}}\, d\tilde{A}$$

and

$$\hat{I}(a,r) := \frac{1}{r^n - a^n} \int_{\hat{E}(x_0,t_0;r)\setminus \hat{E}(x_0,t_0;a)} \mathcal{Q}(\hat{K})\, d\mu\, d\tau$$

for $\hat{E}(x_0,t_0;r) \subset \widetilde{U}$. Here $d\tilde{A}$ is the area element of $\partial \hat{E}(x_0,t;r)$ with respect to $\tilde{g}$. Then $\hat{J}(r) \le \hat{I}(a,r)$, both $\hat{J}(r)$ and $\hat{I}(a,r)$ are monotone non-increasing in r. $\hat{I}(a,r)$ is also monotone non-increasing in a. If $\hat{J}(r_2) = \hat{J}(r_1)$ for some $r_2 > r_1$, then $g(t)$ is a gradient shrinking soliton in $\hat{E}(x_0,t_0;r_2) \setminus \hat{E}(x_0,t_0;r_1)$ (in fact it satisfies that $R_{ij} + \ell_{ij} - \frac{1}{2\tau} g_{ij} = 0$). If $\hat{I}(a,r_2) = \hat{I}(a,r_1)$ then $g(t)$ is a gradient shrinking soliton on $\hat{E}(x_0,t_0;r_2) \setminus \hat{E}(x_0,t_0;a)$. Moreover on a gradient shrinking soliton, if t_0 is the terminating time, then both $\hat{I}(a,r)$ and $\hat{J}(r)$ are constant and equal to the 'reduced volume' $\theta(\tau)$.

The quantity $\mathcal{Q}(\hat{K})$ can also be expressed in terms of the trace LYH expression (also called the trace of the matrix differential Harnack) [P], modeling the gradient shrinking solitons. Please see (5.12) for details. The part on $\hat{I}(a,r)$ of the above result, in the special case $a = 0$, has been established earlier in [EKNT]. (Here our discussion focuses on the smooth situation while [EKNT] allows the Lipschitz functions.) The similar result holds for the mean curvature flow and the monotonicity of J-quantity in terms of 'heat spheres' gives a new local monotone (non-decreasing) quantity for the mean curvature flow. Please see Section 3 and 5 for details. There is a still open question of ruling out the grim reaper (cf. [W2] for an affirmative answer in the mean-convex case) as a possible singularity model for the finite singularity of mean curvature flow of embedded hypersurfaces. It is interesting to find out whether or not the new monotone quantities of this paper can play any role in understanding this problem. One also expect that the new monotonicity formulae can give another proof of the no local collapsing result of Perelman [P].

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2. The mean value theorem for Laplace equation. In this section we shall derive a mean value theorem for harmonic functions on a fairly large class of Riemannian manifolds. Let (M, g) be a Riemannian manifold. Let Ω be a bounded domain in it. Denote by $G_\Omega(x,y)$ the Green's function with Dirichlet boundary condition on $\partial\Omega$. By the definition

$$\Delta_y G_\Omega(x,y) = -\delta_x(y).$$

By the maximum principle $G_\Omega(x,y) > 0$ for any $x, y \in \Omega$. By Sard's theorem we know that for almost every r, the 'G_Ω-sphere'

$$\Psi_r := \{y \,|G_\Omega(x,y) = r^{-n}\}$$

is a smooth hypersurface in Ω. Let $\phi_r(y) = G_\Omega(x,y) - r^{-n}$ and let

$$\Omega_r = \{y \,|\phi_r(y) > 0\}$$

be the 'G_Ω-ball'. We have the following result.

PROPOSITION 2.1. *Let v be a smooth function on Ω. For every $r > 0$*

$$v(x) = \frac{1}{r^n}\int_{\Omega_r} |\nabla \log G_\Omega|^2 v\, d\mu - \frac{n}{r^n}\int_0^r \eta^n \int_{\Omega_\eta} \phi_\eta \Delta v\, d\mu \frac{d\eta}{\eta}. \tag{2.1}$$

Proof. We first show that for almost every $r > 0$

$$v(x) = \int_{\Psi_r} |\nabla G_\Omega| v\, dA_y - \int_{\Omega_r} \phi_r \Delta v\, d\mu_y. \tag{2.2}$$

By Green's second identity

$$\int_{\Omega_r} ((\Delta G_\Omega) v - G_\Omega(\Delta v))\; d\mu = \int_{\Psi_r} \left(\frac{\partial G_\Omega}{\partial \nu} v - \frac{\partial v}{\partial \nu} G_\Omega\right) dA$$

we have that

$$v(x) = -\int_{\Psi_r} \left(\frac{\partial G_\Omega}{\partial \nu} v - \frac{\partial v}{\partial \nu} G_\Omega\right) dA - \int_{\Omega_r} G_\Omega(\Delta v)\, d\mu. \tag{2.3}$$

Notice that on Ψ_r

$$\frac{\partial G_\Omega}{\partial \nu} = -|\nabla G_\Omega| \tag{2.4}$$

and

$$\begin{aligned} \int_{\Psi_r} \frac{\partial v}{\partial \nu} G_\Omega\, dA &= \frac{1}{r^n}\int_{\Psi_r} \frac{\partial v}{\partial \nu}\, dA \\ &= \frac{1}{r^n}\int_{\Omega_r} \Delta v\, d\mu. \end{aligned} \tag{2.5}$$

The equation (2.2) follows by combing (2.3), (2.4) and (2.5).

The equality (2.1) follows from (2.2) by the co-area formula. In deed, multiplying η^{n-1} on the both sides of (2.2), integrate on $[0, r]$. Then we have that

$$\begin{aligned} \frac{1}{n} r^n v(x) &= \int_0^r \eta^{n-1} \int_{G_\Omega = \eta^{-n}} |\nabla G_\Omega| v\, dA\, d\eta + \int_0^r \eta^n \int_{\Omega_\eta} \phi_\eta \Delta v\, d\mu \frac{d\eta}{\eta} \\ &= \frac{1}{n}\int_{r^{-n}}^{\infty} \frac{1}{\alpha^2}\int_{G_\Omega=\alpha} |\nabla G_\Omega| v\, dA\, d\alpha + \int_0^r \eta^n \int_{\Omega_\eta} \phi_\eta \Delta v\, d\mu \frac{d\eta}{\eta} \\ &= \frac{1}{n}\int_{r^{-n}}^{\infty} \int_{G_\Omega=\alpha} \frac{|\nabla \log G_\Omega|^2 v}{|\nabla G_\Omega|}\, dA\, d\alpha + \int_0^r \eta^n \int_{\Omega_\eta} \phi_\eta \Delta v\, d\mu \frac{d\eta}{\eta} \\ &= \frac{1}{n}\int_{\Omega_r} |\nabla \log G_\Omega|^2 v\, d\mu + \int_0^r \eta^n \int_{\Omega_\eta} \phi_\eta \Delta v\, d\mu \frac{d\eta}{\eta}. \end{aligned}$$

In the last equation we have applied the co-area formula [EG2], for which we have to verify that $|\nabla \log G_\Omega|^2$ is integrable. This follows from the asymptotic behavior of $G_\Omega(x, y)$ near x. It can also be seen, in the case of $n \geq 3$, from a known estimate (cf. [L1] Theorem 6.1) of Cheng-Yau, which asserts that near x, $|\nabla \log G_\Omega|^2(y) \leq A\left(1 + \frac{1}{d^2(x,y)}\right)$. for some $A > 0$. □

It would be more convenient to apply the Proposition 2.1 if we can replace the Green's function $G_\Omega(x, y)$, which depends on Ω and usually hard to estimate/compute, by a canonical positive Green's function. To achieve this we need to assume that the Riemannian manifold (M, g) is non-parabolic. Namely, there exists a minimum positive Green's function $G(x, y)$ on M. However, on general non-parabolic manifolds, it may happen that $G(x, y)$ does not approach to 0 as $y \to \infty$. This would imply that the integrals on the right hand side of (2.2) or (2.1) are over a noncompact domain (a unbounded hyper-surface). Certain requirements or justifications on v are needed to make sense of the integral. It certainly works for the case that v has compact support. If we want to confine ourself with the situation that $\Omega_r = \{y \,|G(x, y) \geq r^{-n}\}$ is compact we have to impose further that

$$\lim_{y\to\infty} G(x, y) = 0. \tag{2.6}$$

Definition 2.2. We call the Riemannian manifold (M, g) is *strongly non-parabolic* if (2.6) holds for the minimal positive Green's function.

Now similarly we can define the 'G-sphere' Ψ_r, the 'G-ball' Ω_r, and the function ϕ_r as before. By verbatim repeating the proof of Proposition 2.1 we have the following global result.

Theorem 2.3. *Assume that (M, g) is a strongly non-parabolic Riemannian manifold. Let v be a smooth function. Then for every $r > 0$*

$$v(x) = \frac{1}{r^n}\int_{\Omega_r} |\nabla \log G|^2 v \, d\mu - \int_0^r \frac{n}{\eta^{n+1}}\int_{\Omega_\eta} \psi_\eta \Delta v \, d\mu d\eta. \tag{2.7}$$

Here $\psi_r = \log(Gr^n)$. Also for almost every $r > 0$

$$v(x) = \int_{\Psi_r} |\nabla G| v \, dA_y - \int_{\Omega_r} \phi_r \Delta v \, d\mu_y. \tag{2.8}$$

Proof. The only thing we need to verify is that

$$\int_0^r \frac{n}{\eta^{n+1}}\int_{\Omega_\eta} \psi_\eta \Delta v \, d\mu d\eta = \frac{n}{r^n}\int_0^r \eta^n \int_{\Omega_\eta} \phi_\eta \Delta v \, d\mu \frac{d\eta}{\eta}$$

which can be checked directly. Please see also (2.14) for a more general equality. □

Remark 2.4. a) If $(M, g) = \mathbb{R}^n$ with $n \geq 3$, then

$$G(x, y) = \frac{1}{n(n-2)\omega_n} d^{2-n}(x, y),$$

with ω_n being the volume of unit Euclidean ball and $d(x,y)$ being the distance between x and y. Clearly M is strongly non-parabolic. If v is a harmonic function, a routine exercise shows that Theorem 2.3 implies the classical mean value theorem

$$v(x) = \frac{1}{nR^{n-1}\omega_n}\int_{\partial B(x,R)} v(y)\, dA.$$

b) The regularity on v can be weaken to, say being Lipschitz. Now we have to understand Δv in the sense of distribution and the integral $\int_{\Omega_r} \phi_r \Delta v$ via a certain suitable approximation [E1, EKNT, N2].

c) Li and Yau [LY] proved that if (M,g) has nonnegative Ricci curvature, then (M,g) is non-parabolic if and only if $\int_r^\infty \frac{\tau}{V_x(\tau)}\, d\tau < \infty$, where $V_x(\tau)$ is the volume of ball $B(x,\tau)$. Moreover, there exists $C = C(n)$ such that

$$C^{-1}\int_{d(x,y)}^\infty \frac{\tau}{V_x(\tau)}\, d\tau \le G(x,y) \le C\int_{d(x,y)}^\infty \frac{\tau}{V_x(\tau)}\, d\tau.$$

This shows that any non-parabolic (M,g) is strongly non-parabolic. In this case by the gradient estimate of Yau [L1] we have that there exists $C_1(n)$ such that if $\Delta v \ge 0$ and $v \ge 0$

$$v(x) \le \frac{C_1(n)}{r^n}\int_{\Psi_r} \frac{v(y)}{d(x,y)}\, dA_y. \tag{2.9}$$

One would expect that this should imply the well-known mean value theorem of Li-Schoen [LS].

d) An example class of strongly non-parabolic Riemannian manifolds are those manifolds with the Sobolev inequality:

$$\left(\int_M f^{\frac{2\nu}{\nu-2}}\, d\mu\right)^{\frac{\nu-2}{\nu}} \le A\int_M |\nabla f|^2\, d\mu \tag{2.10}$$

for some $\nu > 2$, $A > 0$, for any smooth f with compact support. In deed, by [Da] and [Gr], the Sobolev inequality (2.10) implies that there exist B and $D > 0$ such that

$$H(x,y,t) \le Bt^{-\frac{\nu}{2}} \exp\left(-\frac{d^2(x,y)}{Dt}\right).$$

This implies the estimate

$$G(x,y) \le C(B,\nu,D)d^{-\nu+2}(x,y)$$

for some C. Therefore (M,g) is strongly non-parabolic.

e) Another set of examples are the Riemannian manifolds with positive lower bound on the spectrum with respect to the Laplace operator [LW]. On this type of manifolds we usually assume that the Ricci curvature $Ric \ge -(n-1)g$. Since the manifold may contain ends with finite volume (as examples in [LW] show), on which the Green's function certainly does not tend to zero at infinity, further assumptions are needed to ensure the strongly non-parabolicity. If we assume further that there exists a $\delta > 0$ such that $V_x(1) \ge \delta$, we can show that it is strongly non-parabolic (shown in the proposition below). Note that the assumption holds for the universal covers of compact Riemannian manifolds with $Ric \ge -(n-1)g$.

Proposition 2.5. *Assume that (M^n, g) is a complete Riemannian manifold with positive lower bound on the spectrum of the Laplace operator. Suppose that $Ric \geq -(n-1)g$ and that there exists a $\delta > 0$ such that $V_x(1) \geq \delta$ for all $x \in M$. Then (M, g) is strongly non-parabolic.*

Proof. First we have the following upper bound of the heat kernel

$$H(x,y,t) \leq C_1 \exp(-\lambda t) V_x^{-\frac{1}{2}}(\sqrt{t}) V_y^{-\frac{1}{2}}(\sqrt{t}) \exp\left(-\frac{d^2(x,y)}{Dt} + C_2\sqrt{t}\right) \tag{2.11}$$

for some absolute constant $D > 4$, $C_2 = C_2(n)$ and $C_1 = C_1(D, n)$. Please see [L2] for a proof. Here $\lambda > 0$ denotes the greatest lower bound on the spectrum of the Laplace operator. The assumption on the lower bound of the curvature implies that

$$V_x^{-\frac{1}{2}}(\sqrt{t}) V_y^{-\frac{1}{2}}(\sqrt{t}) \leq h(t)$$

where

$$h(t) = \begin{cases} \frac{1}{\delta} & \text{if } t \geq 1 \\ C(\delta, n) t^{-\frac{n}{2}} & \text{if } t \leq 1 \end{cases}.$$

By some elementary computation and estimates we have that

$$\begin{aligned} G(x,y) &= \int_0^\infty H(x,y,t)\, dt \\ &\leq \frac{4}{\delta\lambda} e^{\frac{C_2^2}{2\lambda}} \exp\left(-\sqrt{\frac{\lambda}{D}} d(x,y)\right) \\ &\quad + C(\delta, n, D) e^{\frac{C_2^2}{2\lambda}} d^{-n+2}(x,y) \int_{\frac{d^2(x,y)}{D}}^\infty \exp(-\tau) \tau^{\frac{n}{2}-2}\, d\tau \end{aligned}$$

which goes to 0 as $d(x,y) \to \infty$. In fact one can have the upper bound $G(x,y) \leq C \exp\left(-\sqrt{\frac{\lambda}{D}} d(x,y)\right)$. $\square$

There exist monotonicity formulae related to Proposition 2.1 and Theorem 2.3, which we shall illustrate below. For simplicity we just denote by G for both $G_\Omega(x,y)$, in the discussion concerning a bounded domain Ω, and $G(x,y)$, when it is on the strongly non-parabolic manifolds.

For any smooth (or Lipschitz) v we define

$$I_v(r) := \frac{1}{r^n} \int_{\Omega_r} |\nabla \log G|^2 v\, d\mu$$

and

$$J_v(r) := \int_{\Psi_r} |\nabla G| v\, dA.$$

They are related through the relation $r^n I_v(r) = n \int_0^r \eta^{n-1} J(\eta)\, d\eta$, which can be shown by using Tonelli's theorem and the co-area formula.

Corollary 2.6. *Let $\psi_r = \log(Gr^n)$, which is nonnegative on Ω_r. We have that for almost every $r > 0$*

$$\frac{d}{dr} I_v(r) = \frac{n}{r^{n+1}} \int_{\Omega_r} (\Delta v)\, \psi_r\, d\mu; \tag{2.12}$$

$$\frac{d}{dr} J_v(r) = \frac{n}{r^{n+1}} \int_{\Omega_r} \Delta v \, d\mu. \tag{2.13}$$

In particular, $I_v(r)$ and $J_v(r)$ are monotone non-increasing (non-decreasing) in r, provided that v is super-harmonic (sub-harmonic).

Proof. Differentiate (2.7) with respect to r. Then

$$I'_v(r) = -\frac{n^2}{r^{n+1}} \int_0^r \eta^n \int_{\Omega_\eta} \phi_\eta \Delta v \, d\mu \frac{d\eta}{\eta} + \frac{n}{r} \int_{\Omega_r} \phi_r \Delta v \, d\mu.$$

On the other hand, Tonelli's theorem gives

$$\begin{aligned}\int_0^r \eta^n \int_{\Omega_\eta} \phi_\eta \Delta v \, d\mu \frac{d\eta}{\eta} &= \int_{G \geq r^{-n}} \int_{G^{-\frac{1}{n}}}^r \left(G\eta^{n-1} - \frac{1}{\eta} \right) \Delta v \, d\mu \, d\eta \\ &= \frac{r^n}{n} \int_{\Omega_r} (\Delta v)(G - r^{-n}) \, d\mu - \frac{1}{n} \int_{\Omega_r} (\Delta v) \psi_r \, d\mu.\end{aligned}$$

From the above two equations, we have the claimed (2.12), observing that $\phi_r = G - r^{-n}$. The proof of (2.13) is very similar. □

A by-product of the above proof is that for any $f(y)$ (regular enough to makes sense the integrals)

$$\frac{n}{r^n} \int_0^r \eta^n \int_{\Omega_\eta} f \phi_\eta \, d\mu \frac{d\eta}{\eta} = \int_0^r \frac{n}{\eta^{n+1}} \int_{\Omega_\eta} f \psi_\eta \, d\mu \, d\eta. \tag{2.14}$$

In fact let $F(r)$ be the left hand side, the above proof shows that

$$F'(r) = \frac{n}{r^{n+1}} \int_{\Omega_r} f \psi_r \, d\mu.$$

In [SY], Schoen and Yau proved a mean value inequality for sup-harmonic functions with respect to sufficiently small geodesic balls, when (M, g) is a complete Riemannian manifold (or just a piece of) with nonnegative Ricci curvature. More precisely, they showed that if $x \in M$ and $B(x, R)$ lies inside a normal coordinate centered at x. Let $f \geq 0$ be a Lipschitz function satisfying that $\Delta f \leq 0$ then

$$f(x) \geq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B(x,R)} f(y) \, dA. \tag{2.15}$$

We shall see that a consequence of Theorem 2.3 implies a global version of the above result. First recall the following fact. If (M, g) is a Riemannian manifold such that its Ricci curvature satisfies $Ric \geq -(n-1)k^2 g$, then

$$\Delta_y \hat{G}(d(x, y)) \geq -\delta_x(y). \tag{2.16}$$

We call such $\hat{G}$ a sub-Green's function. Here $\hat{G}(\bar{d}(x, y))$ is the Green's function of the space form $(\overline{M}, \bar{g})$ with constant curvature $-k^2$, where $\bar{d}$ is the distance function of $\overline{M}$. One can refer to Chapter 3 of [SY] for a proof of the corresponding parabolic result (which is originally proved in [CY]). The key facts used for the proof are the standard Laplace comparison theorem [SY] and $\hat{G}' \leq 0$ (which follows from the maximum principle). Now we can define

$$\hat{\Omega}_r = \{ y \,|\, \hat{G} \geq r^{-n} \}, \qquad \hat{\Psi}_r = \{ y \,|\, \hat{G} = r^{-n} \}$$

and let $\hat{\phi}_r = \hat{G} - r^{-n}$.

The virtue of the proof of Proposition 2.1 gives the following general result.

PROPOSITION 2.7. *Let (M, g) be a complete Riemannian manifold with Ric $\geq -(n-1)k^2 g$. In the case $k = 0$ we assume further that $n \geq 3$. Let $v \geq 0$ be a Lipschitz function. Then*

$$v(x) \geq \int_{\hat{\Psi}_r} |\nabla \hat{G}| v \, dA_y - \int_{\hat{\Omega}_r} \hat{\phi}_r \Delta v \, d\mu_y. \tag{2.17}$$

For every $r > 0$

$$v(x) \geq \frac{1}{r^n} \int_{\hat{\Omega}_r} |\nabla \log \hat{G}|^2 v \, d\mu - \frac{n}{r^n} \int_0^r \eta^n \int_{\hat{\Omega}_\eta} \hat{\phi}_\eta \Delta v \, d\mu \frac{d\eta}{\eta}. \tag{2.18}$$

By some straight forward computations one can verify that (2.17), in the case of $k = 0$, implies a global version of (2.15).

Moreover, in this non-exact case there are still monotonicity formulae related to Proposition 2.7.

COROLLARY 2.8. *Let*

$$\hat{I}_v(r) = \frac{1}{r^n} \int_{\hat{\Omega}_r} |\nabla \log \hat{G}|^2 v \, d\mu, \quad \hat{J}_v(r) = \int_{\hat{\Psi}_r} |\nabla \hat{G}| v \, dA.$$

If $v \geq 0$, then for almost every r,

$$\frac{d}{dr} \hat{I}_v(r) \leq \frac{n}{r^{n+1}} \int_{\hat{\Omega}_r} (\Delta v) \hat{\psi}_r \, d\mu \tag{2.19}$$

where $\hat{\psi}_r = \log(\hat{G} r^n)$, and

$$\frac{d}{dr} \hat{J}_v(r) \leq \frac{n}{r^{n+1}} \int_{\hat{\Omega}_r} \Delta v \, d\mu. \tag{2.20}$$

In particular, if $\Delta v \leq 0$, $\hat{I}_v(r)$ and $\hat{J}_v(r)$ are monotone non-increasing in r. Moreover, if the equality holds in the inequality (2.19) (or (2.20)) for some $v > 0$ and some $r > 0$ then $B(x, R)$, the biggest ball contained in $\hat{\Omega}_r$, is isometric to the corresponding ball in the space form.

Proof. Here we adapt a different scheme from that of Corollary 2.6. For the convenience we let $\hat{\psi} = \log \hat{G}$. Differentiate $\hat{I}_v(r)$ we have

$$\hat{I}'_v(r) = -\frac{n}{r^{n+1}} \int_{\hat{\Omega}_r} |\nabla \hat{\psi}_r|^2 v \, d\mu + \frac{n}{r^{n+1}} \int_{\hat{\Psi}_r} |\nabla \hat{\psi}| v \, dA. \tag{2.21}$$

Using that $\frac{\partial}{\partial \nu} \hat{\psi} = -|\nabla \hat{\psi}|$ on $\hat{\Psi}_r$, we have that

$$\begin{aligned}
\int_{\hat{\Psi}_r} |\nabla \hat{\psi}| v \, dA &= -\int_{\hat{\Psi}_r} \left(\frac{\partial}{\partial \nu} \hat{\psi} \right) v \, dA \\
&= \int_{\hat{\Omega}_r} \Big((\Delta v) \hat{\psi} - (\Delta \hat{\psi}) v \Big) \, d\mu - \int_{\hat{\Psi}_r} \left(\frac{\partial}{\partial \nu} v \right) \hat{\psi} \, dA \\
&= \int_{\hat{\Omega}_r} \Big((\Delta v) \hat{\psi} - (\Delta \hat{\psi}) v \Big) \, d\mu + \log r^n \int_{\hat{\Omega}_r} \Delta v \, d\mu.
\end{aligned}$$

Using (2.16), and the fact that $\frac{1}{\hat{G}} \to 0$ as $y \to x$, we have that

$$-\int_{\hat{\Omega}_r} (\Delta\hat{\psi}) v \, d\mu \le \int_{\hat{\Omega}_r} |\nabla \hat{\psi}|^2 v \, d\mu.$$

Therefore

$$\int_{\hat{\Psi}_r} |\nabla \hat{\psi}| v \, dA \le \int_{\hat{\Omega}_r} |\nabla \hat{\psi}|^2 v \, d\mu + \int_{\hat{\Omega}_r} (\Delta v)\hat{\psi}_r \, d\mu.$$

Together with (2.21) we have the claimed inequality (2.19). The equality case follows from the equality in the Laplace comparison theorem.

To prove (2.20), for any $r_2 > r_1$, the Green's second identity implies that

$$\begin{aligned} J_v(r_2) - J_v(r_1) &= \int_{\hat{\Psi}_{r_2}} \left(-\frac{\partial \hat{G}}{\partial \nu}\right) v \, dA + \int_{\hat{\Psi}_{r_1}} \frac{\partial \hat{G}}{\partial \nu} v \, dA \\ &= -\int_{\hat{\Omega}_{r_2} \setminus \hat{\Omega}_{r_1}} \left((\Delta \hat{G}) v - (\Delta v)\hat{G}\right) \, d\mu \\ &\quad - \int_{\hat{\Psi}_{r_2}} G \frac{\partial v}{\partial \nu} \, dA + \int_{\hat{\Psi}_{r_1}} G \frac{\partial v}{\partial \nu} \, dA \\ &\le \int_{\hat{\Omega}_{r_2}} \Delta v (\hat{G} - r_2^{-n}) - \int_{\hat{\Omega}_{r_1}} \Delta v (\hat{G} - r_1^{-n}) \, d\mu. \end{aligned}$$

Dividing by $(r_2 - r_1)$, the claimed result follows by co-area formula and taking limit $(r_2 - r_1) \to 0$. $\square$

The general result can also be applied to a somewhat opposite situation, namely to Cartan-Hardamard manifolds and minimal sub-manifolds in such manifolds. Recall that (M^n, g) is called a Cartan-Hardamard manifold if it is simply-connected with the sectional curvature $K_M \le 0$. Let $\tilde{G}(x,y) = \frac{1}{n(n-2)\omega_n} d^{2-n}(x,y)$ $(n \ge 3)$. Then we have that [GW]

$$\Delta_y \tilde{G}(x,y) \le -\delta_x(y). \tag{2.22}$$

Similarly we call such $\tilde{G}$ a sup-Green's function. Let N^k $(k \ge 3)$ be a minimal (immersed) submanifold in M^n (where M is the Cartan-Hardamard manifold as the above. Let $\bar{\Delta}$ be the Laplace operator of N^k. It is easy to check that

$$\bar{\Delta} \bar{G} \le -\delta_x(y) \tag{2.23}$$

where $\bar{G}(x,y) = \frac{1}{n(n-2)\omega_k} d^{2-k}(x,y)$. Here $d(x,y)$ is the extrinsic distance function of M. Similarly we can define $\tilde{\Omega}_r \subset M$ for the first case and $\overline{\Omega}_r \subset N$ for the minimum submanifold case.

Proposition 2.9. *Let (M,g) be Cartan-Hardamard manifold. Let $v \ge 0$ be a Lipschitz function. Then*

$$v(x) \le \int_{\tilde{\Psi}_r} |\nabla \tilde{G}| v \, dA_y - \int_{\tilde{\Omega}_r} \tilde{\phi}_r \Delta v \, d\mu_y. \tag{2.24}$$

For every $r > 0$

$$v(x) \le \frac{1}{r^n} \int_{\tilde{\Omega}_r} |\nabla \log \tilde{G}|^2 v \, d\mu - \frac{n}{r^n} \int_0^r \eta^n \int_{\tilde{\Omega}_\eta} \tilde{\phi}_\eta \Delta v \, d\mu \frac{d\eta}{\eta}. \tag{2.25}$$

We state the result only for $\tilde{G}$ since the exactly same result holds $\bar{G}$. The mean value inequality (2.24) of $\bar{G}$ recovers the well-known result of Michael and Simon [MS] if N is a minimal sub-manifold in the Euclidean spaces. See also [CLY] for another proof of Michael-Simon's result via a heat kernel comparison theorem.

Corollary 2.10. *Let*

$$\tilde{I}_v(r) = \frac{1}{r^n}\int_{\tilde{\Omega}_r} |\nabla \log \tilde{G}|^2 v\, d\mu, \quad \tilde{J}_v(r) = \int_{\tilde{\Psi}_r} |\nabla \tilde{G}| v\, dA.$$

If $v \geq 0$, then for almost every r,

$$\frac{d}{dr}\tilde{I}_v(r) \geq \frac{n}{r^{n+1}}\int_{\tilde{\Omega}_r} (\Delta v)\tilde{\psi}_r\, d\mu \tag{2.26}$$

where $\tilde{\psi}_r = \log(\tilde{G}r^n)$, and

$$\frac{d}{dr}\tilde{J}_v(r) \geq \frac{n}{r^{n+1}}\int_{\tilde{\Omega}_r} \Delta v\, d\mu. \tag{2.27}$$

In particular, if $\Delta v \geq 0$, $\tilde{I}_v(r)$ and $\tilde{J}_v(r)$ are monotone non-decreasing in r. Moreover, if the equality holds in the inequality (2.26) (or (2.27)) for some $v > 0$ at some $r > 0$ then $B(x, R)$, the biggest ball contained in $\tilde{\Omega}_r$, is isometric to the corresponding ball in the Euclidean space.

3. Spherical mean value theorem for heat equation with changing metrics. In [EKNT], mean value theorems in terms of 'heat balls' were proved for heat equations with respect to evolving metrics. A mean value theorem in terms of 'heat spheres' was also established for the heat equation with respect to a fixed Riemannian metric. Since the mean value theorems in terms of 'heat balls' are usually consequences of the ones in terms of 'heat spheres' it is desirable to have the mean value theorem in terms of 'heat spheres' for the heat equation with respect to evolving metrics too. Deriving such a mean value theorem is the main result of this section. The proof is an adaption of the argument of [FG] for operators of divergence form on Euclidean case (see also [Fu]) to the evolving metrics case. The key is the Green's second identity applied to the space-time.

Let $(M, g(t))$ be a family of metrics evolved by the equation

$$\frac{\partial}{\partial t} g_{ij} = -2\Upsilon_{ij}. \tag{3.1}$$

We start by setting up some basic space-time notions. First we fix a point (x_0, t_0) in the space time. For the simplicity we assume that $t_0 = 0$. Assume that $g(t)$ is a solution to (3.1) on $M \times (\alpha, \beta)$ with $\alpha < 0 < \beta$. Denote $M \times (\alpha, \beta)$ by $\widetilde{M}$, over which we define the metric $\tilde{g}(x,t) = g_{ij}(x,t)dx^i dx^j + dt^2$, where t is the global coordinate of (α, β). We consider the heat operator $\left(\frac{\partial}{\partial t} - \Delta\right)$ with respect to the changing metric $g(t)$. Now the *conjugate heat operator* is $\frac{\partial}{\partial t} + \Delta - R(y,t)$, where $R(y,t) = g^{ij}\Upsilon_{ij}$. We need some elementary space-time computations for $(\widetilde{M}, \tilde{g})$. In [CC] (see also [P]), a similar consideration was originated, but for some degenerate metrics satisfying the generalized Ricci flow equation instead. In the following indices i, j, k are between 1 and n, A, B, C are between 0 and n. The index 0 denotes the t direction.

Lemma 3.1. *Let Γ_{ij}^k be the Christoffel symbols of g_{ij}, and let $\widetilde{\Gamma}_{AB}^C$ be the ones for $\widetilde{g}$. Then*

$$\widetilde{\Gamma}_{ij}^0 = \Upsilon_{ij}, \tag{3.2}$$

$$\widetilde{\Gamma}_{0,k}^0 = \widetilde{\Gamma}_{00}^0 = 0, \tag{3.3}$$

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k, \tag{3.4}$$

$$\widetilde{\Gamma}_{0k}^i = -\widetilde{g}^{il}\Upsilon_{lk} = -\Upsilon_k^i. \tag{3.5}$$

Proof. It follows from straight forward computations. ◻

Lemma 3.2. *Let X be a time dependent vector field on M (namely for each t, $X(t)$ is a vector field of M). Let $\widetilde{X} = X + X^0\frac{\partial}{\partial t}$ for some function X^0 on $\widetilde{M}$. Let $\widetilde{\mathrm{div}}$ be the divergence operator with respect to $\tilde{g}$, and let* div *be the divergence operator with respect to g (on $M \times \{t\}$). Then*

$$\widetilde{\mathrm{div}}\widetilde{X} = \mathrm{div}(X) - X^0 R + \frac{\partial}{\partial t}X^0. \tag{3.6}$$

Proof. Follows from Lemma 3.1 and routine computations. ◻

Let $H(y,t;x_0,0)$ be a fundamental solution to the *conjugate heat operator* centered at $(x_0,0)$. Let $\tau = -t$. By the abuse of the notation we sometime also write the fundamental solution as $H(y,\tau;x_0,0)$, by which we mean the fundamental solution with respect to the operator $\frac{\partial}{\partial\tau} - \Delta + R$. Now we define the 'heat ball' by $E_r = \{(y,\tau) \,|\, H(y,\tau;x_0,0) \geq r^{-n}\}$. By the asymptotics of the fundamental solution [GL2] we know that E_r is compact for r sufficiently small. Following [FG] we define $E_r^s = \{(y,\tau) \in E_r,\, t < s\}$ and two portions of its boundary $P_1^s = \{(y,\tau) \,|\, H(y,\tau;x_0,0) = r^{-n},\, t < s\}$ and $P_2^s = \{(y,\tau) \in \overline{E_r^s},\, t = s\}$. $P_1^0 = \partial E_r$ is the 'heat sphere', which is the boundary of the 'heat ball'. Let $\psi_r = H(y,\tau;x_0,0) - r^{-n}$. We then have the following spherical mean value theorem.

Theorem 3.3. *Let v be a smooth function on $\widetilde{M}$. Then*

$$\begin{aligned} v(x_0,0) = &\int_{\partial E_r} v\frac{|\nabla H|^2}{\sqrt{|\nabla H|^2 + |H_t|^2}}\, d\tilde{A} + \frac{1}{r^n}\int_{E_r} Rv\, d\mu\, dt \\ &+ \int_{E_r} \phi_r \left(\frac{\partial}{\partial t} - \Delta\right) v\, d\mu\, dt. \end{aligned} \tag{3.7}$$

Here $d\tilde{A}$ is the n-dimensional measure induced from $\tilde{g}$.

Proof. This follows as in, for instance, Fabes-Garofalo [FG], by applying a divergence theorem in $\widetilde{M}$ and the above Lemma 3.6 for the changing metrics. More precisely, let $\widetilde{X} = v\nabla H - H\nabla v + Hv\frac{\partial}{\partial t}$. Applying the divergence theorem to $\tilde{X}$ on E_r^s we have that

$$\begin{aligned} \int_{E_r^s} \left(v(\frac{\partial}{\partial t} + \Delta - R)H + H(\frac{\partial}{\partial t} - \Delta)v\right) d\mu\, dt &= \int_{E_r^s} \widetilde{\mathrm{div}}(\widetilde{X})\, d\mu\, dt \\ &= \int_{\partial E_r^s} \langle \widetilde{X}, \tilde{\nu}\rangle d\tilde{A}. \end{aligned} \tag{3.8}$$

Here $\tilde{\nu}$ is the normal ∂E_r^s with respect to $(\widetilde{M}, \tilde{g})$. On P_1^s it is given by

$$\tilde{\nu} = \left(-\frac{\nabla H}{\sqrt{|\nabla H|^2 + |H_t|^2}}, -\frac{H_t}{\sqrt{|\nabla H|^2 + |H_t|^2}} \right).$$

On P_2^s it is just $\frac{\partial}{\partial t}$. Compute

$$\begin{aligned}
\int_{\partial E_r^s} \langle \widetilde{X}, \tilde{\nu} \rangle d\tilde{A} &= \int_{P_1^s} \left(-v\langle \nabla H, \tilde{\nu}\rangle - H\langle \nabla v, \tilde{\nu}\rangle + Hv\langle \frac{\partial}{\partial t}, \tilde{\nu}\rangle \right) d\tilde{A} \\
&\quad + \int_{P_2^s} vH \, d\mu_s \\
&= \int_{P_1^s} \left(-v \frac{|\nabla H|^2}{\sqrt{|\nabla H|^2 + |H_t|^2}} - H\langle \nabla v, \tilde{\nu}\rangle \right. \\
&\quad \left. + Hv\langle \frac{\partial}{\partial t}, \tilde{\nu}\rangle \right) d\tilde{A} + \int_{P_2^s} vH \, d\mu_s \\
&= \int_{P_1^s} \left(-v \frac{|\nabla H|^2}{\sqrt{|\nabla H|^2 + |H_t|^2}} \right) d\tilde{A} + \int_{P_2^s} vH \, d\mu_s \\
&\quad + r^{-n} \int_{P_1^s} \left(-\langle \nabla v, \tilde{\nu}\rangle + v\langle \frac{\partial}{\partial t}, \tilde{\nu}\rangle \right) d\tilde{A}.
\end{aligned} \tag{3.9}$$

Similarly, applying the divergence theorem to $\widetilde{Y} = -\nabla v + v\frac{\partial}{\partial t}$ gives that

$$\begin{aligned}
\int_{E_r^s} \left(-\Delta v - Rv + \frac{\partial v}{\partial t} \right) d\mu \, dt &= \int_{P_1^s} \left(-\langle \nabla v, \tilde{\nu}\rangle + v\langle \frac{\partial}{\partial t}, \tilde{\nu}\rangle \right) d\tilde{A} \\
&\quad + \int_{P_2^s} v \, d\mu_s.
\end{aligned} \tag{3.10}$$

By (3.8)–(3.10) we have that

$$\begin{aligned}
\int_{P_2^s} v(H - r^{-n}) \, d\mu_s &= \int_{E_r^s} \phi_r \left(\frac{\partial v}{\partial t} - \Delta v \right) d\mu \, dt + r^{-n} \int_{E_r^s} Rv \, d\mu \, dt \\
&\quad + \int_{P_1^s} \left(v \frac{|\nabla H|^2}{\sqrt{|\nabla H|^2 + |H_t|^2}} \right) d\tilde{A}.
\end{aligned}$$

Letting $s \to 0$, the claimed result follows from the asymptotics

$$\lim_{s \to 0} \int_{P_2^s} v(H - r^{-n}) \, d\mu_s = v(x_0, 0) \tag{3.11}$$

which can be checked directly using the asymptotics of H, or simply the definition of H. ∎

Remark 3.4. Theorem 3.3 is the parabolic analogue of (2.2) and (2.8). In the spacial case of the metrics being fixed, namely $\Upsilon = 0$, Theorem 3.3 gives a manifold version of the spherical mean value theorem for solutions to the heat equation. Please see [Fu] and [FG] for earlier results when $M = \mathbb{R}^n$.

As a corollary of Theorem 3.3 one can obtain the ‘heat ball’ mean value theorem proved in [EKNT] by integrating (3.7) for r and applying the co-area formula and Tonelli’s theorem. Indeed multiplying η^{n-1} on both sides of (3.7), then integrating from 0 to r as in Proposition 2.1, we have that

$$\begin{aligned} v(x_0, 0) = \frac{1}{r^n} \int_{E_r} \left(|\nabla \log H|^2 + R\psi_r\right) v \, d\mu dt \\ + \frac{n}{r^n} \int_0^r \eta^n \int_{E_\eta} \phi_\eta \left(\frac{\partial}{\partial t} - \Delta\right) v \, d\mu \, dt \, \frac{d\eta}{\eta}. \end{aligned} \tag{3.12}$$

By an argument similar to the proof of (2.14) we have that

$$\begin{aligned} &\frac{n}{r^n} \int_0^r \eta^n \int_{E_\eta} \phi_\eta \left(\frac{\partial}{\partial t} - \Delta\right) v \, d\mu \, dt \, \frac{d\eta}{\eta} \\ &= \int_0^r \frac{n}{\eta^{n+1}} \int_{E_\eta} \psi_\eta \left(\frac{\partial}{\partial t} - \Delta\right) v \, d\mu \, dt \, d\eta. \end{aligned}$$

Therefore (3.12) is the same as the formula in [EKNT]. Similar to Corollary 2.6, we can recover the monotonicity formulae of [EKNT] from (3.12). Moreover, from (3.7) we have

$$\frac{d}{dr} J_v(r) = -\frac{n}{r^{n+1}} \int_{E_r} \left(\frac{\partial}{\partial t} - \Delta\right) v \, d\mu \, dt$$

where

$$J_v(r) := \int_{\partial E_r} v \frac{|\nabla H|^2}{\sqrt{|\nabla H|^2 + |H_t|^2}} \, d\tilde{A} + \frac{1}{r^n} \int_{E_r} R v \, d\mu \, dt.$$

When $\Upsilon_{ij} = R_{ij}$, namely we are in the situation of the Ricci flow, thanks to [P], there exists a ‘sub-heat kernel’ to the *conjugate heat equation* constructed using the *reduced distance* function discovered by Perelman [P]. For fixed point $(x_0, 0)$, $\tau = -t$ as above, recall that

$$L(y, \bar{\tau}) = \inf_{\gamma} \mathcal{L}(\gamma)$$

where the infimum is taken for all arcs $\gamma(\tau)$ joining $x_0 = \gamma(0)$ to $y = \gamma(\bar{\tau})$, and

$$\mathcal{L}(\gamma) = \int_0^{\bar{\tau}} \sqrt{\tau}(|\gamma'|^2 + R(\gamma(\tau), \tau)) \, d\tau \tag{3.13}$$

called the $\mathcal{L}$-length of γ. The ‘reduced distance’ is defined by $\ell(y, \tau) = \frac{L(y,\tau)}{2\sqrt{\tau}}$. The ‘sub-heat kernel’ is $\hat{K}(y, \tau; x_0, 0) = \frac{e^{-\ell(y,\tau)}}{(4\pi\tau)^{\frac{n}{2}}}$. One can similarly define the ‘pseudo heat ball’ $\hat{E}_r$. An important result of [P] asserts that

$$\left(\frac{\partial}{\partial \tau} - \Delta + R\right) \hat{K} \leq 0.$$

Restricted in a sufficiently small parabolic neighborhood of $(x_0, 0)$, we know that $\hat{K}(y, \tau; x_0, 0)$ is smooth, and for small r one can check the compactness of the pseudo-heat ball (cf. [EKNT], also the next section). A corollary of Theorem 3.3 is the following result.

COROLLARY 3.5. *Let $v \geq 0$ be a smooth function on $\widetilde{M}$. Let $\hat{\phi}_r = \hat{K} - r^{-n}$. Then*

$$v(x_0, 0) \geq \int_{\partial \hat{E}_r} v \frac{|\nabla \hat{K}|^2}{\sqrt{|\nabla \hat{K}|^2 + |\hat{K}_t|^2}} \, d\tilde{A} + \frac{1}{r^n} \int_{\hat{E}_r} Rv \, d\mu \, dt$$
$$+ \int_{\hat{E}_r} \hat{\phi}_r \left(\frac{\partial}{\partial t} - \Delta \right) v \, d\mu \, dt. \tag{3.14}$$

A related new monotonicity quantity is

$$\hat{J}_v(r) = \int_{\partial \hat{E}_r} v \frac{|\nabla \hat{K}|^2}{\sqrt{|\nabla \hat{K}|^2 + |\hat{K}_t|^2}} \, d\tilde{A} + \frac{1}{r^n} \int_{\hat{E}_r} Rv \, d\mu \, dt.$$

The virtue of the proof of Corollary 2.8 as well as that of Theorem 3.3 proves the following

COROLLARY 3.6.

$$\frac{d}{dr} \hat{J}_v(r) \leq -\frac{1}{r^{n+1}} \int_{E_r} \left(\frac{\partial}{\partial t} - \Delta \right) v \, d\mu \, dt. \tag{3.15}$$

This, in the case $v = 1$ (or more generally any v with $\left(\frac{\partial}{\partial t} - \Delta\right) v \geq 0$), gives a new monotonicity quanity/formula for the Ricci flow. In [EKNT], the monotonicity was proved for the quantity

$$\hat{I}_v(r) := \frac{1}{r^n} \int_{\hat{E}_r} \left(|\nabla \log \hat{K}|^2 + R\hat{\psi}_r \right) v \, d\mu dt$$

where $\hat{\psi} = \log(\hat{K} r^n)$. $\hat{I}_v(r)$ is related to $\hat{J}_v(r)$ by the relation $r^n \hat{I}_v(r) = n \int_0^r \eta^{n-1} \hat{J}_v(\eta) \, d\eta$. The previous known property that $\hat{I}_v(r)$ is monotone non-increasing does follows from that of $\hat{J}_v(r)$. In fact, we can rewrite

$$\hat{I}_v(r) = \frac{\int_0^r \eta^{n-1} \hat{J}_v(\eta) \, d\eta}{\int_0^r \eta^{n-1} \, d\eta}.$$

Then the result follows from the elementary fact that $\frac{\int_a^r f(\eta) d\eta}{\int_a^r g(\eta) d\eta}$ is monotone non-increasing for any $a \in [0, r]$, provided that $\frac{f(r)}{g(r)}$ is non-increasing. In fact, we have the monotonicity (non-increasing in both r and a) of a more general quantity

$$\hat{I}_v(a, r) := \frac{1}{r^n - a^n} \int_{\hat{E}_r \setminus \hat{E}_a} \left(|\nabla \log \hat{K}|^2 + R\hat{\psi}_r \right) v \, d\mu dt.$$

Concerning the spherical mean value theorem, it seems more natural to consider the fundamental solution of the *backward heat equation* $\frac{\partial}{\partial t} + \Delta$ and a solution to the *forward conjugate heat equation* $\frac{\partial}{\partial t} - \Delta - R$, since if $H(y, \tau; x_0, 0)$ is the fundamental solution to $\frac{\partial}{\partial \tau} - \Delta$ and we define the 'heat ball', 'heat sphere', ϕ_r and ψ_r in the same way as before we can have the following cleaner result.

THEOREM 3.7. *Let v be a smooth function on $\widetilde{M}$. Then*

$$v(x_0, 0) = \int_{\partial E_r} v \frac{|\nabla H|^2}{\sqrt{|\nabla H|^2 + |H_t|^2}} \, d\tilde{A} + \int_{E_r} \phi_r \left(\frac{\partial}{\partial t} - \Delta - R \right) v \, d\mu \, dt. \tag{3.16}$$

This looks nicer if v is a solution to the *forward conjugate heat equation*. We also have the following related result.

Corollary 3.8. *Let*

$$J_v^f(r) = \int_{\partial E_r} v \frac{|\nabla H|^2}{\sqrt{|\nabla H|^2 + |H_t|^2}}\, d\tilde{A}, \quad I_v^f(r) = \frac{1}{r^n}\int_{E_r} |\nabla \log H|^2 v \, d\mu \, dt.$$

Then

$$\frac{d}{dr} J_v^f(r) = -\frac{n}{r^{n+1}} \int_{E_r} \left(\frac{\partial v}{\partial t} - \Delta v - Rv\right) d\mu\, dt \tag{3.17}$$

and

$$\frac{d}{dr} I_v^f(r) = -\frac{n}{r^{n+1}} \int_{E_r} \psi_r \left(\frac{\partial v}{\partial t} - \Delta v - Rv\right) d\mu\, dt. \tag{3.18}$$

In particular, $I_v^f(r)$ and $J_v^f(r)$ are monotone non-increasing if v is a sup-solution to the forward conjugate heat equation.

Since most of the above discussion works for any family of metrics satisfying (3.1) we can also apply it to the mean curvature flow setting. The mean value theorem and related monotonicity formulae, with respect to the 'heat balls', have been studied in [E1]. Here we just outline the result with respect to the 'heat spheres'. First recall that a family $(M_t)_{t\in(\alpha,\beta)}$ of a n-dimensional submanifolds of $\mathbb{R}^{n+k}$ moves by mean curvature if there exist immersions $y_t = y(\cdot, t) : M^n \to \mathbb{R}^{n+k}$ of an n-dimensional manifold M^n with images $M_t = y_t(M^n)$ satisfying the equation

$$\frac{\partial y}{\partial t} = \vec{H}. \tag{3.19}$$

Here $\vec{H}(p,t)$ denotes the mean curvature vector of M_t at $y(p,t)$. (We shall still use H to denote the fundamental solutions.) It was known (see for example [Hu, E1]) that the induced metric $g_{ij}(p,t) = \langle \nabla_i y, \nabla_j y\rangle$ on M satisfying the equation

$$\frac{\partial}{\partial t} g_{ij} = -2\vec{H}\vec{H}_{ij}$$

where $\vec{H}_{ij}(p,t)$ is the second fundamental form of M_t at $y(p,t)$. (In this case $R = |\vec{H}|^2$.) By the virtue of [H1] (see also [E1]) we know that the 'sup-heat kernel'

$$\bar{K}(y,\tau; x_0, 0) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \exp\left(-\frac{|x_0 - y|^2}{4\tau}\right) \tag{3.20}$$

viewed as a function on M via the immersion y_t, satisfies that

$$\left(\frac{\partial}{\partial \tau} - \Delta + |\vec{H}|^2\right)\bar{K} = \bar{K}\left|\vec{H} - \frac{\nabla^\perp \bar{K}}{\bar{K}}\right|^2 \geq 0. \tag{3.21}$$

Here $|x_0 - y|$ denotes the Euclidean distance of $\mathbb{R}^{n+k}$. The proof of Theorem 3.3 gives the following corollary.

COROLLARY 3.9. *Let $v \geq 0$ be a smooth function on $\mathbb{R}^{n+k} \times (\alpha, \beta)$. Assume that M_0 meets $y_0(x_0)$. Let $\bar{\phi}_r = \bar{K} - r^{-n}$. Then*

$$\begin{aligned} v(x_0, 0) = &\int_{\partial \bar{E}_r} v \frac{|\nabla \bar{K}|^2}{\sqrt{|\nabla \bar{K}|^2 + |\bar{K}_t|^2}} \, d\tilde{A} + \frac{1}{r^n} \int_{\bar{E}_r} |\vec{H}|^2 v \, d\mu \, dt \\ &+ \int_{\bar{E}_r} \bar{\phi}_r \left(\frac{\partial}{\partial t} - \Delta \right) v \, d\mu \, dt - \int_{\bar{E}_r} v \bar{K} \left| \vec{H} - \frac{\nabla^{\perp} \bar{K}}{\bar{K}} \right|^2 d\mu dt. \end{aligned} \tag{3.22}$$

Here we denote by the same symbols on the functions on $\mathbb{R}^{n+k} \times (\alpha, \beta)$ and their pull-backs on $M \times (\alpha, \beta)$.

Integrating (3.22) one can recover the mean value formula in [E1], noticing, along with (2.14), that for any function f (regular enough)

$$\frac{n}{r^n} \int_0^r \eta^{n-1} \int_{\bar{E}_\eta} \bar{K} f \, d\mu \, dt \, d\eta = \int_0^r \frac{n}{\eta^{n+1}} \int_{\bar{E}_\eta} f \, d\mu \, dt \, d\eta.$$

If we let

$$\bar{J}_v(r) = \int_{\partial \bar{E}_r} v \frac{|\nabla \bar{K}|^2}{\sqrt{|\nabla \bar{K}|^2 + |\bar{K}_t|^2}} \, d\tilde{A} + \frac{1}{r^n} \int_{\bar{E}_r} |\vec{H}|^2 v \, d\mu \, dt$$

we have that

$$\frac{d}{dr} \bar{J}_v(r) = \frac{n}{r^{n+1}} \left(- \int_{\bar{E}_r} \left(\frac{\partial}{\partial t} - \Delta \right) v \, d\mu \, dt + \int_{\partial \bar{E}_r} \frac{v \bar{K}}{|\tilde{\nabla} \bar{K}|} \left| \vec{H} - \frac{\nabla^{\perp} \bar{K}}{\bar{K}} \right|^2 d\tilde{A} \right). \tag{3.23}$$

Here $\tilde{\nabla} \bar{K} = \langle \nabla \bar{K}, \frac{\partial \bar{K}}{\partial t} \rangle$. The equation (3.23) gives a new monotonicity formula in case that v is a nonnegative sub-solution to the heat equation. Note that it was previously proved in [E1] that

$$\bar{I}(r) = \frac{1}{r^n} \int_{\bar{E}_r} \left(|\nabla \log \bar{K}|^2 + |\vec{H}|^2 \psi_r \right) d\mu \, dt$$

is monotone non-decreasing. As the above Ricci flow case, the monotonicity on $\bar{I}(r)$ follows from the monotonicity of $\bar{J}_v(r)$ for $v \equiv 1$. In fact we also have that

$$\bar{I}(a, r) = \frac{1}{r^n - a^n} \int_{\bar{E}_r \setminus \bar{E}_a} \left(|\nabla \log \bar{K}|^2 + |\vec{H}|^2 \psi_r \right) d\mu \, dt$$

is monotone non-decreasing in both r and a.

4. Local regularity theorems. The monotonicity formula proved in [EKNT] (as well as the one proved in Proposition 5.4 of [N2]), together with Hamilton's compactness theorem [H2] and the arguments in Section 10 of [P], allows us to formulate a ϵ-regularity theorem for Ricci flow Theorem 4.4 (Theorem 4.5). The closest results of this sort are the ones for mean curvature flow [E2, W1]. The result here is influenced by [E2].

We first recall some elementary properties of the so-called 'reduced geometry' (namely the geometry related to the functional $\mathcal{L}(\gamma)$). Let $(M, g(t))$ be a solution to Ricci flow on $M \times [0, T]$. Let (x_0, t_0) be a fixed point with $\frac{T}{2} \leq t_0 \leq T$. We can define the $\mathcal{L}$-length functional for any path originated from (x_0, t_0). Let $\ell^{(x_0,t_0)}(y, \tau)$ be the 'reduced distance' with respect to (x_0, t_0).

Here we mainly follow [N2] (Appendix), where the simpler case for a fixed Riemannian metric was detailed. It seems more natural to consider $\widetilde{M} = M \times [0, t_0]$. Perelman defined the $\mathcal{L}$-exponential map as follows. First one defines the $\mathcal{L}$-geodesic to be the critical point of the above $\mathcal{L}$-length functional in (3.13). The geodesic equation

$$\nabla_X X - \frac{1}{2}\nabla R + \frac{1}{2\tau}X + 2Ric(X, \cdot) = 0,$$

where $X = \frac{d\gamma}{d\tau}$, is derived in [P], which has (regular) singularity at $\tau = 0$. It is desirable to change the variable to $\sigma = 2\sqrt{\tau}$. Now we can define $\bar{g}(\sigma) = g(\tau) = g(\frac{\sigma^2}{4})$. It is easy to see that $\frac{\partial}{\partial \sigma}\bar{g} = \sigma Ric$. The $\mathcal{L}$-length has the form

$$\mathcal{L}(\gamma) = \int_0^{\bar{\sigma}} \left(\left| \frac{d\gamma}{d\sigma} \right|^2 + \frac{\sigma^2}{4} R \right) d\sigma.$$

The $\mathcal{L}$-geodesic has the form

$$\nabla_{\bar{X}} \bar{X} + \sigma Ric(\bar{X}, \cdot) - \frac{\sigma^2}{8} \nabla R = 0$$

where $\bar{X} = \frac{d\gamma}{d\sigma}$. By the theory of ODE we know that for any $v \in T_{x_0}M$ there exists a $\mathcal{L}$-geodesic $\gamma(\sigma)$ such $\frac{d}{d\sigma}\left(\gamma(\sigma)\right)|_{\sigma=0} = v$, where $\sigma = 2\sqrt{\tau}$. Then as in [P] one defines the $\mathcal{L}$-exponential map as

$$\mathcal{L}\exp_v(\bar{\sigma}) := \gamma_v(\bar{\sigma})$$

where $\gamma_v(\sigma)$ is a $\mathcal{L}$-geodesic satisfying that $\lim_{\sigma \to 0} \frac{d}{d\sigma}\left(\gamma_v(\sigma)\right) = v$. The space-time exponential map $\widetilde{\exp} : T_{x_0}M \times [0, 2\sqrt{t_0}] \to \widetilde{M}$ is defined as

$$\widetilde{\exp}(\tilde{v}^a) = (\mathcal{L}\exp_{\frac{v^a}{a}}(a), a),$$

where $\tilde{v}^a = (v^a, a)$. (Here we abuse the notation $\widetilde{M}$ since in terms of σ, $\widetilde{M} = M \times [0, 2\sqrt{t_0}]$.) Denote $\frac{v^a}{a}$ simply by v^1 and $(v^1, 1)$ by $\tilde{v}^1$. Also let $\tilde{\gamma}_{\tilde{v}^a}(\eta) = \widetilde{\exp}(\eta \tilde{v}^a)$. It is easy to see that

$$\tilde{\gamma}_{\tilde{v}^a}(\eta) = \tilde{\gamma}_{\tilde{v}^1}(\eta a). \tag{4.1}$$

This implies the following lemma.

Lemma 4.1.

$$d\widetilde{\exp}|_{(0,0)} = identity.$$

In particular, there exists a neighborhood $U \subset T_{x_0}M \times [0, 2\sqrt{t_0}]$ near $(x_0, 0)$ so that restricted on $U \cap \left(T_{x_0}M \times (0, 2\sqrt{t_0}]\right)$, $\widetilde{\exp}$ is a diffeomorphism.

Proof. The first part follows from taking derivative on (4.1). Tracing the proof of the implicit function theorem (see for example [Ho]) gives the second part. ◻

Based on this fact we can show the following result analogous to the Guass lemma.

COROLLARY 4.2. *The $\mathcal{L}$-geodesic $\gamma_v(\sigma)$ from $(x_0, 0)$ to $(y, \bar{\sigma})$ is $\mathcal{L}$-length minimizing for $(y, \bar{\sigma})$ close to $(x_0, 0)$, with respect to metric $\tilde{g}(x, \sigma)$ on $\widetilde{M}$, where $\tilde{g}(x, \sigma) = \bar{g}(x, \sigma) + d\sigma^2$ (or any other compatible topology).*

Proof. We adapt the parameter τ for this matter. By the above lemma we can define function $\underline{\mathcal{L}}(y, \tau)$ to be the length of the $\mathcal{L}$-geodesic jointing from $(x_0, 0)$ to (y, τ). For (y, τ) close to $(x_0, 0)$ it is a smooth function. It can be shown by simple calculation as in [P] that

$$\nabla \underline{\mathcal{L}}(y, \tau) = 2\sqrt{\tau_1} X \tag{4.2}$$

where ∇ is the spatial gradient with respect to $g(\tau)$, $X = \gamma'(\tau)$ with $\gamma(\tau)$ being the $\mathcal{L}$-geodesic joining $(x_0, 0)$ to (y, τ), and

$$\frac{\partial}{\partial \tau} \underline{\mathcal{L}}(y, \tau) = \sqrt{\tau}(R - |X|^2). \tag{4.3}$$

Now for any curve $(\eta(\tau), \tau)$ joining $(x_0, 0)$ to some (y, τ_1), we have that

$$\begin{aligned}
\underline{\mathcal{L}}(\eta(\tau_1), \tau_1) &= \int_0^{\tau_1} \frac{d\underline{\mathcal{L}}}{d\tau}\, d\tau \\
&= \int_0^{\tau_1} \langle \nabla \underline{\mathcal{L}}, \eta' \rangle + \frac{\partial \underline{\mathcal{L}}}{\partial \tau}\, d\tau \\
&= \int_0^{\tau_1} 2\sqrt{\tau} \langle X, \eta' \rangle + \sqrt{\tau}(R - |X|^2)\, d\tau \\
&\le \int_0^{\tau_1} \tau \left(R + |\eta'|^2\right)\, d\tau \\
&= \mathcal{L}(\eta).
\end{aligned}$$

This shows that the $\mathcal{L}$-geodesic is the $\mathcal{L}$-length minimizing path joining $(x_0, 0)$ with (y, τ_1). ∎

Lemma 4.1 particularly implies that $\mathcal{L}\exp_v(\tau)$ (which is defined as $\gamma_v(\tau)$ with $v = \lim_{\tau \to 0} 2\sqrt{\tau}\gamma'(\tau)$) is a diffeomorphism if $|v| \le 1$, for $\tau \le \epsilon$, provided that ϵ is small enough. Here we identify $T_{x_0}M$ with the Euclidean space using the metric $g(x_0, t_0)$. One can similarly define the cut point (locus) and the first conjugate point (locus) with respect to the $\mathcal{L}$-length functional. For example, $y = \gamma_v(\bar{\sigma})$ (with $\bar{\sigma} < 2\sqrt{t_0}$) is a cut point if the $\mathcal{L}$-geodesic $\gamma_v(\sigma)$ is minimizing up to $\bar{\sigma}$ and fails to be so for any $\sigma > \bar{\sigma}$ (the case that $\bar{\sigma} = 2\sqrt{t_0}$ needs a different definition which shall be addressed later). Similarly one defines the first $\mathcal{L}$-conjugate point. Now we let

$$D(\bar{\sigma}) \subseteq T_{x_0}M$$

to be the collection of vectors v such that $(\mathcal{L}\exp_v(\sigma), \sigma)$ is a $\mathcal{L}$-geodesic along which there is no conjugate for all $\sigma < \bar{\sigma}$. Similarly we let

$$\Sigma(\bar{\sigma}) \subseteq T_{x_0}M$$

to be the collection of vectors v such that $(\mathcal{L}_v(\sigma), \sigma)$ is a minimizing $\mathcal{L}$-geodesic up to $\bar{\sigma}$. One can see easily that $\Sigma(\sigma) \subset D(\sigma)$, and both $D(\sigma)$ and $\Sigma(\sigma)$ decrease (as sets) as σ increases. For any measurable subset $A \subset T_{x_0}M$ we can define

$$D_A(\sigma) = A \cap D(\sigma) \quad \text{and} \quad \Sigma_A(\sigma) = A \cap \Sigma(\sigma).$$

It is exactly the same argument as the classical case to show that for a cut point $(y, \bar{\sigma})$ with $\bar{\sigma} < 2\sqrt{t_0}$, either it is a conjugate point (namely $(y, \bar{\sigma})$ is a critical value of $\widetilde{\exp}$) or there are two minimizing $\mathcal{L}$-geodesics $\gamma_{v_1}(\sigma)$ and $\gamma_{v_2}(\sigma)$ joining x_0 with y at $\bar{\sigma}$. For $\bar{\sigma} = 2\sqrt{t_0}$, we can use this property to define the cut points. Namely $(y, 2\sqrt{t_0})$ is called a cut point if either it is a critical value of $\widetilde{\exp}$ or there exists two minimizing $\mathcal{L}$-geodesics joining $(x_0, 0)$ with $(y, 2\sqrt{t_0})$. For a fixed $v \in T_{x_0}M$ we can define $\bar{\sigma}(v)$ to be the first $\bar{\sigma}$, if it is smaller than $2\sqrt{t_0}$, such that $\gamma_v(\sigma)$ is $\mathcal{L}$-minimizing for all $\sigma \leq \bar{\sigma}$ and no longer so or not defined for $\sigma > \bar{\sigma}$. If the original solution to the Ricci flow $g(t)$ is an ancient solution, then $\widetilde{M} = M \times [0, \infty)$. We define $\bar{\sigma} = \infty$ if $\gamma_v(\sigma)$ is always $\mathcal{L}$-minimizing, in which case we call $\widetilde{\gamma}_{\tilde{v}^1}(\sigma)$, with $\tilde{v}^1 = (v, 1)$, a $\mathcal{L}$-geodesic ray in $\widetilde{M}$. For the case $\widetilde{M} = M \times [0, 2\sqrt{t_0}]$ with finite t_0, define $\bar{\sigma}(v) = 2\sqrt{t_0}+$ (slightly bigger than $2\sqrt{t_0}$) if $\gamma_v(\sigma)$ is minimizing up to $2\sqrt{t_0}$ and it is the unique $\mathcal{L}$-geodesic joining to $(\gamma_v(2\sqrt{t_0}), 2\sqrt{t_0})$ from $(x_0, 0)$. One can show that $\bar{\sigma}(v)$ is a continuous function on $T_{x_0}M$. Define

$$\widetilde{\Sigma} = \{(\sigma v, \sigma) \mid v \in T_{x_0}M, 0 \leq \sigma < \bar{\sigma}(v)\}.$$

It can also be shown that $\widetilde{\Sigma}$ is open (relatively in $T_{x_0}M \times [0, 2\sqrt{t_0}]$) and can be checked that $\widetilde{\exp}|_{\widetilde{\Sigma}}$ is a diffeomorphism. The cut locus (in the tangent half space $T_{x_0}M \times \mathbb{R}_+$, or more precisely $T_{x_0}M \times [0, 2\sqrt{t_0}]$)

$$\mathcal{C} = \{(\bar{\sigma}(v)v, \bar{\sigma}(v)) \in T_{x_0}M \times [0, 2\sqrt{t_0}] \subset T_{x_0}M \times \mathbb{R}_+\}.$$

It is easy to show, by Fubini's theorem, that $\mathcal{C}$ has zero ($n+1$-dimensional) measure. As in the classical case $\widetilde{\exp}(\widetilde{\Sigma} \cup \mathcal{C}) = \widetilde{M}$. Here comparing with the classical Riemannian geometry, $T_{x_0}M$ which can be identified with $T_{x_0}M \times \{1\} \subset T_{x_0}M \times \mathbb{R}_+$, plays the same role as the unit sphere of the tangent space in the Riemannian geometry.

However, there are finer properties on the cut locus. Let $C(\sigma_0)$ be the cut points at σ_0 and $\mathcal{C}(\sigma_0)$ the corresponding vectors in $T_{x_0}M$. Namely $\mathcal{C}(\sigma_0) = \{v \mid \bar{\sigma}(v) = \sigma_0\}$ and $C(\sigma_0) = \mathcal{L}\exp_{\mathcal{C}(\sigma_0)}(\sigma_0)$. Then $\widetilde{\exp}\left(\left(\widetilde{\Sigma} \cap (T_{x_0}M \times \{\sigma\})\right) \cup (\sigma\mathcal{C}(\sigma) \times \{\sigma\})\right) = M \times \{\sigma\}$. More importantly, for any $\sigma < 2\sqrt{t_0}$, $C(\sigma) \times \{\sigma\} \subseteq M \times \{\sigma\}$ has (n-dimensional) measure zero, provided that the metrics $g(x, \sigma)$ are sufficiently smooth on $\widetilde{M}$. By Sard's theorem, we know that the set of the conjugate points in $C(\sigma)$ has zero measure. Hence to prove our claim we only need to show the second type cut points (the ones having more than one minimizing $\mathcal{L}$-geodesics ending at) in $C(\sigma)$ has measure zero in $M \times \{\sigma\}$. These exactly are the points on which $\nabla L(\cdot, \sigma)$ is not well-defined. (Namely $L(\cdot, \sigma)$ fails to be differentiable.) Let $\bar{L} = \sigma L$. It was proved in [Y1] (see also [CLN]) that $\bar{L}(y, \sigma)$ is locally Lipschitz on $M \times [0, \bar{\sigma}]$. In particular, for every $\sigma > 0$, $\ell(y, \sigma)$ is locally Lipschitz as a function of y. The claimed result that $C(\sigma)$ has measure zero follows from Rademacher's theorem on the almost everywhere smoothness of a Lipschitz function. Once we have this fact, if defining that

$$\widetilde{\Sigma}(\sigma) = \{v \mid (\sigma v, \sigma) \in \widetilde{\Sigma}\},$$

for any integrable function $f(y, \sigma)$ we have that

$$\int_M f(y, \sigma)\, d\mu_\sigma = \int_{\widetilde{\Sigma}(\sigma)} f(y, \sigma)\, J(v, \sigma)\, d\mu_{euc}(v)$$

where y is a function of v through the relation $v = (\mathcal{L}\exp_{(\cdot)}(\sigma))^{-1}(y)$, $J(v, \sigma)$ is the Jacobian of $\mathcal{L}\exp_{(\cdot)}(\sigma)$ and $d\mu_{euc}$ is the volume form of $T_{x_0}M = \mathbb{R}^n$ (via the metric of $g(t_0)$). It is clear that $\widetilde{\Sigma}(\sigma) \subset \Sigma(\sigma)$ and $\widetilde{\Sigma}(\sigma)$ decreases as σ increases.

The above discussion can be translated in terms of τ. Let

$$\hat{K}(y,\tau;x_0,t_0)=\frac{1}{(4\pi\tau)^{\frac{n}{2}}}\exp\left(-\ell^{(x_0,t_0)}(y,\tau)\right). \tag{4.4}$$

It was proved in [P] that if y lies in $\mathcal{L}\exp_{\widetilde{\Sigma}(\tau)}(\tau)$,

$$\frac{d}{d\tau}\left(\hat{K}(y,\tau;x_0,t_0)J(v,\tau)\right)\le 0$$

where $y=\mathcal{L}\exp_v(\tau)$, which further implies that the 'reduced volume'

$$\theta(\tau):=\int_M \hat{K}(y,\tau;x_0,t_0)\,d\mu_\tau$$

is monotone non-increasing in τ. Let $\hat{E}_r$ be the 'pseudo heat ball' and

$$\hat{I}^{(x_0,t_0)}(r):=\frac{1}{r^n}\int_{\hat{E}_r}\left(|\nabla\log\hat{K}|^2+R\hat{\psi}_r\right)d\mu dt$$

where $\hat{\psi}_r=\log(\hat{K}r^n)$. The following result, which can be viewed as a localized version of Perelman's above monotonicity of $\theta(\tau)$, was proved in [EKNT]. (This also follows from Corollary 3.5).

THEOREM 4.3. *a) Let $(M,g(t))$ be a solution to the Ricci flow as above. Let $\ell^{(x_0,t_0)}(y,\tau)$, $\hat{K}$ and $\hat{I}^{(x_0,t_0)}(r)$ be defined as above. Then $\frac{d}{dr}\hat{I}^{(x_0,t_0)}(r)\le 0$. If the equality holds for some r, it implies that $(M,g(t))$ is a gradient shrinking Ricci soliton on $\hat{E}_r$.*

b) If (M,g) be a complete Riemannian manifold with $Ric\ge-(n-1)k^2g$. Let $\overline{H}(x,y,\tau)=\overline{H}(\bar{d}(x,y),\tau)$ be the fundamental solution of backward heat equation $\frac{\partial}{\partial\tau}-\Delta$ on the space form $\bar{M}$ with constant curvature $-k^2$, where $\bar{d}(\cdot,\cdot)$ is the distance function of $\bar{M}$. Define $\hat{I}^{(x_0,t_0)}(r)$ similarly using the 'sub heat kernel' $\hat{H}(y,\tau;x_0,0)=\overline{H}(d(x,y),\tau)$. Then $\frac{d}{dr}\hat{I}^{(x_0,t_0)}(r)\le 0$. The equality for some r implies that (M,g) is isometric to $\bar{M}$ on $\hat{E}_r$.

In [EKNT], it was shown that for a solution $g(x,t)$ to Ricci flow defined on $M\times[0,T]$, $\hat{I}^{(x,t)}(r,g)$ is well-defined if $t\ge\frac{3}{4}T$ and r is sufficiently small. We adapt the notation of [P] by denoting the parabolic neighborhood $B_t(x,r)\times[t+\Delta t,t]$ (in the case $\Delta t<0$) of (x,t) by $P(x,t,r,\Delta t)$. Let $\widetilde{U}=U\times[t_1,t_0]$ be a parabolic neighborhood of (x_0,t_0). Let r_0 and ρ_0 are sufficient small so that $P(x_0,t_0,r_0,-r_0^2)\subset\widetilde{U}$ and $\hat{I}^{(x,t)}(\rho)$ is defined for all $\rho\le\rho_0$ and $(x,t)\in P(x_0,t_0,r_0,-r_0^2)$. Motivated by Perelman's pseudo-locality theorem, we prove the following local curvature estimate result for the Ricci flow.

THEOREM 4.4. *There exist positive ϵ_0 and C_0 such that that if $g(t)$ is a solution of Ricci flow on a parabolic neighborhood $\widetilde{U}$ of (x_0,t_0) and if*

$$\hat{I}^{(x,t)}(\rho_0)\ge 1-\epsilon_0, \tag{4.5}$$

for $(x,t)\in P(x_0,t_0,r_0,-r_0^2)$ then

$$|Rm|(y,t)\le\frac{C_0}{r_0^2} \tag{4.6}$$

for any $(y,t) \in P(x_0, t_0, \frac{1}{2}r_0, -\frac{1}{4}r_0^2)$.

Proof. We prove the result by contradiction argument via Hamilton's compactness theorem [H2]. Notice that both the assumptions and the conclusion are scaling invariant. So we can assume $r_0 = 1$ without the loss of the generality. Assume the conclusion is not true. We then have a sequence of counter-examples to the theorem. Namely there exist $(\widetilde{U}_j, g_j(t))$, parabolic neighborhood of (x_0^j, t_0^j) and $P_1^j = P(x_0^j, t_0^j, 1, -1)$ with

$$\hat{I}^{(x,t)}(\rho_0, g_j) \geq 1 - \frac{1}{j}$$

for all $(x,t) \in P_1^j$ (here we write $\hat{I}$ explicitly on its dependence of the metric) but

$$Q_j := \sup_{P^j_{\frac{1}{2}}} |Rm(g_j)|(y,t) \geq j \to \infty$$

where $P^j_{\frac{1}{2}} = P^j(x_0^j, t_0^j, \frac{1}{2}, -\frac{1}{4})$.

By the argument in Section 10 of [P] we can find $(\bar{x}_j, \bar{t}_j) \in P^j_{\frac{3}{4}}$ such that $|Rm|(y,t) \leq 4\bar{Q}_j$, for any $(y,t) \in \bar{P}_j := P(\bar{x}_j, \bar{t}_j, H_j\bar{Q}_j^{-\frac{1}{2}}, -H_j^2\bar{Q}_j^{-1})$, with $H_j \to \infty$ as $j \to \infty$. Here $\bar{Q}_j = |Rm|(\bar{x}_j, \bar{t}_j)$. This process can be done through two steps as in Claim 1 and 2 of Section 10 of [P]. Let $\tilde{g}_j(t) := \bar{Q}_j g(\bar{t}_j + \frac{t}{\bar{Q}_j})$. Now we consider two cases.

Case 1: The injectivity radius of $(\widetilde{U}_j, \tilde{g}_j(0))$ is bounded from below uniformly at $\bar{x}_j$. In this case by Hamilton's compactness theorem [H2] we can conclude that $(\bar{P}_j, \bar{x}_j, \tilde{g}_j)$ converges to $(M_\infty, x_\infty, g_\infty)$, an ancient solution to Ricci flow with $|Rm|(x_\infty, 0) = 1$. On the other hand, for any $\rho \leq \rho_0$, by the monotonicity

$$1 \geq \hat{I}^{(\bar{x}_j, \bar{t}_j)}(\rho, g_j) \geq 1 - \frac{1}{j}$$

which then implies that for any $\rho \leq \rho_0 \bar{Q}_j$,

$$1 \geq \hat{I}^{(\bar{x}_j, 0)}(\rho, \tilde{g}_j) \geq 1 - \frac{1}{j}.$$

This would imply that for any $\rho > 0$,

$$\hat{I}^{(x_\infty, 0)}(\rho, g_\infty) \equiv 1.$$

By Theorem 4.3, it then implies that there exists a function f such that

$$R_{ij} + f_{ij} + \frac{1}{2t} g_{ij} = 0.$$

Namely, (M_∞, g_∞) is a non-flat gradient shrinking soliton, which become singular at $t = 0$. This is a contradiction to $|Rm|(x_\infty, 0) = 1$.

Case 2: The injectivity radius of $(\widetilde{U}_j, \tilde{g}_j(0))$ is not bounded from below uniformly at $\bar{x}_j$. By passing to a subsequence, we can assume that the injectivity radius of $(\widetilde{U}_j, \tilde{g}_j(0))$ at $\bar{x}_j$, which we denote by λ_j, goes to zero. In this case we re-scale the metric $\tilde{g}_j(t)$ further by letting $\tilde{g}_j^*(t) = \frac{1}{\lambda_j^2}\tilde{g}_j(\lambda_j^2 t)$. The new sequence $(\bar{P}_j, \bar{x}_j, \tilde{g}_j^*)$ will have the required injectivity radius lower bound, therefore converges to a flat limit

$(M^*_\infty, x^*_\infty, g^*_\infty(t))$ with the injectivity radius at x^*_∞ being equal to one. On the other hand, by the monotonicity of $\hat{I}^{(\bar{x}_j,0)}(\rho, \tilde{g}^*_j)$ and the similar argument as in *Case 1* we can conclude that $\hat{I}^{(x^*_\infty,0)}(\rho, g^*_\infty) \equiv 1$ for any $\rho > 0$. Therefore, by Theorem 4.3, we conclude that that (M^*_∞, g^*_∞) must be isometric to $\mathbb{R}^n$. This is a contradiction! □

If t_0 is singular time we may define the 'density function' at (x_0, t_0) by

$$\hat{I}^{(x_0,t_0)}(\rho) := \liminf_{(x,t)\to(x_0,t_0)} \hat{I}^{(x,t)}(\rho).$$

Then we may conclude that (x_0, t_0) is a smooth point if $\hat{I}^{(x_0,t_0)}(\rho) \geq 1 - \epsilon_0$ for some $\rho_0 > 0$.

Same result can be formulated for the localized entropy. Recall that in [P], for any (x_0, t_0) one can look at u the fundamental solution to the *backward conjugate heat equation* $\left(\frac{\partial}{\partial\tau} - \Delta + R\right) u(x,\tau) = 0$ centered at (x_0, t_0), one has that

$$\left(\frac{\partial}{\partial\tau} - \Delta + R\right)(-v) = 2\tau |R_{ij} + f_{ij} - \frac{1}{2\tau}|^2 u$$

where $\tau = t_0 - t$, $v = \left[\tau\left(2\Delta f - |\nabla f|^2 + R\right) + f - n\right] u$ and f is defined by $u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}$. In particular, when M is compact one has the entropy formula

$$\frac{d}{dt}\int_M (-v)\, d\mu_t = -2\int_M \tau |R_{ij} + f_{ij} - \frac{1}{2\tau}|^2 u\, d\mu_t.$$

In [N2] we observed that the above entropy formula can be localized. In deed, let $\bar{L}^{(x_1,t_1)}(y,t) = 4(t_1 - t)\ell^{(x_1,t_1)}(y, t_1 - t)$ and let

$$\psi^{(x_1,t_1)}_{t_2,\rho}(x,t) := \left(1 - \frac{\bar{L}^{(x_1,t_1)}(x,t) + 2n(t - t_2)}{\rho^2}\right)_+.$$

It is easy to see that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\psi^{(x_1,t_1)}_{t_2,\rho}(x,t) \leq 0.$$

The Proposition 5.5 of [N2] asserts that

$$\frac{d}{dt}\left(\int_M -v\psi^{(x_1,t_1)}_{t_2,\rho}\, d\mu_t\right) \leq -2\int_M \tau\left(|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}|^2\right) u\psi^{(x_1,t_1)}_{t_2,\rho}\, d\mu_t. \tag{4.7}$$

One can then define

$$\nu^{(x_0,t_0)}(\rho,\tau,g) = \int_M -v(y,\tau)\psi^{(x_0,t_0)}_{t_0,\rho}(y,\tau)\, d\mu_\tau(y).$$

The similar argument as in the Section 10 of [P] (or the above proof of Theorem 4.4) shows the following result. First we assume that $g(t)$ is a solution of Ricci flow on $U \times [0, T)$. Fix a space-time point (x_0, t_0) with $t_0 < T$. Let $\rho_0 > 0$ τ_0 and r_0 be positive constants which are small enough such that $\nu^{(x,t)}(\rho,\tau,g)$, with $\rho \leq \rho_0$ and $\tau \leq \tau_0$, is defined for all (x,t) in $P(x_0, t_0, r_0, -r_0^2)$.

THEOREM 4.5. *There exist positive constants ϵ_0 and C_0 such that for $(U, g(t))$ a solution to the Ricci flow, (x_0, t_0) and $P(x_0, t_0, r_0, -r_0^2)$ as above, if*

$$\nu^{(x,t)}(\rho_0, \tau_0, g) \leq \epsilon_0, \tag{4.8}$$

for $(x,t) \in P(x_0, t_0, r_0, -r_0^2)$ *then*

$$|Rm|(y,t) \le \frac{C_0}{r_0^2} \tag{4.9}$$

for any $(y,t) \in P(x_0, t_0, \frac{1}{2}r_0, -\frac{1}{4}r_0^2)$.

In the proof one needs to replace Theorem 4.3 by a similar rigidity type result (Corollary 1.3 of [N1] on page 371), formulated in terms of the entropy, to obtain the contradiction for the collapsing case.

In the view of the works [E2, W1], one can think both $1 - \hat{I}^{(x,t)}(\rho, g)$ and $\nu^{(x,t)}(\rho, \tau, g)$ as local 'densities' for Ricci flow.

5. Further discussions. First we show briefly that if $\hat{J}(r_1) = \hat{J}(r_2)$ for some $r_2 > r_1$, it implies that $g(t)$ is a gradient shrinking soliton in $\hat{E}_{r_2} \setminus \hat{E}_{r_1}$. By the proof of Theorem 3.3, the equality implies that on $\hat{E}_{r_2} \setminus \hat{E}_{r_1}$, $(\frac{\partial}{\partial \tau} - \Delta + R)\hat{K} = 0$, which is equivalent to

$$\ell_\tau - \Delta \ell + |\nabla \ell|^2 - R + \frac{n}{2\tau} = 0. \tag{5.1}$$

By Proposition 9.1 of [P] we have that

$$\left(\frac{\partial}{\partial \tau} - \Delta + R\right) v = -2\tau \left| R_{ij} + \ell_{ij} - \frac{1}{2\tau} g_{ij} \right|^2 \hat{K} \tag{5.2}$$

where

$$v = \left(\tau(2\Delta \ell - |\nabla \ell|^2 + R) + \ell - n\right) \hat{K}.$$

On the other hand, (7.5) and (7.6) of [P] implies that ℓ satisfies the first order PDE:

$$-2\ell_\tau - |\nabla \ell|^2 + R - \frac{1}{\tau}\ell = 0. \tag{5.3}$$

This together with (5.1) implies that $v = 0$ on $\hat{E}_{r_2} \setminus \hat{E}_{r_1}$. The result now follows from (5.2).

The space-time divergence theorem from Section 3, Lemma 3.2, allows us to write monotonic quantities $J_v(r)$, $I_v(r)$ ($\hat{J}_v(r)$, $\hat{I}_v(r)$ as well), defined in Section 3 and 4, in a nicer form when $v = 1$.

Applying the space time divergence theorem to $\tilde{X} = \frac{\partial}{\partial t}$ we have that

$$\int_{E_r} R \, d\mu \, dt = -\int_{\partial E_r} \langle \frac{\partial}{\partial t}, \tilde{\nu} \rangle \, d\tilde{A} = \int_{\partial E_r} \frac{H_t}{\sqrt{|\nabla H|^2 + |H_t|^2}} \, d\tilde{A}. \tag{5.4}$$

Therefore we have that ($J(r) := J_1(r)$)

$$J(r) = \int_{\partial E_r} \frac{|\nabla H|^2 - H H_\tau}{\sqrt{|\nabla H|^2 + |H_\tau|^2}} \, d\tilde{A}. \tag{5.5}$$

Now the quantity is expressed solely in terms of the surface integral on ∂E_r. For the Ricci flow case we then have that

$$\hat{J}(r) = \int_{\partial \hat{E}_r} \frac{|\nabla \hat{K}|^2 - \hat{K}\hat{K}_\tau}{\sqrt{|\nabla \hat{K}|^2 + |\hat{K}_\tau|^2}} \, d\tilde{A} \tag{5.6}$$

is monotone non-increasing, where $\hat{K}$ is the 'sub-heat kernel' defined in (4.4). For the mean curvature flow case we have that

$$\bar{J}(r) = \int_{\partial \bar{E}_r} \frac{|\nabla \bar{K}|^2 - \bar{K}\bar{K}_\tau}{\sqrt{|\nabla \bar{K}|^2 + |\bar{K}_\tau|^2}} \, d\tilde{A} \tag{5.7}$$

is monotone non-decreasing, where $\bar{K}$ is 'sup-heat kernel' defined by (3.20). Notice that the numerator $|\nabla H|^2 - HH_\tau$, after dividing H^2, is the expression of Li-Yau in their celebrated gradient estimate [LY].

Applying Lemma 3.2 to the vector field $\tilde{X} = \psi_r \frac{\partial}{\partial t}$ we have that

$$\int_{E_r} R\psi_r \, d\mu \, dt = \int_{\partial E_r} \frac{\partial \psi_r}{\partial t} \, d\mu \, dt = \int_{\partial E_r} \frac{\partial}{\partial t} \log H \, d\mu \, dt. \tag{5.8}$$

Here we have used that $\psi_r = 0$ on ∂E_r and

$$\lim_{s \to 0} \int_{P_s^2} \psi_r \, d\mu_s = 0.$$

This was also observed in [EKNT]. Hence

$$\begin{aligned} I(r) &= \frac{1}{r^n} \int_{E_r} \left(|\nabla \log H|^2 + R\psi_r \right) \, d\mu \, dt \\ &= \frac{1}{r^n} \int_{E_r} \left(|\nabla \log H|^2 - (\log H)_\tau \right) \, d\mu \, dt. \end{aligned} \tag{5.9}$$

In particular, for the Ricci flow we have the non-increasing monotonicity of

$$\hat{I}(r) = \frac{1}{r^n} \int_{\hat{E}_r} \left(|\nabla \log \hat{K}|^2 - (\log \hat{K})_\tau \right) \, d\mu \, dt \tag{5.10}$$

and for the mean curvature flow there exists the monotone non-decreasing

$$\bar{I}(r) = \frac{1}{r^n} \int_{\bar{E}_r} \left(|\nabla \log \bar{K}|^2 - (\log \bar{K})_\tau \right) \, d\mu \, dt, \tag{5.11}$$

which is just a different appearance of Ecker's quantity [E1]. Remarkably, the integrands in the above monotonic quantities are again the Li-Yau's expression. Notice that (5.9)-(5.11) also follow from (5.5) and its cousins by the integrations. In the case of Ricci flow, it was shown by Perelman in [P] that

$$|\nabla \log \hat{K}|^2 - (\log \hat{K})_\tau = |\nabla \ell|^2 + \ell_\tau + \frac{n}{2\tau} = \frac{n}{2\tau} - \frac{1}{2\tau^{\frac{3}{2}}} \mathcal{K}(y, \tau) \tag{5.12}$$

where $\mathcal{K}(y, \bar{\tau}) = \int_0^{\bar{\tau}} \tau^{\frac{3}{2}} H(\frac{d\gamma}{d\tau}) \, d\tau$ with γ being the minimizing $\mathcal{L}$-geodesic joining to $(y, \bar{\tau})$ (assuming that $(y, \bar{\tau})$ lies out of cut locus), $H(X) = -\frac{\partial R}{\partial \tau} - \frac{R}{\tau} - 2\langle X, \nabla R \rangle + 2Ric(X, X)$. In the case of the mean curvature flow (see [E1]) we have that

$$|\nabla \log \bar{K}|^2 - (\log \bar{K})_\tau = \frac{n}{2\tau} + \langle \vec{H}, \nabla^\perp \log \bar{K} \rangle - |\nabla^\perp \log \bar{K}|^2. \tag{5.13}$$

When $g(t)$ become singular or degenerate at $(x_0, 0)$, special cares are needed in justifying the above identities. Interesting cases are the gradient shrinking solitons for the Ricci flow/homothetically shrinking solutions for mean curvature flow respectively, which have singularity at $\tau = 0$. For the case of the mean curvature flow, one can work in terms of the image of $y_t(\cdot)$. A homothetically shrinking solution satisfies that $\vec{H}(y, \tau) = -\frac{(y-x_0)^\perp}{2\tau}$. As in [E1] denote the space time track of (M_t) by

$$\mathcal{M} = \cup_{t\in(\alpha,\beta)} M_t \times \{t\} \subset \mathbb{R}^{n+k} \times \mathbb{R}.$$

It can be checked (cf. [E1]) that on a shrinking soliton

$$\left(\frac{\partial}{\partial \tau} - \Delta + |\vec{H}|^2\right) \bar{K}(y, \tau; x_0, 0) = 0.$$

For homothetically shrinking solutions, it was shown in [E1] that

$$\bar{I}(r) = \int_{M_1} \bar{K}(y, 1; x_0, 0)\, d\mu_1.$$

Apply the space-time divergence theorem to $\widetilde{X} = \nabla \bar{K} + \bar{K}\frac{\partial}{\partial t}$. We can check that $\lim_{s\to 0}\int_{P_2^s\cap\mathcal{M}} d\mu_s = 0$, if the shrinking soliton is properly embedded near$(x_0, 0)$, by Proposition 3.25 of [E2]. By the virtue of Theorem 3.3 we have that

$$\bar{J}(r) = \lim_{s\to 0}\int_{P_2^s\cap\mathcal{M}} \bar{K}(y, s; x_0, 0)\, d\mu_s.$$

Using the scaling invariance of $\mathcal{M}$ one can show that

$$\lim_{s\to 0}\int_{P_2^s\cap\mathcal{M}} \bar{K}(y, s; x_0, 0)\, d\mu_s = \int_{M_1} \bar{K}(y, 1; x_0, 0)\, d\mu_1.$$

Here, with a little abuse of the notation, we denote $M_1 = \mathcal{M} \cap \{\tau = 1\}$. Summarizing we have that both $\bar{I}(r)$ and $\bar{J}(r)$ are equal to the the so-called Gaussian density $\Theta(\mathcal{M}, x_0, 0) = \int_{M_t} \bar{K}\, d\mu_t$.

The Ricci flow case is similar. Assume that $(M, g(t))$ is a gradient shrinking soliton (see for example [CLN] for a precise definition, and [FIK] for the new examples of noncompact shrinkers). In particular, there exists a smooth function $f(x, \tau)$ $(\tau = -t)$ so that

$$R_{ij} + f_{ij} - \frac{1}{2\tau} g_{ij} = 0.$$

Moreover, the metric $g(\tau) = \tau\varphi^*(\tau)g(1)$, where $\varphi(\tau)$ is a one parameter family of diffeomorphisms, generated by $-\frac{1}{\tau}\nabla_{g(1)} f(x, 1)$. We further assume that there exists a 'attracting' sub-manifold S such that $(\nabla f)(x_0, \tau) = 0$ (which is equivalent to that $(\nabla f)(x_0, 1) = 0$), for every $x_0 \in S$, f is constant on S and the integral curve $\sigma(\tau)$ of the vector field ∇f flows into S as $\tau \to 0$. (All known examples of gradient shrinking Ricci soliton seem to satisfy the above assumptions). In this case for any $x_0 \in S$ we can define the *reduce distance* as before. However, we require that the competing curves $\gamma(\tau)$ satisfies that $\lim_{\tau\to 0}\gamma'(\tau)\sqrt{\tau}$ exists (this is the case if $(x_0, 0)$ is a regular time and $\gamma'(\tau)$ is a $\mathcal{L}$-geodesic. We may define $\ell(y, \tau) = \inf_{x_0\in S} \ell^{(x_0,0)}(y, \tau)$. Under this assumptions, after suitable normalization on f (by adding a constant), we can show that $f(y, \tau) = \ell(y, \tau)$ (see for example, [CHI] and [CLN]). It can also be easily checked that $\hat{K}$ is a solution to the *conjugate heat equation*. Then the very similar argument as above shows that

$$\hat{J}(r) = \lim_{s\to 0} \int_{P_2^s} \hat{K}(y, s; x_0, 0)\, d\mu_s$$

which then implies that $\hat{J}(r) = \int_{M_1} \hat{K}\, d\mu_1$, a constant independent of r. Using the relation that $\hat{I}(r) = \frac{\int_0^r \eta^{n-1} \hat{J}(\eta)\, d\eta}{\int_0^r \eta^{n-1}\, d\eta}$, we have that $\hat{I}(r)$ also equals to the 'reduced volume' of Perelman $\theta^{(x_0,0)}(g,\tau) = \int_M \hat{K}\, d\mu_\tau$ (cf. [EKNT]).

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