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CHERN'S WORK IN GEOMETRY*

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It's fair to say that E. Cartan is the grandfather of differential geometry and S.S. Chern is the father of modern differential geometry.

Together they have created a beautiful and rich subject that have reached out to every branch of mathematics and physics.

Right before he died, Chern said that he is going to see the great Greek geometers. There is no doubt that he had reached the same status as these great geometers.

Now, we would like to review the major events in the glorious history of geometry.

Pythagoras (524-480 B.C.) found the Pythagoras theorem for triangles.

Euclid (325-265 B.C.) formulated axioms for Euclidean geometry.

Archimedes (287-212 B.C.) initiated the use of infinite processes and launched the study of conics.

Descartes (1596-1650) introduced coordinates, this is the birth of analytic geometry where algebra and geometry are merged.

G. Desargues (1591-1661) invented projective geometry.

Fermat (1601-1665) shaped variational principle when studying optics.

Newton (1642-1727) and Leibniz(1646-1716) independently created the great calculus.

Euler (1707-1783) studied combinational geometry and developed the method of calculus of variation.

Gauss (1777-1885) pioneered the study of intrinsic geometry.

Riemann (1826-1866) announced Riemannian geometry in 1854 in his Habitationschrift.

Sophus Lie (1842-1899) created the theory of transformation groups and discovered contact geometry.

F. Klein (1849-1925) announced the Erlangen programm which defines geometry as the study of a space with a group of transformation in 1872.

The group of projective collineations is the most encompassing group and the resulting geometry is projective geometry. Contributors include: J.V. Poncelet (1788-1867), A.F. Möbius (1790-1868), M. Chasles (1793-1880), J. Stenier (1796-1863).

There are many other geometries, such as affine geometry and conformal geometry whose corresponding groups are respectively the affine group and the conformal group.

A. Weil wrote in his preface to *Selected Papers of S.S. Chern*:

"The psychological aspects of true geometric intuition will perhaps never be cleared up. ... Whatever the truth of the matter, mathematics in our century would not have made such impressive progress without the geometric sense of Elie Cartan, Heinz Hopf, Chern and a very few more. It seems safe to predict that such men will always be needed if mathematics is to go on as before."

*This is the speech given on the Harvard Memorial Conference for S.S. Chern.

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Birth of modern differential geometry. Cartan completed the foundational works since Gauss-Riemann. Combining his development of Lie group theory and invariant theory of differential system, he introduced modern gauge theory.

Cartan defined generalized spaces which includes both Klein's homogeneous spaces and Riemann's local geometry. In modern terms, it is called "a connection in a fiber bundle". It generalizes the Levi-Civita parallelism.

In general, we have a fiber bundle $\pi : E \to M$, whose fibers $\pi^{-1}(x)$, $x \in M$, are homogeneous spaces acted on by a Lie group G. A connection is an infinitesimal transport of the fibers compatible with the group action by G.

While Grassmann introduced exterior forms, Cartan introduced the operation of exterior differentiation. His theory of Pfaffian system and theory of prolongation created invariants for solving equivalence problem in geometry.

Cartan's view of building invariants by moving frame had deep influence on Chern.

H. Hopf initiated the study of differential topology, e.g. vector fields on manifold. His student Stiefel (1936) generalized Hopf's theorem to obtain Stiefel-Whitney Class.

H. Hopf did the hypersurface case of Gauss-Bonnet in 1925 in his thesis. In 1932, Hopf emphasized that the integrand can be written as a polynomial of components of Riemann curvature tensor.

These works inspired the work of Allendoerfer-Weil-Chern on Gauss-Bonnet formula.

Chern: Father of global intrinsic geometry. Chern: "Riemannian geometry and its generalization in differential geometry are local in character. It seems a mystery to me that we do need a whole space to piece the neighborhood together. This is achieved by topology."

Both Cartan and Chern saw the importance of fiber bundle on problems in differential geometry.

It is certainly true that global differential geometry was studied by other great mathematics: Cohn-Vossen, Minkowski, Hilbert, Weyl ...

But most of their works are focus on global surfaces in three dimensional Euclidean space.

Chern was the first mathematician who succeeded to build a bridge between intrinsic geometry and algebraic topology. (Besides his work on symmetric space, Cartan's work is more local in nature.)

Chern's education at Tsing Hua University. Chern spent his undergraduate days at Tsing Hua University. He studied: Coolidge's books on non-Euclidean geometry and geometry of the circle and sphere; Salmon's book on Conic sections and analytic geometry of three dimensions; Castelnuovo's book about Analytic and projective geometry; Otto Stande's book *Fadenkonstruktionen*.

His teacher Professor Dan Sun studied projective differential geometry, this subject was founded by E.J. Wilczynski in 1901 and followed by G. Fubini, E. Cěch.

Chern's master thesis was on projective line geometry which studied hypersurface in the space of all lines in three dimensional projective space. He studied line congruences: two dimensional submanifold of lines and their oscillation by quadratic line complex.

Chern's education with Blaschke. In 1932, Blaschke visited Peking. He lectured on "topological questions in differential geometry". He discussed pseudogroup of diffeomorphisim and their local invariants.

Chern started to think about global differential geometry and realized the importance of algebraic topology. He read Veblen's book "Analysis Situs" which was published in 1922.

In 1934, he studied in Hamburg under Blaschke. Artin, Hecke and Kähler were all there. Blaschke worked on web geometry and integral geometry at that time. Chern started to read Seifert-Thrilfall (1934) and Alexandroff-Hopf (1935).

Chern's education with Kähler and Cartan. In Hamburg, Kähler lectured on Cartan-Kähler theory – "*Einführung in die Theorie der systeme von Differentialeichungen*". Chern was a faithful student.

In 1936 to 1937, Chern went to Paries to study with E. Cartan on moving frames and the method of equivalence. He also explored more detailed research on Cartan-Kähler theory. He spent ten months in Paries and met Cartan every two weeks.

Chern went back to China in the summer of 1937. In the next few years, he spent full time to study Cartan's work. He said that Cartan wrote more than six thousand pages in his whole life and he read at least seventy to eighty percent of these works. He read some of the works over and over again. During the War, it is much easier to spend full time to read and think in isolation.

Chern's comment on Cartan:

"Undoubtedly one of the greatest mathematician of this century, his career was characterized by a rare harmony of genius and modesty.

In 1940, I was struggling in learning Elie Cartan. I realized the central role to be played by the notion of a connection and wrote several papers associating a connection to a given geometrical structures."

Chern was almost the only geometer who can master Cartan's work so well. Even the great master Weyl found Cartan difficult to read.

Weyl: "Cartan is undoubtedly the greatest living master in differential geometry...I must admit that I found the book, like most of Cartan's papers, hard reading..."

Equivalence problem. Most of the works of Chern are related to the problem of equivalence.

In 1869, E. Christoffel and R. Lipschitz solved the fundamental problem in Riemannian geometry. It was called the form problem:

> To decided when two ds^2's differ by a change of coordinate, Christoffel introduced the covariant differentiation which is now called Levi-Civita connection.

Cartan's equivalence problem:

> Given two sets of linear differential forms θ^i, θ^{*j} in the coordinates x^k, x^{*l} respectively, $1 \leq i, j, k, l \leq n$, both linearly independent.
> Given a Lie group $G \in GL(n, \mathbb{R})$, find the conditions that there are functions
>
> $$x^{*i} = x^{*i}(x^1, x^2, ..., x^n)$$
>
> such that θ^{*j}, after the substitution of these functions, differ from θ^i by a transformation of G.

The problem generally involves local invariants, and Cartan gave a procedure to generate such invariants.

Chern (1932-1943). In this period, Chern studied web geometry, projective line geometry, invariants of contact of pairs of submanifolds in projective space, transformations of surfaces which is related to Bäcklund transform in soliton theory. Chern continued this line of research in the seventies with Griffiths and Terng.

Projective differential geometry:

> Find a complete system of local invariants of a submanifold under the projective group and interpret them geometrically through osculation by simple geometrical figures.

Another typical problem in projective differential geometry is to study the geometry of path structure by normal projective connections. For example, Tresse (a student of Sophis Lie) studied pathes defined by integral curves of

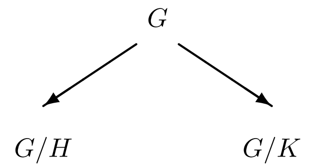
$$y'' = F(x, y, y')$$

by normal projective connections in space (x, y, y').

Chern generalized this to n-dimension. Given $2(n-1)$ dimensional family of curves satisfying a differential system such that through any point and tangent to any direction at the point, there is exactly one such curve. Chern associated such geometry with a normal projective connection and then extended such construction to families of submanifolds.

Between 1940 to 1942, Chern started to generalize the theory of integral geometry as was developed by Crofton, Blaschke. He observed that such theory can be best understood in terms of two homogeneous spaces with the same Lie group G.

Hence there are two subgroups H and K



Two cosets aH and bK is incident to each other if they intersect in G.

In this way, he was able to generalized many important formula of Crofton. In 1952, He generalized the kinematic formula of Poincare, Santalo and Blaschke.

Weil: "It lifted the whole subject at one stroke to a higher plane than where Blaschke's school had lift it. I was impressed by the unusual talent and depth of understanding that shone through it."

Chern's visit of Princeton (1943-1945). In 1943, Chern went from Kunming to Princeton, invited by Veblen and Weyl. Weyl was his hero. Fiber bundle theory was evolving starting from the works of Cartan and Whitney. Stiefel-Whitney classes were only defined mod two. Weil just published his work on Gauss-Bonnet formula and told Chern the works of Todd and Eger on "canonical classes" in algebraic geometry. These works were done in the spirit of Italian geometry and rested on some unproved assumptions.

Chern considered his best work to be his intrinsic proof of Gauss-Bonnet formula.

F. Gauss was the first one to derive the formula for geodesic triangle (1827): *Disquistiones Circa superficies Curvas*. He considered surface in $\mathbb{R}^3$ and used the Gauss map.

O. Bonnet (1948) generalized the formula to any simply connected domain bounded by an arbitrary curve: *Mémoire sur la théorie générale des surfaces*, J. de l'Ecole Poly. Tome 19, Cahier **32**(1848)1-146.

W. Dyck generalized it to surfaces with arbitrary genus: *Beiträge zur analysis situs*, Math Annalan **32**(1888)457-512.

H. Hopf generalized the formula to codimensional one hypersurface in $\mathbb{R}^n$ in his thesis.

C.B. Allendoerfer (1940) and W. Fenchel studied closed orientable Riemannian manifold which can be embedded in Euclidean space.

C.B. Allendoerfer and A. Weil (1843) extended the formula to closed Riemannian polyhedron and hence to general closed Riemannian manifold: *The Gauss-Bonnet theorem for Riemannian polyhedra*, Trans. Amer. Math. Soc., **53**(1943)101-129.

The proof rested on the embedding of the manifold in Euclidean space.

Weil: "Following the footsteps of H. Weyl and other writers, the latter proof, resting on the consideration of 'tubes', did depend (although this was not apparent at that time) on the construction of an sphere-bundle, but of a non-intrinsic one, viz. the transversal bundle for a given immersion."

Weil: "Chern's proof operated explicitly for the first time with an intrinsic bundle, the bundle of tangent vectors of length one, thus clarifying the whole subject once and for all."

One century ago, Gauss established the concept of intrinsic geometry. Chern's proof of Gauss-Bonnet opened up a new horizon. Global topology is linked with intrinsic geometry through the concept of fiber bundle and transgression on the intrinsic tangent sphere bundle. A new era of global intrinsic geometry was created.

In terms of moving frame, the structure equation for surface is

$$\begin{aligned} d\omega_1 &= \omega_{12} \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_{12} \\ d\omega_{12} &= -K\omega_1 \wedge \omega_2. \end{aligned}$$

Here ω_{12} is the connection form and K is the Gauss curvature.

If the unit vector e_1 is given by a globally defined vector field V by

$$e_1 = \frac{V}{\parallel V \parallel}$$

at points where $V \neq 0$. Then we can apply Stoke's formula to obtain

$$-\int_M K\omega_1 \wedge \omega_2 = \sum_i \int_{\partial B(x_i)} \omega_{12}$$

where $B(x_i)$ is a small disk around x_i where $V(x_i) = 0$. Moreover $\int_{\partial B(x_i)} \omega_{12}$ can be computed from the index of the vector field V at x_i.

According to the theorem of H. Hopf, summation of indices of a vector field is the Euler number. In this way, integral of curvature expresses Euler number.

This is the proof of Chern for the Gauss-Bonnet formula. It is new even in two dimension. In higher dimension, the bundle is the unit tangent sphere bundle.

Curvature form Ω_{ij} is a skew symmetric matrix. Its Pfaffian is

$$\mathrm{Pf} = \sum \epsilon_{i_1 \dots i_{2n}} \Omega_{i_1 i_2} \wedge \dots \wedge \Omega_{i_{2n-1} i_{2n}}.$$

The Gauss-Bonnet formula is

$$(-1)^n \frac{1}{2^{2n}\pi^n n!} \int \mathrm{Pf} = \chi(M).$$

Chern has to find a form Π on the unit sphere bundle so that $d\Pi$ is the lift of Pf. This is the birth of transgression.

Chern Class. Chern: "My introduction to characteristic class was through the Gauss-Bonnet formula, known to every student of surfaces theory. Long before 1943, when I gave an intrinsic proof of the n-dimensional Gauss-Bonnet formula, I know, by using orthonormal frames in surface theory, that the classical Gauss-Bonnet is but a global consequence of the Gauss formula which expresses the 'theorime egregium'. The algebraic aspect of this proof is the first instance of a construction later known as transgression, which is destined to play a fundamental role in the homology theory of fiber bundle, and in other problems."

Cartan's work on frame bundles and de Rham's theorem have been always behind Chern's thinking.

Fiber bundle stands at the very heart of modern mathematics. It's a central unifying notion for many important objects in mathematics and physics.

E. Stiefel (1936) and Whitney (1937) introduced Stiefel-Whitney Classes. It is only defined mod two.

J. Feldbau (1939), C. Ehresmam (1941, 1942, 1943), Chern (1944, 1945), N. Steenord (1944) made a systematical study of topology of fiber bundles.

Pontrjagin (1942) introduced Pontrjagin Class. He also associated topological invariants to curvature of Riemannian manifolds in 1944. The paper was published on the Doklady journal. It depends on embedding of manifolds and he did not know that these invariants are Pontrjagin Classes.

In the proof of Gauss-Bonnet formula, we can look for k vector fields $s_1, ..., s_k$ in general position. The points where they are linearly independent form a $(k-1)$ dimensional cycle independent of the choice of s_i. This was done by E. Stiefel in his thesis (1936).

H. Whitney (1937) considered sections for more general sphere bundle and looked at it from the point of view of obstruction theory.

Whitney noticed the importance of the universal bundle over the Grassmannian $G(q, n)$ of q planes in $\mathbb{R}^N$. He (1937) showed that any rank q bundle over the manifold can be induced by a map $f : M \to G(q, N)$ from this bundle.

When N is large, Pontrjagin (1942) and Steenrod (1944) observed that the map f is defined up to homotopy. The characteristic classes of the bundle is given by

$$f^* H^*(Gr(q, N)) \subset H^*(M).$$

The cohomology of $H^*(Gr(q, N))$ was studied by C. Ehresmann (1936) and they are generated by Schubert Cells.

Chern: "It was a trivial observation and a stroke of luck, when I saw in 1944 that the situation for complex vector bundles is far simpler, because most of the classical complex spaces, such as the classical complex Grassmann manifolds, the complex Stiefel manifolds, etc. have no torsion."

For a complex vector bundle E, the Chern Classes $c_i(E) \in H^{2i}(M, \mathbb{Z})$.

Chern defined it in three different ways: by obstruction theory, by Schubert Cells and by curvature forms of a connection on the bundle. And he proved their equivalence.

The fundamental paper of Chern (1946). In the paper *Characteristic classes of Hermitian manifolds*, Ann. of Math., (2) **47**(1946) 85-121, Chern also laid the foundation of Hermitian geometry on complex manifolds. The concept of Hermitian connections was first introduced by him, for example.

If Ω is the curvature form of the vector bundle, one defines

$$\det\left(I+\frac{\sqrt{-1}}{2\pi}\Omega\right)=1+c_1(\Omega)+\cdots+c_q(\Omega).$$

The advantage of defining Chern Classes by differential forms have tremendous importance in geometry and in modern physics.

An example is the concept of transgression created by Chern.

Let φ be the connection form defined on the frame bundle associated to the vector bundle. Then the curvature form is

$$\Omega=d\varphi-\varphi\wedge\varphi.$$

Hence

$$c_1(\Omega)=\frac{\sqrt{-1}}{2\pi}\mathrm{Tr}\Omega=\frac{\sqrt{-1}}{2\pi}d(\mathrm{Tr}\varphi).$$

Similarly,

$$\mathrm{Tr}(\Omega\wedge\Omega)=d(\mathrm{Tr}(\varphi\wedge\varphi)+\frac{1}{3}\mathrm{Tr}(\varphi\wedge\varphi\wedge\varphi))=d(CS(\varphi)).$$

This term $CS(\varphi)$ is called Chern-Simons form and has played a fundamental role in three dimensional manifolds, in anomoly cancellation in string theory and in solid state physics.

The idea of doing transgression on form level also gives rise to secondary operation on homology, e.g. Massey product, which appeared in K.T. Chen's work on iterated integral.

When the manifold is a complex manifold, we can write

$$d=\partial+\bar\partial.$$

In a fundamental paper, Bott-Chern (1965) found that there is canonically constructed $(i-1,i-1)$-form $\tilde{T}c_i(\Omega)$ so that $c_i(\Omega)=\bar\partial\partial(\tilde{T}(c_i(\Omega)))$.

Chern made use of this theorem to generalize Nevanlinna theory of value distribution to holomorphic maps between higher dimensional complex manifolds.

The form $\tilde{T}c_i(\Omega)$ plays a fundamental role in later developed Arakelov theory.

Donaldson used the case of $i=2$ to prove the Donaldson-Uhlenbeck-Yau theorem for existence of Hermitian Yang-Mills connection on algebraic surfaces.

For $i=1$,

$$c_1=\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log\det(h_{i\bar{j}})$$

where $h_{i\bar{j}}$ is the Hermitian metric.

The right hand side is the Ricci tensor of the metric. The simplicity of the first Chern form motivates the Calabi conjecture.

The simplicity and beauty of geometry over complex number can not be exaggerated.

Chern (Chicago days). After the fundamental paper on Chern Class in 1946, he explored more detail on the multiplicative structure of the characteristic classes.

In 1951, he had a paper with E. Spanier on the Gysin sequence on fiber bundle. They proved the Thom isomorphism independently of Thom.

In the paper *On the characteristic classes of complex sphere bundle and algebraic varieties*, Amer. J. Math., **75**(1953) 565-597, Chern showed that by considering an associated bundle with the flag manifold as fibers the characteristic classes can be defined in terms of line bundles. As a consequence the dual homology class of a characteristic class of an algebraic manifold contains a representative of algebraic cycle.

This paper provides the splitting principle in K-theory and coupled with Thom isomorphism allows one to give the definition of Chern classes on the associated bundle as was done by Grothendick later.

Hodge has considered the problem of representing homology classes by algebraic cycles. He considered the above theorem of Chern and was only able to prove it when the manifold is complete intersection of nonsingular hypersufaces in a projective space.

Chern's theorem is the first and the only known general statement for the "Hodge conjecture". It also gives the first direct link between holomorphic K-theory and algebraic cycles.

Chern (Berkeley days and return to China). Chern's ability to create invariants for important geometric structure is unsurpassed by any mathematicians that I have ever known. His works on projective differential geometry, on affine geometry and on Chern-Moser invariants for pseudo-convex domains demonstrate his strength. The intrinsic norm on cohomology of complex manifold that he defined with Levine and Nirenberg has not been fully exploited yet. Before he died, a major program for him was to carry out Cartan-Kähler system for more general geometric situation.

Chern once said: "The importance of complex numbers in geometry is a mystery to me. It is well-organized and complete."

Chern always regret that ancient Chinese mathematicians never discovered complex numbers. Chern's everlasting works in complex geometry make up the loss of Chinese mathematics for the last two thousand years.