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GEOGRAPHY AND THE NUMBER OF MODULI OF SURFACES OF GENERAL TYPE*

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To the memory of Andrei Nikolaevich Tyurin

Abstract. The paper considers a relationship between the Chern numbers K_X^2, $c_2(X)$ of a smooth minimal surface X of general type and the dimension of the space of infinitesimal deformations of X, i.e. $h^1(\Theta_X)$, where Θ_X is the holomorphic tangent bundle of X. We prove that if the ratio of the Chern numbers $\alpha(X) = \frac{c_2(X)}{K_X^2} \leq \frac{3}{8}$ and K_X is ample then

$$h^1(\Theta_X) \leq 9(3c_2 - K^2).$$

On the geometric side it is shown that a smooth surface of general type X with $\alpha(X) \leq \frac{3}{8}$ and $h^1(\Theta_X) \geq 3$ has two distinguished effective divisors F and E such that $H^1(\Theta_X)$ admits a direct sum decomposition $H^1(\Theta_X) = V_1 \oplus V_0$, where V_1 is identified with a subspace of $H^0(\mathcal{O}_X(F))$ while V_0 is identified with a subspace of $H^0(\Theta_X \otimes \mathcal{O}_E(E))$. This gives a geometric interpretation of the cohomology classes in $H^1(\Theta_X)$ and allows to bound the dimension of V_0 (resp. V_1) in terms of geometry of the divisor E (resp. F).

The main idea of the paper is to use the natural identification

$$H^1(\Theta_X) = Ext^1(\Omega_X, \mathcal{O}_X)$$

where Ω_X is the holomorphic cotangent bundle of X. Then the "universal" extension gives rise to a certain vector bundle whose study constitutes the essential part of the paper.

Key words. Surfaces of general type, vector bundles, stability

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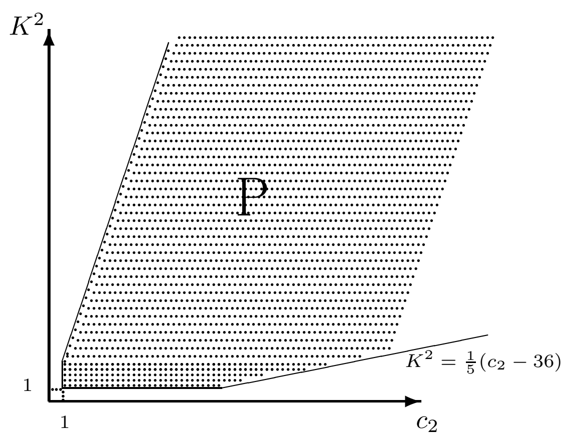
0. Introduction. Two questions occupy an important place in the theory of surfaces of general type: the problem of geography and the problem of moduli. Recall, if X is a smooth minimal surface of general type, then its Chern numbers $K_X^2, c_2(X)$ are two fundamental discrete invariants of X. The problem of geography asks for which pairs of integers (m, n) there exists a smooth minimal surface of general type X with $c_2(X) = m$ and $K_X^2 = n$. The well-known restrictions on the Chern numbers for surfaces of generaral type give us the region P in (c_2, K^2)-plane



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where the integral points (c_2, K^2) with $K^2 + c_2 \equiv 0(\mod 12)$ in the shaded area are called admissible points (see [16]). The ground-braking work of U.Persson,[15], followed by the works of G.Xiao, [18],[20] and Z.Chen,[8], have shown the existence of surfaces for every admissible pair in the part of P subject to $K^2 \leq 2c_2$ as well as filled the vast part of the remaining sector.

On the other hand the problem of moduli of surfaces of general type has been actively developed in the last 25 years or so (see e.g., [5],[6],[7]). In particular, F.Catanese in [6] revives a classical problem of determining an upper bound on the number of moduli of a surface of general type in terms of its Chern numbers.

The main purpose of this paper is to explore the relations between the Chern numbers K_X^2, $c_2(X)$ of a surface of general type X and the dimension of the space of the infinitesimal deformations of the complex structure of X, i.e. $h^1(\Theta_X)$, where Θ_X is the holomorphic tangent bundle of a surface X.

Such relationships have been implicitly used in studying surfaces with Chern numbers close to the lower limiting line of P: these surfaces and their moduli are amendable to an explicit description because they are genus 2 fibrations(see [10] and [18], [19], for general methods for studying fibered surfaces). About the other extreme of P one knows from Yau's work on Kähler-Einstein metric,[21], that surfaces with $K^2 = 3c_2$ are compact quotients of a unit ball in $\mathbf{C^2}$. This together with a result of Calabi and Vesentini,[4], yields the infinitesimal rigidity of these surfaces. One of the results of this paper is the following.

Theorem 0.1 (= Corollary 4.10). *Let X be a smooth surface with K_X ample and let $K_X^2 \geq \frac{8}{3}c_2(X)$. Then $h^1(\Theta_X) \leq 9(3c_2 - K^2)$.*

An upper bound of the above nature can be viewed as a conceptual reason for difficulties in constructing surfaces with Chern numbers which are close to the upper limiting line of P. Our approach also suggests where one should look for such surfaces since in deriving the above bound we give a geometric interpretation of the space of the infinitesimal deformations $H^1(\Theta_X)$. From this interpretation it follows that *all such surfaces infinitesimally look as though they either come as ramified covers of some other surfaces or as divisors in a 3-fold.*

To explain our approach to the study of the space $H^1(\Theta_X)$ let us consider the following hypothetical situation. Let $\pi : (\mathcal{X}, X) \longrightarrow (B, b_0)$ be the universal family of deformations of a smooth minimal surface of general type $X = \pi^{-1}(b_0)$, with the base B smooth. Then the total tangent bundle $\Theta_{\mathcal{X}}$ of $\mathcal{X}$ fits into the following exact sequence

$$0 \longrightarrow \Theta_{\mathcal{X}/B} \longrightarrow \Theta_{\mathcal{X}} \longrightarrow \pi^*\Theta_B \longrightarrow 0 \qquad (0.1)$$

where $\Theta_{\mathcal{X}/B}$ is the relative tangent bundle of π. Taking the restriction of (0.1) to X we obtain

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathcal{X}} \otimes \mathcal{O}_X \longrightarrow \Theta_{B,b_0} \otimes \mathcal{O}_X \longrightarrow 0. \qquad (0.2)$$

By the Kodaira-Spencer theory of deformation of complex structure the coboundary map $\Theta_{B,b_0} \longrightarrow H^1(\Theta_X)$ arising from (0.2) is the identity. Thus the sequence (0.2) can be viewed as the element of the group of extensions $Ext^1(\Theta_{B,b_0} \otimes \mathcal{O}_X, \Theta_X)$ corresponding to the identity endomorphism of $H^1(\Theta_X)$ under the natural identification $Ext^1(\Theta_{B,b_0} \otimes \mathcal{O}_X, \Theta_X) = End(H^1(\Theta_X))$. Of course, such an extension can be considered independently of the geometric argument above, i.e. as long as $H^1(\Theta_X) \neq 0$

we have the group of extensions $Ext^1(\Omega_X, H^1(\Theta_X)^* \otimes \mathcal{O}_X) = End(H^1(\Theta_X))$, where Ω_X is the cotangent bundle of X, and the identity endomorphism $id_{H^1(\Theta_X)}$ gives rise to the following short exact sequence of sheaves on X

$$0 \longrightarrow H^1(\Theta_X)^* \otimes \mathcal{O}_X \longrightarrow \mathcal{T} \longrightarrow \Omega_X \longrightarrow 0. \tag{0.3}$$

Besides its naturality the sheaf $\mathcal{T}$ is a good place to look for a relationship between $h^1(\Theta_X)$ and the Chern numbers of X since the rank of $\mathcal{T}$ is $rk(\mathcal{T}) = h^1(\Theta_X) + 2$ and its Chern invariants are $c_1(\mathcal{T}) = K_X,\ c_2(\mathcal{T}) = c_2(X)$. A study of the sheaf $\mathcal{T}$ is the essential point of our approach.

Set $\alpha = \dfrac{c_2(X)}{K_X^2}$. The first immediate observation is that for surfaces with $\alpha < \dfrac{1}{2}$ (i.e. surfaces with positive index) we have a "topological" upper bound on $h^1(\Theta_X)$ coming from the semistability of $\mathcal{T}$: *if $\mathcal{T}$ is semistable with respect to some polarization on X then* $h^1(\Theta_X) \leq \dfrac{1}{1-2\alpha} - 2$.

If the above inequality fails then $\mathcal{T}$ is unstable with respect to *any* numerically effective nonzero divisor D on X. This can be used to obtain some geometric information. First of all the fact that $\mathcal{T}$ is D-unstable implies that $\mathcal{T}$ contains the D-maximal destabilizing subsheaf $\mathcal{T}_1^D$ which gives rise to a nontrivial decomposition of the canonical divisor of X

$$K_X = L_1 + E_1 \tag{0.4}$$

where $L_1 = c_1(\mathcal{T}_1^D)$ and $E_1 = c_1(\mathcal{T}/\mathcal{T}_1^D)$. This decomposition becomes especially meaningful geometrically once we take $D = K_X$ and $\alpha \leq \dfrac{3}{8}$. We show that in this case L_1 is in the positive cone of the Néron-Severi group $NS(X)$ of X and E_1 is an effective nonzero divisor whose degree with respect to K_X is bounded by a function depending on $(3c_2 - K^2)$ (see Corollary 1.7). Furthermore, the rank of K_X-maximal destabilizing subsheaf $\mathcal{T}_1$ turns out to be 2 or 3 and the inclusion $\mathcal{T}_1 \longrightarrow \mathcal{T}$ combined with the defining sequence (0.3) gives rise to a generically surjective morphism

$$\mu_1 : \mathcal{T}_1 \longrightarrow \Omega_X.$$

This morphism looks as if it were the codifferential of a morphism $f : X \longrightarrow Y$ which is generically of maximal rank and where $dimY = 2$ or 3. Of course, there is no reason for μ_1 to come from such a geometric situation. However, it gives rise to a decomposition of $H^1(\Theta_X)$ as a direct sum of its subspaces $H^1(\Theta_X) = V_0 \oplus V_1$. Each of these subspaces has features characteristic to the aforementioned geometric situation (see Proposition 2.2):

1. V_0 is a subspace of $H^1(\Theta_X)$ contained in the kernel of the obvious morphism
$$H^1(\Theta_X) \longrightarrow H^1(\Theta_X(E))$$
for some component E of E_1 in the decomposition (0.4), i.e. it looks as though E is the ramification divisor of some morphism of X onto another surface;
2. V_1 injects into $H^0(\mathcal{O}_X(F))$, where F is again a component of E_1, i.e. V_1 looks as a subspace of infinitesimal deformations of a divisor in a 3-fold.

The above result gives a geometric interpretation of the cohomology classes in $H^1(\Theta_X)$ as global sections of $\mathcal{O}_X(F)$ or $\Theta_X \otimes \mathcal{O}_E(E)$. It also allows to obtain upper bounds on the dimensions of the subspaces V_0 and V_1 in terms of geometry of the divisors E and F respectively (see e.g., Corollary 2.4, Corollary 3.9). Putting these bounds together with the estimate of $E_1.K_X$ as a function of $(3c_2 - K^2)$ we derive the upper bound for $h^1(\Theta_X)$ as a function of $(3c_2 - K^2)$ (see Corollary 4.10). We also point out that the nature of the bound as a linear function of $(3c_2 - K^2)$ can not be improved in view of examples which we discuss in Example 4.11.

The paper is organized as follows.
In §1 we define the extension bundle $\mathcal{T}$ as in (0.3) and consider its properties from the point of view of stability.
In §2 we introduce the notion of divisorial and locally supported moduli and show that surfaces with $\alpha = \dfrac{c_2(X)}{K_X^2} \leq \dfrac{3}{8}$ and $h^1(\Theta_X) > 2$ have the property that the space of the infinitesimal deformations $H^1(\Theta_X)$ admits a vector space decomposition $H^1(\Theta_X) = V_0 \oplus V_1$ where V_1 is divisorial moduli and V_0 is locally supported moduli of X. We also derive an upper bound on the dimension of the divisorial moduli V_1 of X (Corollary 2.4).
The sections §3 and §4 are devoted to a study of the subspace V_0 of locally supported moduli and, particularly, to a study of the divisor on which V_0 is supported.
In section §5 we consider surfaces whose Chern numbers are subject to $3c_2 - K^2 \leq \frac{1}{2}\sqrt{K^2}$. This "quadratic" condition emerges naturally in view of the bound on the degree of E_1 obtained in Corollary 1.7. The point is that the Hodge index together with the "quadratic" condition implies that the intersection form restricted to the sublattice of $NS(X)$ generated by the irreducible components of E_1 is negative semidefinite. This allows us to give a detailed description of these components (Lemma 5.4) as well as to deduce conditions for these surfaces to be fibred by curves of genus $\leq (3c_2 - K^2)$.

1. Extension construction. Let X be a smooth minimal surface of general type. The holomorphic tangent (resp. cotangent) bundle of X will be denoted by Θ_X (resp. Ω_X). Throughout the paper, unless said otherwise, we assume $H^1(\Theta_X) \neq 0$. For a nonzero subspace V of $H^1(\Theta_X)$ we consider the extension

$$0 \longrightarrow V^* \otimes \mathcal{O}_X \longrightarrow \mathcal{T}_V \longrightarrow \Omega_X \longrightarrow 0 \tag{1.1}$$

corresponding to the natural inclusion $V \subset H^1(\Theta_X)$ where the following natural identifications are used:

$$Ext^1(\Omega_X, V^* \otimes \mathcal{O}_X) = V^* \otimes H^1(\Theta_X) = Hom_{\mathbf{C}}(V, H^1(\Theta_X)).$$

We will often refer to the sheaf $\mathcal{T}_V$ sitting in the middle of (1.1) as extension corresponding to V. If $V = H^1(\Theta_X)$, then the corresponding sheaf will be denoted by $\mathcal{T}$.

The invariants of $\mathcal{T}$ are easily computed from the defining sequence (1.1):

$$rk(\mathcal{T}) = h^1(\Theta_X) + 2, \ \ c_1(\mathcal{T}) = c_1(\Omega_X) = K_X, \ \ c_2(\mathcal{T}) = c_2(\Omega_X) = c_2(X).$$

If no ambiguity is likely we will omit X in the above notation. Set $\alpha = \dfrac{c_2(X)}{K_X^2}$ and assume $\alpha < \frac{1}{2}$. The semistability of $\mathcal{T}$ gives "the topological" upper bound for $h^1(\Theta_X)$ as a function of α.

Proposition 1.1. *Let $\alpha < \frac{1}{2}$. If $\mathcal{T}$ is semistable with respect to some ample divisor H on X then $h^1(\Theta_X) \leq \dfrac{1}{1-2\alpha} - 2$.*

Proof. The Bogomolov-Gieseker inequality (see, e.g., [13]) applied to $\mathcal{T}$ gives

$$2(h^1(\Theta_X)+2)c_2 \geq (h^1(\Theta_X)+1)K_X^2 \Leftrightarrow 2(h^1(\Theta_X)+2)\alpha \geq h^1(\Theta_X)+1.$$

Solving for $h^1(\Theta_X)$ yields the asserted inequality. □

From now on we assume that $\alpha < \frac{1}{2}$ and $h^1(\Theta_X) > \frac{1}{1-2\alpha} - 2$. In view of Proposition 1.1 the vector bundle $\mathcal{T}$ is unstable with respect to any numerically effective (nef) divisor D on X. We consider D-destabilizing filtration of $\mathcal{T}$ for D nef and big ($D^2 > 0$):

$$\mathcal{T} = \mathcal{T}_s^D \supset \cdots \supset \mathcal{T}_1^D \supset \mathcal{T}_0^D = 0. \tag{1.2}$$

Associated to this filtration we have:

$$\mathcal{G}_i^D = \mathcal{T}_i^D/\mathcal{T}_{i-1}^D,\ L_i = c_1(\mathcal{G}_i^D),\ d_i = c_2(\mathcal{G}_i^D),\ r_i = rk(\mathcal{G}_i^D),\ \alpha_i = \frac{L_i.D}{r_iK.D}. \tag{1.3}$$

The main properties of the filtration (1.2) are (see [13]):

- the graded sheaves $\mathcal{G}_i^D$ are D-semistable,
- $\alpha_1 > \ldots > \alpha_s$

The subsheaf $\mathcal{T}_1^D = \mathcal{G}_1^D$ is called the D-maximal destabilizing subsheaf of $\mathcal{T}$.

Following Miyaoka,[13], we have the notion of semipositivity of a torsion-free sheaf.

Definition 1.2.

a). *A torsion-free sheaf $\mathcal{F}$ on X is called semipositive with respect to a nef divisor D(also referred to as D-semipositive) if for any torsion-free quotient $\mathcal{Q}$ of $\mathcal{F}$ one has $c_1(\mathcal{Q}).D \geq 0$.*

b). *$\mathcal{F}$ is called generically semipositive if it is so for any nef divisor on X.*

Lemma 1.3. *The sheaf $\mathcal{T}_V$ is generically semipositive.*

Proof. Suppose $\mathcal{T}_V$ is not D-semipositive. Then there exists a subsheaf $\mathcal{G}$ of $\mathcal{T}$ whose quotient $\mathcal{Q}$ is torsion-free and $c_1(\mathcal{Q}).D < 0$ or, equivalently, $c_1(\mathcal{G}).D > K.D$. Consider the diagram

$$\begin{array}{ccccccccc} & & & & \mathcal{G} & & & & \\ & & & & \downarrow & \overset{\mu}{\searrow} & & & \\ 0 & \longrightarrow & V^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T}_V & \longrightarrow & \Omega_X & \longrightarrow & 0 \end{array} \tag{1.4}$$

The induced morphism $\mu : \mathcal{G} \longrightarrow \Omega_X$ must be generically of maximal rank, since otherwise $Im(\mu)$ is a subsheaf of Ω_X such that $c_1(Im(\mu)).D \geq c_1(\mathcal{G}).K > K.D$ contradicting generic semipositivity of Ω_X. In particular, the rank $r = rk(\mathcal{G}) \geq 2$. Taking the r-th exterior power of (1.4) we obtain

$$\begin{array}{ccccccccc} & & & & \det \mathcal{G} & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \bigwedge^r \mathcal{T}_V & \longrightarrow & \bigwedge^{r-2} V^* \otimes \mathcal{O}_X(K_X) & \longrightarrow & 0 \end{array} \tag{1.5}$$

The slanted arrow in (1.5) must be zero since $c_1(\mathcal{G}).D > K.D$. This yields a nonzero morphism $\det \mathcal{G} \longrightarrow \mathcal{F}$. But $\mathcal{F}$ fits into the following exact sequence

$$0 \longrightarrow \bigwedge^r V^* \otimes \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow \bigwedge^{r-1} V^* \otimes \Omega_X \longrightarrow 0$$

which gives a nonzero morphism $\det \mathcal{G} \longrightarrow \Omega_X$ contradicting generic semipositivity of Ω_X ▯

Using the D-destabilizing filtration (1.2) and the notation in (1.3) we obtain

$$2c_2 = 2\sum_i d_i + 2\sum_{i<j} L_i.L_j = 2\sum_i d_i + K^2 - \sum_i L_i^2. \tag{1.6}$$

Since $\mathcal{G}_i$'s are D-semistable the Bogomolov-Gieseker inequality gives $2d_i \geq \frac{r_i-1}{r_i}L_i^2$. Substituting into (1.6) we obtain

$$2c_2 \geq 2d_1 - L_1^2 - \sum_{i\geq 2} \frac{1}{r_i} L_i^2 + K^2. \tag{1.7}$$

By Hodge Index $L_i^2 \leq \frac{(L_i.D)^2}{D^2}$. This gives

$$\frac{1}{r_i} L_i^2 \leq \frac{L_i.D}{r_i} \frac{L_i.D}{D^2} = \alpha_i \frac{L_i.D}{D^2}(K.D).$$

Substituting in (1.7)

$$2c_2 \geq 2d_1 - L_1^2 - \left(\sum_{i\geq 2} \alpha_i L_i.D \right) \frac{K.D}{D^2} + K^2. \tag{1.8}$$

Remark 1.4. *From Lemma 1.3 it follows that* $\alpha_s \geq 0$. *This implies that* $\alpha_i,\ L_i.D \geq 0$ *for all* i *and the inequality is strict for* $1 \leq i < s$.

From now on we assume $D = K_X$. Substituting this in (1.8) we obtain

$$2c_2 \geq 2d_1 - L_1^2 - \sum_{i\geq 2} \alpha_i L_i.K + K^2. \tag{1.9}$$

Lemma 1.5. *If* $L_1^2 \leq 0$, *then* $\alpha > \frac{1}{2} - \frac{1}{8r_1}$.

Proof. The Bogomolov-Gieseker inequality for $\mathcal{G}_1$ together with $L_1^2 \leq 0$ yields

$$2c_2 \geq -\sum_{i\geq 2} \alpha_i L_i.K + K^2 > -\alpha_1(K^2 - L_1.K) + K^2 = -\alpha_1(1 - r_1\alpha_1)K^2 + K^2$$

where the second inequality follows from Remark 1.4 and the fact that the sequence $\{\alpha_i\}$ is strictly decreasing. Dividing by K^2 we obtain

$$2\alpha > -\alpha_1(1 - r_1\alpha_1) + 1,$$

or, equivalently,

$$r_1\alpha_1^2 - \alpha_1 + (1-2\alpha) < 0.$$

In particular, the discriminant of the quadratic polynomial on the left-hand side must be positive:

$$1 - 4r_1(1-2\alpha) > 0 \Leftrightarrow \alpha > \frac{1}{2} - \frac{1}{8r_1}.$$

□

PROPOSITION 1.6. *If $\alpha \leq \frac{3}{8}$, then $L_1^2 > 0$ and the morphism $\mu_1 : \mathcal{T}_1 \longrightarrow \Omega_X$ induced from*

$$\begin{array}{ccccccccc} & & & & \mathcal{T}_1 & & & & \\ & & & & \downarrow & \searrow^{\mu_1} & & & \\ 0 & \longrightarrow & H^1(\Theta_X)^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T} & \longrightarrow & \Omega_X & \longrightarrow & 0 \end{array} \tag{1.10}$$

is generically surjective. In particular, the rank r_1 of $\mathcal{T}_1$ is 2 or 3.

Proof. From Lemma 1.5 it follows that $L_1^2 > 0$. Since Ω_X can not have subsheaves of rank 1 of D-dimension 2 we deduce that $r_1 \geq 2$ and μ_1 is generically surjective. To obtain an upper bound on r_1 we use (1.8)

$$2c_2 \geq -\alpha_1 K^2 + K^2 \Leftrightarrow r_1 < \frac{1}{1-2\alpha} \leq 4.$$

□

COROLLARY 1.7. *If $\alpha \leq \frac{3}{8}$ and $h^1(\Theta_X) > 2$ then K_X has a distinguished decomposition $K_X = L_1 + E_1$ where $L_1 = c_1(\mathcal{T}_1)$ is in the positive cone $C^+(X)$ of the Néron-Severi group $NS(X)$ of X and E_1 is an effective nonzero divisor. Furthermore, the rank r_1 of $\mathcal{T}_1$ is equal to 2 or 3 and*

$$\frac{E_1.K}{K^2} < \begin{cases} 2(3\alpha - 1), & if \;\; r_1 = 3 \\ \frac{8(3\alpha-1)}{1+\sqrt{1+16(3\alpha-1)}}, & if \;\; r_1 = 2 \end{cases}.$$

Proof. If $\alpha \leq \frac{3}{8}$ then the "topological" bound $\dfrac{1}{1-2\alpha} - 2 \leq 2$. So the Bogomolov-Gieseker inequality for $\mathcal{T}$ fails as soon as $h^1(\Theta_X) > 2$. In particular, we have the K_X-maximal destabilizing subsheaf $\mathcal{T}_1$ which is subject to Proposition 1.6. Hence the assertion about L_1 and r_1. To see the properties of E_1 we consider the diagram (1.10) according the values of r_1.

1) $\mathbf{r_1 = 3}$: in this case the above mentioned diagram has the following form

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}^* & \longrightarrow & \mathcal{T}_1 & \longrightarrow & \mathcal{T}_1' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(\Theta_X)^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T} & \longrightarrow & \Omega_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{Q}_1 & \longrightarrow & \mathcal{S}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{1.11}$$

where $\mathcal{T}_1' = Im\mu_1,\ \mathcal{K}^* = ker\mu_1,\ \mathcal{S}_1 = coker\mu_1,\ \mathcal{Q}_1 = \mathcal{T}/\mathcal{T}_1,\ \mathcal{R} = (H^1(\Theta_X)^* \otimes \mathcal{O}_X)/\mathcal{K}^*$. From (1.11) we obtain that $\mathcal{K}^*$ is a line bundle whose dual $\mathcal{K}$ is generated by global sections outside of a subscheme of codim ≥ 2. More precisely, dualizing the column on the left we obtain

$$0 \longrightarrow \mathcal{R}^* \longrightarrow H^1(\Theta_X) \otimes \mathcal{O}_X \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}xt^1(\mathcal{R}, \mathcal{O}_X) \longrightarrow 0 \tag{1.12}$$

where the sheaf $\mathcal{E}xt^1(\mathcal{R}, \mathcal{O}_X)$ is supported on a subscheme of codim ≥ 2. Putting $F = c_1(\mathcal{K})$ we conclude that F is an effective divisor and $L_1 = c_1(\mathcal{T}_1') - F = K_X - c_1(\mathcal{S}_1) - F$. This implies

$$E_1 = K_X - L_1 = c_1(\mathcal{S}_1) + F. \tag{1.13}$$

Since both $c_1(\mathcal{S}_1)$ and F are effective and they can not vanish simultaneously we obtain that E_1 is an effective nonzero divisor.

To obtain the asserted upper bound on the degree (with respect to K_X) of E_1 we use (1.8) to obtain

$$2c_2 > -\alpha_1 K^2 + K^2 \Leftrightarrow \frac{L_1.K}{3K^2} > 1 - 2\alpha \Leftrightarrow \frac{E_1.K}{K^2} < 2(3\alpha - 1).$$

2) $\mathbf{r_1 = 2}$: the diagram (1.10) has the following form

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{T}_1 & = & \mathcal{T}_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(\Theta_X)^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T} & \longrightarrow & \Omega_X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(\Theta_X)^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{Q}_1 & \longrightarrow & \mathcal{S}_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{1.14}
$$

This implies that $E_1 = K_X - L_1 = c_1(\mathcal{S}_1)$ is effective and it must be nonzero (otherwise the extension defining $\mathcal{T}$ splits). To bound the degree of E_1 in this case we use a result of Miyaoka which says that $3d_1 = 3c_2(\mathcal{T}_1) \geq c_1^2(\mathcal{T}_1) = L_1^2$ (see Remark 4.18,[12]). Substituting this in (1.8) we obtain

$$
\begin{aligned}
2c_2 &\geq -\frac{1}{3}L_1^2 - \sum_{i\geq 2} \alpha_i K.L_i + K^2 > -\frac{1}{3}L_1^2 - \alpha_1 E_1.K + K^2 \\
&= -\frac{1}{3}(K - E_1)^2 - \frac{K^2 - E_1.K}{2K^2} E_1.K + K^2. \qquad (1.15)
\end{aligned}
$$

Dividing by K^2 and using the Hodge Index $\frac{E_1^2}{K^2} \leq \left(\frac{E_1.K}{K^2}\right)^2$ one obtains the following inequality

$$
\left(\frac{E_1.K}{K^2}\right)^2 + \frac{E_1.K}{K^2} - 4(3\alpha - 1) < 0. \tag{1.16}
$$

Solving it for $\frac{E_1.K}{K^2}$ yields

$$
\frac{E_1.K}{K^2} < \frac{-1 + \sqrt{1 + 16(3\alpha - 1)}}{2} = \frac{8(3\alpha - 1)}{1 + \sqrt{1 + 16(3\alpha - 1)}}.
$$

□

2. Two types of infinitesimal deformations. The situation encountered in the proof of Corollary 1.7 looks as though our surface X admits a morphism $f : X \longrightarrow Y$ which is generically of maximal rank and where $dimY = r_1 = 2$ or 3. With this hypothetical geometric interpretation in mind one could say that the case $r_1 = 2$ corresponds to the situation where all infinitesimal deformations of X come from the infinitesimal deformations of the ramification divisor of f and the case $r_1 = 3$ would generally brake into two parts: the infinitesimal deformations of the divisor $X' = Im(f)$ in the 3-fold Y and the infinitesimal deformations of the ramification divisor of f. Of course, there is no reason for the morphism $\mu_1 : \mathcal{T}_1 \longrightarrow \Omega_X$ in (1.10) to come from geometry. However, the infinitesimal deformations of X have all the features of such geometric situations: in the case $r_1 = 2$ all elements of $H^1(\Theta_X)$ are supported on the divisor E_1 as in Corollary 1.7 and in the case $r_1 = 3$ we can brake $H^1(\Theta_X)$ into two parts: $V_0 = H^0(\mathcal{R}^*)$ (see (1.12)) and V_1, a subspace of $H^1(\Theta_X)$ complementary to V_0. The subspace V_0 is as in the case $r_1 = 2$ while the subspace V_1 injects into $H^0(X, \mathcal{K})$ (see (1.11) for notation), i.e. it looks like infinitesimal deformations of a divisor in a 3-fold.

The following definition is motivated by the above discussion.

Definition 2.1. *1) A subspace V in $H^1(\Theta_X)$ is called a divisorial moduli of X if there exists a divisor D on X and an injective linear map $V \longrightarrow H^0(\mathcal{O}_X(D))$.*
2) A subspace V in $H^1(\Theta_X)$ is called a locally supported moduli of X if there exists a nonzero effective divisor E on X such that the sequence

$$0 \longrightarrow H^0(\Theta_X \otimes \mathcal{O}_E(E)) \longrightarrow H^1(\Theta_X) \xrightarrow{e} H^1(\Theta_X(E))$$

is exact and $V \subset ker(e)$, where e is a section defining E. In this case we will say that V is locally supported on the divisor E.
3) We say that $H^1(\Theta_X)$ admits a decomposition into divisorial and locally supported moduli if there exists a vector space decomposition $H^1(\Theta_X) = V \oplus V'$ such that V (resp. V') is a divisorial (resp. locally supported) moduli.
If $V = 0$ (resp. $V' = 0$), we say that X has locally supported (resp. divisorial) moduli only.

Let us show that the space of the infinitesimal deformations of X subject to the conditions of Corollary 1.7 admits a decomposition into divisorial and locally supported moduli. In order to do this we return to (1.12) and consider the decomposition

$$H^1(\Theta_X) = V_0 \oplus V_1 \tag{2.1}$$

where we put $V_0 = H^0(\mathcal{R}^*)$ and V_1, a subspace of $H^1(\Theta_X)$ complementary to V_0.

Proposition 2.2. *The decomposition (2.1) is a decomposition into divisorial and locally supported moduli. The subspace V_1 injects into $H^0(\mathcal{O}_X(F))$ (recall: $F = c_1(\mathcal{K})$) and V_0 is locally supported on $E' = c_1(\mathcal{S}_1)$, i.e. V_0 injects into $H^0(\Theta_X \otimes \mathcal{O}_{E'}(E'))$. Furthermore, if $r_1 = 2$ then X has locally supported moduli only, and if $r_1 = 3$ and $V_0 = H^0(\mathcal{R}^*) = 0$ then X has divisorial moduli only.*

Proof. The injection $V_1 \longrightarrow H^0(\mathcal{O}_X(F))$ follows from the definition of V_1 and (1.12). To see the assertion about V_0 we assume it to be nonzero and consider the extension of Ω_X corresponding to the natural inclusion $V_0 \subset H^1(\Theta_X)$

$$0 \longrightarrow V_0^* \otimes \mathcal{O}_X \longrightarrow \mathcal{T}_{V_0} \longrightarrow \Omega_X \longrightarrow 0.$$

The dual of the bottom sequence (1.11) implies that V_0 injects into $H^0(\mathcal{E}xt^1(\mathcal{S}_1, \mathcal{O}_X))$. The latter space is contained in the kernel of $H^1(\Theta_X) \longrightarrow H^1(\mathcal{T}_1'^{\,*})$ (this is seen by taking the dual of the column on the right-hand side of (1.11)). From this it follows that the morphism $(\mathcal{T}_1')^{**} \longrightarrow \Omega_X$ induced by $\mathcal{T}_1' \longrightarrow \Omega_X$ in (1.11) lifts to a morphism to $\mathcal{T}_{V_0}$, i.e. we have

$$\begin{array}{ccccccccc}
 & & & & & & (\mathcal{T}_1')^{**} & & \\
 & & & & & \swarrow & \downarrow & & \\
0 & \longrightarrow & V_0^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T}_{V_0} & \longrightarrow & \Omega_X & \longrightarrow & 0
\end{array}$$

Factoring out by the torsion part of $\mathcal{T}_{V_0}/\,(\mathcal{T}_1')^{**}$ we arrive to the following situation

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{G} & = & \mathcal{G} & & \\
 & & & & \downarrow{\scriptstyle \tilde{\mu}} & & \downarrow{\scriptstyle \mu} & & \\
0 & \longrightarrow & V_0^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T}_{V_0} & \longrightarrow & \Omega_X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_0^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{Q} & \longrightarrow & \mathcal{S} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{2.2}
$$

where $\mathcal{G}$ is a locally free subsheaf of Ω_X and $\mathcal{Q}$ is torsion-free. Let $L = c_1(\mathcal{G})$ and $E = c_1(\mathcal{S})$. So E is effective nonzero divisor. This divisor is related to E_1 in (1.13) and E' as follows

$$E_1 = F + c_1(\mathcal{S}_1) = F + E' = F + c_1(\mathrm{Tor}\left(\mathcal{T}_{V_0}/\,(\mathcal{T}_1')^{**}\right)) + E \tag{2.3}$$

In particular, E is a component of E'. We will show that the subspace V_0 is locally supported on E and hence on E' as asserted in the proposition.
Let e be a global section of $\mathcal{O}_X(E)$ defining E and consider the short exact sequence

$$0 \longrightarrow \Theta_X \xrightarrow{\ e\ } \Theta_X(E) \longrightarrow \Theta_X \otimes \mathcal{O}_E(E) \longrightarrow 0.$$

Observe that $\Theta_X(E) = \Omega_X(-L)$. Since $L = c_1(\mathcal{G}) = c_1((\mathcal{T}_1')^{**}) + c_1(\mathrm{Tor}(\mathcal{T}_{V_0}/\,(\mathcal{T}_1')^{**})) = L_1 + F + c_1(\mathrm{Tor}(\mathcal{T}_{V_0}/\,(\mathcal{T}_1')^{**}))$ we have that L has D-dimension 2. This implies that $H^0(\Theta_X(E)) = 0$ and the sequence

$$0 \longrightarrow H^0(\Theta_X \otimes \mathcal{O}_E(E)) \longrightarrow H^1(\Theta_X) \xrightarrow{\ e\ } H^1(\Theta_X(E))$$

is exact (the same argument holds for E' as well). It remains to check that V_0 is contained in the kernel of $H^1(\Theta_X) \xrightarrow{e} H^1(\Theta_X(E))$. This can be seen by taking the second exterior power of (2.2) and tensoring it with $\mathcal{O}_X(-L)$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{O}_X & \searrow & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{F}(-L) & \longrightarrow & \bigwedge^2 \mathcal{T}_{V_0}(-L) & \longrightarrow & \mathcal{O}_X(E) & \longrightarrow & 0.
\end{array}
\tag{2.4}
$$

The sheaf $\mathcal{F}(-L)$ fits into the following exact sequence

$$0 \longrightarrow \bigwedge^2 V_0^* \otimes \mathcal{O}_X(-L) \longrightarrow \mathcal{F}(-L) \longrightarrow V_0^* \otimes \Theta_X(E) \longrightarrow 0. \tag{2.5}$$

From (2.4) we deduce that $e \in H^0(\mathcal{O}_X(E))$ lies in the kernel of the coboundary morphism

$$H^0(\mathcal{O}_X(E)) \longrightarrow H^1(\mathcal{F}(-L)).$$

This morphism and (2.5) give the linear map

$$H^0(\mathcal{O}_X(E)) \longrightarrow V_0^* \otimes H^1(\Theta_X(E)) = Hom(V_0, H^1(\Theta_X(E))) \tag{2.6}$$

which is induced by the obvious cup-product

$$H^0(\mathcal{O}_X(E)) \otimes H^1(\Theta_X) \longrightarrow H^1(\Theta_X(E)).$$

Since e is mapped to zero in (2.6) we deduce that V_0 is contained in $ker(H^1(\Theta_X) \xrightarrow{e} H^1(\Theta_X(E)))$. $\square$

Remark 2.3. *From Proposition 2.2 it follows that the inclusion $V_0 \subset H^1(\Theta_X)$ factors through $H^0(\Theta_X \otimes \mathcal{O}_E(E))$ as follows*

$$\begin{array}{ccccccc} & & & & V_0 & & \\ & & & \swarrow^{i} & \big\downarrow & & \\ 0 \longrightarrow & H^0(\Theta_X \otimes \mathcal{O}_E(E)) & \longrightarrow & & H^1(\Theta_X) & \xrightarrow{e} & H^1(\Theta_X(E)). \end{array}$$

In fact one can be more precise. First remark that the fact that V_0 is annihilated by e implies that the morphism $\Omega_X(-E) \longrightarrow \Omega_X$ lifts to $\mathcal{T}_{V_0}$ and this lift $\Omega_X(-E) \longrightarrow \mathcal{T}_{V_0}$ factors through $\mathcal{G}$ in (2.2). This gives the following diagram

$$\begin{array}{ccccccccc} & & 0 \longrightarrow & \Omega_X(-E) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{S}' & \longrightarrow 0 \\ & & & \downarrow & & \downarrow & & & \\ & & & \Omega_X & = & \Omega_X & & & \\ & & & \downarrow & & \downarrow & & & \\ 0 \longrightarrow & \mathcal{S}' & \longrightarrow & \Omega_X \otimes \mathcal{O}_E & \longrightarrow & \mathcal{S} & \longrightarrow 0 & & \\ & & & \downarrow & & \downarrow & & & \\ & & & 0 & & 0 & & & \end{array}$$

Dualizing the bottom sequence we obtain

$$0 \longrightarrow \mathcal{E}xt^1(\mathcal{S}, \mathcal{O}_X) \longrightarrow \Theta_X \otimes \mathcal{O}_E(E) \longrightarrow \mathcal{E}xt^1(\mathcal{S}', \mathcal{O}_X) \longrightarrow 0$$

and the injection $V_0 \xrightarrow{i} H^0(\Theta_X \otimes \mathcal{O}_E(E))$ factors through $H^0(\mathcal{E}xt^1(\mathcal{S}, \mathcal{O}_X))$

$$\begin{array}{ccccccc} & & & & V_0 & & \\ & & & \swarrow^{j} & \big\downarrow i & & \\ 0 \longrightarrow & H^0(\mathcal{E}xt^1(\mathcal{S}, \mathcal{O}_X)) & \longrightarrow & & H^0(\Theta_X \otimes \mathcal{O}_E(E)) & \longrightarrow & H^0(\mathcal{E}xt^1(\mathcal{S}', \mathcal{O}_X)) \longrightarrow 0 \end{array} \tag{2.7}$$

Corollary 2.4. *Let V_1 be as in Proposition 2.2, the subspace of divisorial moduli of X. Then $dimV_1 \leq \frac{1}{2}(3c_2 - K^2)$.*

Proof. From Proposition 2.2 we know that $dimV_1 \leq h^0(\mathcal{O}_X(F))$. Thus we need to give an upper bound on the space of sections of the line bundle $\mathcal{O}_X(F)$. We may assume $F \neq 0$ (otherwise the assertion is obvious since the hypothesis $H^1(\Theta_X) \neq 0$ implies, by Yau's result,[21], and a theorem of Calabi-Vesentini,[4], that $3c_2 - K^2 \geq 4$). Then we know that the linear system $\mid F \mid$ has at most finite number of base points. Blowing-up X along the base locus of $\mid F \mid$ we obtain

$$\begin{array}{ccl} \tilde{X} & \xrightarrow{\ \tilde{\phi}\ } & \Sigma \hookrightarrow \mathbf{P}(H^0(\mathcal{O}_X(F))^*) \\ {\scriptstyle\sigma}\downarrow & \nearrow{\scriptstyle\phi} & \\ X & & \end{array}$$

where Σ is the image of the rational map ϕ defined by $\mathcal{O}_X(F)$ and $\sigma : \tilde{X} \longrightarrow X$ is a sequence of blowing-ups. We consider two cases according to the dimension of the image Σ.

1). $dim\Sigma = 2$. In this case we have the following estimate

$$h^0(\mathcal{O}_X(F)) \leq \frac{1}{2}F^2 + 2. \tag{2.8}$$

Indeed, if Σ is not a ruled surface then it is well-known that $h^0(\mathcal{O}_X(F)) \leq \frac{1}{2}deg\Sigma + 2$ (see e.g., Lemma 1.4,[2]). This combined with $deg\Sigma \leq \frac{F^2}{deg\phi}$ implies $h^0(\mathcal{O}_X(F)) \leq \frac{1}{2}\frac{F^2}{deg\phi} + 2 \leq \frac{1}{2}F^2 + 2$.

If Σ is ruled then we use $h^0(\mathcal{O}_X(F)) \leq deg\Sigma + 2 \leq \frac{F^2}{deg\phi} + 2$. Since the degree of ϕ must be at least 2 we obtain the inequality (2.8) as well.

By Hodge Index $F^2 \leq \frac{(F.K)^2}{K^2}$. Substituting this into (2.8) and using (1.13) together with Corollary 1.7 we obtain

$$dimV_1 \leq h^0(\mathcal{O}_X(F)) < 2(3\alpha - 1)(3c_2 - K^2) + 2 \leq \frac{1}{4}(3c_2 - K^2) + 2.$$

Since $3c_2 - K^2 = 4(9\chi(\mathcal{O}_X) - K^2)$ is divisible by 4 it follows $dimV_1 \leq \frac{1}{4}(3c_2 - K^2) + 1 \leq \frac{1}{2}(3c_2 - K^2)$, where the last inequality follows from the assumption $H^1(\Theta_X) \neq 0$.

2). $dim\Sigma = 1$. The morphism $\tilde{\phi} : X \longrightarrow \Sigma$ factors through the normalization Σ' of Σ. Taking the Stein factorization we arrive to the following diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ \psi\ } & B \\ & {\scriptstyle\phi'}\searrow \quad \swarrow{\scriptstyle f} & \\ & \Sigma' & \end{array} \tag{2.9}$$

where f is finite and ψ is a surjective morphism with connected fibres. In particular, the strict transform $\tilde{F}$ of F has the form $\phi'^*(D)$ for a divisor D on Σ' with

$deg(D) = deg(\Sigma)$. Putting $\tilde{C}$ to be the class of a smooth fibre of ψ we obtain $\tilde{F} = deg(\Sigma)deg(f)\tilde{C}$. This implies that $F.K = \tilde{F}.\sigma^* K = deg(\Sigma)deg(f)C.K$, where $C = \sigma_*(\tilde{C})$. From this it follows

$$dimV_1 \leq h^0(\mathcal{O}_X(F)) \leq deg(\Sigma) + 1 = \frac{F.K}{deg(f)C.K} + 1. \tag{2.10}$$

Combining this with (1.13) and Corollary 1.7 we obtain

$$dimV_1 < \frac{2(3c_2 - K^2)}{deg(f)C.K} + 1 \leq \frac{2(3c_2 - K^2)}{C.K} + 1.$$

The asserted inequality follows from the following.

Claim. $C.K \geq 4$.

Proof of the Claim. The hypothesis $\alpha \leq \frac{3}{8}$ is equivalent to $K^2 \geq \frac{96}{11}\chi(\mathcal{O}_X)$. In particular, $K^2 \geq 9$ with equality holding if $\chi(\mathcal{O}_X) = 1$. By Yau's theorem ([21]) such a surface must be a compact quotient of a unit ball in $\mathbf{C}^2$. By a result of Calabi-Vesentini ([4]) such surfaces are infinitesimally rigid which contradicts our assumption $H^1(\Theta_X) \neq 0$. So $K^2 \geq 10$.
Assume $C.K \leq 3$. The Hodge Index and the inequality $K^2 \geq 10$ imply $C^2 \leq 0$. Since C is nef divisor it follows that $C^2 = 0$ and $C.K = 2$. Furthermore, $C^2 = 0$ implies that the linear system $\mid F \mid$ is base point free. So $X = \tilde{X} \longrightarrow B$ in (2.9) is a genus 2 fibration. However, by a result of Xiao (see [18]) such surfaces are subject to $\alpha \geq \frac{1}{2}$. Hence $C.K \geq 4$. □

3. A study of locally supported moduli. Let V_0 be as in Proposition 2.2. Our study of this subspace of $H^1(\Theta_X)$ goes via considerations of the extension $\mathcal{T}_{V_0}$. In particular, we go back to the diagram (2.2). The divisor $E = c_1(\mathcal{S})$ has a stratification according to the rank of the morphism $\mu : \mathcal{G} \longrightarrow \Omega_X$ in (2.2). Let Γ be the component of E where μ vanishes. Then we can decompose the morphism μ as follows

$$\mathcal{G} \xrightarrow{\mu_1} \Omega_X(-\Gamma) \xrightarrow{\gamma} \Omega_X$$

where μ_1 is a morphism which vanishes at most in codimension 2 and $\Gamma = (\gamma = 0)$. This factorization yields the the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{G} & \xrightarrow{\mu_1} & \Omega_X(-\Gamma) & \longrightarrow & \mathcal{S}_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \gamma} & & & & \\
 & & \Omega_X & = & \Omega_X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{S} & \longrightarrow & \Omega_X \otimes \mathcal{O}_\Gamma & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.1}$$

where $\mathcal{S}_2 = coker(\mu_1)$. From this diagram we deduce

$$0 \longrightarrow \mathcal{S}_2 \longrightarrow \mathcal{S} \longrightarrow \Omega_X \otimes \mathcal{O}_\Gamma \longrightarrow 0. \tag{3.2}$$

This implies

$$E = c_1(\mathcal{S}) = c_1(\mathcal{S}_2) + 2\Gamma. \tag{3.3}$$

Dualizing (3.2) we obtain

$$0 \longrightarrow \Theta_X \otimes \mathcal{O}_\Gamma(\Gamma) \longrightarrow \mathcal{E}xt^1(\mathcal{S}, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^1(\mathcal{S}_2, \mathcal{O}_X) \longrightarrow 0.$$

This together with (2.7) imply

$$\begin{array}{ccccc} & & V_0 & & \\ & & \big\downarrow{\scriptstyle j} & & \\ 0 \longrightarrow H^0(\Theta_X \otimes \mathcal{O}_\Gamma(\Gamma)) & \longrightarrow & H^0(\mathcal{E}xt^1(\mathcal{S}, \mathcal{O}_X)) & \longrightarrow & H^0(\mathcal{E}xt^1(\mathcal{S}_2, \mathcal{O}_X)). \end{array} \tag{3.4}$$

Let $V_0'' = V_0 \cap H^0(\Theta_X \otimes \mathcal{O}_\Gamma(\Gamma))$ and let V_0' be the image of V_0 in $H^0(\mathcal{E}xt^1(\mathcal{S}_2, \mathcal{O}_X))$. Then we have

$$dimV_0 = dimV_0'' + dimV_0' \leq h^0(\Theta_X \otimes \mathcal{O}_\Gamma(\Gamma)) + dimV_0'. \tag{3.5}$$

We will now investigate the spaces V_0' and V_0''. An understanding of the latter one goes via the study of the divisor Γ. We begin by observing the following.

LEMMA 3.1. *The sheaf $\mathcal{O}_\Gamma(-L)$ is generated by global sections outside of a subscheme of dimension 0.*

Proof. Restricting the diagram (2.2) to Γ we obtain

$$\begin{array}{ccccccc} & & & \mathcal{G} \otimes \mathcal{O}_\Gamma & & & \\ & & & \big\downarrow & & & \\ 0 \longrightarrow & V_0^* \otimes \mathcal{O}_\Gamma & \longrightarrow & \mathcal{T}_{V_0} \otimes \mathcal{O}_\Gamma & \longrightarrow & \Omega_X \otimes \mathcal{O}_\Gamma & \longrightarrow 0 \end{array}$$

where the morphism $\mathcal{G} \otimes \mathcal{O}_\Gamma \longrightarrow \mathcal{T}_{V_0} \otimes \mathcal{O}_\Gamma$ factors through $V_0^* \otimes \mathcal{O}_\Gamma$. This gives a monomorphism $\mathcal{G} \otimes \mathcal{O}_\Gamma \longrightarrow V_0^* \otimes \mathcal{O}_\Gamma$. Taking determinant and dualizing yield the assertion. □

To understand further properties of Γ we let $\Gamma = \sum m_C C$ be the decomposition into distinct irreducible components. Put Λ_Γ to be the sublattice of $NS(X)$ generated by the irreducible components C's.

LEMMA 3.2. *The intersection pairing restricted to Λ_Γ is negative definite.*

Proof. Let $L = L^- + L^+$ be the Zariski decomposition of L with L^+ (resp. L^-) its positive (resp. negative) part. By Corollary 1.7 the positive part $L^+ \neq 0$. From Lemma 3.1 it follows $C.L \leq 0$ for every irreducible component C of Γ. This implies that either C is in the support of L^- and then $L^+.C = 0$, or C is not in the support of

L^- and then $C.L^{\pm} = 0$. Thus, we have $L^+.C = 0$ for every irreducible component of Γ. This implies that the sublattice Λ_Γ is orthogonal to L^+. By Hodge Index theorem, the intersection pairing is negative definite on Λ_Γ. ∎

This lemma will enable us to show that the contribution of V_0'' amounts to counting certain rational curves in Γ.

LEMMA 3.3. *Assume $H^0(\Theta_X \otimes \mathcal{O}_\Gamma(\Gamma)) \neq 0$. Then the following holds.*
a) There exists a decreasing sequence of components of Γ

$$\Gamma \supset \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_{N-1} \supset \Gamma_N$$

such that $h^0(\Theta_X \otimes \mathcal{O}_\Gamma(\Gamma)) = h^0(\Theta_X \otimes \mathcal{O}_{\Gamma_0}(\Gamma_0)) > h^0(\Theta_X \otimes \mathcal{O}_{\Gamma_1}(\Gamma_1)) > \cdots > h^0(\Theta_X \otimes \mathcal{O}_{\Gamma_{N-1}}(\Gamma_{N-1})) > h^0(\Theta_X \otimes \mathcal{O}_{\Gamma_N}(\Gamma_N)) = 0$.
b) For each $i \in \{0, \ldots, N-1\}$ there exists a rational curve $C_{i+1} \subset \Gamma_i - \Gamma_{i+1}$ such that $C_{i+1}.\Gamma_i = -1$ or -2.
c) $h^0(\Theta_X \otimes \mathcal{O}_{\Gamma_i}(\Gamma_i)) - h^0(\Theta_X \otimes \mathcal{O}_{\Gamma_{i+1}}(\Gamma_{i+1})) \leq 3 + C_{i+1}.\Gamma_i$ for every $i \in \{0, \ldots, N-1\}$. In particular,

$$h^0(\Theta_X \otimes \mathcal{O}_\Gamma(\Gamma)) \leq 3N + \sum_{i=0}^{N-1} C_{i+1}.\Gamma_i \leq 2N.$$

Proof. Let Γ_0 be a smallest component of Γ with the property $H^0(\Theta_X \otimes \mathcal{O}_\Gamma(\Gamma)) = H^0(\Theta_X \otimes \mathcal{O}_{\Gamma_0}(\Gamma_0))$. Choose a reduced irreducible component C_1 of Γ_0 such that

$$C_1.\Gamma_0 = min\{C.\Gamma_0 \mid C \text{ is an irreducible component of } \Gamma_0\}.$$

From Lemma 3.2 it follows $C_1.\Gamma_0 < 0$. The definition of Γ_0 implies that the restriction morphism $H^0(\Theta_X \otimes \mathcal{O}_{\Gamma_0}(\Gamma_0)) \longrightarrow H^0(\Theta_X \otimes \mathcal{O}_{C_1}(\Gamma_0))$ is nonzero. Consider the normal sequence of C_1 tensored with $\mathcal{O}_{C_1}(\Gamma_0)$

$$0 \longrightarrow \Theta_{C_1} \otimes \mathcal{O}_{C_1}(\Gamma_0) \longrightarrow \Theta_X \otimes \mathcal{O}_{C_1}(\Gamma_0) \longrightarrow \mathcal{O}_{C_1}(C_1 + \Gamma_0).$$

Since $H^0(\mathcal{O}_{C_1}(C_1 + \Gamma_0)) = 0$ it follows $H^0(\Theta_{C_1} \otimes \mathcal{O}_{C_1}(\Gamma_0)) = H^0(\Theta_X \otimes \mathcal{O}_{C_1}(\Gamma_0)) \neq 0$. This and $C_1.\Gamma_0 < 0$ imply that C_1 is rational and $C_1.\Gamma_0 = -1$ or -2. Set $\Gamma_0' = \Gamma_0 - C_1$. From the exact sequence

$$0 \longrightarrow \Theta_X \otimes \mathcal{O}_{\Gamma_0'}(\Gamma_0') \longrightarrow \Theta_X \otimes \mathcal{O}_{\Gamma_0}(\Gamma_0) \longrightarrow \Theta_X \otimes \mathcal{O}_{C_1}(\Gamma_0) \longrightarrow 0$$

we deduce $h^0(\Theta_X \otimes \mathcal{O}_{\Gamma_0}(\Gamma_0)) - h^0(\Theta_X \otimes \mathcal{O}_{\Gamma_0'}(\Gamma_0')) \leq h^0(\Theta_X \otimes \mathcal{O}_{C_1}(\Gamma_0)) \leq 3 + C_1.\Gamma_0$. If $h^0(\Theta_X \otimes \mathcal{O}_{\Gamma_0'}(\Gamma_0')) = 0$, then we set $\Gamma_1 = \Gamma_0'$ and the sequence $\Gamma_0 \supset \Gamma_1$ has the required properties. If $h^0(\Theta_X \otimes \mathcal{O}_{\Gamma_0'}(\Gamma_0')) \neq 0$, then we repeat the above argument with Γ_0' in place of Γ. This will define Γ_1 and C_2. The process will terminate after a finite number of steps yielding a decreasing sequence $\Gamma \supset \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_{N-1} \supset \Gamma_N$ with the asserted properties. In particular, summing up all the inequalities in c) of the lemma we obtain

$$h^0(\Theta_X \otimes \mathcal{O}_\Gamma(\Gamma)) \leq 3N + \sum_{i=0}^{N-1} C_{i+1}.\Gamma_i \leq 2N.$$

∎

Remark 3.4. *Dualizing the bottom sequence in (2.2) we obtain the morphism*

$$V_0 \otimes \mathcal{O}_X \longrightarrow \mathcal{E}xt^1(\mathcal{S}, \mathcal{O}_X)$$

which is surjective outside a subscheme of dimension 0. This combined with the proof of Lemma 3.3 implies that $c_1(\mathcal{S}_2)$ *in (3.3) is nonzero and the morphism* $V_0 \otimes \mathcal{O}_X \longrightarrow \mathcal{E}xt^1(\mathcal{S}_2, \mathcal{O}_X)$ *is surjective outside a subscheme of dimension 0. Hence, the space* V_0' *in (3.5) is nonzero.*

Next we turn to a study of V_0'. From the short exact sequence on the top of (3.1) it follows that V_0' injects into the kernel of $H^1(\Theta_X(\Gamma)) \longrightarrow H^1(\mathcal{G}^*)$. This implies that the morphism μ_1 in (3.1) lifts to the extension of $\Omega_X(-\Gamma)$ corresponding to V_0' (viewed as a subspace of $H^1(\Theta_X(\Gamma))$), i.e. we have the following diagram

$$\begin{array}{ccccccccc}
 & & & & \mathcal{G} & = & \mathcal{G} & & \\
 & & & & \downarrow{\scriptstyle \tilde{\mu}_1} & & \downarrow{\scriptstyle \mu_1} & & \\
0 & \longrightarrow & (V_0')^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T}_{V_0'} & \longrightarrow & \Omega_X(-\Gamma) & \longrightarrow & 0.
\end{array}$$

Factoring out by the torsion of the quotient $\mathcal{T}_{V_0'}/\mathcal{G}$ we obtain the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{G}_1 & = & \mathcal{G}_1 & & \\
 & & & & \downarrow{\scriptstyle \tilde{\eta}} & & \downarrow{\scriptstyle \eta} & & \\
0 & \longrightarrow & (V_0')^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T}_{V_0'} & \longrightarrow & \Omega_X(-\Gamma) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (V_0')^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{Q}_2 & \longrightarrow & \mathcal{P} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.6}$$

analogous to the one in (2.2). As before the morphism $\eta : \mathcal{G}_1 \longrightarrow \Omega_X(-\Gamma)$ is generically an isomorphism. But this time it vanishes at most in codimension 2. This implies that the sheaf $\mathcal{P}$ has rank 1 outside of a subscheme of dimension 0 on the support of $\mathcal{P}$. Set $E_2 = c_1(\mathcal{P})$ and $L_1' = c_1(\mathcal{G}_1)$. Since the extension $\mathcal{T}_{V_0'}$ is nontrivial it follows that E_2 is an effective nonzero divisor. By definition of $\mathcal{G}_1$ its first Chern class $L_1' = c_1(\mathcal{G}) + c_1(\mathrm{Tor}\left(\mathcal{T}_{V_0'}/\mathcal{G}\right)) = L + c_1(\mathrm{Tor}\left(\mathcal{T}_{V_0'}/\mathcal{G}\right))$. We have seen that L is a divisor of D-dimension 2. So the same holds for L_1'. Furthermore, dualizing the bottom sequence in (3.6) we obtain

$$dim V_0' \leq h^0(\mathcal{E}xt^1(\mathcal{P}, \mathcal{O}_X)). \tag{3.7}$$

To obtain the upper bound on the dimension of V_0' we compare the extension $\mathcal{T}_{V_0'}$ with a one-dimensional extension.

Fix a nonzero vector ξ in V_0'. Viewing it as a cohomology class in $H^1(\Theta_X(\Gamma))$

(recall: we identify V'_0 with its image in $H^1(\Theta_X(\Gamma))$) we consider the extension corresponding to ξ

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X(-\Gamma) \longrightarrow 0.$$

This extension and $\mathcal{T}_{V'_0}$ are related by the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & (V'_0/\langle\xi\rangle)^* \otimes \mathcal{O}_X & = & (V'_0/\langle\xi\rangle)^* \otimes \mathcal{O}_X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & (V'_0)^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T}_{V'_0} & \longrightarrow & \Omega_X(-\Gamma) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X(-\Gamma) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.8}$$

where $\langle\xi\rangle$ is the one-dimensional subspace of $H^1(\Theta_X(\Gamma))$ spanned by ξ. Combining this with the middle column in (3.6) we obtain

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{G}_1 & = & \mathcal{G}_1 & & \\
 & & & & \downarrow{\scriptstyle \tilde{\eta}} & & \downarrow{\scriptstyle \tilde{\eta}_\xi} & & \\
0 & \longrightarrow & (V'_0/\langle\xi\rangle)^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T}_{V'_0} & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (V'_0/\langle\xi\rangle)^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{Q}_2 & \longrightarrow & \mathcal{Q}_\xi & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.9}$$

where $\mathcal{Q}_\xi$ is the cokernel of $\tilde{\eta}_\xi$.

LEMMA 3.5. *For a general ξ in V'_0 the sheaf $\mathcal{Q}_\xi$ is torsion-free.*

Proof. It is enough to show that the morphism $\tilde{\eta}_\xi$ in (3.9) drops its rank at most at finite set of points. Observe that $\tilde{\eta}_\xi$ drops its rank where
(i) $\tilde{\eta}$ drops its rank
and where
(ii) $Im(\tilde{\eta})$ intersects nontrivially the subbundle $(V'_0/\langle\xi\rangle)^* \otimes \mathcal{O}_X$.

The set of points in (i) is finite since $\mathcal{Q}_2 = coker(\tilde{\eta})$ is torsion-free. Turning to the points in (ii) we use the fact that $\eta\,|_{E_2}\colon \mathcal{G}_1 \otimes \mathcal{O}_{E_2} \longrightarrow \Omega_X(-\Gamma) \otimes \mathcal{O}_{E_2}$ drops its rank precisely by 1 outside of a finite set of points. Then the kernel of $\eta\,|_{E_2}$ gives rise to a one-dimensional subscheme in $\mathbf{P}(V_0'^{\,*})$ and the points in (ii) are the points of the intersection of the hyperplane $\mathbf{P}((V_0'/\langle\xi\rangle)^*)$ with this one dimensional subscheme. It is clear that this intersection is finite for a general choice of ξ. □

We are now in the position to give an upper bound on the dimension of V_0'. From Lemma 3.5 it follows that $\mathcal{Q}_\xi$ has the form $\mathcal{I}_{Z_\xi}(E_2)$, where $\mathcal{I}_{Z_\xi}$ is the sheaf of ideals of some 0-dimensional subscheme Z_ξ. The right-hand column in (3.9) combined with the defining sequence of $\mathcal{T}_\xi$ gives the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{G}_1 & = & \mathcal{G}_1 & & \\
 & & & & \downarrow{\scriptstyle \tilde{\eta}_\xi} & & \downarrow{\scriptstyle \eta} & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X(-\Gamma) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{I}_{Z_\xi}(E_2) & \longrightarrow & \mathcal{P} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.10}$$

Dualizing the bottom sequence we have

$$0 \longrightarrow \mathcal{O}_X(-E_2) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}xt^1(\mathcal{P}, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^1(\mathcal{I}_{Z_\xi}(E_2), \mathcal{O}_X) \longrightarrow 0.$$

This implies $h^0(\mathcal{E}xt^1(\mathcal{P}, \mathcal{O}_X)) \le h^0(\mathcal{O}_{E_2}) + h^0(\mathcal{E}xt^1(\mathcal{I}_{Z_\xi}(E_2), \mathcal{O}_X)) = h^0(\mathcal{O}_{E_2}) + \chi(\mathcal{E}xt^1(\mathcal{I}_{Z_\xi}(E_2), \mathcal{O}_X))$. Substituting into (3.7) we obtain

$$dim V_0' \le h^0(\mathcal{O}_{E_2}) + \chi(\mathcal{E}xt^1(\mathcal{I}_{Z_\xi}(E_2), \mathcal{O}_X)). \tag{3.11}$$

Proposition 3.6. *Let $E_2' = E_2 + 2\Gamma$. Then the following inequality holds:*

$$dim V_0' \le \frac{1}{3}(3c_2 - K^2) + \frac{1}{6}\left(K.E_2' + (E_2')^2\right) - \Gamma^2.$$

Proof. The asserted inequality is obtained by bounding two terms in (3.11). We begin by estimating the second term. Dualizing the column in the middle of (3.10) we obtain

$$\chi(\mathcal{E}xt^1(\mathcal{I}_{Z_\xi}(E_2), \mathcal{O}_X)) = \chi(\mathcal{G}_1^*) - \chi(\mathcal{T}_\xi^*) + \chi(\mathcal{O}_X(-E_2)).$$

Applying the Riemann-Roch to the right-hand side yields

$$\begin{aligned}
\chi(\mathcal{E}xt^1(\mathcal{I}_{Z_\xi}(E_2), \mathcal{O}_X)) &= \frac{1}{2}\left(L_1'\right)^2 - c_2(\mathcal{G}_1) + \frac{1}{2}E_2^2 - \frac{1}{2}(K - 2\Gamma)^2 + (c_2 - K.\Gamma + \Gamma^2) \\
&= \frac{1}{2}\left(L_1'\right)^2 - c_2(\mathcal{G}_1) + \frac{1}{2}E_2^2 - \frac{1}{2}K^2 + c_2 + K.\Gamma - \Gamma^2. \qquad (3.12)
\end{aligned}$$

By Miyaoka $c_2(\mathcal{G}_1) \geq \frac{1}{3}(L_1')^2$ (Remark 4.18,[12]). Substituting in (3.12) we obtain

$$\begin{aligned}\chi(\mathcal{E}xt^1(\mathcal{I}_{Z_\xi}(E_2),\mathcal{O}_X)) &\leq \frac{1}{6}(L_1')^2 + \frac{1}{2}E_2^2 - \frac{1}{2}K^2 + c_2 + K.\Gamma - \Gamma^2 \\ &= \frac{1}{6}(K - 2\Gamma - E_2)^2 + \frac{1}{2}E_2^2 - \frac{1}{2}K^2 + c_2 + K.\Gamma - \Gamma^2 \\ &= \frac{1}{3}(3c_2 - K^2) - \frac{1}{3}K.E_2' + \frac{1}{6}(E_2')^2 + \frac{1}{2}E_2^2 + K.\Gamma - \Gamma^2\end{aligned}$$

where $E_2' = E_2 + 2\Gamma$. This together with (3.11) imply

$$dim V_0' \leq h^0(\mathcal{O}_{E_2}) + \frac{1}{3}(3c_2 - K^2) - \frac{1}{3}K.E_2' + \frac{1}{6}(E_2')^2 + \frac{1}{2}E_2^2 + K.\Gamma - \Gamma^2. \quad (3.13)$$

To bound the first term in in the above inequality we use the following.

LEMMA 3.7. *Let X be a smooth minimal surface of general type. Assume $K_X = D + D'$, where D is effective and D' has the Zariski decomposition whose positive part is nonzero. Then $h^0(\mathcal{O}_D) \leq \dfrac{K.D - D^2}{2}$.*

Let us assume this result and complete the proof of the proposition.
From the middle column of (3.10) it follows $K - 2\Gamma = L_1' + E_2$. This gives the decomposition $K = E_2 + (L_1' + 2\Gamma)$ which satisfies the hypothesis of Lemma 3.7. So we deduce $h^0(\mathcal{O}_{E_2}) \leq \frac{1}{2}(K.E_2 - E_2^2)$. Substituting in (3.13) gives

$$\begin{aligned}dim V_0' &\leq \frac{1}{3}(3c_2 - K^2) - \frac{1}{3}K.E_2' + \frac{1}{2}K.E_2 + K.\Gamma + \frac{1}{6}(E_2')^2 - \Gamma^2 \\ &= \frac{1}{3}(3c_2 - K^2) + \frac{1}{6}K.E_2' + \frac{1}{6}(E_2')^2 - \Gamma^2\end{aligned}$$

which is the inequality asserted in the proposition. We turn now to a proof of Lemma 3.7.

Proof of Lemma 3.7. We may assume $D \neq 0$ (otherwise there is nothing to prove). If D is nef and big then by Ramanujam's vanishing theorem (Theorem 8.1, IV, [1]) $h^0(\mathcal{O}_D) = 1$. On the other hand $K.D - D^2 = D.D'$. Writing $D' = P' + N'$, the Zariski decomposition of D', where P' (resp. N') is its positive (resp. negative) part, we have $D.P' > 0$ (since $P' \neq 0$ and D is nef and big). So we have $D.D' > 0$. Since $K.D - D^2 = D.D'$ is even we obtain $\dfrac{K.D - D^2}{2} \geq 1 = h^0(\mathcal{O}_D)$.

If D is nef and $D^2 = 0$, then one has the following decomposition $D = \sum_{i=1}^{N} D_i$ where $D_i.D_j = 0$, for every i, j and $h^0(\mathcal{O}_{D_i}) = 1$, for every i. In particular,

$$h^0(\mathcal{O}_D) \leq N \leq \frac{1}{2}\sum_{i=1}^{N} D_i.K = \frac{1}{2}K.D.$$

The general case can be reduced to the case of a nef divisor as follows. By Riemann-Roch for $\mathcal{O}_D$ we have :

$$h^0(\mathcal{O}_D) - h^1(\mathcal{O}_D) = -\frac{K.D + D^2}{2}. \quad (3.14)$$

By Serre duality $h^1(\mathcal{O}_D) = h^0(\mathcal{O}_D(D + K_X))$. Set $D = D_0$ and let C_0 be a reduced, irreducible component of D_0 for which $D_0.C_0 < 0$. Put $D_1 = D_0 - C_0$ and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{D_1}(D_1 + K_X) \longrightarrow \mathcal{O}_{D_0}(D_0 + K_X) \longrightarrow \mathcal{O}_{C_0}(D_0 + K_X) \longrightarrow 0.$$

From the cohomology sequence we deduce

$$h^0(\mathcal{O}_{D_0}(D_0+K_X)) - h^0(\mathcal{O}_{D_1}(D_1+K_X)) \leq h^0(\mathcal{O}_{C_0}(D_0+K_X)) \leq K.C_0. \quad (3.15)$$

Repeating the above procedure a finite number of times we obtain the decomposition $D = D_n + \sum_{i=0}^{n-1} C_i$, where D_n is a nef divisor. Summing up the inequalities of type (3.15) we obtain

$$h^0(\mathcal{O}_D(D+K_X)) - h^0(\mathcal{O}_{D_n}(D_n+K_X)) \leq \sum_{i=0}^{n-1} K.C_i. \quad (3.16)$$

From the Riemann-Roch applied to D_n we have

$$h^0(\mathcal{O}_{D_n}(D_n+K_X)) = h^0(\mathcal{O}_{D_n}) + \frac{K.D_n + {D_n}^2}{2}$$

Substituting this in (3.16) we obtain

$$h^1(\mathcal{O}_D) = h^0(\mathcal{O}_D(D+K_X)) \leq h^0(\mathcal{O}_{D_n}) + \frac{K.D_n + {D_n}^2}{2} + \sum_{i=0}^{n-1} K.C_i.$$

This and (3.14) imply

$$\begin{aligned} h^0(\mathcal{O}_D) &\leq h^0(\mathcal{O}_{D_n}) + \frac{K.D_n + {D_n}^2}{2} + \sum_{i=0}^{n-1} K.C_i - \frac{K.D + D^2}{2} \\ &= h^0(\mathcal{O}_{D_n}) + \frac{1}{2}\sum_{i=0}^{n-1} K.C_i + \frac{{D_n}^2 - D^2}{2} \end{aligned} \quad (3.17)$$

If ${D_n}^2 = 0$, then by the first part of the argument $h^0(\mathcal{O}_{D_n}) \leq \frac{1}{2}K.D_n$. Substituting this in (3.17) we obtain the assertion.
If ${D_n}^2 > 0$, then $h^0(\mathcal{O}_{D_n}) = 1$ and (3.17) imply

$$h^0(\mathcal{O}_D) \leq 1 + \frac{1}{2}\sum_{i=0}^{n-1} K.C_i + \frac{{D_n}^2 - D^2}{2}.$$

Since ${D_n}^2 \leq D_n.D = D_n.(K - D') = D_n.K - D_n.D'$ we obtain

$$h^0(\mathcal{O}_D) \leq 1 + \frac{1}{2}(K.D - D^2) - \frac{1}{2}D_n.D'.$$

Using the Zariski decomposition of $D' = P' + N'$ we have $D_n.D' = D_n.P' + D_n.N' \geq D_n.P' > 0$. So $D_n.D' \geq 1$ and we have

$$h^0(\mathcal{O}_D) \leq \frac{1}{2}(K.D - D^2) + \frac{1}{2}.$$

Since $\frac{1}{2}(K.D - D^2)$ is an integer it follows $h^0(\mathcal{O}_D) \leq \frac{1}{2}(K.D - D^2)$. ☐

Corollary 3.8. *Let V_0' be as in Proposition 3.6 then*

$$dim V_0' < \begin{cases} 2\alpha(3c_2 - K^2) - \Gamma^2 - \frac{1}{6}\left(1 + 2\frac{K.E_2'}{K^2} + \frac{K.F}{K^2}\right)K.F, & if\ r_1 = 3 \\ 3c_2 - K^2 - \Gamma^2, & if\ r_1 = 2 \end{cases}$$

where F is as in Proposition 2.2, Γ and E'_2 are as in Proposition 3.6

Proof. If $r_1 = 3$, then from (1.13), (3.3) and the definition of E'_2 in Proposition 3.6 it follows

$$K.E'_2 \leq K.E_1 - K.F.$$

This and the Hodge index imply

$$(E'_2)^2 \leq \frac{(K.E_1 - K.F)^2}{K^2} = \frac{(K.E_1)^2}{K^2} - 2\frac{K.E_1}{K^2}K.F + \frac{(K.F)^2}{K^2}.$$

Substituting the two inequalities above in the inequality of Proposition 3.6 we obtain

$$\begin{aligned} dimV'_0 &\leq \frac{1}{3}(3c_2 - K^2) + \frac{1}{6}\left(K.E_1 + \frac{(K.E_1)^2}{K^2}\right) - \frac{1}{6}\left(K.F + 2\frac{K.E_1}{K^2}K.F - \frac{(K.F)^2}{K^2}\right) - \Gamma^2 \\ &\leq \frac{1}{3}(3c_2 - K^2) + \frac{1}{6}\left(K.E_1 + \frac{(K.E_1)^2}{K^2}\right) - \frac{1}{6}\left(1 + 2\frac{K.E'_2}{K^2} + \frac{K.F}{K^2}\right)K.F - \Gamma^2 \quad (3.18) \end{aligned}$$

This and the bound on $K.E_1$ in Corollary 1.7 imply

$$\begin{aligned} dimV'_0 &< \frac{1}{3}(3c_2 - K^2) + \frac{1}{3}(1 + 2(3\alpha - 1))(3c_2 - K^2) - \frac{1}{6}\left(1 + 2\frac{K.E'_2}{K^2} + \frac{K.F}{K^2}\right)K.F - \Gamma^2 \\ &= 2\alpha(3c_2 - K^2) - \frac{1}{6}\left(1 + 2\frac{K.E'_2}{K^2} + \frac{K.F}{K^2}\right)K.F - \Gamma^2 \quad (3.19) \end{aligned}$$

If $r_1 = 2$, then we use the inequality (1.16) with E_1 replaced by E'_2. This yields

$$K.E'_2 + (E'_2)^2 < 4(3c_2 - K^2).$$

Substituting into the inequality in Proposition 3.6 we obtain the assertion. ∎

Corollary 3.9. *Let V_0 be as in Proposition 2.2, the subspace of the locally supported moduli of X. Then*

$$dimV_0 < \begin{cases} 2\alpha(3c_2 - K^2) - \Gamma^2 + 2N - \frac{1}{6}\left(1 + 2\frac{K.E'_2}{K^2} + \frac{K.F}{K^2}\right)K.F, & if\ r_1 = 3 \\ 3c_2 - K^2 - \Gamma^2 + 2N, & if\ r_1 = 2 \end{cases}$$

where Γ and N are as in Lemma 3.3, F and E'_2 are as in Corollary 3.8.

Proof. The inequalities of Corollary 3.8 and the last inequality in Lemma 3.3,c) substituted in (3.5) yield the assertion. ∎

Combining the result of Corollary 3.9 with our analysis of the divisorial moduli of X in the proof of Corollary 2.4 will yield a bound on the dimension of the space of infinitesimal deformations of X.

Corollary 3.10. *Let $\alpha \leq \frac{3}{8}$ and $h^1(\Theta_X) > 2$ and let Γ and N be as in Lemma 3.3. Then*

$$h^1(\Theta_X) \leq 3c_2 - K^2 - \Gamma^2 + 2N.$$

Proof. If $r_1 = 3$, then combining Proposition 2.2 with Corollary 3.9 we obtain

$$h^1(\Theta_X) < 2\alpha(3c_2 - K^2) - \Gamma^2 + 2N - \frac{1}{6}\left(1 + 2\frac{K.E_2'}{K^2} + \frac{K.F}{K^2}\right) K.F + dimV_1. \quad (3.20)$$

If $F = 0$, then $dimV_1 = 1$ and we obtain

$$h^1(\Theta_X) < 2\alpha(3c_2 - K^2) - \Gamma^2 + 2N + 1.$$

Combining this with $\alpha \leq \frac{3}{8}$ we obtain the asserted inequality.

If $F \neq 0$, we argue as in the proof of Corollary 2.4.

1). The linear system $\mid F \mid$ is not composed of a pencil. Then (2.8) implies

$$dimV_1 \leq \frac{1}{2}F^2 + 2 \leq \frac{1}{2}\frac{(K.F)^2}{K^2} + 2.$$

Substituting the last expression above in (3.20) yields

$$\begin{aligned} h^1(\Theta_X) &< 2\alpha(3c_2 - K^2) - \Gamma^2 + 2N - \frac{1}{6}\left(1 + 2\frac{K.E_2'}{K^2} + \frac{K.F}{K^2}\right) K.F + \frac{1}{2}\frac{(K.F)^2}{K^2} + 2 \\ &= 2\alpha(3c_2 - K^2) - \Gamma^2 + 2N + 2 + \frac{1}{6}\left(2\frac{K.F}{K^2} - 1 - 2\frac{K.E_2'}{K^2}\right) K.F \\ &\leq 2\alpha(3c_2 - K^2) - \Gamma^2 + 2N + 2 + \frac{1}{6}\left(2\frac{K.E_1}{K^2} - 1 - 4\frac{K.E_2'}{K^2}\right) K.F. \quad (3.21) \end{aligned}$$

Using the upper bound for $\frac{K.E_1}{K^2}$ from Corollary 1.7 we obtain

$$h^1(\Theta_X) < 2\alpha(3c_2 - K^2) - \Gamma^2 + 2N + 2 + \frac{1}{6}\left(12\alpha - 5 - 4\frac{K.E_2'}{K^2}\right) K.F. \quad (3.22)$$

This combined with $\alpha \leq \frac{3}{8}$ gives

$$h^1(\Theta_X) < \frac{3}{4}(3c_2 - K^2) - \Gamma^2 + 2N + 2 - \frac{1}{6}\left(\frac{1}{2} + 4\frac{K.E_2'}{K^2}\right) K.F$$

which implies $h^1(\Theta_X) \leq 3c_2 - K^2 - \Gamma^2 + 2N$ as asserted.

2). The linear system $\mid F \mid$ is composed of a pencil. Then (2.10) together with **Claim** in the proof of Corollary 2.4 imply $dimV_1 \leq \frac{1}{4}K.F + 1$. Substituting in (3.20) we obtain

$$\begin{aligned} h^1(\Theta_X) &< 2\alpha(3c_2 - K^2) - \Gamma^2 + 2N + 1 + \frac{1}{12}K.F - \left(2\frac{K.E_2'}{K^2} + \frac{K.F}{K^2}\right) K.F \\ &= 2\alpha(3c_2 - K^2) - \Gamma^2 + 2N + 1 + \frac{1}{12}K.E_1 - \frac{1}{12}K.E_2' - \left(2\frac{K.E_2'}{K^2} + \frac{K.F}{K^2}\right) K.F \\ &< (2\alpha + \frac{1}{6})(3c_2 - K^2) - \Gamma^2 + 2N + 1 - \frac{1}{12}K.E_2' - \left(2\frac{K.E_2'}{K^2} + \frac{K.F}{K^2}\right) K.F \\ &\leq \frac{11}{12}(3c_2 - K^2) - \Gamma^2 + 2N + 1 - \frac{1}{12}K.E_2' - \left(2\frac{K.E_2'}{K^2} + \frac{K.F}{K^2}\right) K.F. \quad (3.23) \end{aligned}$$

From this one immediately deduces the inequality $h^1(\Theta_X) \leq 3c_2 - K^2 - \Gamma^2 + 2N$.

If $r_1 = 2$, then X has locally supported moduli only, i.e. $H^1(\Theta_X) = V_0$. This together with the second inequality of Corollary 3.9 imply $h^1(\Theta_X) < 3c_2 - K^2 - \Gamma^2 + 2N$. □

REMARK 3.11. *One can give a straightforward upper bound on $-\Gamma^2$ in terms of geometry of the irreducible components of Γ: write the decomposition $\Gamma = \sum_i m_i C_i$ into reduced irreducible components C_i. The adjunction formula applied to the component C_i gives $K.C_i + C_i^2 \geq 2g_i - 2$, where $g_i = g(\tilde{C}_i)$ is the genus of the normalization $\tilde{C}_i$ of C_i. This together with*

$$-\Gamma^2 = -\sum_i m_i^2 C_i^2 - \sum_{i \neq j} m_i m_j C_i.C_j$$

yield

$$-\Gamma^2 \leq \sum_i m_i^2 K.C_i - \sum_i m_i^2 (2g_i - 2) - \sum_{i \neq j} m_i m_j C_i.C_j. \tag{3.24}$$

The divisor 2Γ is a component of E_1 in Corollary 1.7. This implies that $K.\Gamma < c(3c_2 - K^2)$, where $c = 1$, if $r_1 = 3$ and $c = 2$,if $r_1 = 2$. From this we obtain $m_i K.C_i < c(3c_2 - K^2) - \sum_{j \neq i} m_j K.C_j$. Substituting into (3.24) we obtain

$$-\Gamma^2 < c\left(\sum_i m_i\right)(3c_2 - K^2) - \sum_i m_i^2(2g_i - 2) - \sum_{i \neq j} m_i m_j C_i.C_j - \sum_{i \neq j} m_i m_j K.C_j.$$

This implies

$$-\Gamma^2 < c\left(\sum_i m_i\right)(3c_2 - K^2) + 2{\sum_i}^{\circ} m_i^2$$

where the second sum is taken over all rational components of Γ. In the next section we will derive an upper bound on $-\Gamma^2$ using the technique of extension construction.

4. More on locally supported moduli. In this section we consider more closely a one-dimensional extension corresponding to an element in $V_0 \setminus V_0''$ (see (3.5) for notation).

Fix a cohomology class ξ in $V_0 \setminus V_0'' \subset H^1(\Theta_X)$ and consider the corresponding extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X \longrightarrow 0.$$

This one-dimensional extension is related to the extension $\mathcal{T}_{V_0}$ in (2.2) by a diagram analogous to the one in (3.8)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & (V_0/\langle\xi\rangle)^* \otimes \mathcal{O}_X & = & (V_0/\langle\xi\rangle)^* \otimes \mathcal{O}_X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & (V_0)^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T}_{V_0} & \longrightarrow & \Omega_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where $\langle\xi\rangle$ is the one-dimensional subspace of $H^1(\Theta_X)$ spanned by ξ. The morphism $\tilde{\mu} : \mathcal{G} \longrightarrow \mathcal{T}_{V_0}$ in (2.2) composed with the morphism $\mathcal{T}_{V_0} \longrightarrow \mathcal{T}_\xi$ in the above diagram yields the following

$$
\begin{array}{ccccccccc}
 & & & & \mathcal{G} & = & \mathcal{G} & & \\
 & & & & \downarrow{\scriptstyle \tilde{\mu}_\xi} & & \downarrow{\scriptstyle \mu} & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X & \longrightarrow & 0.
\end{array}
$$

Factoring out by the torsion of the $coker(\tilde{\mu}_\xi)$ we obtain the following

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{F}_\xi & = & \mathcal{F}_\xi & & \\
 & & & & \downarrow{\scriptstyle \tilde{\nu}_\xi} & & \downarrow{\scriptstyle \nu_\xi} & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{I}_{Z_\xi}(D_\xi) & \longrightarrow & \mathcal{S}_\xi & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{4.1}
$$

where $\mathcal{F}_\xi$ is locally free, $\mathcal{I}_{Z_\xi}$ is the sheaf of ideals of some 0-dimensional subscheme Z_ξ on X and D_ξ is a nonzero effective divisor.

Remark 4.1. *By an argument analogous to the one in the proof of Lemma 3.5 we obtain that for a general choice of* $\xi \in V_0 \setminus V_0''$ *the divisor* $D_\xi = \Gamma + E_2$*, where the components* Γ *and* E_2 *are as in (3.10) while* $c_1(\mathrm{Tor}(coker(\tilde{\mu}_\xi))) = \Gamma$*. In particular, this implies*

$$L = c_1(\mathcal{G}) = K - D_\xi - \Gamma = K - E_2 - 2\Gamma. \tag{4.2}$$

We will now consider the properties of the divisor D_ξ. For this we take the second exterior power of (4.1)

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{O}_X(K-D_\xi) & & & & \\ & & & & \downarrow{\scriptstyle \det\tilde{\nu}_\xi} & \searrow^{\det\nu_\xi} & & & \\ 0 & \longrightarrow & \Omega_X & \longrightarrow & \bigwedge^2 \mathcal{T}_\xi & \longrightarrow & \mathcal{O}_X(K) & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & \mathcal{I}_{Z_\xi}(D_\xi)\otimes\mathcal{F}_\xi & & & & \end{array}$$

The divisor D_ξ is the subscheme where $\det\nu_\xi$ vanishes. This implies that the restriction of $\det\tilde{\nu}_\xi$ to any reduced irreducible component C of D_ξ factors through $\Omega_X\otimes\mathcal{O}_C$, i.e. we obtain a nonzero morphism $\mathcal{O}_C(K-D_\xi)\longrightarrow\Omega_X\otimes\mathcal{O}_C$. Tensoring with $\mathcal{O}_X(-K)$ gives a nonzero morphism $s_C:\mathcal{O}_C(-D_\xi)\longrightarrow\Theta_X\otimes\mathcal{O}_C$ and its zero-locus is $Z^C_\xi = Z_\xi\cap C$.

Remark 4.2. *The above considerations can be applied to any component D of D_ξ. This gives a nonzero morphism*

$$s_D:\mathcal{O}_D(-D_\xi)\longrightarrow\Theta_X\otimes\mathcal{O}_D.$$

To understand properties of the morphism s_C we combine it with the normal sequence for $C\subset X$:

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ & & & & \Theta_C \\ & & & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_C(-D_\xi) & \xrightarrow{s_C} & \Theta_X\otimes\mathcal{O}_C \\ & & & \searrow_{\phi_C} & \downarrow \\ & & & & \mathcal{O}_C(C) \end{array} \tag{4.3}$$

We distinguish two types of irreducible components in D_ξ according to whether the morphism ϕ_C in (4.3) is zero or not.

Type I. $\phi_C\neq 0$. Then the zero-locus Z^C_ξ of s_C is contained in the zero-locus of ϕ_C. This implies

$$deg Z^C_\xi \leq C^2 + C.D_\xi. \tag{4.4}$$

Type II. $\phi_C = 0$. Then s_C factors through Θ_C. Let $\eta_C:\tilde{C}\longrightarrow C$ be the normalization of C. Then we have the following exact sequence of sheaves on $\tilde{C}$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \Theta_{\tilde{C}} & & & & \\
 & & & & \downarrow{\scriptstyle d\eta_C} & & & & \\
0 & \longrightarrow & \eta_C^*\mathcal{O}_C(-D_\xi)\otimes\mathcal{O}_{\tilde{C}}(\eta_C^* Z_\xi^C) & \longrightarrow & \eta_C^*\Theta_X & \longrightarrow & \eta_C^*\mathcal{O}_C(-K+D_\xi)\otimes\mathcal{O}_{\tilde{C}}(-\eta_C^* Z_\xi^C) & \longrightarrow & 0
\end{array}
\tag{4.5}
$$

We claim that the injective morphism in the horizontal sequence of (4.5) factors through $\Theta_{\tilde{C}}$. Indeed, the normalization morphism gives $\eta_{C*}:\mathcal{O}_C\longrightarrow\eta_{C*}\mathcal{O}_{\tilde{C}}$. Tensoring it with Θ_X and combining with the differential of η_C gives the following

$$
\begin{array}{ccccc}
0 & \longrightarrow & \Theta_C & \longrightarrow & \Theta_X\otimes\mathcal{O}_C \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow \\
0 & \longrightarrow & \eta_{C*}\Theta_{\tilde{C}} & \longrightarrow & \Theta_X\otimes\eta_{C*}\mathcal{O}_{\tilde{C}}
\end{array}
$$

where the induced morphism $\Theta_C\longrightarrow\Theta_X\otimes\eta_{C*}\mathcal{O}_{\tilde{C}}$ factors through $\eta_{C*}\Theta_{\tilde{C}}$ (see Theorem 11.9,[5]) as indicated in the above diagram. The natural isomorphism $Hom_{\mathcal{O}_C}(\Theta_C,\eta_{C*}\Theta_{\tilde{C}}) = Hom_{\mathcal{O}_{\tilde{C}}}(\eta_C^*\Theta_C,\Theta_{\tilde{C}})$ implies that the morphism $\eta_C^*\Theta_C \longrightarrow \eta_C^*\Theta_X$ factors through $\Theta_{\tilde{C}}$. On the other hand $\eta_C^*(s_C):\eta_C^*\mathcal{O}_C(-D_\xi)\longrightarrow\eta_C^*\Theta_X$ factors through $\eta_C^*\Theta_C$. So we deduce that $\eta_C^*(s_C)$ factors through $\Theta_{\tilde{C}}$ as claimed. Thus the diagram (4.5) has the following form

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & \eta_C^*\Theta_C & \overset{\eta^*\psi}{\dashrightarrow} & \Theta_{\tilde{C}} & & & & \\
 & & \uparrow & \nearrow{\scriptstyle \tilde{j}} & \downarrow{\scriptstyle d\eta_C} & & & & \\
0 & \longrightarrow & \eta_C^*\mathcal{O}_C(-D_\xi)\otimes\mathcal{O}_{\tilde{C}}(\eta_C^* Z_\xi^C) & \overset{j}{\longrightarrow} & \eta_C^*\Theta_X & \longrightarrow & \eta_C^*\mathcal{O}_C(-K+D_\xi)\otimes\mathcal{O}_{\tilde{C}}(-\eta_C^* Z_\xi^C) & \longrightarrow & 0
\end{array}
\tag{4.6}
$$

Since cokernel of j in the above diagram is locally free it follows that $\tilde{j}$ is an isomorphism as well as all the oblique arrows in (4.6). This implies

$$deg Z_\xi^C \leq deg\left(\eta_C^*(Z_\xi^C)\right) = C.D_\xi - (2g_{\tilde{C}}-2) \tag{4.7}$$

where $g_{\tilde{C}}$ is the genus of the normalization of C.

Remark 4.3. *The fact that $\tilde{j}$ is an isomorphism and j is injective implies that the differential $d\eta_C$ in (4.6) is of maximal rank everywhere, i.e. the morphism $\eta_C:\tilde{C}\longrightarrow C\subset X$ is an immersion.*

Let us decompose the divisor D_ξ as follows

$$D_\xi = D'_\xi + D''_\xi \tag{4.8}$$

where D'_ξ (resp. D''_ξ) is the part of D_ξ composed of the reduced irreducible components of type I (resp. type II). We will be concerned with the part D''_ξ. Let $\left(D''_\xi\right)_{\text{red}}$ be the reduced part of D''_ξ.

Lemma 4.4. *The singular locus of* $\left(D''_\xi\right)_{\text{red}}$ *is contained in* $Z_\xi \cap D''_\xi$.

Proof. The morphism $\mathcal{O}_{\left(D''_\xi\right)_{\text{red}}}(-D_\xi) \longrightarrow \Theta_{\left(D''_\xi\right)_{\text{red}}}$ induced by the morphism $s_{\left(D''_\xi\right)_{\text{red}}}$ is an isomorphism away from $Z_\xi \cap D''_\xi$. This implies that $\Theta_{\left(D''_\xi\right)_{\text{red}},p}$ is a locally free $\mathcal{O}_{\left(D''_\xi\right)_{\text{red}},p}$-module for any $p \in \left(D''_\xi\right)_{\text{red}} \setminus Z_\xi$. By a result of Lipman,[11], this implies that p is a smooth point of $\left(D''_\xi\right)_{\text{red}}$. □

Lemma 4.5. *Let C be a reduced irreducible curve in X such that $C.D_\xi < 0$. Then C is a smooth rational curve with $C.D_\xi = -2$ or -1.*

Proof. The assumption $C.D_\xi < 0$ implies that C is a component of D_ξ with $C^2 < 0$. This implies that C is of type II. From (4.7) and $C.D_\xi < 0$ it follows that $g_{\tilde{C}} = 0$ and $C.D_\xi = -1$ or -2. This yields that $deg(\eta_C^*(Z^C)) \leq 1$. So Z^C must be contained in the smooth locus of C and s_C induces an isomorphism $\mathcal{O}_C(Z^C - D_\xi) \cong \Theta_C$. Hence Θ_C is locally free. Again Lipman's result in [11] gives the smoothness of C. □

The above lemma implies the following.

Proposition 4.6. *If X has no smooth rational curves then the divisor D_ξ is nef.*

Remark 4.7. *If the subspace $V''_0 \neq 0$ and we take the extension $\mathcal{T}_\xi$ corresponding to a nonzero $\xi \in V''_0$, then the divisor D_ξ is a component of Γ (see (3.2) for notation) and we will denote it by Γ_ξ. The results of Lemma 4.5 and Lemma 3.2 imply that the irreducible curves intersecting Γ_ξ negatively are smooth rational curves having intersection -1 or -2 with Γ_ξ. In particular, if X has no smooth rational curves then $V''_0 = 0$ and $V_0 = V'_0$. This combined with Corollary 3.8 yields*

$$dimV_0 < \begin{cases} 2\alpha(3c_2 - K^2) - \Gamma^2 - \frac{1}{6}\left(1 + 2\frac{K.E'_2}{K^2} + \frac{K.F}{K^2}\right)K.F, & if\ r_1 = 3 \\ 3c_2 - K^2 - \Gamma^2, & if\ r_1 = 2 \end{cases}.$$

We will now apply the above considerations to obtain an upper bound for $(-\Gamma^2)$ in the inequality of Corollary 3.10 (compare with Remark 3.11).

First we recall that by Remark 4.1 the divisor $D_\xi = \Gamma + E_2$ for a general $\xi \in V_0 \backslash V''_0$. Let N_2 be the number of rational curves (counted with their multiplicities) subject to Lemma 4.5 and contained in E_2. Then we have the following bound on $-\Gamma^2$.

Lemma 4.8. *Let Γ, E_2 and N_2 be as above. Then the following inequality holds.*

$$-\Gamma^2 \leq (\Gamma + E_2)^2 - K.\Gamma + 2N_2.$$

Proof. From Lemma 4.5 it follows $D_\xi.E_2 = (\Gamma + E_2).E_2 \geq -2N_2$. From (4.2) and Lemma 3.1 we obtain that $(2\Gamma + E_2).\Gamma \geq K.\Gamma$. Taking the sum of these two inequalities we obtain

$$\Gamma^2 + (\Gamma + E_2)^2 \geq K.\Gamma - 2N_2.$$

Hence the assertion of the lemma. □

COROLLARY 4.9. *Let X, Γ, N be as in Corollary 3.10 and let E_2, N_2 be as in Lemma 4.8 then the following holds.*

$$h^1(\Theta_X) \leq \begin{cases} \frac{3}{2}(3c_2 - K^2) + 2(N + N_2), & if\ r_1 = 3 \\ 5(3c_2 - K^2) + 2(N + N_2) - K.E_1 - K.\Gamma, & if\ r_1 = 2 \end{cases}.$$

Proof. By Hodge index $(\Gamma + E_2)^2 \leq \dfrac{((\Gamma + E_2).K)^2}{K^2}$. Using the fact that $\Gamma + E_2$ is a component of E_1 in Corollary 1.7 and arguing as in the beginning of the proof of Corollary 3.8 we have

$$\begin{aligned}(\Gamma + E_2)^2 \leq \frac{((\Gamma + E_2).K)^2}{K^2} &\leq \frac{(K.E_1 - K.(F + \Gamma))^2}{K^2} \\ &\leq \frac{(K.E_1)^2}{K^2} - 2\frac{K.(\Gamma + E_2)}{K^2}K.(F + \Gamma) - \frac{(K.(\Gamma + F))^2}{K^2}.\end{aligned}$$

Substituting this in the inequality of Lemma 4.8 we obtain

$$-\Gamma^2 \leq \frac{(K.E_1)^2}{K^2} - 2\frac{K.(\Gamma + E_2)}{K^2}K.(F + \Gamma) - \frac{(K.(\Gamma + F))^2}{K^2} - K.\Gamma + 2N_2. \tag{4.9}$$

If $r_1 = 3$ then substituting (4.9) in (3.20) and treating the cases according to the properties of the linear system $\mid F \mid$ as it is done in the proof of Corollary 3.10 we obtain

$$h^1(\Theta_X) < \begin{cases} (14\alpha - 4)(3c_2 - K^2) + 2(N + N_2) + 1, & if\ \text{F=0}, \\ (14\alpha - 4)(3c_2 - K^2) + 2(N + N_2), & if\ \mid F \mid\ is\ not\ composed\ of\ a\ pencil, \\ (14\alpha - 4 + \frac{1}{6})(3c_2 - K^2) + 2(N + N_2) + 1, & if\ \mid F \mid\ is\ composed\ of\ a\ pencil. \end{cases} \tag{4.10}$$

This together with $\alpha \leq \frac{3}{8}$ give $h^1(\Theta_X) \leq \dfrac{5}{4}(3c_2 - K^2) + 2(N + N_2)$ in the first two cases and $h^1(\Theta_X) < \dfrac{17}{12}(3c_2 - K^2) + 2(N + N_2) + 1$ in the last case. These two inequalities clearly imply

$$h^1(\Theta_X) \leq \frac{3}{2}(3c_2 - K^2) + 2(N + N_2).$$

If $r_1 = 2$ then we use (1.16) to obtain

$$(\Gamma + E_2)^2 < 4(3c_2 - K^2) - K.E_1.$$

This combined with Lemma 4.8 yields

$$-\Gamma^2 < 4(3c_2 - K^2) + 2N_2 - K.E_1 - K.\Gamma.$$

Substituting this in the inequality of Corollary 3.10 we obtain

$$h^1(\Theta_X) < 5(3c_2 - K^2) + 2(N + N_2) - K.E_1 - K.\Gamma.$$

□

COROLLARY 4.10. *1) If the canonical divisor K of X in Corollary 4.9 is ample then*

$$h^1(\Theta_X) \leq \begin{cases} \frac{21}{4}(3c_2 - K^2), & if\ r_1 = 3 \\ 9(3c_2 - K^2) - 3K.\Gamma, & if\ r_1 = 2 \end{cases}.$$

In particular, $h^1(\Theta_X) \leq 9(3c_2 - K^2)$.
2) If X in Corollary 4.9 does not contain smooth rational curves then

$$h^1(\Theta_X) \leq \begin{cases} \frac{3}{2}(3c_2 - K^2), & if\ r_1 = 3 \\ 5(3c_2 - K^2) - K.E_1 - K.\Gamma, & if\ r_1 = 2 \end{cases}.$$

In particular, $h^1(\Theta_X) \leq 5(3c_2 - K^2)$.

Proof. The ampleness of K implies

$$N + N_2 \leq K.(\Gamma + E_2) \leq K.E_1 - K.(\Gamma + F). \tag{4.11}$$

If $r_1 = 3$ then by Corollary 1.7 $N + N_2 < 2(3c_2 - K^2) - K.(\Gamma + F)$. Treating the cases according the properties of the linear system $\mid F \mid$ as it is done in the proof of Corollary 3.10 we obtain

$$h^1(\Theta_X) < \begin{cases} 14\alpha(3c_2 - K^2) + 1 - 2K.\Gamma, & if\ \text{F=0}, \\ 14\alpha(3c_2 - K^2) - 2K.(\Gamma + F), & if\ \mid F \mid\ is\ not\ composed\ of\ a\ pencil, \\ 14\alpha(3c_2 - K^2) + 1 - \frac{23}{12}K.F - 2K.\Gamma, & if\ \mid F \mid\ is\ composed\ of\ a\ pencil. \end{cases} \tag{4.12}$$

This combined with $\alpha \leq \frac{3}{8}$ implies $h^1(\Theta_X) \leq \frac{21}{4}(3c_2 - K^2)$.

If $r_1 = 2$ then combining (4.11) with the inequality of Corollary 4.9 we obtain

$$h^1(\Theta_X) < 5(3c_2 - K^2) + K.E_1 - 3K.\Gamma.$$

Using the bound on $K.E_1$ in Corollary 1.7 we deduce

$$h^1(\Theta_X) < 9(3c_2 - K^2) - 3K.\Gamma.$$

The second part of the corollary follows from Corollary 4.9 and the fact $N = N_2 = 0$ which is guarantied by Remark 4.7 and the absence of smooth rational curves on X.
□

We close this section by observing that the upper bound for $h^1(\Theta_X)$, as a linear function of $(3c_2 - K^2)$, can not be improved since for every rational α in the interval $]\frac{1}{3}, \frac{1}{2}[$ we have examples of surfaces with Chern numbers c_2, K^2 whose ratio is equal to α and having $h^1(\Theta_X) = c(3c_2 - K^2)$, for some universal constant c.

EXAMPLE 4.11. *The following construction is due to Sommese (see [17]).*

Let X be a smooth minimal surface of general type with $\alpha = \dfrac{c_2}{K^2} = \dfrac{1}{3}$ and a morphism $\pi : X \longrightarrow C$ onto a smooth curve C of genus $g_C \geq 1$ (such surfaces were constructed by Hirzebruch, Inoue, Livné; see [9]).

Take n sheeted unramified cover $\tau : C_n \longrightarrow C$ with C_n connected (this can be done for any $n \geq 1$, since $g_C \geq 1$) follow it by a 2 sheeted branched cover $\sigma : C_{m,n} \longrightarrow C_n$ having $2m$ branch points (this can be done for any $m \geq 1$). Taking the fibre products we obtain

$$\begin{array}{ccccc} X_{m,n} & \xrightarrow{\tilde{\sigma}} & X_n & \xrightarrow{\tilde{\tau}} & X \\ {\scriptstyle \pi''}\downarrow & & {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ C_{m,n} & \xrightarrow{\sigma} & C_n & \xrightarrow{\tau} & C \end{array}$$

where X_n is n sheeted unramified cover of X and $X_{m,n}$ is a double cover of X_n branched along $2m$ fibres of π' which lie over the branch points of σ. Choosing the branch points of σ away from the critical values of π' we obtain $X_{m,n}$ smooth and minimal. The Chern invariants of $X_{m,n}$ are as follows:

$$K^2_{X_{m,n}} = 2nK^2_X + 8m(g-1), \ \ c_2(X_{m,n}) = 2nc_2(X) + 4m(g-1)$$

where g is the genus of a smooth fibre of π. This implies $3c_2(X_{m,n}) - K^2_{X_{m,n}} = 4m(g-1)$ while $h^1(\Theta_{X_{m,n}}) = 2m$, i.e. we obtain surfaces X subject to the following

$$h^1(\Theta_X) = \frac{1}{2(g-1)}(3c_2 - K^2).$$

Furthermore, as Sommese shows in [17], for any rational α in the interval $]\frac{1}{3}, \frac{1}{2}[$ an appropriate choice of m and n in the above construction gives a surface $X_{m,n}$ with $\alpha_{X_{m,n}} = \alpha$.

5. Surfaces with small values of $3c_2 - K^2$. In this section we consider surfaces subject to the following conditions:

$$\begin{array}{ll} 1) & \alpha < \dfrac{1}{2} \\ 2) & h^1(\Theta_X) \geq 2 \\ 3) & 3c_2 - K^2 \leq \dfrac{1}{2}\sqrt{K^2}. \end{array} \tag{5.1}$$

The last numerical assumption arises naturally in view of the inequality in Corollary 1.7 since combined with the Hodge Index it implies that the intersection pairing is negative semidefinite on the sublattice of $NS(X)$ spanned by the irreducible components of the divisor E_1 (see Corollary 1.7 for notation). This, as we will see shortly, imposes strong restrictions on the geometry of E_1 as well as on X.

First we check that the inequality $3c_2 - K^2 \leq \frac{1}{2}\sqrt{K^2}$ insures that $\alpha \leq \frac{3}{8}$. Indeed, if $\alpha > \frac{3}{8}$ then $\frac{1}{2}\sqrt{K^2} \geq 3c_2 - K^2 > \frac{1}{8}K^2$ implying $K^2 < 16$. On the other hand the condition $H^1(\Theta_X) \neq 0$ yields $3c_2 - K^2 \geq 4$. This and 3) of (5.1) give $K^2 \geq 64$. Hence we must have $\alpha \leq \frac{3}{8}$.

From the condition $3c_2 - K^2 \leq \frac{1}{2}\sqrt{K^2}$ and $K^2 \geq 64$ it also follows that the "topological" bound in Proposition 1.1 is $\dfrac{1}{1-2\alpha} - 2 \leq \dfrac{10}{7}$. Hence the Bogomolov-Gieseker inequality fails for the extension bundle $\mathcal{T}$ in (1.1) as soon as $h^1(\Theta_X) \geq 2$. So the conditions (5.1) imply the decomposition of the canonical divisor $K_X = L_1 + E_1$ as in Corollary 1.7 with

$$E_1.K < \begin{cases} 2(3c_2 - K^2) \leq \sqrt{K^2}, & r_1 = 3 \\ \frac{8(3c_2-K^2)}{1+\sqrt{1+16(3\alpha-1)}} \leq \frac{4\sqrt{K^2}}{1+\sqrt{1+\frac{8}{\sqrt{K^2}}}} < 2\sqrt{K^2}, & r_1 = 2 \end{cases}. \tag{5.2}$$

LEMMA 5.1. *Let E be a component of E_1. Then $E^2 \leq 0$.*

Proof. If $r_1 = 3$ then the first inequality in (5.2) implies that $E.K < \sqrt{K^2}$. This combined with the Hodge Index implies $E^2 \leq 0$.

If $r_1 = 2$ then following through the same argument we obtain $E^2 \leq 3$. To improve this estimate we return to the inequality (1.15) rewriting it as follows

$$2(3c_2 - K^2) > E_1.K + 3\frac{(E_1.K)^2}{K^2} - 2E_1^2. \tag{5.3}$$

This together with $3c_2 - K^2 \leq \frac{1}{2}\sqrt{K^2}$ and $E_1^2 \leq 3$ imply $E_1.K < \sqrt{K^2} + 3$. Using the Hodge Index and $K^2 \geq 64$ we obtain $E_1^2 \leq 1$. Substituting this in (5.3) yields $E_1.K < \sqrt{K^2}$. This implies $E.K < \sqrt{K^2}$ and $E^2 \leq 0$. ∎

In what follows we consider the geometric consequences of Lemma 5.1.

COROLLARY 5.2. $E_1.K \leq 3c_2 - K^2$.

Proof. The definition of K_X-destabilizing filtration in (1.2) and the notation in (1.3) imply $E_1 = \sum_{i\geq 2} L_i$. From the proof of Lemma 5.1 and Remark 1.4 it follows that $L_i.K < \sqrt{K^2}$, for every $i \geq 2$. This combined with the Hodge index yields $L_i^2 \leq 0$, for every $i \geq 2$. Substituting in (1.7) we obtain

$$2c_2 \geq 2d_1 - L_1^2 + K^2. \tag{5.4}$$

We know, by Corollary 1.7, that the rank of $\mathcal{T}_1$ is 2 or 3. In both cases we obtain $d_1 \geq \dfrac{1}{3}L_1^2$ (see the proof of Corollary 1.7 for details). Substituting in (5.4) yields

$$2c_2 \geq -\frac{1}{3}L_1^2 + K^2 = -\frac{1}{3}(K - E_1)^2 + K^2 = \frac{2}{3}K^2 + \frac{2}{3}K.E_1 - \frac{1}{3}E_1^2.$$

This and Lemma 5.1 imply $E_1.K \leq 3c_2 - K^2$. ∎

PROPOSITION 5.3. *Let $r_1 = 3$ and $c_1(\mathcal{K}) \neq 0$, where the sheaf $\mathcal{K}$ is as in (1.11). Then the line bundle $\mathcal{K} = \mathcal{O}_X(F)$ is generated by global sections and defines a morphism*

$$f : X \longrightarrow B$$

where B is a smooth projective curve and f is a surjective morphism with connected fibres. Let F_0 be the class of a smooth fibre of f and write $F = f^*(Z)$ for some divisor Z on B. Then $F = (degZ)F_0$ and $(degZ)F_0.K \leq 3c_2 - K^2$.

Proof. We already know that $\mathcal{O}_X(F)$ has at most 0-dimensional base locus (see a) of the proof of Corollary 1.7). From (1.13) it also follows that F is a component of E_1. This and Lemma 5.1 imply that $F^2 \leq 0$. Since F is nef we conclude that $F^2 = 0$. This also implies that $\mathcal{O}_X(F)$ is generated by global sections and the image of the morphism defined by $\mathcal{O}_X(F)$ is a curve. Taking its normalization and then Stein factorization we obtain the asserted morphism.

The last inequality of the proposition follows from the bound on $E_1.K$ in Corollary 5.2. □

Next we turn to the case when $r_1 = 3$ and $\mathcal{K} = \mathcal{O}_X$ or $r_1 = 2$. In this case the subspace V_0 of locally supported moduli of X is of codimension at least 1 in $H^1(\Theta_X)$. Thus V_0 is nonzero and we consider the extension $\mathcal{T}_\xi$ for a general ξ in V_0. This gives rise to the divisor D_ξ which is a nonempty component of E_1 with $D_\xi^2 \leq 0$ and $D_\xi.K \leq 3c_2 - K^2$ (Lemma 5.1 and Corollary 5.2, respectively). We decompose D_ξ into two parts

$$D_\xi = D'_\xi + D''_\xi \tag{5.5}$$

according to type as in (4.8).

LEMMA 5.4. *Assume X has no smooth rational curves. Then $D_\xi^2 = (D'_\xi)^2 = (D''_\xi)^2 = 0$ and the divisors D'_ξ and D''_ξ have the following properties.*

- (i') *Each irreducible component C of D'_ξ is smooth with $\Omega_X \otimes \mathcal{O}_C = \mathcal{O}_C(-C) \oplus \mathcal{O}_C(K + C)$,*
- (ii') *for any two curves C and C' in D'_ξ the intersection $C.C' = 0$,*
- (iii') *Let m_C be the multiplicity of the component C in D'_ξ. Then $\mathcal{O}_C((m_C+1)C) = \mathcal{O}_C$, i.e. $\mathcal{O}_C(C)$ is a torsion point of $Pic^\circ(C)$ and its order of torsion t_C divides $(m_C + 1)$.*
- (i'') *Each irreducible component C of D''_ξ is a rational curve with a single double point,*
- (ii'') *any two distinct irreducible components of D''_ξ are either disjoint or intersect at their (common) double point.*
- (iii'') *Let $p_1, \dots, p_{M''}$ be the set of distinct doubles points as above. Then the decomposition $D''_\xi = \sum_i^{M''} D''_\xi(p_i)$ of D''_ξ into connected components is such that every connected component $D''_\xi(p_i) = \sum_{j=1}^{M''_i} m_{ij}C_{ij}$, where C_{ij}'s are rational curves with a single double point at p_i.*

Proof. Since X has no smooth rational curves Proposition 4.6 implies that D_ξ is nef. This yields $D_\xi.C \geq 0$ for any reduced irreducible component C of D_ξ. From this it follows $D_\xi^2 \geq 0$ which combined with Lemma 5.1 gives $D_\xi^2 = 0$ and $D_\xi.C = 0$ for any reduced irreducible component C of D_ξ.

Let C be a reduced irreducible component of D'_ξ. Then the inequality (4.4) combined with $C^2 \leq 0$ and $C.D_\xi = 0$ yields $C^2 = 0$ and the morphism ϕ_C in (4.3) must be an isomorphism. This implies the assertions $(i') - (iii')$ as well as $(D'_\xi)^2 = D'_\xi.D''_\xi = (D''_\xi)^2 = 0$.

We turn now to the components of D''_ξ. From the above argument we deduce that $C.D''_\xi = 0$ for every reduced irreducible component C of D''_ξ. This together with (4.7) implies $g_{\tilde{C}} \leq 1$. We claim that $g_{\tilde{C}} = 0$. In fact, if $g_{\tilde{C}} = 1$ then (4.7) implies that the the zero-locus of the morphism s_C in (4.3) is empty. This and Lemma 4.4 yield that C is a smooth curve of genus 1 and C does not meet any other component of D''_ξ. This combined with $(D''_\xi).C = 0$ yields $C^2 = 0$. By the adjuction formula for C we obtain $K.C = C^2 = 0$ which contradicts the fact that X is a minimal surface of general type. Thus C is a rational curve. By assumption it can not be smooth. From (4.7) and Remark 4.3 it follows that Z^C_ξ must be a double point of C. This proves (i'') of the lemma.

From Lemma 4.4 it follows that if two distinct irreducible components of D''_ξ intersect then they intersect at their (common) double point. Hence the assertion (ii'') of the lemma.

Turning to (iii'') we put $p_1, \ldots, p_{M''}$ to be distinct double points as above. Then (iii'') follows from (i'') and (ii''). □

Proposition 5.5. *Let D'_ξ and D''_ξ be as in Lemma 5.4 and let M' (resp. M'') be the number of connected components of D'_ξ (resp. D''_ξ). If $M' + M'' \geq 2$, then X admits a morphism*

$$f : X \longrightarrow B$$

where B is a smooth projective curve and f is a surjective morphism with connected fibres. Furthermore, the divisors D'_ξ and D''_ξ are contained in the fibres of f. More precisely, let F_0 be the class of a fibre of f, then $F_0 = t_C C$, where C is an irreducible curve in D'_ξ and t_C is as in (iii') of Lemma 5.4, and $F_0 = \delta_i D''_\xi(p_i)$, where $\delta_i \in \mathbf{Q}^+$ and $D''_\xi(p_i)$ is a connected component of D''_ξ as in (iii'') of Lemma 5.4.

Proof. Consider the set $\{C \mid \text{irreducible curve in } D'_\xi\} \cup \{D''_\xi(p_i) \mid i = 1, \ldots, M''\}$ (see Lemma 5.4 for notation). From the assumption $M' + M'' \geq 2$ it follows that the set contains at least two distinct and hence disjoint divisors, say Σ_1 and Σ_2. From $((K.\Sigma_1)\Sigma_2 - (K.\Sigma_2)\Sigma_1).K = 0$ and the Hodge Index it follows that $((K.\Sigma_1)\Sigma_2 - (K.\Sigma_2)\Sigma_1)^2 \leq 0$. Since $\Sigma_1.\Sigma_2 = (\Sigma_1)^2 = (\Sigma_2)^2 = 0$ it follows that $(K.\Sigma_1)\Sigma_2 - (K.\Sigma_2)\Sigma_1 = 0$ in $NS(X)$. This implies that some power of $\mathcal{O}_X((K.\Sigma_1)\Sigma_2 - (K.\Sigma_2)\Sigma_1)$ lies in the kernel of $H^1(\mathcal{O}^*_X) \longrightarrow H^2(X, \mathbf{Z})$. If $H^1(\mathcal{O}_X) = 0$, then $\mathcal{L} = \mathcal{O}_X(m(K.\Sigma_1)\Sigma_2) = \mathcal{O}_X(m(K.\Sigma_2)\Sigma_1)$ for some positive integer m. This implies that the divisors $m(K.\Sigma_1)\Sigma_2$, $m(K.\Sigma_2)\Sigma_1$ give a base point free pencil in the linear system $\mid \mathcal{L} \mid$, i.e. we have a morphism $X \longrightarrow \mathbf{P}^1$. Taking the Stein factorization yields the assertion.

We turn now to the case $H^1(\mathcal{O}_X) \neq 0$. If $M'' \neq 0$, then the Albanese map of X contracts the divisors $D''_\xi(p_i)$'s to points. From this and $(D''_\xi(p_i))^2 = 0$ it follows that the image of the Albanese map is a curve. This gives the asserted morphism. So we may assume $M'' = 0$. Then $M' \geq 2$ and we take two distinct curves C_1 and C_2 in D'_ξ. Arguing as in the first part of the the proof we obtain that there exist positive integers a_1, a_2 such that $\mathcal{L} = \mathcal{O}_X(a_1C_1 - a_2C_2) \in Pic^\circ(X)$. We may assume that $\mathcal{L}$ is not of finite order (otherwise we are done by the first part of the proof). The restriction $\mathcal{L} \otimes \mathcal{O}_{C_1} = \mathcal{O}_{C_1}(a_1C_1)$ is, by Lemma 5.4, (iii'), a torsion point of $Pic^\circ(C_1)$. Thus the restriction morphism $r_{C_1} : Pic^\circ(X) \longrightarrow Pic^\circ(C_1)$ contains an infinite subgroup of the cyclic group $\{\mathcal{L}^n\}_{n \in \mathbf{Z}}$. Hence $ker(r_{C_1})$ is an abelian subvariety of $Pic^\circ(X)$ of dimension ≥ 1. This implies that the kernel of the differential of r_{C_1} at 0 is nonzero. But this differential is $H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_{C_1})$. By Ramanujam's Lemma (see TheoremA,[3]) nC_1, for some $n \in \mathbf{N}$, moves in an irrational pencil. More precisely, the inclusion $C_1 \subset X$ gives the morphism of the Albanese varieties $\phi : Alb(C_1) \longrightarrow Alb(X)$ and the image of ϕ is a proper abelian subvariety of $Alb(X)$ since the differential of ϕ at 0 is dual to $H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_{C_1})$. Consider $\psi : X \longrightarrow Alb(X)/Im(\phi)$. This morphism contracts C_1 to a point. Since $C_1^2 = 0$ the image of ψ must be a curve. This yields the asserted morphism.

From the construction of the morphism f it follows that the class of its fibre F_0 is a positive rational multiple of either $D''_\xi(p_i)$ or any irreducible C in D'_ξ. Furthermore, in the latter case Lemma 5.4,(iii') and Lemma 8.3,III,[1], imply that C is the support of the multiple fibre of multiplicity t_C, i.e. $t_C C = F_0$. ▯

Corollary 5.6. *Let X be a smooth surface subject to (5.1) and let V_0 be the subspace of locally supported moduli of X. If X contains no rational curves, then*

$$dimV_0 \leq \frac{1}{2} K.D_\xi$$

where D_ξ is as in Lemma 5.4 and $\xi \in V_0$ is general.

Proof. For a general $\xi \in V_0$ the subspace $V_0 \subset H^1(\Theta_X)$ is supported on D_ξ, i.e. $V_0 \subset ker(H^1(\Theta_X) \longrightarrow H^1(\Theta_X(D_\xi)))$. This implies that $dimV_0 \leq h^0(\Theta_X \otimes \mathcal{O}_{D_\xi}(D_\xi))$. Since X has no rational curves Lemma 5.4 implies that D_ξ is composed of the irreducible components of type I only, i.e. $D_\xi = D'_\xi$ (see (5.5)). From this it follows

$$dimV_0 \leq h^0(\Theta_X \otimes \mathcal{O}_{D_\xi}(D_\xi)) \leq \sum_C h^0(\Theta_X \otimes \mathcal{O}_{m_C C}(m_C C)) \tag{5.6}$$

where the sum is taken over the reduced irreducible components of D'_ξ.

Let t_C be the order of torsion of $\mathcal{O}_C(C)$ (see Lemma 5.4, (iii')). Using Lemma 5.4,(i') we obtain

$$h^0(\Theta_X \otimes \mathcal{O}_{m_C C}(m_C C)) \leq \begin{cases} \frac{m_C+1}{t_C}, & if\ t_C \geq 2 \\ m_C, & if\ t_C = 1 \end{cases} .$$

This combined with (5.6) yields

$$dimV_0 \leq \sum_C{}^1 m_C + \sum_C{}^2 \frac{m_C+1}{t_C} \tag{5.7}$$

where the first (resp. second) sum is taken over the reduced irreducible components C of D_ξ with $t_C = 1$ (resp. $t_C \geq 2$). Observing that $K.C \geq 2$ for every irreducible component C of D_ξ we obtain

$$\sum_C^1 m_C \leq \frac{1}{2}K.D_\xi^{(1)} \ and \ \sum_C^2 \frac{m_C}{t_C} \leq \frac{1}{4}K.D_\xi^{(2)} \tag{5.8}$$

where $D_\xi^{(1)}$ (resp. $D_\xi^{(2)}$) is the part of D_ξ composed of the irreducible curves C with $t_C = 1$ (resp. $t_C \geq 2$). Furthermore, $t_C K.C \geq 4$, for every irreducible curve C in $D_\xi^{(2)}$. This yields $\sum_C^2 \frac{1}{t_C} \leq \frac{1}{4}K.D_\xi^{(2)}$. Combining this with the second inequality in (5.8) gives

$$\sum_C^2 \frac{m_C + 1}{t_C} \leq \frac{1}{2}K.D_\xi^{(2)}.$$

Substituting this and the first inequality in (5.8) into (5.7) we obtain

$$dim V_0 \leq \frac{1}{2}K.D_\xi^{(1)} + \frac{1}{2}K.D_\xi^{(2)} = \frac{1}{2}K.D_\xi.$$

□

Corollary 5.7. *Let X be as in Corollary 5.6. Then*

$$h^1(\Theta_X) \leq \begin{cases} \frac{1}{2}(3c_2 - K^2), & if\ r_1 = 2 \\ \frac{1}{2}(3c_2 - K^2) + 1, & if\ r_1 = 3 \end{cases}.$$

Proof. If $r_1 = 2$ then $h^1(\Theta_X) = dim V_0$. This combined with Corollary 5.6 and Corollary 5.2 gives $h^1(\Theta_X) \leq \frac{1}{2}(3c_2 - K^2)$.

If $r_1 = 3$ then $h^1(\Theta_X) = dim V_0 + dim V_1$ and we consider two case according to properties of the linear system $\mid F \mid$, where F is as in Proposition 5.3.

Case: $F = 0$. Then $dim V_1 = 1$ and $h^1(\Theta_X) = dim V_0 + 1$. Arguing as in the case $r_1 = 2$ we obtain $h^1(\Theta_X) \leq \frac{1}{2}(3c_2 - K^2) + 1$.

Case: $F \neq 0$. Then by Proposition 5.3 the linear system $\mid F \mid$ defines a fibration $f : X \longrightarrow B$ with the class of a smooth fibre F_0 and $F = deg(Z)F_0$ for some divisor Z on B. In particular, $dim V_1 \leq deg(Z) + 1 = \frac{K.F}{K.F_0} + 1$. From this it follows

$$h^1(\Theta_X) \leq \frac{K.F}{K.F_0} + 1 + dim V_0. \tag{5.9}$$

To bound $dim V_0$ we use the argument in the proof of Corollary 5.6 with the following modification. By Proposition 5.5 the irreducible components C of D_ξ are related to the class of the fibre F_0 as follows: $t_C C = F_0$, where t_C is the order of torsion of $\mathcal{O}_C(C)$. Using this relation and the notation in (5.8) we obtain

$$\sum_C^1 m_C + \sum_C^2 \frac{m_C}{t_C} = \frac{K.D_\xi^{(1)}}{K.F_0} + \frac{K.D_\xi^{(2)}}{K.F_0} = \frac{K.D_\xi}{K.F_0} \ and \ \sum_C^2 \frac{1}{t_C} = \sum_C^2 \frac{K.C}{K.F_0} \leq \frac{K.D_\xi^{(2)}}{K.F_0}. \tag{5.10}$$

Substituting this in (5.7) we obtain

$$dimV_0 \leq \frac{1}{K.F_0}\left(K.D_\xi + K.D^{(2)}_\xi\right).$$

This together with (5.9) yield

$$\begin{aligned} h^1(\Theta_X) &\leq \frac{1}{K.F_0}\left(K.F + K.D_\xi + K.D^{(2)}_\xi\right) + 1 \\ &= \frac{1}{K.F_0}\left(K.F + 2K.D_\xi - K.D^{(1)}_\xi\right) + 1 \\ &\leq \frac{1}{K.F_0}\left(2K.E_1 - K.F - K.D^{(1)}_\xi\right) + 1 \end{aligned} \tag{5.11}$$

where the last inequality follows from the fact that $F + D_\xi$ is a component of E_1 (see (1.13), (2.3), (3.3) and Remark 4.1). Combining (5.11) with Corollary 5.2 we obtain

$$h^1(\Theta_X) \leq \frac{2}{K.F_0}(3c_2 - K^2) + 1 - deg(Z)$$

where Z is as in Proposition 5.3. Recalling that $K.F_0 \geq 4$ (see **Claim** in the proof of Corollary 2.4) we deduce $h^1(\Theta_X) \leq \frac{1}{2}(3c_2 - K^2)$ □

Proposition 5.8. *Let X be a smooth surface subject to (5.1) and let Ω_X be ample. Then X has divisorial moduli only. More precisely, let $\mathcal{O}_X(F)$ be as in Proposition 5.3 then there is an isomorphism*

$$H^0(\mathcal{O}_X(F)) \xrightarrow{\sim} H^1(\Theta_X). \tag{5.12}$$

Furthermore, the K_X-maximal destabilizing subsheaf $\mathcal{T}_1$ is given by the extension

$$0 \longrightarrow \mathcal{O}_X(-F) \longrightarrow \mathcal{T}_1 \longrightarrow \Omega_X \longrightarrow 0 \tag{5.13}$$

and the cup-product with this extension (viewed as an element of $H^1(\Theta_X(-F))$) defines the isomorphism (5.12).

Proof. The ampleness of Ω_X implies that X has neither rational curves nor the curves subject to (i') of Lemma 5.4. This yields $V_0 = 0$, i.e. X has the divisorial moduli only and the sheaf $\mathcal{T}_1$ in (1.11) fits into the following exact sequence

$$0 \longrightarrow \mathcal{O}_X(-F) \longrightarrow \mathcal{T}_1 \longrightarrow \mathcal{T}'_1 \longrightarrow 0. \tag{5.14}$$

Since the linear system $\mid F \mid$ is base point free and $F^2 = 0$ we obtain that the sequence (1.12) has the form

$$0 \longrightarrow \mathcal{R}^* \longrightarrow H^1(\Theta_X) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(F) \longrightarrow 0. \tag{5.15}$$

Furthermore, the vanishing of V_0 implies that the homomorphism

$$H^1(\Theta_X) \longrightarrow H^0(\mathcal{O}_X(F)) \tag{5.16}$$

arising from the cohomology sequence of (5.15) is injective. Thus we have the first assertion of the proposition.

In view of the sequence (5.14) the remaining part of the proposition follows as soon as we show that $\mathcal{T}_1' = \Omega_X$ or, equivalently, $c_1(\mathcal{S}_1) = 0$, where $\mathcal{S}_1$ is as in (1.11). In order to do this we consider the extension

$$0 \longrightarrow V^* \otimes \mathcal{O}_X \longrightarrow \mathcal{T}_V \longrightarrow \Omega_X \longrightarrow 0 \tag{5.17}$$

defined by a general 2-dimensional subspace V of $H^1(\Theta_X)$.

The inclusion $\mathcal{T}_1 \longrightarrow \mathcal{T}$ induces a monomorphism $\mathcal{T}_1 \longrightarrow \mathcal{T}_V$ which gives rise to the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-F) & \longrightarrow & \mathcal{T}_1 & \longrightarrow & \mathcal{T}_1' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T}_V & \longrightarrow & \Omega_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(F) & \longrightarrow & \mathcal{Q}_V & \longrightarrow & \mathcal{S}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.18}$$

Factoring out by the torsion of $\mathcal{Q}_V$ we obtain

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-F) & \longrightarrow & \mathcal{F}_V & \longrightarrow & \mathcal{F}_V' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T}_V & \longrightarrow & \Omega_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(F) & \longrightarrow & \mathcal{I}_{Z_V}(D_V) & \longrightarrow & \mathcal{S}_V' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.19}$$

where $\mathcal{I}_{Z_V}$ is the sheaf of ideals of some 0-dimensional subscheme Z_V of X.

Claim. $\mathcal{F}_V' = \Omega_X$.

Let us assume this and complete the proof of the proposition.
From the claim it follows that the sheaf $\mathcal{F}_V$ fits into the following exact sequence

$$0 \longrightarrow \mathcal{O}_X(-F) \longrightarrow \mathcal{F}_V \longrightarrow \Omega_X \longrightarrow 0\,. \tag{5.20}$$

This sequence can be viewed as an (nonzero) element of $Ext^1(\Omega_X, \mathcal{O}_X(-F)) = H^1(\Theta_X(-F))$. Denote the corresponding cohomology class in $H^1(\Theta_X(-F))$ by τ.

The cup-product with τ gives rise to the linear map

$$H^0(\mathcal{O}_X(F)) \xrightarrow{\tau} H^1(\Theta_X)\,. \tag{5.21}$$

The ampleness of Ω_X guaranties that this homomorphism is injective. Combining this with the injectivity in (5.16) we deduce that the cup-product in (5.21) is an isomorphism. Furthermore, this also implies that $\mathcal{F}_V$ lifts to $\mathcal{T}$, i.e. we have a morphism of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(-F) & \longrightarrow & \mathcal{F}_V & \longrightarrow & \Omega_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H^1(\Theta_X)^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T} & \longrightarrow & \Omega_X & \longrightarrow & 0
\end{array}$$

The maximality of $\mathcal{T}_1$ yields the equality $\mathcal{T}_1 = \mathcal{F}_V$.

Proof of Claim. Take a nonzero element ξ in V and consider the corresponding one-dimensional extension $\mathcal{T}_\xi$

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X \longrightarrow 0.$$

This extension is related to $\mathcal{T}_V$ by the following exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{T}_V \longrightarrow \mathcal{T}_\xi \longrightarrow 0.$$

Putting this together with vertical sequence in the middle of (5.19) we obtain

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{F}_V & = & \mathcal{F}_V & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{T}_V & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{I}_{Z_V}(D_V) & \longrightarrow & \mathcal{P}_\xi & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{5.22}$$

In particular, we obtain a distinguished divisor in the linear system $\mid D_V \mid$ which we continue to denote D_V. From (5.19) we deduce

$$D_V = F_V + E_V \tag{5.23}$$

where F_V is the divisor in the linear system $\mid F \mid$ corresponding to the line $(V/\langle\xi\rangle)^*$ in $V^* \subset H^0(\mathcal{O}_X(F))$, and $E_V = c_1(\mathcal{S}'_V)$, where $\mathcal{S}'_V$ is as in (5.19). We will now establish the following.

(i) D_V is nef

(ii) $Z_V = \emptyset$

The argument is essentially the same as in the considerations of (4.1). Namely, we take the third exterior power of (5.22)

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{O}_X(K-D_V) & \searrow & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \bigwedge^2 \mathcal{T}_\xi & \longrightarrow & \bigwedge^3 \mathcal{T}_V & \longrightarrow & \mathcal{O}_X(K) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{I}_{Z_V}(D_V) \otimes \bigwedge^2 \mathcal{F}_V & & & &
\end{array}
\tag{5.24}
$$

The restriction of the above diagram to any reduced irreducible component C of D_V gives a nonzero morphism $\mathcal{O}_C(K - D_V) \longrightarrow \bigwedge^2 \mathcal{T}_\xi \otimes \mathcal{O}_C = \mathcal{T}^*_\xi(K) \otimes \mathcal{O}_C$. Tensoring with $\mathcal{O}_X(-K)$ we obtain a nonzero morphism $\mathcal{O}_C(-D_V) \longrightarrow \mathcal{T}^*_\xi \otimes \mathcal{O}_C$. Combining this with the defining sequence for $\mathcal{T}_\xi$ yields

$$
\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
 & & & & \Theta_X \otimes \mathcal{O}_C \\
 & & & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_C(-D_V) & \longrightarrow & \mathcal{T}^*_\xi \otimes \mathcal{O}_C \\
 & & & {\scriptstyle \phi_C} \searrow & \downarrow \\
 & & & & \mathcal{O}_C \\
 & & & & \downarrow \\
 & & & & 0
\end{array}
\tag{5.25}
$$

Observe that $D_V.C \geq 0$ since otherwise $\phi_C = 0$ and we obtain a nonzero morphism $\mathcal{O}_C(-D_V) \longrightarrow \Theta_X \otimes \mathcal{O}_C$ which contradicts the ampleness of Ω_X. This proves that D_V is nef. But D_V is a component of E_1 which implies, by Lemma 5.1, that $D_V^2 \leq 0$. Thus we must have $D_V.C = 0$ for every reduced irreducible component C of D_V. Returning to (5.25) we see that the morphism ϕ_C must be nonzero and, hence, it is an isomorphism for every reduced irreducible component C of D_V. This implies, in particular, that $Z_V = \emptyset$.

Once we know that $Z_V = \emptyset$ the diagram (5.19) becomes as follows

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(-F) & \longrightarrow & \mathcal{F}_V & \longrightarrow & \mathcal{F}'_V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V^* \otimes \mathcal{O}_X & \longrightarrow & \mathcal{T}_V & \longrightarrow & \Omega_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(F) & \longrightarrow & \mathcal{O}_X(D_V) & \longrightarrow & \mathcal{S}'_V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{5.26}$$

Assume $E_V = c_1(\mathcal{S}'_V) = D_V - F \neq 0$. We will show that this produces a nonzero subspace of locally supported moduli which is impossible by the first assertion of the proposition. To this end dualize the sequence at the bottom of the above diagram to obtain

$$0 \longrightarrow \mathcal{O}_X(-D_V) \longrightarrow \mathcal{O}_X(-F) \longrightarrow \mathcal{E}xt^1(\mathcal{S}'_V, \mathcal{O}_X) \longrightarrow 0.$$

This implies that $\mathcal{E}xt^1(\mathcal{S}'_V, \mathcal{O}_X) = \mathcal{O}_{E_V}(-F)$. Furthermore, since F_V is a component of D_V we have $D_V.F = 0 = E_V.F$ which implies $\mathcal{E}xt^1(\mathcal{S}'_V, \mathcal{O}_X) = \mathcal{O}_{E_V}$. Now dualizing the column on the right in (5.26) we obtain

$$0 \longrightarrow \Theta_X \longrightarrow (\mathcal{F}'_V)^* \longrightarrow \mathcal{O}_{E_V} \longrightarrow 0$$

which yields that $ker(H^1(\Theta_X) \longrightarrow H^1(\mathcal{F}'_V)) \neq 0$. But this is a subspace of locally supported moduli which must be zero by the first part of the proof of the proposition. Thus $c_1(\mathcal{S}'_V) = 0$ and we obtain the equality $\mathcal{F}'_V = \Omega_X$. □

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