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A SECOND-ORDER INVARIANT OF THE NOETHER-LEFSCHETZ LOCUS AND TWO APPLICATIONS*

CATRIONA MACLEAN†

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1. Introduction and statement of results. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d cut out by a polynomial

$$F \in k[X_0, \ldots X_3].$$

We will be interested in the following questions. What curves does X contain? Can these curves be classified?

For a generic X of degree $d \geq 4$, this question was answered in the 20's, when the Noether-Lefschetz theorem was proved by Lefschetz.

Theorem 1 (Lefschetz). *If X is a generic smooth surface of degree $d \geq 4$ in $\mathbb{P}^3$ then for any curve $C \subset X$ there exists a surface Y such that $C = X \cap Y$.*

A curve C which has the property that $C = X \cap Y$ for some surface Y will be said to be a complete intersection in X.

This theorem says essentially that if X is generic then the set of curves contained in X is well understood and is as simple as possible. In this article we will study the distribution of surfaces for which the conclusion of Theorem 1 does not hold — or in other words, surfaces containing curves which are not well understood.

Throughout the rest of this article, we will denote by U_d the space parameterising smooth degree d surfaces in $\mathbb{P}^3$. We define the *Noether-Lefschetz locus*, which we denote by NL_d, as follows:

$$X \in NL_d \Leftrightarrow X \text{contains a curve } C \text{ which is not a complete intersection in } X$$

which, by the Leftschetz $(1,1)$ theorem, can alternatively be written as

$$X \in NL_d \Leftrightarrow H^{1,1}_{\text{prim}}(X, \mathbb{Z}) \neq 0.$$

Theorem 1 says that NL_d is a countable union of proper subvarieties of U_d. Throughout the rest of the article, NL will denote one of these subvarieties. Ciliberto et al. showed in [3] that NL_d is dense in the Zariski and complex topologies.

It is interesting to have an idea of the size of the components of NL_d, since this gives us some idea of how rare badly-behaved curves are. An initial (very rough) estimate comes out of Hodge theory. Any component NL can be expressed as the

*Received October 1, 2004; accepted for publication June 15, 2005.
†Institut Fourier, 100 Rue des Mathématiques, Domaine Universitaire, Saint Martin d'Héres, Isére, France (Catriona.MacLean@ujf-grenoble.fr).

zero locus of a section of a vector bundle of dimension $\binom{d-1}{3}$. Hence the codimension of NL is at most $\binom{d-1}{3}$, and we expect that it will in fact be *equal* to $\binom{d-1}{3}$. If a component NL has codimension strictly less than $\binom{d-1}{3}$, then we will say it is *exceptional*.

Since the dimension of U_d is $\binom{d+3}{3} - 1$, we expect NL to be very small compared with U_d. Unfortunately, this bound is highly unsatisfactory, because in the simplest examples it fails to be exact by a very large margin. For example, the set of all surfaces containing a line is a Noether-Lefschetz locus of co-dimension $d-3$.

The principle that has guided much of the work on NL_d is that *very large components should be geometrically predictable.* More precisely, the codimension estimate of $\binom{d-1}{3}$ was based on *cohomological* arguments, which do not take into account geometric information. Suppose X contains a curve C of low degree which is not a complete intersection in X. The Noether-Lefschetz locus corresponding to C (which will be precisely defined in section 2.1), then has codimension $\leq H^0(\mathcal{O}_C(d))$. This is much less than $\binom{d-1}{3}$ when $d \gg \deg(C)$.

The hope is that when a component NL is large, this should always be explained by the presence of low-degree curves in the corresponding surfaces. Harris conjectured that the number of exceptional loci should be finite: Green and Ciliberto went further, proposing the following conjecture (which implies Harris's) and which we will call henceforth the Green-Ciliberto conjecture.

Conjecture 1 (Green-Ciliberto). *If* $\mathrm{codim}(NL) < \binom{d-1}{3}$ *, and X is a point of NL then there exists a curve $C \in X$ and a surface $Y \in \mathbb{P}^3$ of degree $\leq d-4$ such that*

1. $C \subset X \cap Y$
2. *C is not a complete intersection in X.*

We will discuss the motivation for this conjecture in section 5. It has been proved by Voisin [16] that Harris's conjecture (and a fortiori the Green-Ciliberto conjecture) does not hold. However, it is interesting to ask whether a weakened version of the conjeture may hold. The main results which have been proved in this direction so far are the following.

- Voisin [13] and Green [6] prove that for $d \geq 5$ every exceptional NL component has codimension at least $d-3$, and for $d \geq 5$ this bound is obtained only for the component of surfaces containing a line.
- Voisin, [14] proves that for $d \geq 5$, the second largest NL component of U_d has codimension $2d-7$, and this bound is achieved only by the space of surfaces containing a conic.
- Otwinowska, [11] and [12], defines an analogue of NL_d for hypersurfaces X of a variety Y of dimension $2n+1$. She then proves that for any b, and for $d \gg b$, if $X \in NL$ has codimension $\leq \frac{bd^n}{n!}$, then X contains an n-cycle of degree $\leq b$.

All of this work relies on a fundamental paper of Carlson and Griffiths [1], in which they give an algebraic expression for the tangent space of NL_d.

Our aim in this paper is to extend the results of Carlson and Griffiths via a second-order infinitesimal study of NL. After summarising the results of Carlson and Griffiths in section 2, we calculate in section 3 an invariant which, together with the work of Carlson and Griffiths, describes the infinitesimal geometry of NL at X up to second order. This is the second-order invariant mentioned in the title.

This new invariant gives rise to a new family of equations when X is a singular point of NL or NL is exceptional. In section 4 we will use these equations to prove Theorem 2, which completes the classification of exceptional Noether-Lefschetz loci in U_5 by finding all non-reduced components. (The reduced exceptional loci were determined by Voisin in [14]). In section 5 we will use them to prove Theorem 3, which shows that a weakened version of the Green-Ciliberto conjecture holds for reduced Noether-Lefschetz loci.

THEOREM 2. *Let NL be a non-reduced Noether-Lefschetz locus in U_5. The reduction of NL is the space of all surfaces X with the property that there exists a hyperplane H such that $H \cap X$ contains two lines.*

In Proposition 1 of section 4, we show that it is indeed the case that if X has this property then X lies on certain non-reduced Noether-Lefschetz loci. More precisely, if L_1 and L_2 are the two lines in question, and $\gamma = \alpha[L_1]_{\text{prim}} + \beta[L_2]_{\text{prim}}$, where α and β are distinct non-zero rational numbers, then $NL(\gamma)$ is non-reduced.
In fact we will prove a stronger result, which is given in detail on page 12 (Theorem 8).

THEOREM 3. *Suppose that $e \leq \frac{d-1}{2}$. There exists an integer, $\phi_e(d)$ such that if NL is reduced, $X \in NL$ and codim(NL) $\leq \phi_e(d)$, then there exists a curve $C \in X$ and a surface $Y \in \mathbb{P}^3$ of degree e such that*

1. $C \subset X \cap Y$

2. C is not a complete intersection in X.

Further, $\phi_{\frac{d-1}{2}}(d) = O(d^3)$.

Again, the result actually proved is somewhat stronger (see page 25, Theorem 9), but rather complicated to state.

2. Preliminaries.

2.1. Notation. Throughout the rest of this article, γ will be a non-zero element of $H^{1,1}_{\text{prim}}(X, \mathbb{Z})$, and O will be some contractible neighbourhood of X in U_d. When C is a curve in X we will denote by $[C]_{\text{prim}}$ the primitive part of the cohomology class of C. When γ is of the form $\sum_i \lambda_i [C_i]$ and $D \subset X$ has the property that $C_i \subset D$ for all i, we will say that γ is supported on D. Unless otherwise stated, we will work over O. We now define $NL(\gamma)$, the Noether-Lefschetz locus associated to γ.

Let $\mathcal{H}^i$ be the vector bundle whose fibre over the point X is $H^i(X, \mathbb{C})$. This vector bundle is equipped with the flat Gauss-Manin connection ∇ and has a holomorphic structure. The Hodge filtration on $H^i(X, \mathbb{C})$ gives rise to a descending filtration $F^p(\mathcal{H}^i) \subset \mathcal{H}^i$ by holomorphic sub-vector bundles. We write $F^p/F^{p+1} = \mathcal{H}^{p,q}$. We denote by $\overline{\gamma}$ the section of $\mathcal{H}^2|_O$ induced by flat transport of γ. There is a projection $\pi : \mathcal{H}^2 \to \mathcal{H}^{0,2}$ and we denote $\pi(\overline{\gamma})$ by $\overline{\gamma}^{0,2}$. We now define:

DEFINITION 1. *The space $NL(\gamma)$ is the zero locus in O of the section $\overline{\gamma}^{0,2}$.*

By the Noether-Lefschetz locus associated to a curve C, we mean $NL([C]_{\text{prim}})$. Any Noether-Lefschetz locus is locally equal to $NL(\gamma)$ for some γ. The Zariski tangent space to $NL(\gamma)$ was described by Carlson and Griffiths in [1].

2.2. The work of Carlson and Griffiths. In this section, we summarise the results of [8] and [1]. A summary of this work may also be found in [17].

Griffiths showed in [7] that

$$\nabla\left(F^p\mathcal{H}^i\right) \subset F^{p-1}(\mathcal{H}^i) \otimes \Omega_{U_d}.$$

Quotienting, it follows that ∇ induces an $\mathcal{O}_{U_d}$-linear map

$$\overline{\nabla} : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega_{U_d}.$$

For any n, S^n will denote the space of degree n homogeneous polynomials in variables X_0, X_1, X_2, X_3. Choose $P \in S^{pd-4}$ and let Ω be the canonical section of the bundle $K_{\mathbb{P}^3}(4)$. The form $\frac{P\Omega}{F^p}$ is then a holomorphic 3-form on $\mathbb{P}^3 - X$ and has a class in $H^3(\mathbb{P}^3 - X, \mathbb{C})$. The group $H^3(\mathbb{P}^3 - X, \mathbb{C})$ maps via the residue mapping res_X to $H^2_{\mathrm{prim}}(X, \mathbb{C})$: there is therefore in particular a composed mapping

$$\mathrm{res}_X : S^{pd-4} \to H^2_{\mathrm{prim}}(X, \mathbb{C}),$$

given by

$$\mathrm{res}_X(P) = \mathrm{res}_X\left(\left[\frac{P\Omega}{F^p}\right]\right).$$

It is proved in [8] (see also [1] and [17]) that

$$\mathrm{Im}(\mathrm{res}_X) = F^{3-p}H^2_{\mathrm{prim}}(X, \mathbb{C}),$$

and that

$$\mathrm{res}_X(Q) \in F^{2-p}H^2(X, \mathbb{C}) \text{ if and only if } Q \in \left\langle \frac{\partial F}{\partial X_0}, \dots, \frac{\partial F}{\partial X_3} \right\rangle.$$

We denote by J_F (the Jacobian ideal of F) the homogeneous ideal $\left\langle \frac{\partial F}{\partial X_0}, \dots, \frac{\partial F}{\partial X_3} \right\rangle$. We further denote by R_F (the Jacobian ring of F) the graded ring $k[X_0 \dots X_3]/J_F$. The results above can be summarised as follows.

Theorem 4 (Carlson, Griffiths). *The map* res_X *induces a natural isomorphism between* R_F^{pd-4} *and* $H^{3-p,p-1}_{\mathrm{prim}}(X, \mathbb{C})$.

In [1], the infinitesimal variation of this Hodge structure with variations of the hypersurface X was also calculated. We have a map

$$\overline{\nabla} : \mathcal{H}^{p,q}_{\mathrm{prim}} \to \mathrm{Hom}(T_{U_d}, \mathcal{H}^{p-1,q+1}_{\mathrm{prim}}).$$

Carlson and Griffiths showed that after making the following identifications

1. $T_{U_d}(F) = S^d / \langle F \rangle$,
2. $\mathcal{H}^{p,q}_{\mathrm{prim}}(F) = R_F^{(3-p)d-4}$,
3. $\mathcal{H}^{p-1,q+1}_{\mathrm{prim}}(F) = R_F^{(4-p)d-4}$,

we have the following result.

Theorem 5 (Carlson, Griffiths). *Up to multiplication by a constant,* $\overline{\nabla}_F(\mathrm{res}_X P)$ *is identified with the multiplication map*

$$\cdot P : R_F^d \to R_F^{(4-p)d-4}.$$

Henceforth, P will denote an element of S^{2p-4} such that $\mathrm{res}_X(P) = \gamma$. We have the following description of the tangent space to $NL(\gamma) = \mathrm{zero}(\overline{\gamma}^{0,2})$.

$$T_{NL(\gamma)}(X) = \mathrm{Ker}(\cdot P : R_F^d \to R_F^{3d-4}).$$

or in other words

$$H \in T_{NL(\gamma)}(X) \text{ if and only if } \text{ there exist } Q_i \in S^{2d-3} \text{ such that } PH = \sum_{i=0}^{3} Q_i \frac{\partial F}{\partial X_i}.$$

We will lean heavily in what follows on the following classical result, due to Macaulay (which may be found in [4], for example).

Theorem 6 (Macaulay). *The ring* R_F *is a Gorenstein graded ring. In other words,* $R_F^{4d-8} = \mathbb{C}$ *and the multiplication map*

$$R_F^a \otimes R_F^{4d-8-a} \to R_F^{4d-8} = \mathbb{C}$$

is a perfect pairing.

3. The second order invariant of IVHS. Throughout the rest of this article, G and H will be degree d polynomials contained in $T_{NL(\gamma)}(X)$, and $\{Q_i\}_{i=0}^3$, $\{R_i\}_{i=0}^3$ will be degree $2d-3$ polynomials such that

$$PG = \sum_{i=0}^{3} Q_i \frac{\partial F}{\partial X_i} \text{ and } PH = \sum_{i=0}^{3} R_i \frac{\partial F}{\partial X_i}.$$

We will extend the work of Carlson and Griffiths to second order using the fundamental quadratic form of a section of a vector bundle— a generalisation of the Hessian, which we now briefly recall.

Let M be a smooth m-dimensional complex scheme, V a rank-r vector bundle on M and σ a section of V. We denote by W the zero scheme of σ and choose a point x of W. We choose also holomorphic co-ordinates, $z_1, \ldots, z_m$, on some neighbourhood of x and a trivialisation of V near x. Having picked such trivialisations, σ becomes an r-tuple of holomorphic functions $(\sigma_1, \sigma_2 \ldots \sigma_r)$. We define the map

$$d\sigma_x : T_U(x) \to V_x$$

by

$$d\sigma_x(\sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial z_i}) = \sum_{i=1}^{n} \alpha_i \frac{\partial \sigma}{\partial z_i}. \tag{1}$$

It can be shown that this map is independent of the choice of trivialisation and of local co-ordinates. The space $\mathrm{Ker}(d\sigma_x)$ is the Zariski tangent space to W at x. We define the fundamental quadratic form, $q_{\sigma,x}$, of σ at x as follows.

$$q_{\sigma,x} : T_W(x) \otimes T_W(x) \to V_x/\mathrm{Im}(d\sigma_x)$$

is defined by

$$q_{\sigma,x}\left(\sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial z_i}, \sum_{j=1}^{m} \beta_j \frac{\partial}{\partial z_j}\right) = \sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial z_i}\left(\sum_{j=1}^{m} \beta_j \frac{\partial}{\partial z_j}(\sigma)\right).$$

This, similarly, is independent of the choice of local trivialisation of V and the choice of local co-ordinates z_j.

Remark 1. *If x is a smooth point of W_{red} and $\mathrm{rk}(\mathrm{Ker}(d\sigma))$ is constant in a neighbourhood of x, then $q(u,w) = 0$ for any $u \in T_{W_{\mathrm{red}}}$. Indeed, we may choose local co-ordinates on U in such a way that $w = \frac{\partial}{\partial z_1}$ and $\frac{\partial \sigma}{\partial z_1}|_{W_{\mathrm{red}}} = 0$.*

As an example, if M is the space $\mathbb{C}^2$, V is the trivial vector bundle $\mathbb{C}$ and σ is the section xy, then the space $V_x/\mathrm{Im}(d\sigma_x)$ is non-zero only at the point $x = (0,0)$ and the form $q_{\sigma,x} : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}$ is given by

$$q((a,b),(c,d)) = ac\frac{\partial^2 xy}{\partial x \partial x} + ad\frac{\partial^2 xy}{\partial x \partial y} + bc\frac{\partial^2 xy}{\partial y \partial x} + bd\frac{\partial^2 xy}{\partial y \partial y} = ad + bc.$$

We are now in a position to state our result.

Theorem 7. *The fundamental quadratic form*

$$q_{\overline{\gamma},X} : \mathrm{Sym}^2(T_{NL(\gamma)}(X)) \to R_F^{3d-4}/\mathrm{Im}(\cdot P)$$

is given by

$$q(G,H) = \sum_{i=0}^{3}\left(H\frac{\partial Q_i}{\partial X_i} - R_i\frac{\partial G}{\partial X_i}\right).$$

The attentive reader will be surprised to see that this form is apparently not symmetric in G and H. This is, however, only apparent: we have the following lemma.

Lemma 1. *For all H and G in $T_{NL(\gamma)}(X)$,*

$$q(G,H) = q(H,G).$$

Proof of Lemma 1. We know that

$$\sum_{i=0}^{3} GR_i\frac{\partial F}{\partial X_i} = GHP = \sum_{i=0}^{3} HQ_i\frac{\partial F}{\partial X_i}.$$

Rearranging, we get that

$$\sum_{i=0}^{3}(GR_i - HQ_i)\frac{\partial F}{\partial X_i} = 0.$$

Since the $\frac{\partial F}{\partial X_i}$ form a regular sequence, there exist $A_{i,j}$, polynomials, such that

1. $A_{i,j} = -A_{j,i}$,
2. $GR_i - HQ_i = \sum_{j=0}^{3} A_{i,j} \frac{\partial F}{\partial X_i}$.

Deriving this second equation and summing over i, we get that

$$\sum_{i=0}^{3}\left(G\frac{\partial R_i}{\partial X_i} + R_i\frac{\partial G}{\partial X_i}\right) - \sum_{i=0}^{3}\left(H\frac{\partial Q_i}{\partial X_i} - Q_i\frac{\partial H}{\partial X_i}\right) = \sum_{i,j}\left(\frac{\partial A_{i,j}}{\partial X_i}\frac{\partial F}{\partial X_i} + A_{i,j}\frac{\partial F}{\partial X_i \partial X_j}\right).$$

From this we deduce that

$$\sum_{i=0}^{3}\left(G\frac{\partial R_i}{\partial X_i} + R_i\frac{\partial G}{\partial X_i}\right) - \sum_{i=0}^{3}\left(H\frac{\partial Q_i}{\partial X_i} + Q_i\frac{\partial H}{\partial X_i}\right) \in \left\langle \frac{\partial F}{\partial X_i}\right\rangle.$$

This completes the proof of Lemma 1. □

3.1. The fundamental quadratic form: an explicit description (proof of theorem 7). Recall that G, H are elements of $T_{NL(\gamma)}(X)$. When f is a section of a vector bundle vanishing at X, we will denote by $\frac{\partial f}{\partial G}(X)$ the derivative of f along the tangent vector G at the point X. We have that:

$$q_{\overline{\gamma}^{0,2},X}(G,H) = \frac{\partial(d\overline{\gamma}^{0,2}(H))}{\partial G}(X),$$

where $d\overline{\gamma}^{0,2}$ is as defined in 1 This equation is an equality between elements of the space $H^{0,2}(X,\mathbb{C})/\mathrm{Im}(d\overline{\gamma}^{0,2})$.

We choose s a section of $S^{2d-4} \otimes \mathcal{O}_{NL(\gamma)}$ such that $\mathrm{res}_{\tilde{X}}(s(\tilde{X})) = \overline{\gamma}(\tilde{X})$. After identification of $\mathcal{H}^{3-p,p-1}_{\mathrm{prim}}$ and R_F^{pd-4} we have that

1. $\mathrm{Im}(d\overline{\gamma}^{0,2}(X)) = \mathrm{Im}(\cdot P)$
2. $d\overline{\gamma}^{0,2}(H)(\tilde{X}) = Hs(\tilde{X})$.

and hence

$$q_{\overline{\gamma}^{0,2},X}(G,H) = \frac{\partial(\mathrm{res}_{\tilde{X}}(Hs(\tilde{X})))}{\partial G}(X), \tag{2}$$

this last equation being an equality between elements of $R_F^{3p-4}/\mathrm{Im}(\cdot P)$.

Let us explain more precisely what we mean by the formula (2). Since $Hs(\tilde{X})$ is a degree $3d-4$ polynomial, it has a residue class $\mathrm{res}_{\tilde{X}}(Hs(\tilde{X}))$ in $H^{0,2}(\tilde{X})$. This class disappears at X, and $\frac{\partial(\mathrm{res}_{\tilde{X}}(Hs(\tilde{X})))}{\partial G}(X)$ denotes its derivation along the tangent vector $G \in T_{U_d}(X)$. We note that

$$\frac{\partial(\mathrm{res}_{\tilde{X}}(Hs(\tilde{X})))}{\partial G}(X) = \mathrm{res}_X\left(H\frac{\partial s(\tilde{X})}{\partial G}(X)\right) + \frac{\partial(\mathrm{res}_{\tilde{X}}(HP))}{\partial G}(X).$$

Lemma 2. *We have* $\frac{\partial(\mathrm{res}_{\tilde{X}}(HP))}{\partial G}(X) = -\mathrm{res}_X(\sum_{i=0}^{3} R_i \frac{\partial G}{\partial X_i})$.

Proof of Lemma 2. If X_ϵ is the variety cut out by the polynomial $F + \epsilon G$, then we have

$$\frac{\partial(\mathrm{res}_{X_\epsilon}(HP))}{\partial \epsilon}(0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\mathrm{res}_{X_\epsilon}(HP).$$

We know that $HP = \sum_{i=0}^{3} R_i \frac{\partial F}{\partial X_i}$, whence we see that

$$HP = \sum_{i=0}^{3} \left(R_i \frac{\partial F + \epsilon G}{\partial X_i} - \epsilon R_i \frac{\partial G}{\partial X_i} \right).$$

Therefore,

$$\mathrm{res}_{X_\epsilon}(HP) = \mathrm{res}_{X_\epsilon} \left(-\epsilon \sum_{i=0}^{3} R_i \frac{\partial G}{\partial X_i} \right),$$

and hence

$$\frac{\partial(\mathrm{res}_{\tilde{X}}(HP))}{\partial G}(X) = \frac{\partial(\mathrm{res}_{X_\epsilon}(HP))}{\partial \epsilon}(X) = \lim_{\epsilon \to 0} \mathrm{res}_{X_\epsilon} \left(- \sum_{i=0}^{3} R_i \frac{\partial G}{\partial X_i} \right).$$

From this we get that

$$\frac{\partial(\mathrm{res}_{\tilde{X}}(HP))}{\partial G}(X) = \mathrm{res}_X \left(- \sum_{i=0}^{3} R_i \frac{\partial G}{\partial X_i} \right).$$

This completes the proof of Lemma 2. ☐

It remains to calculate $\dfrac{\partial s}{\partial G}(X)$.

Lemma 3. *The section s can be chosen in such a way that* $\dfrac{\partial s}{\partial G}(X) = \sum_{i=0}^{3} \dfrac{\partial Q_i}{\partial X_i}$.

Proof of Lemma 3. By definition

$$\mathrm{res}_X(P) = \mathrm{res}_X \left[\frac{P\Omega}{F^2} \right].$$

The polynomial $s(\tilde{X})$ is chosen such that the section $\mathrm{res}_{\tilde{X}}(s(\tilde{X})) = \mathrm{res}_{\tilde{X}} \frac{s(\tilde{X})\Omega}{\tilde{F}^2}$ of $\mathcal{H}^2 \otimes \mathcal{O}_{NL(\gamma)}$ is flat with respect to the Gauss-Manin connection. In particular,

$$\frac{\partial(\mathrm{res}_{\tilde{X}}(s(\tilde{X})))}{\partial G}(X) = 0$$

and hence

$$\mathrm{res}_X \left(\frac{\partial \frac{s\Omega}{F^2}}{\partial G}(X) \right) = 0.$$

On deriving this formula, we obtain that

$$\mathrm{res}_X \left(\frac{(\frac{\partial s}{\partial G}(X))\Omega}{F^2} - 2\frac{GP\Omega}{F^3} \right) = 0. \tag{3}$$

It is proved in [2] that (3) only holds if there is an $\alpha \in H^0(\Omega^2_{\mathbb{P}^3}(2Y))$ such that

$$\frac{\frac{\partial s}{\partial G}(X)\Omega}{F^2} - 2\frac{GP\Omega}{F^3} = d\alpha.$$

Any $\alpha \in H^0(\Omega^2_{\mathbb{P}^3}(2Y))$ may be written in the form

$$\alpha = \frac{\sum_{i=0}^{3} S_i \mathrm{int}(\frac{\partial}{\partial X_i})\Omega}{F^2},$$

where the S_i are degree $2d-3$ polynomials. Here, the operation int $T_Y \otimes \Omega^2_Y \to \Omega^1_Y$ is defined for any smooth variety Y by $\mathrm{int}(t,\omega)(v) = (\omega(t,v))$. We now show that

$$d\alpha = \frac{-2}{F^3}\sum_{i=0}^{3} S_i \frac{\partial F}{\partial X_i}\Omega + \frac{1}{F^2}\sum_{i=0}^{3}\frac{\partial S_i}{\partial X_i}\Omega.$$

We shall do this by calculation on $\mathbb{C}^4$. There is a natural application $\pi : \mathbb{C}^4 \to \mathbb{P}^3$ given by $(x_0, \ldots, x_3) \to [x_0, \ldots, x_3]$. The pullback $\pi^*(\Omega)$ is given by

$$\pi^*(\Omega) = \mathrm{int}(\sum_{j=0}^{3} x_j \frac{\partial}{\partial x_j}, dx_0 \wedge \ldots \wedge dx_3)$$

and the pullback $\pi^*\alpha$ is given by

$$\pi^*(\alpha) = \sum_{i=0}^{3} \frac{S_i}{F^2}\mathrm{int}(\frac{\partial}{\partial x_i}, \mathrm{int}(\sum_{j=0}^{3} x_j \frac{\partial}{\partial x_j}, dx_0 \wedge \ldots \wedge dx_3)).$$

We now consider (for example) U_3, the open set of $\mathbb{P}^3$ given by $X_3 \neq 0$, and we map it into $\mathbb{C}^4$ via the map

$$s : [X_0, \ldots, X_3] \to (X_0/X_3, X_1/X_3, X_2/X_3, 1).$$

The coordinates $X_0/X_3, X_1/X_3, X_2/X_3$ on U_3 will be denoted by $x_0, \ldots, x_2$. The map s is a section of π. We therefore have that $s^* \circ \pi^*(\alpha) = \alpha|_{U_3}$. Therefore

$$\alpha|_{U_3} = (\sum_{i=0}^{2} -(1)^{i+1}\frac{S_i}{F^2} + (-1)^i X_i \frac{S_3}{F^2})(x_0, \ldots, x_2, 1)dx_0 \wedge \ldots \hat{dx_i} \ldots \wedge dx_2.$$

It follows that

$$d\alpha|_{U_3} = \sum_{i=0}^{2}(-\frac{\partial \frac{S_i}{F^2}}{\partial X_i} + \frac{\partial \frac{X_i S_3}{F^2}}{\partial X_i})(x_0, \ldots, x_2, 1)dx_0 \wedge \ldots \wedge dx_2,$$

and hence

$$d\alpha|_{U_3} = (\sum_{i=0}^{3} -\frac{\partial \frac{S_i}{F^2}}{\partial X_i} + \sum_{i=0}^{3} X_i \frac{\partial \frac{S_3}{F^2}}{\partial X_i} + \sum_{i=0}^{2} \frac{S_3}{F^2})(x_0, \ldots, x_2, 1)dx_0 \wedge \ldots \wedge dx_2.$$

By the Euler relationship, plus the fact that the degree of $\frac{S_3}{F^2}$ is -3, it follows that

$$d\alpha|_{U_3} = \sum_{i=0}^{3} -\frac{\partial \frac{S_i}{F^2}}{\partial X_i}(x_0, \ldots, x_2, 1)dx_0 \wedge \ldots \wedge dx_2$$

$$d\alpha|_{U_3} = \sum_{i=0}^{3} \frac{\partial \frac{S_i}{F^2}}{\partial X_i}\Omega.$$

and hence, as required

$$d\alpha = \frac{-2}{F^3}\sum_{i=0}^{3} S_i \frac{\partial F}{\partial X_i}\Omega + \frac{1}{F^2}\sum_{i=0}^{3}\frac{\partial S_i}{\partial X_i}\Omega.$$

Recall that

$$\sum_{i=0}^{3} Q_i \frac{\partial F}{\partial X_i} = GP.$$

Therefore, the equation

$$\frac{((\frac{\partial s}{\partial G}(X))\Omega)}{F^2} - 2\frac{HP\Omega}{F^3} = d\alpha$$

is satisfied whenever

$$\frac{\partial s}{\partial G}(X) = \sum_{i=0}^{3}\frac{\partial Q_i}{\partial X_i}$$

and

$$\alpha = \frac{\sum_{i=0}^{3} Q_i \mathrm{int}(\frac{\partial}{\partial X_i})\Omega}{F^2}.$$

Since the kernel of the map $S^6 \otimes \mathcal{O}_{U_d} \to \mathcal{H}^2$ is of constant rank, it follows that we may choose s such that $\frac{\partial s}{\partial G}(X) = \sum_{i=0}^{3} \frac{\partial Q_i}{\partial X_i}$. This completes the proof of Lemma 3. □

It follows that

$$\frac{\partial d_H(\overline{\gamma}^{0,2})}{\partial G}(X) = \mathrm{res}_X\left(\sum_{i=0}^{3}\left(\frac{\partial Q_i}{\partial X_i}H - R_i\frac{\partial G}{\partial X_i}\right)\right).$$

Therefore $q_{\overline{\gamma}^{0,2},X}(H,G)$ is equal to

$$\sum_{i=0}^{3}\left(\frac{\partial Q_i}{\partial X_i}H - R_i\frac{\partial G}{\partial X_i}\right).$$

As always, this is of course an equality of elements of $R_F^{3d-4}/\mathrm{Im}(\cdot \mathrm{P})$. This completes the proof of Theorem 7. □

4. Non-reduced Noether-Lefschetz loci in U_5 (proof of theorem 2). We will actually prove the following, which is slightly more precise.

THEOREM 8. *Let $NL(\gamma) \subset U_5$ be non-reduced. Let X be a point of $NL(\gamma)$. Then there exist H a hyperplane, L_1, L_2 distinct lines in $X \cap H$ and α, β distinct non-zero rational numbers such that*

$$\gamma = \alpha[L_1]_{\mathrm{prim}} + \beta[L_2]_{\mathrm{prim}}.$$

Traditionally, non-reduced Noether-Lefschetz components have been hard to study, since the much-used technique of degenerating X relies on being able to integrate vector fields. We will use a different approach. The equations arising from the fundamental quadratic form allow us to directly construct harmonic forms on the complement of a special hyperplane section of X. The existence of such harmonic forms implies this section is reducible.

When $d = 5$, any component of the Noether-Lefschetz locus has codimension at most 4. It was proved in [13], [6] that the codimension of $NL(\gamma)$ is ≥ 2 and this bound is achieved only if γ is a multiple of $[L_1]_{\text{prim}}$ for some line $L_1 \subset X$. Further, it was shown in [14] that if $NL(\gamma)$ is of codimension 3, then γ is a multiple of $[C_1]_{\text{prim}}$ for some conic $C_1 \subset X$.

The only other Noether-Lefschetz loci in U_5 which may have tangent spaces with exceptional codimension are non-reduced components, whose reductions are of codimension 4.

PROPOSITION 1. *Assume there exists a hyperplane H whose intersection with X has 3 components L_1, L_2, C such that L_1 and L_2 are distinct lines and C is a cubic. If α and β are distinct non-zero integers, then the cohomology class*

$$\gamma = \alpha[L_1]_{\text{prim}} + \beta[L_2]_{\text{prim}}$$

is such that $NL(\gamma)$ has a non-reduced component.

Proof of Proposition 1. Since α, β are distinct and non-zero, γ is neither the (primitive part of a) class of a line nor the (primitive part of a) class of a conic. We know by the work of Voisin in [15] that $\operatorname{codim}(T_{NL_\gamma}(X)) > 3$, and hence codim $T_{NL(\gamma)_{\text{red}}}(X)) = 4$. We now show that $NL(\gamma)$ has a non-reduced component.

The space $NL(\gamma)$ contains the space $NL(L_{1_{\text{prim}}}) \cap NL(L_{2_{\text{prim}}})$. Since this set has codimension $\leq 2+2 = 4$, it follows that $NL(L_{1_{\text{prim}}}) \cap NL(L_{2_{\text{prim}}})$ is a component of $NL(\gamma)$.

A dimension count shows that for all $Y \in NL(C_{1_{\text{prim}}})$ there is a line $L_1^Y \in Y$ such that $\overline{[L_1]}_{\text{prim}}(Y) = [L_1^Y]_{\text{prim}}$. The intersection number of L_1^Y and L_2^Y in Y is 1: hence, there is a point $p_Y \in L_1^Y \cap L_2^Y$. It follows that there is a plane H_Y in $\mathbb{P}^3$ containing $L_1^Y \cup L_2^Y$. Hence, in particular, there is a hyperplane H_Y in $\mathbb{P}^3$ on which γ_Y is supported.

In [10] (p. 212, observation 4.a.4) (see also [17], p. 408, proposition 17.19) it is shown that if there exists a holomorphic form ω on Y such that γ is supported on the zero locus of ω then $\operatorname{codim}(T_{NL(\gamma)}(Y)) < \binom{d-1}{3}$. Since $K_Y = \mathcal{O}_Y(1)$, there exists such a holomorphic form, and

$$\operatorname{codim}(T_{NL(\gamma)}(Y)) < 4$$

at every point of $NL(\gamma)$. The space $NL(\gamma)$ is therefore non-reduced. This completes the proof of Proposition 1. □

We will now prove Theorem 8, which says that this is the only possible type of non-reduced Noether-Lefschetz locus in U_5.

We assume that X is a sufficiently general smooth point of $NL(\gamma)_{\rm red}$. Recall that P is a degree 6 polynomial such that $\text{res}_X(P) = \gamma$. Since codim $T_{NL(\gamma)}(X) < 4$, it follows from the definition of $T_{NL(\gamma)}(X) = \text{Ker}(\cdot P)$ that the map

$$\cdot P : S^5 \to R_F^{11}$$

is not surjective. By Macaulay duality there is an $X_0 \in S^1$ such that

$$X_0 PH = 0 \text{ for all } H \in R_F^5,$$

whence we deduce that $X_0 P = 0$ in R_F. We define H to be the plane $X_0 = 0$. There exist cubics, $P_i \in S^3$, such that

$$X_0 P = \sum_{i=0}^{3} P_i \frac{\partial F}{\partial X_i}.$$

We now use the fundamental quadratic form to obtain relations on the P_i and $\frac{\partial F}{\partial X_i}$ which will imply that $X \cap H$ is reducible.

4.1. Relationships between P_i and $\frac{\partial F}{\partial X_i}$. We will now use the fundamental quadratic form to derive some special relationships between the P_is and the $\frac{\partial F}{\partial X_i}$s (proposition 2). In the following sections, we will use these relationships to prove that $X \cap H$ is reducible.

PROPOSITION 2. *We have*

$$\sum_{i=1}^{3} P_i \frac{\partial F}{\partial X_i}|_H = 0 \tag{4}$$

$$\sum_{i=1}^{3} \frac{\partial P_i}{\partial X_i}|_H = 0. \tag{5}$$

Equation 4 implies immediately that $X \cap H$ is singular. We will prove that in fact the space of triples P_1, P_2, P_3 satisfying (4) and (5) has dimension at most $(j-1)$, where j is the number of components of $X \cap H$.

Proof of Proposition 2. We will begin by proving the following lemma.

LEMMA 4. *There is a non-zero L contained in S^1 such that in R_F^4*

$$L\left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0\right) = 0. \tag{6}$$

Proof of Lemma 4. We know that codim$T_{NL(\gamma)}(X) \geq 2$ by the result of Voisin and Green, and codim $T_{NL(\gamma)_{\rm red}}(X)$ =4, since X is a smooth point of $NL(\gamma)_{\rm red}$. We treat first the case where the codimension of $T_{NL(\gamma)_{\rm red}}(X)$ in $T_{NL(\gamma)}(X)$ is 1. We have

$$(X_0 H)P = \sum_{i=0}^{3} P_i H \frac{\partial F}{\partial X_i}$$

and similarly

$$(X_0G)P = \sum_{i=0}^{3} P_i G \frac{\partial F}{\partial X_i}.$$

Now, suppose that $G \in S^4$ is such that $X_0G \in T_{NL(\gamma)_{\mathrm{red}}}(X)$. Then for any $H \in S^4$, we have, by remark 1, that

$$q_{\overline{\gamma}^{0,2},X}(X_0H, X_0G) = 0.$$

Hence, the following equations hold in R_F

$$X_0G \sum_{i=0}^{3} \frac{\partial(P_iH)}{\partial X_i} - \sum_{i=0}^{3} P_i G \frac{\partial(X_0H)}{\partial X_i} \in \mathrm{Im}(\cdot P).$$

Rearranging, we get that

$$GH\left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0\right) \in \mathrm{Im}(\cdot P).$$

Multiplying by X_0, we get that

$$X_0GH\left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0\right) = 0,$$

and finally, by Macaulay duality, we have

$$X_0G\left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0\right) = 0. \tag{7}$$

This last equation holds for any G in the space E defined by

$$E = \{G \in S^4 \text{ such that } X_0G \in T_{NL(\gamma)_{\mathrm{red}}}(X)\}.$$

We have that $\mathrm{codim}(E) \leq 1$ (since we have supposed that the codimension of $T_{NL(\gamma)_{\mathrm{red}}}(X)$ in $T_{NL(\gamma)}(X)$ is 1). Straightforward algebraic manipulations show that the ideal generated in R_F by E contains R_F^5. Hence for any $J \in R_F^5$ we have

$$JX_0\left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0\right) = 0,$$

and hence by Macaulay duality

$$X_0\left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0\right) = 0.$$

Hence Lemma 4 is proved in this case.

We now treat the case where the codimension of $T_{NL(\gamma)_{\mathrm{red}}}(X)$ in $T_{NL(\gamma)}(X)$ is

2. In this case, there are two distinct elements of S^1, X_0 and X_1, such that $X_0P = 0$ and $X_1P = 0$. Once again, we define E by

$$E = \{G \in S^4 \text{ such that } X_0G \in T_{NL(\gamma)_{\mathrm{red}}}(X)\},$$

and we then obtain that

$$X_0G\left(X_0\sum_{i=0}^{3}\frac{\partial P_i}{\partial X_i} - P_0\right) = 0,$$

and similarly

$$X_1G\left(X_0\sum_{i=0}^{3}\frac{\partial P_i}{\partial X_i} - P_0\right) = 0.$$

The codimension of E is at most 2. There are 2 maps,

$$\phi_0 \text{ and } \phi_1 : S^4/E \to \mathrm{Ker}(\cdot E) \subset R_F^8$$

given by multiplication by $X_0(X_0\sum_{i=0}^{3}\frac{\partial P_i}{\partial X_i} - P_0)$ and $X_1(X_0\sum_{i=0}^{3}\frac{\partial P_i}{\partial X_i} - P_0)$ respectively. Here by $\mathrm{Ker}(\cdot E)$, we mean the set of all elements in R_F^8 which give 0 on multiplying with any element of E. If ϕ_0 is not an isomorphism then (7) holds for all $G \in \phi_0^{-1}(0)$, which is a hyperplane, and the lemma follows as in the previous case.

Only the case where ϕ_0 is invertible remains. But in this case $\phi_0^{-1} \circ \phi_1$ has an eigenvalue, λ. The multiplication map

$$\cdot(X_0 - \lambda X_1)\left(X_0\sum_{i=0}^{3}\frac{\partial P_i}{\partial X_i} - P_0\right) : R_F^4 \to R_F^8$$

has a kernel of codimension at most 1, from which we conclude as before that $(X_0 - \lambda X_1)(X_0\sum_{i=0}^{3}\frac{\partial P_i}{\partial X_i} - P_0) = 0$. This concludes the proof of Lemma 4. ∎

We will now attempt to prove that this implies that $X_0\sum_{i=0}^{3}\frac{\partial P_i}{\partial X_i} - P_0 = 0$. We start with the following technical lemma.

Lemma 5. *If W' is defined to be the space $S^3 \times S^1 \times \{\mathbb{C}^4/0\} \times S^5$, then the map $\phi : W' \to S^4$ given by $\phi(P, L, \alpha_0, \alpha_1, \alpha_2, \alpha_3, F) = PL - \sum_{i=0}^{3}\alpha_i\frac{\partial F}{\partial X_i}$ is submersive.*

Proof of Lemma 5. Let $(Y_0, \ldots, Y_3)$ be co-ordinates on $\mathbb{P}^3$, such that $\sum_{i=0}^{3}\alpha_i\frac{\partial F}{\partial X_i} = \frac{\partial F}{\partial Y_0}$. Then

$$\frac{\partial \phi}{\partial F}(G) = \frac{\partial G}{\partial Y_0}.$$

Hence $d\phi : T_{W'} \to T_{S^4}$ is surjective. This completes the proof of Lemma 5. ∎

From this lemma we will deduce the following:

Lemma 6. *If $U' \subset U_5$ is defined by*

$$\{F \text{ such that } \exists L_1 \in R_F^1, L_2 \in R_F^3 \text{ such that } L_1 \neq 0, L_2 \neq 0 \text{ and } L_1L_2 = 0 \text{ in } R_F^4,\},$$

then codim $U' \geq 6$.

Proof of Lemma 6. We now define W to be the subset of W' consisting of all septuples $(P, L, \alpha_0, \alpha_1, \alpha_2, \alpha_3, F)$ such that

$$PL = \sum_{i=0}^{3} \alpha_i \frac{\partial F}{\partial X_i}.$$

It follows that the codimension of W in W' is $\dim(S^4)$= 35, whence we see that

$$\dim(W) = \dim S^5 + 4 + 4 + 20 - 35 = \dim(S^5) - 7.$$

It follows that the codimension of the image of W under projection to U_5 is ≥ 6. This completes the proof of Lemma 6. □

And finally, this gives us the following.

LEMMA 7. *In* R_F *we have*

$$X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0 = 0. \tag{8}$$

Proof of Lemma 7. Indeed, it follows immediately from Lemma 6, and the fact that

$$\operatorname{codim}(NL(\gamma)_{\text{red}}) = 4,$$

that for a generic point of $NL(\gamma)$ (6) implies that

$$X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0 = 0.$$

So Lemma 7 follows from Lemma 6. This completes the proof of Lemma 7. □

Equation (4) of Proposition 2 now follows from the two equations

$$P_0 = X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} \tag{9}$$

and

$$\sum_{i=0}^{3} P_i \frac{\partial F}{\partial X_i} = X_0 P.$$

We turn now to the equation (5), which follows when we differentiate (9) with respect to X_0 to obtain

$$\frac{\partial P_0}{\partial X_0} = \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} + X_0 \frac{\partial (\sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i})}{\partial X_0}.$$

Re-arranging, we get that

$$-X_0 \frac{\partial \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i}}{\partial X_0} = \sum_{i=1}^{3} \frac{\partial P_i}{\partial X_i}.$$

This completes the proof of Proposition 2. □

Now, let us consider the quintic plane curve, $D = X \cap H$. In the next section, we will denote by $\tilde{F}$ the restriction of F to H. We define $D_1, \ldots, D_j$ to be the components of D and d_i to be the degree of D_i.

4.2. The cohomology class γ is a linear combination of $[D_1], \ldots, [D_j]$. We will show that the dimensions of the following two spaces are the same :

1. Triples P_i satisfying the equations of Proposition 2,
2. Primitive cohomology classes supported on D.

From this, it will not be too hard to show that γ is supported on D.
We now prove the following proposition.

PROPOSITION 3. *The cohomology class γ is a linear combination of $[D_1], \ldots, [D_j]$.*

Proof of Proposition 3. It will be enough to show that

$$\dim (\langle \gamma, [D_1]_{\text{prim}}, \ldots, [D_{j-1}]_{\text{prim}} \rangle) \leq j - 1. \tag{10}$$

We denote this space by V'. We denote by V the space of all triplets of cubics (P_1, P_2, P_3) in variables X_1, X_2, X_3 such that

$$\sum_{i=1}^{3} P_i \frac{\partial \tilde{F}}{\partial X_i} = 0 \tag{11}$$

and

$$\sum_{i=1}^{3} \frac{\partial P_i}{\partial X_i} = 0. \tag{12}$$

Of course, these are simply the equations of Proposition 2. We will first show that the dimension of V is less than or equal to $(j - 1)$ and then construct an injective linear map $V' \to V$, from which (10) will follow.

LEMMA 8. *The dimension of V is $\leq j - 1$.*

Proof of Lemma 8. For this, we will need to interpret the equations (11) and (12) geometrically. We consider the maps

$$f : V \to H^0(T_{\mathbb{P}^2}(2))$$

and

$$g : H^0(T_{\mathbb{P}^2}(2)) \to H^0(\Omega_{\mathbb{P}^2}(D))$$

which are given by

$$f(P_1, P_2, P_3) = \sum_{i=1}^{3} P_i \frac{\partial}{\partial X_i}$$

and

$$g(\alpha) = \frac{\mathrm{int}(\alpha)\Omega}{\tilde{F}}.$$

The map int is as given on page 10. In this case, Ω is the canonical section of $K_{\mathbb{P}^2}(3)$. The map g is an isomorphism. We will show the following lemma.

LEMMA 9. *The map f is injective.*

Proof of Lemma 9. Suppose that the triple (P_1, P_2, P_3) were such that $f(P_1, P_2, P_3) = 0$. There would then be P' such that

$$(P_1, P_2, P_3) = (X_1 P', X_2 P', X_3 P').$$

However we would then have

$$\sum_{i=1}^{3} P_i \frac{\partial \tilde{F}}{\partial X_i} = P' \tilde{F}$$

and hence (11) implies that $P' = 0$. This completes the proof of Lemma 9. ◻

We now consider the image of $g \circ f$ in $H^0(\Omega_{\mathbb{P}^2}(D))$. We will use the following lemma.

LEMMA 10. *If $(P_1, P_2, P_3) \in V$ then $g \circ f(P_1, P_2, P_3) \in H^0(\Omega^{1,c}_{\mathbb{P}^2}(\log D))$.*

Here, $\Omega^{1,c}_{\mathbb{P}^2}(\log D)$ denotes the sheaf of closed differential forms with logarithmic singularities along D. We note that, since differential forms with logarithmic singularities can be characterised as being those differential forms with simple poles along D whose differential also has logarithmic poles along D, it is automatic that any d-closed member of $H^0(\Omega^{1,c}_{\mathbb{P}^2}(D)$ has in fact a logarithmic singularity along D.

Proof of Lemma 10. It is enough to show that $d(g \circ f(P_1, P_2, P_3)) = 0$. But

$$d\left(\frac{\sum_{i=1}^{3}\left(P_i \mathrm{int}(\frac{\partial}{\partial X_i})(\Omega)\right)}{\tilde{F}}\right) = \sum_{i=1}^{3} \frac{(-P_i \frac{\partial \tilde{F}}{\partial X_i} + \tilde{F}\frac{\partial P_i}{\partial X_i})\Omega}{\tilde{F}^2}.$$

By (11) and (12), the right hand side is 0. This completes the proof of Lemma 10. ◻

We now complete the proof of Lemma 8. By the above, V injects into $H^0(\Omega^{1,c}_{\mathbb{P}^2}(\log D))$. Note that D, being the intersection of a smooth surface and a plane, is reduced.

We define U to be $\mathbb{P}^2 - D_{\mathrm{sing}}$. By the above comment, U is $\mathbb{P}^2$ minus a codimension 2 subset. There is an exact sequence on U,

$$0 \to \Omega^{1,c}_U \to \Omega^{1,c}_U(\log D) \stackrel{\mathrm{res}}{\to} \mathbb{C}_{D-D_{\mathrm{sing}}} \to 0,$$

from which we get an associated long exact sequence,

$$H^0(\Omega^{1,c}_U) \to H^0(\Omega^{1,c}_U(\log D)) \stackrel{p}{\to} H^0(D/D_{\mathrm{sing}}, \mathbb{C}) \stackrel{\delta}{\to} H^1(\Omega^{1,c}_U).$$

However, since $\Omega^1_{\mathbb{P}^2}$ is free and $\mathbb{P}^2 - U$ is of codimension 2, it follows by Levi's extension theorem that

$$H^0(\Omega^1_U) \simeq H^0(\Omega^1_{\mathbb{P}^2}) = 0.$$

Hence,

$$H^0(\Omega^{1,c}_U(\log D)) \simeq \text{Ker } \delta.$$

Since $\dim(H^0(D/D_{\text{sing}}, \mathbb{C})) = j$, it will be enough to show that $\text{Im}(p) \neq H^0(D - D_{\text{sing}}, \mathbb{C})$. But if $u \in H^0(\Omega^{1,c}_U(\log D))$ then we have that

$$p(u)(D_i) = \text{res}_{D_i}(u)$$

where $\text{res}_{D_i}(u)$ is the residue of the form u along D_i. But we know that $\sum_{i=1}^{j} d_i \text{res}_{D_i} u = 0$ and from this it follows that

$$\dim(H^0(\Omega^{1,c}_{\mathbb{P}^2}(\log D))) \leq j - 1.$$

This completes the proof of Lemma 8. ∎

We now prove the following lemma.

Lemma 11. *The space V' has dimension $\leq j-1$.*

Proof of Lemma 11. We will construct a map $L : V' \to V$ which we will then show to be injective. We choose a basis $(e_1 \ldots, e_m)$ for V', such that

1. $e_1 = \gamma$
2. $e_2, \ldots, e_m \in \langle [D_1]_{\text{prim}}, \ldots, [D_{j-1}]_{\text{prim}} \rangle$.

We will show that the argument presented in the proof of Proposition 2 is also valid for polynomials representing classes in the space

$$\langle [D_1]_{\text{prim}}, \ldots, [D_{j-1}]_{\text{prim}} \rangle .$$

For each e_l, we choose Q^l, a degree 6 polynomial such that $\text{res}_X(Q^l) = e_l$. By the choice of basis, we have the following.

Lemma 12. *For all l, $X_0 Q^l = 0$ in R^7_F.*

Proof of Lemma 12. This is true for $e_1 = \gamma$ by definition. For $l \geq 2$, it follows from

$$e_l \in \langle [D_1]_{\text{prim}}, \ldots, [D_{j-1}]_{\text{prim}} \rangle$$

that

$$X_0 \cdot S^4 \subset T_{NL(e_l)}(X). \tag{13}$$

This, by Macaulay duality and the results of Carlson and Griffiths, is equivalent to $X_0 Q^l = 0$ in R_F. This completes the proof of Lemma 12. ∎

We now choose polynomials $Q^l_0, Q^l_1, Q^l_2, Q^l_3$ (in four variables) such that

$$X_0 Q^l = \sum_{i=0}^{3} Q^l_i \frac{\partial F}{\partial X_i}.$$

We then have the following lemma.

LEMMA 13. *The equation (6) is valid for* $(Q_0^l, \ldots, Q_3^l)$. *The equations (11) and (12) are valid for the triple* $(Q_1^l|_H, \ldots, Q_3^l|_H)$.

Proof of Lemma 13. For $l = 1$, this is the statement of Proposition 2. For $l \geq 2$, Lemma 12 implies that for all degree 4 polynomials G_1 and G_2,

$$X_0G_1, X_0G_2 \in T_{NL(e_l)_{\text{red}}}(X).$$

Hence we see that for all G_1 and G_2 in S^4,

$$q_{\overline{e_l}^{0,2},X}(X_0G_1, X_0G_2) = 0.$$

Alternatively, as in the proof of Proposition 2

$$G_1G_2\left(X_0\sum_{i=0}^{3}\frac{\partial Q_i^l}{\partial X_i} - Q_0^l\right) \in \text{Im}(\cdot P)$$

and multiplying by X_0 we see that

$$X_0G_1G_2\left(X_0\sum_{i=0}^{3}\frac{\partial Q_i^l}{\partial X_i} - Q_0^l\right) = 0$$

in R_F This time, this relationship is valid for *any* choice of G_1 and G_2, so it follows immediately by Macaulay duality that

$$X_0\left(X_0\sum_{i=0}^{3}\frac{\partial Q_i^l}{\partial X_i} - Q_0^l\right) = 0$$

in R_F. This is precisely equation (6). By Lemma 7, it follows that since X has been chosen general in $NL(\gamma)$

$$\left(X_0\sum_{i=0}^{3}\frac{\partial Q_i^l}{\partial X_i} - Q_0^l\right) = 0$$

in R_F. Indeed, since $\deg(Q_0^l) = 3$, it follows that $(X_0\sum_{i=0}^{3}\frac{\partial Q_i^l}{\partial X_i} - Q_0^l) = 0$. The two equations (11) and (12) now follow as in the proof of Proposition 2. This completes the proof of Lemma 13. ☐

We set $L(e_l) = (Q_1^l|_H, Q_2^l|_H, Q_3^l|_H)$ and extend by linearity. We will now prove the following lemma.

LEMMA 14. *L is injective.*

Proof of Lemma 14. Let v be any element of V'. By linearity, there are cubic polynomials $Q_0^v, Q_1^v, Q_2^v, Q_3^v$ in variables $X_0, \ldots, X_3$ such that

1. $L(v) = (Q_1^v|_H, Q_2^v|_H, Q_3^v|_H)$,
2. The equation (6) is valid for $Q_0^v, \ldots, Q_3^v$,
3. There exists a Q^v such that $\sum_{i=0}^{3} Q_i^v\frac{\partial F}{\partial X_i} = X_0Q^v$,
4. Q^v represents the cohomology class v.

Lemma 14 now follows from the following lemma.

Lemma 15. *Suppose that $\gamma = \mathrm{res}_X(P)$, and there exist $(P_0, \ldots, P_3)$ such that*

$$X_0 P = \sum_{i=0}^{3} P_i \frac{\partial F}{\partial X_i}.$$

Suppose further that (6) is valid and that

$$P_1|_H = P_2|_H = P_3|_H = 0, i \geq 1.$$

Then $\gamma^{1,1} = 0$.

Proof of Lemma 15. We have

$$X_0 P = \sum_{i=0}^{3} P_i \frac{\partial F}{\partial X_i}. \tag{14}$$

By hypothesis, X_0 divides P_i for $i \geq 1$. It follows from (6) that X_0 divides P_0. Therefore, (14) implies that

$$P \in \left\langle \frac{\partial F}{\partial X_i} \right\rangle$$

from which it follows that

$$\mathrm{res}_X P \in F^2(H^2(X, \mathbb{C})).$$

Alternatively, we have that

$$\gamma^{1,1} = 0.$$

This completes the proof of Lemma 15. ◻

Since all elements of V' are Hodge $(1,1)$ classes, the injectivity of L follows immediately. This completes the proof of Lemma 19. ◻

This completes the proof of Lemma 11. ◻

This completes the proof of Proposition 3. ◻

4.3. The curve D is generically the union of two lines and a cubic. To complete the theorem, it will be enough to show that D is necesarily the union of two lines and a (possibly reducible) cubic. This will follow from a simple dimension count.

Lemma 16. *The curve $X \cap H$ must have at least 3 components.*

Proof of Lemma 16. We know that γ is a linear combination of classes of curves contained on $X \cap H$. If $X \cap H$ contains only two reducible components, then γ is either the linear combination of

1. a line and a hyperplane section or
2. a conic and a hyperplane section.

This is not possible, since all such cohomology classes have reduced associated Noether-Lefschetz loci. This completes the proof of Lemma 16. □

There are now two possibilities:

1. γ is a linear combination of the cohomology classes of two lines and a hyperplane section,
2. X belongs to S, the space of all quintic hypersurfaces possessing a hyperplane section which is the union of two conics and a line.

The codimension of S is 5 and the codimension of $NL(\gamma)$ is at most 4, so the general element of $NL(\gamma)$ cannot be contained in S.

It remains only to exclude the cases $\gamma = \alpha([L_1 + L_2]_{\text{prim}})$ or $\gamma = \alpha([L_1]_{\text{prim}})$. In the first case, γ is (a multiple of) the primitive part of the cohomology class of a conic, and in the second case γ is (a multiple of) the primitive part of the cohomology class of a line. In either case, γ has a reduced Noether-Lefschetz locus.

This concludes the proof of Theorem 2. □

5. A weaker form of the Green-Ciliberto conjecture holds (proof of Theorem 3). Let us begin by summarising the motivation for the Green-Ciliberto conjecture. We recall that the tangent space $T_{NL(\gamma)}(X)$ is simply the kernel of the map

$$\cdot P : S^d/F \to R_F^{3d-4}$$

which is multiplication by P. If $NL(\gamma)$ is exceptional, then the multiplication map $\cdot P : R_F^d \to R_F^{3d-4}$ is not onto. Since the multiplication map

$$R_F^{d-4} \otimes R_F^{3d-4} \to R_F^{4d-8}$$

is a perfect pairing this is equivalent to saying that there exists $Q \in S^{d-4}$ such that $QP = 0$ in R_F. This is equivalent to saying that

$$Q \cdot S^4 \subset T_{NL(\gamma)}(X).$$

There is one case in which it is clear this will be the case— namely when γ is supported on $Z \cap X$, where Z is the surface defined by Q. (In this case, we will say that γ is supported on Q). The Green-Ciliberto conjecture says that this should be the only possibility. The main theorem of this section is as follows.

THEOREM 9. *Suppose that $e \leq \frac{d-1}{2}$ and $j \leq \binom{e+3}{3}$. There exists an integer, $\phi_{e,j}(d)$ such that if $NL(\gamma)$ is reduced and codim$(NL(\gamma)) \leq \phi_{e,j}(d)$ then the dimension of the space $\{Q \in S^e$ such that γ is supported on $Q\}$ is $\geq j$.*

Further, $\phi_{\frac{d-1}{2},1}(d) = O(d^3)$.

On setting $j = 1$ in this statement, we obtain the result given in the introduction.

5.1. Integrating along special sub-bundles of $T_{NL(\gamma)}$. One way in which one might thnk of trying to prove that the class γ is supported on Q would be to try to show that $F + GQ$ is contained in $NL(\gamma)$. From this it would follow by a degeneration argument— due to Griffiths and Harris for smooth Q, and Voisin for general Q— that γ is supported on Q.

This is equivalent to showing that under small perturbation of F in the direction tGQ the tangent vector GQ does not leave the tangent space $T_{NL(\gamma)}$. Unfortunately, this is false. However, in what follows, we show that under the condition that $Q \cdot S^{d-e} \subset T_{NL(\gamma)}(X)$, with $e \leq \frac{d-1}{2}$, we have that $F + GQ^2$ is contained in $NL(\gamma)$ for any G.

The theorem will follow immediately from the following two propositions.

Proposition 4. *Suppose that $NL(\gamma)$ is reduced and for all Y in some neighbourhood of X, a general element of $NL(\gamma)$, the space*

$$V = \{Q \in S^e | Q \cdot S^{d-e} \subset T_{NL(\gamma)}(Y)\}$$

is of dimension $j > 0$. Suppose further that $e \leq \frac{d-1}{2}$. Then, for all $Q \in V$ and $G \in S^{d-2e}$ such that $F + GQ^2 \in O$ we have $F + GQ^2 \in NL(\gamma)$.

Proposition 5. *Let X be an element of $NL(\gamma)$. We can construct $\phi_{e,j}(d)$ as above such that if*

$$codim(NL(\gamma)) \leq \phi_{e,j}(d)$$

then $\dim\{Q \in S^e | Q \cdot S^{d-e} \subset T_{NL(\gamma)}(X)\} \geq j$.

Given these two propositions, it follows by the argument given in section 2 of [15], (pp 56-59), that γ is supported on $Q^2 = 0$— and hence on $Q = 0$.

Proof of Proposition 4. We assume, since the question was dealt with for $d = 6, 7$ in [15], that $d \geq 8$. We construct a space W as follows:

$$W = \{(Y, A) \in NL(\gamma) \times S^e | A \cdot S^{d-e} \subset T_{NL(\gamma)}(Y)\}.$$

If X is a sufficiently general smooth point of $NL(\gamma)$, then the space

$$V_Y = \{A \in S^e | A \cdot S^{d-e} \subset T_{NL(\gamma)}(Y)\}$$

is of constant dimension near X. The space W will be a smooth over some neighbourhood of X. We will prove the following lemma.

Lemma 17. *At any point (Y, A) of W we have $(GA^2, 0) \in T_W(Y, A)$ for all G.*

Proof of Lemma 17. We know that there exists some B such that $(GA^2, B) \in T_W(Y, A)$, since the map $W \to NL(\gamma)$ locally induces a surjection on the tangent spaces. Denote this tangent vector by χ. Let us derive the equation

$$AP = \sum_i L_i \frac{\partial F}{\partial X_i}$$

in the direction χ. By Lemma 3, we can choose to have that

$$\chi(P) = \sum_i \frac{\partial(L_i GA)}{\partial X_i}.$$

By definition of χ we have $\chi(A) = B$ and $\chi(F) = (GA^2)$. Hence we have

$$A \sum_i \left(\frac{\partial(L_i GA)}{\partial X_i} \right) + BP = \sum_i \left(L_i \frac{\partial GA^2}{\partial X_i} + \chi(L_i) \frac{\partial F}{\partial X_i} \right).$$

Rearranging, we get that in R_F

$$GA\sum_i \left(A\frac{\partial L_i}{\partial X_i} - L_i\frac{\partial A}{\partial X_i}\right) = -BP.$$

We will now prove the following result.

LEMMA 18. *We have*

$$A\sum_i \left(A\frac{\partial L_i}{\partial X_i} - L_i\frac{\partial A}{\partial X_i}\right) = 0 \text{ in } R_F.$$

Proof of Lemma 18. It is in the proof of this key lemma that we will use the fundamental quadratic form. Note that for all $H_1, H_2 \in S^{d-e}$,

$$AH_1 \text{ and } AH_2 \in T_{NL(\gamma)}(X),$$

and further,

$$q_{\overline{\gamma}^{0,2},X}(AH_1, AH_2) = 0.$$

Hence, for all H_1, H_2 the following equality holds in R_F

$$\sum_i \left(AH_1\frac{\partial (H_2L_i)}{\partial X_i} - H_1L_i\frac{\partial (AH_2)}{\partial X_i}\right) \in \operatorname{Im}(\cdot P).$$

Rearranging, we get that

$$H_1H_2\sum_i \left(A\frac{\partial L_i}{\partial X_i} - L_i\frac{\partial A}{\partial X_i}\right) \in \operatorname{Im}(\cdot P).$$

From this we see that for all $H \in S^{2d-2e}$,

$$HA\sum_i \left(A\frac{\partial L_i}{\partial X_i} - L_i\frac{\partial A}{\partial X_i}\right) = 0$$

in R_F. We know that

$$\deg\, HA\sum_i \left(A\frac{\partial L_i}{\partial X_i} - L_i\frac{\partial A}{\partial X_i}\right) = 3d - 4 + e \leq 4d - 8.$$

In the last inequality we have used the fact that $d \geq 8$. It follows that

$$A\sum_i \left(A\frac{\partial L_i}{\partial X_i} - L_i\frac{\partial A}{\partial X_i}\right) = 0$$

in R_F. This completes the proof of Lemma 18. □

Returning to the proof of Lemma 17, we see that $BP = 0$. Hence

$$(0, B) \in T_W(Y, A)$$

and therefore

$$(GA^2, 0) \in T_W(Y, A) \text{ for all } G \in S^{d-2e}.$$

This completes the proof of Lemma 17. ∎

We now complete the proof of Proposition 4. We have just shown there is a field of tangent vectors on W which we denote by τ_G given by

$$\tau_G(Y, A) = (GA^2, 0).$$

We may now integrate along the tangent field τ_G, at least locally. (Here, we have used the fact that (Y, A) is a smooth point of W). Hence $F + \epsilon Gp^2$ is contained in $NL(\gamma)$ for all sufficiently small ϵ. This completes the proof of Proposition 4. ∎

We must now construct the integer $\phi_{e,j}(d)$ such that if $\operatorname{codim}(NL(\gamma)) \leq \phi_{e,j}(d)$ then the dimension of the space

$$V = \{Q \in S^e \text{ such that } Q \cdot S^{d-e} \in T_{NL(\gamma)}(X)\}$$

is at least j. In what follows, when $W \subset S^n$ is a sub-vector space, $\langle V\rangle^{n+m}$ will denote the subspace of S^{n+m} generated by W.

Proof of Proposition 5. This theorem is essentially a statement about multiplication in a certain polynomial ring. We will rely on the following theorem, due to Macaulay and Gotzmann which may be found in [5] (pp. 64-65).

Theorem 10 (Macaulay, Gotzmann). *Given an integer, d, any other integer c may be written in a unique way as*

$$c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \dots \binom{k_i}{i},$$

for some integer i. where $k_d > k_{d-1} \dots > k_i$. We define $c^{<d>}$ by

$$c^{<d>} = \binom{k_d + 1}{d+1} + \binom{k_{d-1} + 1}{d} + \dots \binom{k_i + 1}{i+1}.$$

Let V be a subvector space of S^d of codimension c. Then, the codimension of $\langle V\rangle^{d+1}$ in S^{d+1} is $\leq c^{<d>}$ and if equality holds then for all j we have

$$\operatorname{codim}\,(\langle V\rangle^{d+j}) = (((c^{<d>})^{<d+1>})\dots)^{<d+j-1>}.$$

Here, $\langle V\rangle^i$ denotes the degree i part of the ideal generated by V in $\mathbb{C}[X_0, \dots X_3]$. We now define a set of functions, $g_i(n)$. The function $g_i(n)$ should be thought of as "the maximal codimension of $\langle V\rangle^{d+i}$ in S^{d+i} if V is a subvector space of S^d of codimension n containing $\langle \frac{\partial F}{\partial X_0}, \dots, \frac{\partial F}{\partial X_3}\rangle$." We define

- $g_0(n) = n$,
- $g_{i+1}(n) = g_i(n)^{<d+i>} - 1$.

Lemma 19. *If $V \subset S^d$ has codimension n and $S^1 \cdot \left\langle \dfrac{\partial F}{\partial X_i} \right\rangle \subset V$, then for any integer j the subspace generated by V in S^{d+j} has codimension $\leq g_j(n)$.*

Proof of Lemma 19. This follows from Theorem 10 by induction on noting that the inclusion $S^1 \cdot \left\langle \frac{\partial F}{\partial X_i} \right\rangle \subset V$ implies that V generates S^{4d-7}, and hence it is not possible to have

$$\operatorname{codim}(\langle V \rangle^{d+j+1}) = (\operatorname{codim}(\langle V \rangle^{d+j}))^{<d+j>}$$

for any $j \leq 3d - 8$. This completes the proof of Lemma 19. □

We are now in a position to define the integer $\phi_{e,j}(d)$.

DEFINITION 2. *The integer $\phi_{e,j}(d)$ is the smallest integer n having the property that*

$$g_{2d-4-e}(n) \leq \binom{e+3}{3} - j.$$

The above work can be combined to prove Theorem 9 with this definition of $\phi_{e,j}$. It will be enough to show that if codim $(NL(\gamma)) \leq \phi_{e,j}(d)$ then dim Ker$(\cdot P) \geq j$. But the ring

$$S_F = R_F / \operatorname{Ker}(\cdot P)$$

is a Gorenstein graded ring of rang $2d - 4$. It follows by duality that

$$\dim\, (S_F)^e = \dim\, (S_F)^{2d-4-e}$$

and hence that

$$\dim(R_F/\operatorname{Ker}(\cdot P))^e \leq \binom{e+3}{3} - j$$

by the definition of $\phi_{e,j}(d)$. Hence we have

$$\dim\, (\operatorname{Ker}(\cdot P))^e \geq j.$$

REMARK 2. When we choose $e = 1$, $j = 2$, we recover the result of [13] and [6]—albeit with the additional hypothesis that $NL(\gamma)$ should be reduced.

It remains only to prove that $\phi_{\frac{d-1}{2}}(d)$ is indeed a cubic function of d.

PROPOSITION 6. *There exists $\alpha > 0$ such that*

$$\phi_{\frac{d-1}{2}}(d) \geq \alpha d^3$$

for d sufficiently large.

Proof of Proposition 6. Since $\binom{\frac{d-1}{2}+3}{3}$ is a cubic in d, there exists $\beta < 1$ such that for d large

$$\binom{\frac{d-1}{2}+3}{3} - 1 \geq (\beta d + 1)\binom{\frac{3d-1}{2}+2}{2}.$$

Hence

$$\binom{\frac{d-1}{2}+3}{3}-1 \geq \sum_{i=0}^{\lceil \beta d\rceil} \binom{\frac{3d-1}{2}-i+2}{2},$$

and it follows that

$$g_{\frac{d+1}{2}}(\sum_{i=0}^{\lceil \beta d\rceil} \binom{d-i+2}{2}) \leq \binom{\frac{d-1}{2}+3}{3}-1.$$

Hence we have

$$\phi_{\frac{d-1}{2},1}(d) \geq \sum_{i=0}^{\lceil \beta d\rceil} \binom{d-i+2}{2}.$$

But we know that

$$\sum_{i=0}^{\lceil \beta d\rceil} \binom{d-i+2}{2} > \frac{\beta(1-\beta)}{2}d^3.$$

and this completes the proof of Proposition 6. □

Theorem 8 follows immediately. □

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