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A SECOND MAIN THEOREM ON PARABOLIC MANIFOLDS*

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Abstract. In [St], [WS], Stoll and Wong-Stoll established the Second Main Theorem of meromorphic maps $f : M \to \mathbb{P}^N(\mathbb{C})$ intersecting hyperplanes, under the assumption that f is linear non-degenerate, where M is a m-dimensional affine algebraic manifold(the proof actually works for more general category of Stein parabolic manifolds). This paper deals with the degenerate case. Using P. Vojta's method, we show that there exists a finite union of proper linear subspaces of $\mathbb{P}^N(\mathbb{C})$, depending only on the given hyperplanes, such that for every (possibly degenerate) meromorphic map $f : M \to \mathbb{P}^N(\mathbb{C})$, if its image is not contained in that union, the inequality of Wong-Stoll's theorem still holds (without the ramification term). We also carefully examine the error terms appearing in the inequality.

Key words. second main theorem, parabolic manifolds, value distribution theory

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In [WS], W. Stoll and Pit-Mann Wong established the Second Main Theorem of meromorphic maps $f : M \to \mathbb{P}^N(\mathbb{C})$ intersecting hyperplanes, under the assumption that f is linear non-degenerate, where M is a m-dimensional affine algebraic manifold(the proof actually works for more general category of Stein parabolic manifolds). This paper deals with the degenerate case. Motivated by the works of Vojta (see [Vo2], [Vo3]), we show that, for a finite set of hyperplanes in $\mathbb{P}^N(\mathbb{C})$, there exists a finite union of proper linear subspaces, depending only on the given hyperplanes, such that for every meromorphic map $f : M \to \mathbb{P}^N(\mathbb{C})$, if its image is not contained in that union, then the inequality of Wong-Stoll's theorem still holds, except that the ramification term is lost. Here the exceptional subspaces(i.e. the subspaces which the image $f(M)$ is not contained in) depend only on the given hyperplanes and can be determined explicitly. We note that the Second Main Theorem for linearly degenerated maps was also studied by W.X. Chen(see [Chen]). The estimate in the Theorem of Chen holds without exceptions. However, his estimate is weaker than the estimate of the current paper(which allows a finite number exceptions).

Throughout this paper, we shall use the standard notation in the value distribution theory of meromorphic maps on parabolic manifolds (see [WS] or [St]). An affine algebraic manifold M can be represented as a finite branch cover over $\mathbb{C}^m$, $\pi : M \to \mathbb{C}^m$. Let κ be the sheet number of the projection π and d_π be the degree of the branching divisor of τ.

Our main theorem is stated as follows:

Main Theorem. *Let M be an affine algebraic manifold of complex dimension m. Let $\pi : M \to \mathbb{C}^m$ be a finite branched covering. Let $\mathcal{H} = \{H_1, ..., H_q\}$ be a finite collection of hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. Then there exists a*

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finite union $\mathcal{R}$ of proper linear subspaces of $\mathbb{P}^N(\mathbb{C})$ depending only on $\mathcal{H}$ such that if $f : M \to \mathbb{P}^N(\mathbb{C})$ is a meromorphic map whose image does not lie in $\mathcal{R}$, then, for every $\epsilon > 0$,

$$\sum_{j=1}^{q} m_f(H_j, r) . \leq . (N+1)T_f(r, s_0) + \frac{N(N+1)}{2} d_\pi \log r$$
$$+ \kappa \frac{N(N+1)}{2} [\log^+ T_f(r, s_0) + (2+\epsilon) \log^+ \log^+ T_f(r, s_0) + O(\log^+ r)],$$

where κ is the sheet number of π, d_π is the degree of the branching divisor of π, and $. \leq .$ means that the inequality holds for all $r \in [s_0, +\infty)$ outside a union of intervals of finite total length.

The proof of the main theorem also works for more general category of Stein parabolic manifolds. See the "Second Main Theorem for parabolic manifolds" in section 6.

We organize our paper as follows: In section 1, we recall the Cartan-Ahlfors theory for meromorphic maps on parabolic manifolds(see [St] or [WS]). In section 2, we give a slight generalization of the theorem of Wong-Stoll [WS] to the case where the hyperplanes $H_1, \ldots, H_q$ in $\mathbb{P}^n(\mathbb{C})$ are not necessarily in general position. In section 3, we recall the concept of the associate cycle $C_{\mathcal{H}}$ to the given set of hyperplanes $\mathcal{H} = \{H_1, \ldots, H_q\}$. We then study the relationship between the distance function of $f(z)$ to $C_{\mathcal{H}}$ and to $H_j, 1 \leq j \leq q$. In this section, we also recall the concept of Möbius inversion of cycles and express the associate cycle $C_{\mathcal{H}}$ in terms of the Möbius inversion of cycles. In section 4, we extend the Second Main Theorem which we established in section 2 to the case where linear subspaces E of $\mathbb{P}^n(\mathbb{C})$ are involved. In section 5, we recall an algebraic lemma, due to P. Vojta, which plays an essential role. In section 6, we adapt Vojta's method in [Vo2] to prove our main theorem.

1. Preliminaries. In this section, we recall some basic results in the theory of meromorphic maps on parabolic manifolds. For reference, see [St] and [WS].

1.1. Parabolic manifolds and affine algebraic manifolds. Let M be a connected complex manifold of dimension m. Let $\tau \geq 0$ be a non-negative, unbounded function of class C^∞ on M. For $0 \leq r \in \mathbb{R}$ and $A \subseteq M$ define

$$A[r] = \{x \in A \mid \tau(x) \leq r^2\}, \ A(r) = \{x \in A \mid \tau(x) < r^2\},$$

$$A\langle r \rangle = \{x \in A \mid \tau(x) = r^2\}, \ A_* = \{x \in A \mid \tau(x) > 0\},$$

$$\upsilon = dd^c \tau, \ \omega = dd^c log \tau, \ \sigma = d^c log \tau \wedge \omega^{m-1}.$$

If $M[r]$ is compact for each $r > 0$, the function τ is then said to be an *exhaustion* of M. The function τ is said to be *parabolic* if

$$\omega \geq 0, \ \omega^m \equiv 0, \ \upsilon^m \not\equiv 0$$

on M_*. Note that this also implies that $\upsilon \geq 0$ on M. If τ is a parabolic exhaustion, (M, τ) is said to be a *parabolic manifold*. Define

$$\hat{\mathbb{R}}_\tau = \{r \in \mathbb{R}^+ \mid d\tau(x) \neq 0 \text{ for all } x \in M\langle r \rangle\}.$$

Then $\mathbb{R}^+\backslash\hat{\mathbb{R}}_\tau$ has measure zero. If $r \in \hat{\mathbb{R}}_\tau$, the boundary $\partial M(r) = M\langle r\rangle$ is a compact, real, $(2m-1)$-dimensional submanifold of class C^∞ of M, oriented to the exterior of $M\langle r\rangle$. By Stoll ([St], p. 133), for all $r \in \hat{\mathbb{R}}_\tau$, $\int_{M\langle r\rangle}\sigma$ is a positive constant, independent of r.

Throughout this paper, we shall assume that M is an (connected) affine algebraic manifold of dimension $m \geq 1$ with $\pi : M \to \mathbb{C}^m$ a finite branched covering map (i.e., π is a surjective holomorphic map such that the number of points of a fiber is finite and that there exists a subvariety S of lower dimension such that the restriction of π to $M - S$ onto $\mathbb{C}^m - \pi(S)$ is a covering map). Here, by "affine algebraic", we mean there exists $k > m$ such that M is a closed complex submanifold of $\mathbb{C}^k$ and the closure $\bar{M}$ of M in $\mathbb{P}^k(\mathbb{C})$ is an analytic subset of $\mathbb{P}^k(\mathbb{C})$. Furthermore there exists a projection $\bar{\pi} : \bar{M} \to \mathbb{P}^m(\mathbb{C})$ such that $\bar{\pi}|_M = \pi$. Here $\mathbb{P}^m(\mathbb{C})$ is the closure of $\mathbb{C}^m$ in $\mathbb{P}^k(\mathbb{C})$. Since M is connected, the closure $\bar{M}$ is irreducible. The set $\tilde{\theta} = \{z \in M|$ the rank of the differential $\partial\pi(z)$ is not maximal$\}$ is an affine algebraic variety (of strictly lower dimension) of M and the image $\theta = \pi(\tilde{\theta})$ is an affine algebraic variety (of strictly lower dimension) of $\mathbb{C}^m$. We shall refer to $\tilde{\theta}$ as the *branching divisor*. The map $\pi : M \to \mathbb{C}^m$ is a finite map and the number of points in $\pi^{-1}(p)$ is independent of $p \in \mathbb{C}^m - \theta$. This common number, denoted by κ, is called the *sheet number*. A point $p \in \mathbb{C}^m - \theta$ is called a *generic* point.

For an (connected) affine algebraic manifold $\pi : M \to \mathbb{C}^m$, we define the exhaustion τ of M as $\tau = \|\pi\|^2$. Then τ is parabolic (cf. [GK], [St]). With this exhaustion function, we can also prove that the sheet number is

$$\kappa = \int_{M\langle r\rangle}\sigma, \tag{1.1.1}$$

and the degree of the branching divisor of π is

$$d_\pi = \lim_{r\to+\infty}\frac{Ric_\tau(r, s_0)}{\log r}. \tag{1.1.2}$$

1.2. Meromorphic maps; reduced representation. Let M be a complex manifold with $\dim M = m$. Let $A \neq \emptyset$ be an open subset of M such that $S = M - A$ is analytic. Then A is dense in M. Let $f : A \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map on A. The closure Γ of the graph $\{(x, f(x))|x \in A\}$ in $M \times \mathbb{P}^n(\mathbb{C})$ is called the *closed graph* of f. The map f is said to be *meromorphic* on M if (i) $\Gamma(f)$ is analytic in $M \times \mathbb{P}^n(\mathbb{C})$ and (ii) $\Gamma(f) \cap (K \times \mathbb{P}^n(\mathbb{C}))$ is compact for each compact subset $K \subseteq M$, i.e. the projection $\rho : \Gamma(f) \to M$ is proper. If f is meromorphic, then the set of *indeterminacy* $I_f = \{x \in M|\#\rho^{-1}(x) > 1\}$ is analytic with $\dim I_f \leq m-2$ and is contained in S. The holomorphic map $f : A \to \mathbb{P}^n(\mathbb{C})$ continues to a holomorphic map $f : M - I_f \to \mathbb{P}^n(\mathbb{C})$ such that we can assume, a posteriori, that $S = I_f$. If $m = 1$, I_f is necessarily empty and $f : M \to \mathbb{P}^n(\mathbb{C})$ is holomorphic.

Given M, A, S and a holomorphic map $f : A \to \mathbb{P}^n(\mathbb{C})$ as above. A holomorphic map $\mathbf{f}(\not\equiv 0) : U \to \mathbb{C}^{n+1}$ on an open and connected subset U of M is said to be a *representation* of f if $f(x) = \mathbb{P}(\mathbf{f}(x))$ for all $x \in A \cap U$ with $\mathbf{f}(x) \neq 0$. A representation $\mathbf{f}$ is said to be *reduced* if $\dim \mathbf{f}^{-1}(0) \leq m-2$. The map f is meromorphic if and only if for every point $p \in M$, there is a representation $\mathbf{f} : U \to \mathbb{C}^{n+1}$ of f with $p \in U$. If so, a representation $\mathbf{f}$ is reduced if and only if $U \cap I_f = \mathbf{f}^{-1}(0)$. There is also a reduced representation at every point $p \in M$.

1.3. The associated map. To define the associated maps, we need to assume that there exists a holomorphic form B of bidegree $(m-1, 0)$ on M. Let $\mathbf{f}$ be a holomorphic vector-valued function on an open subset U of M. If $z = (z_1, \ldots, z_m)$ is a chart with $U_z \cap U \neq \emptyset$, then the B-derivative $\mathbf{f}'_{B,z} = \mathbf{f}'$ on $U \cap U_z$ for z is defined by $d\mathbf{f} \wedge B = \mathbf{f}' dz_1 \wedge \cdots \wedge dz_m$. The operation can be iterated so that the k-th B-derivative $\mathbf{f}^{(k)}$ is defined: $\mathbf{f}^{(k)} = (\mathbf{f}^{(k-1)})'$. Put $\mathbf{f}^{(0)} = \mathbf{f}$. Abbreviate

$$\mathbf{f}_k = \mathbf{f} \wedge \mathbf{f}' \wedge \cdots \wedge \mathbf{f}^{(k)} : U \to \wedge^{k+1}\mathbb{C}^{n+1}.$$

Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map. If $\mathbf{f}_k \not\equiv 0$ for one choice of a reduced representation $\mathbf{f} : U \to \mathbb{C}^{n+1}$ on a chart U_z, then $\mathbf{f}_k \not\equiv 0$ for all possible choices and f is said to be *general of order k for B*. In this case, the k-th associated map $f_k : M \to \mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1})$ is well-defined as a meromorphic map by $f_k|U = \mathbb{P}(\mathbf{f}_k)$ for all possible choices of $\mathbf{f}$ and chart z. We say that f *is general for B* if f is general of order k for B for all k, $1 \le k \le n$.

The basic existence theorem for a holomorphic $(m-1)$-form B on M is due to W. Stoll. He (see [St]) proved the following statement: *Let M be a connected Stein manifold and let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic map. Then there exists a holomorphic $(m-1)$-form B on M such that f is general for B. If* $\dim M = 1$, *we may take $B \equiv 1$. If M is affine algebraic with the exhaustion τ defined as above and* $\dim M \ge 2$, *then the form B can be chosen so that*

$$mi_{m-1}B \wedge \bar{B} \le (1+\tau)^{n-1}(dd^c\tau)^{m-1}$$

where $i_{m-1} = \left(\dfrac{\sqrt{-1}}{2\pi}\right)(m-1)!(-1)^{(m-1)(m-2)/2}$.

Note that a general parabolic manifold M of complex dimension $m \ge 2$ may not be Stein. This is the reason that the theory is developed only for parabolic Stein manifold. For a general parabolic Stein manifold (M, τ), even though the existence of B is assured we do not, in general, have a polynomial type estimate as for affine algebraic manifolds. To overcome this difficulty, Stoll [St] postulates the existence of a majorant function such that

$$mi_{m-1}B \wedge \bar{B} \le Y(r)v^{m-1}$$

on $M[r]$. The theory can be carried out as in the algebraic case, except that the majorant function $Y(r)$ introduces an extra term in the Second Main Theorem. For simplicity, we only prove the theorem for the affine algebraic manifold M in this paper, and state the general theorem at the end.

1.4. Projective distance. Denote by ${\mathbb{C}^*}^{n+1}$ the dual space of $\mathbb{C}^{n+1}$. For $0 \le k \le n$, let $\lfloor$: $\left(\bigwedge^{k+1}\mathbb{C}^{n+1}\right) \times {\mathbb{C}^*}^{n+1} \to \bigwedge^{k}\mathbb{C}^{n+1}$ be the interior product defined in the usual way. Let $x \in \mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1})$ with representative $\xi \in \bigwedge^{k+1}\mathbb{C}^{n+1} - \{0\}$ and let $a \in \mathbb{P}({\mathbb{C}^*}^{n+1})$ with representative $\alpha \in {\mathbb{C}^*}^{n+1} - \{0\}$, the **projective distance**

between x and a is defined by

$$0 \le \|x; a\| = \frac{\|\xi \lfloor \alpha\|}{\|\xi\| \|\alpha\|} \le 1, \tag{1.4.1}$$

where the norm on on $\bigwedge^k \mathbb{C}^{n+1}$ is induced by the standard norm on $\mathbb{C}^{n+1}$. Note that the above definition is independent of choice of the representatives α and ξ. Note that a hyperplane H in $\mathbb{P}^n(\mathbb{C})$ can also be regarded as a point in $\mathbb{P}^n(\mathbb{C}^*)$. Hence, for every meromorphic map $f : M \to \mathbb{P}^n(\mathbb{C})$, $\|f_k(z); H\|$ is defined for $z \in M$. This gives a distance function (from $f_k(z)$ to H) on M.

1.5. The first main theorem. Let M be an (connected) affine algebraic manifold. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is linearly non-degenerate. Then, as we discussed above, a holomorphic $(m-1)$-form B exists on M such that f is general for B and

$$m i_{m-1} B \wedge \bar{B} \le (1+\tau)^{n-1} (dd^c \tau)^{m-1}.$$

Let f_k be the k-th associated map of f. Let Ω_k be the Fubini-Study form on $\mathbb{P}^n(\bigwedge^{k+1} \mathbb{C}^{n+1})$. Define the k-th characteristic function for $0 < s_0 < r$

$$T_{f_k}(r, s_0) = \int_{s_0}^{r} \frac{dt}{t^{2m-1}} \int_{M[t]} f_k^*(\Omega_k) \wedge \upsilon^{m-1}.$$

It is known that $T_{f_n}(r, s_0) \equiv 0$. Denote $T_{f_{-1}}(r, s) \equiv 0$.

Let ν be a divisor on M with $S = \text{supp}\ \nu$. The counting function of ν is defined to be

$$N_\nu(r, s_0) = \int_{s_0}^{r} n_\nu(t) \frac{dt}{t}$$

where

$$n_\nu(t) = t^{2-2m} \int_{S[t]} \nu \upsilon^{m-1} = \int_{S_*[t]} \nu \omega^{m-1} + n_\nu(0), \text{ if } m > 1$$

$$n_\nu(t) = \sum_{z \in S[t]} \nu(z), \text{ if } m = 1.$$

For a hyperplane H in $\mathbb{P}^n(\mathbb{C})$, define an H-divisor $\nu = \mu_{f_k}^H$ as in Stoll [St]. Let $N_{f_k}(r, H) = N_\nu(r, s_0)$ and let

$$m_{f_k}(r, H) = \int_{M\langle r \rangle} \log \frac{1}{\|f_k; H\|} \sigma.$$

Then we have

Theorem 1.5 [First Main Theorem] ([St, (8.21)]). *Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is general for B. Then, for every hyperplane H in $\mathbb{P}^n(\mathbb{C})$ and for every $0 \le k \le n, s_0, r \in \hat{R}_\tau, 0 < s_0 < r$,*

$$T_{f_k}(r, s_0) \ge N_{f_k}(r, H) + m_{f_k}(r, H) - m_{f_k}(s_0, H).$$

1.6. Calculus lemma. Let T be a nonnegative function defined on an interval $[s_0, r]$ with $s_0 \geq 0$. Define the error functions $E(T,r)$ and $\tilde{E}(T,r)$ by

$$\tilde{E}(T,r) = T(r)\log^{1+\epsilon}(1+T(r))\log^{1+\epsilon}[1+r^{2m-1}T(r)\log^{1+t}(1+T(r))] \tag{1.6.1}$$

and

$$E(T,r) = \log^+ \tilde{E}(T,r). \tag{1.6.2}$$

Calculus Lemma. *Let h be a nonnegative measurable function on M such that $h\upsilon^m$ is locally integrable. Let T be a function defined by*

$$T(r) = \int_{s_0}^{r} \frac{dt}{t^{2m-1}} \int_{M[t]} h\upsilon^m.$$

Then $h\sigma$ is integrable over $M\langle r\rangle$ for almost all $r > 0$ and

$$2m\int_{M\langle r\rangle} h\sigma = r^{-(2m-1)}\frac{d}{dr}\left(r^{2m-1}\frac{dT}{dr}\right) \leq \tilde{E}(T,r)$$

holds for all $r \in [s_0, +\infty)$ outside a union of intervals of finite total length.

Proof. By Corollary 2.4 in [WS] using $g(t) = \log^{1+\epsilon}(1+t)$.

1.7. The Plücker formula. Let d_k be the zero divisor of f_k. When $k = n$, we obtain the Wronskian divisor d_n. The divisor $l_k = d_{k-1} - 2d_k + d_{k+1} \geq 0$ is called the k-th stationary index, where we assume that $d_{-1} = 0$. Let I_k be the indeterminacy of f_k. On $M - I_k$, we define

$$\tilde{h}_k = m!\left(\frac{\sqrt{-1}}{2\pi}\right)^{m-1}(-1)^{(m-1)(m-2)/2} f_k^*(\Omega_k)\wedge B\wedge \bar{B}. \tag{1.7.1}$$

It is known that $\tilde{h}_k \geq 0$ (cf. [St]). Define

$$h_k = \tilde{h}_k/\upsilon^m. \tag{1.7.2}$$

For all $r \in \hat{R}_\tau$, define

$$S_k(r) = \frac{1}{2}\int_{M\langle r\rangle} \log\ h_k\ \sigma. \tag{1.7.3}$$

Plücker Formula [St, theorem 7.6]. *For almost all $s_0, r \in \hat{R}_\tau, 0 < s_0 < r$,*

$$N_{l_k}(r,s_0) + T_{f_{k-1}}(r,s_0) - 2T_{f_k}(r,s_0) + T_{f_{k+1}}(r,s_0) = S_k(r) - S_k(s_0) + Ric_\tau(r,s_0).$$

The Plücker formula implies the following result(see [St, (10.23)]):

Theorem 1.7.2. *For $0 \leq k \leq n-1$,*

$$T_{f_k}(r,s) \leq 3^k T_f(r,s) + \frac{1}{2}(3^k - 1)(\kappa(n-1)\log(1+r^2) + Ric_\tau(r,s) + \epsilon\kappa\log r)$$

holds for all $r \in [s_0, \infty)$ outside a union of intervals of finite total length.

1.8. The Ahlfors' estimate.

THE AHLFORS' ESTIMATE [ST, P.160, THEOREM 10.3]. *Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$. Then for any $0<\lambda<1$, $0<s_0<r$, we have*

$$\int_s^r \frac{dt}{t^{2m-1}}\int_{M[t]}\frac{\|f_{k+1};H\|^2}{\|f_k;H\|^{2-2\lambda}}h_k\upsilon^m \le (1+r^2)^{n-1}\left(\frac{4+4\lambda}{\lambda}T_{f_k}(r,s)+\frac{2\kappa}{\lambda^2}\log 2\right)$$

where h_k is defined in (1.7.2), and $\kappa=\int_{M\langle r\rangle}\sigma>0$ (cf. (1.1.1)) is a constant.

THEOREM 1.8.2. *Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$. Let $\epsilon>0$ and $\Lambda(r)=\min_k\{1/(1+T_{f_k}(r,s_0))\}$. Then, for every $0\le k\le n-1$,*

$$\log^+\int_{M\langle r\rangle}\frac{\|f_{k+1};H\|^2}{\|f_k;H\|^{2-2\Lambda(r)}}h_k\sigma .\le .2\log^+T_f(r,s_0)+2(2+\epsilon)\log^+\log^+T_f(r,s_0)$$
$$+4(n-1)\log^+r+3\log^+Ric_\tau(r,s_0)+5\log^+\log^+r+C'$$

where $.\le.$ means that the inequality holds for all $r\in[s_0,+\infty)$ outside a union of intervals of finite total length, and the constant C' is independent of r.

Proof. Let $0<\Lambda(r)<1$ be a decreasing function of $r\ge 0$. Define functions

$$K_k(r,s_0)=\int_s^r\frac{dt}{t^{2m-1}}\int_{M[t]}\frac{\|f_{k+1};H\|^2}{\|f_k;H\|^{2-2\Lambda^*}}h_k\upsilon^m$$

where $\Lambda^*=\Lambda\circ\tau^{1/2}$. By the calculus lemma, we have

$$\int_{M\langle r\rangle}\frac{\|f_{k+1};H\|^2}{\|f_k;H\|^{2-2\Lambda(r)}}h_k\sigma .\le .\tilde{E}(K_k,r).$$

On the other hand, noticing that Λ is a decreasing function, we have $\|f_k;H\|^{\Lambda^*}\le\|f_k;H\|^{\Lambda(r)}$. Hence by Ahlfor's estimate with $\lambda=\Lambda(r)$, we have

$$K_k(r,s_0)=\int_s^r\frac{dt}{t^{2m-1}}\int_{M[t]}\frac{\|f_{k+1};H\|^2}{\|f_k;H\|^{2-2\Lambda^*}}h_k\upsilon^m$$
$$\le(1+r^2)^{n-1}\left(\frac{8}{\Lambda(r)}T_{f_k}(r,s_0)+\frac{2\kappa\log 2}{\Lambda(r)^2}\right).$$

Since $\Lambda(r)=\min_k\{1/(1+T_{f_k}(r,s))\}$,

$$K_k(r,s_0)\le(1+r^2)^{n-1}(b_1T^2_{f_k}(r,s_0)+b_2),$$

where b_1 and b_2 are constants depending only on κ. By choosing a larger constant b_3, we have

$$\tilde{E}(K_k,r)\le\tilde{E}(b_3(1+r^2)^{n-1}T^2_{f_k}(r,s_0),r).$$

Hence we get

$$\int_{M\langle r\rangle}\frac{\|f_{k+1};H\|^2}{\|f_k;H\|^{2-2\Lambda(r)}}h_k\sigma .\le .\tilde{E}(b_3(1+r^2)^{n-1}T^2_{f_k}(r,s_0),r). \tag{1.8.1}$$

By the definition, we have (see (2.8) in [WS], page 1046)

$$\begin{aligned}
&E(b_3(1+r^2)^{n-1}T_{f_k}^2(r,s_0),r)\\
&\leq \log^+(b_3(1+r^2)^{n-1}T_k^2(r,s_0)) + 2(1+\epsilon)\log^+\log^+(b_3(1+r^2)^{n-1}T_k^2(r,s_0))\\
&+ (1+\epsilon)\log^+\log^+\log^+(b_3(1+r^2)^{n-1}T_k^2(r,s_0)) + (1+\epsilon)\log^+\log^+ r + C'\\
&\leq 2\log^+ T_k(r,s_0) + 2(1+\epsilon)\log^+\log^+ T_k(r,s_0)\\
&+ (1+\epsilon)\log^+\log^+\log^+ T_k(r,s_0) + 2\log^+(1+r^2)^{n-1} + 2\log^+\log^+ r + C'.
\end{aligned}$$

By Theorem 1.7.2,

$$T_{f_k}(r,s_0) \leq 3^k T_f(r,s_0) + \frac{1}{2}(3^k-1)(\kappa(n-1)\log(1+r^2) + Ric_\tau(r,s_0) + \epsilon\kappa\log r).$$

Hence

$$\begin{aligned}
E(b_3(1+r^2)^{n-1}T_{f_k}^2(r,s_0),r) \leq{}& 2\log^+ T_f(r,s_0) + 2(2+\epsilon)\log^+\log^+ T_f(r,s_0)\\
&+ 4(n-1)\log^+ r + 3\log^+ Ric_\tau(r,s_0) + 5\log^+\log^+ r + C'.
\end{aligned}$$

This, together with (1.8.1), concludes the proof.

2. A slight generalization of Wong-Stoll's theorem. In this section, we extend the Second Main Theorem of Wong-Stoll (c.f. [WS]) to the case where the given hyperplanes $H_1,\ldots,H_q$ in $\mathbb{P}^n(\mathbb{C})$ are not necessarily in general position.

THEOREM 2.1. *Let M be an affine algebraic manifold of complex dimension m. Let $\pi: M \to \mathbb{C}^m$ be a finite branched covering. Let $\tau = \|\pi\|^2$. Let $f: M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is linearly non-degenerate. Let $\epsilon > 0$ and let $H_1,\ldots,H_q$ be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Then*

$$\begin{aligned}
&\int_{M\langle r\rangle} \max_K \sum_{j\in K} \log\frac{1}{\|f;H_j\|}\sigma .\leq. (n+1)T_f(r,s_0) + \frac{n(n+1)}{2}Ric_\tau(r,s_0)\\
&+ \kappa\frac{n(n+1)}{2}[\log^+ T_f(r,s_0) + (2+\epsilon)\log^+\log^+ T_f(r,s_0)\\
&+ 2\log^+ Ric_\tau(r,s) + 2(n-1)\log^+ r + 3\log^+\log^+ r + O(1)].
\end{aligned}$$

where ". ≤ ." means that the inequality holds for all $r \in [s_0,+\infty)$ outside a union of intervals of finite total length, and the max is taken over all subsets K of $\{1,\ldots,q\}$ such that the linear forms $H_j, j \in K$, are linearly independent.

Proof. Denote by $K \subset \{1,\ldots,q\}$ such that linear forms $\{H_k, k\in K\}$, are linearly independent. Without loss of generality, we may assume $q \geq n+1$ and that $\#K = n+1$. Let T be the set of all the injective maps $\mu: \{0,1,\ldots,n\} \to \{1,\ldots,q\}$ such that $H_{\mu(0)},\ldots,H_{\mu(n)}$ are linearly independent. Denote by $\Gamma = \max_{1\leq j\leq q}\{\sum_{k=0}^{n-1} m_{f_k}(s_0,H_j)\}$ and $\Lambda(r) = \min_k\{1/(1+T_{f_k}(r,s_0))\}$. For any $\mu \in T, z \notin I_f$, the Product to Sum Estimate (see [WS] Lemma 1.12), with $\lambda = \Lambda(r)$, reads

$$\prod_{j=0}^{n} \frac{\|f_{k+1}(z);H_{\mu(j)}\|^2}{\|f_k(z);H_{\mu(j)}\|^{2-2\Lambda(r)}} \leq c_k \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z);H_{\mu(j)}\|^2}{\|f_k(z);H_{\mu(j)}\|^{2-2\Lambda(r)}}\right)^{n-k},$$

where $c_k > 0$ is a constant. Since $\|f_n; H_{\mu(j)}\|$ is a constant for any $0 \le j \le n$, we have

$$\prod_{j=0}^{n} \frac{1}{\|f(z); H_{\mu(j)}\|^2} = \prod_{k=0}^{n-1}\prod_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \cdot \prod_{k=0}^{n-1}\prod_{j=0}^{n} \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}}$$
$$\le c \prod_{k=0}^{n-1} \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \right)^{n-k} \cdot \prod_{k=0}^{n-1}\prod_{j=0}^{n} \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}},$$

where $c > 1$ is a constant. Therefore, for $r > s_0$, we have

(2.1)
$$\int_{M\langle r\rangle} \max_K \sum_{j\in K} \log \frac{1}{\|f(z); H_j\|^2}\, \sigma = \int_{M\langle r\rangle} \max_{\mu\in T} \log \left(\prod_{j=0}^{n} \frac{1}{\|f(z); H_{\mu(j)}\|^2} \right) \sigma$$
$$\le \sum_{k=0}^{n-1} \int_{M\langle r\rangle} \max_{\mu\in T} \log \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \right)^{n-k} \sigma$$
$$+ \sum_{k=0}^{n-1}\sum_{j=0}^{n} \int_{M\langle r\rangle} \max_{\mu\in T} \log \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}} \sigma + O(1)$$
$$= \sum_{k=0}^{n-1} (n-k) \int_{M\langle r\rangle} \log \max_{\mu\in T} \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k \right) \sigma - 2\sum_{k=0}^{n-1} (n-k) S_k(r)$$
$$+ \sum_{k=0}^{n-1}\sum_{j=0}^{n} \int_{M\langle r\rangle} \max_{\mu\in T} \log \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}} \sigma + O(1)$$

where, in above, h_k is defined by (1.7.2), $S_k(r)$ is defined by (1.7.3). We now estimate each term appearing the above inequality. First,

(2.2)
$$\int_{M\langle r\rangle} \log \max_{\mu\in T} \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k \right) \sigma$$
$$= \kappa \int_{M\langle r\rangle} \log \max_{\mu\in T} \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k \right) \frac{\sigma}{\kappa}$$
$$\le \kappa \log \int_{M\langle r\rangle} \max_{\mu\in T} \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k \right) \frac{\sigma}{\kappa}$$
$$\le \kappa \max_{1\le j\le q} \log^+ \int_{M\langle r\rangle} \frac{\|f_{k+1}(z); H_j\|^2}{\|f_k(z); H_j\|^{2-2\Lambda(r)}} h_k \sigma + C'.$$

By Theorem 1.8.2,

$$\max_{1\leq j\leq q}\log^+\int_{M\langle r\rangle}\frac{\|f_{k+1}(z);H_j\|^2}{\|f_k(z);H_j\|^{2-2\Lambda(r)}}h_k\sigma$$
$$.\leq.\ 2\left[\log^+T_f(r,s_0)+(2+\epsilon)\log^+\log^+T_f(r,s_0)\right.$$
$$\left.+2\log^+Ric_\tau(r,s_0)+2(n-1)\log^+r+3\log^+\log^+r+O(1)\right].$$

Hence,

(2.3)

$$\sum_{k=0}^{n-1}\int_{M\langle r\rangle}\log\max_{\mu\in T}\left(\sum_{j=0}^{n}\frac{\|f_{k+1}(z);H_{\mu(j)}\|^2}{\|f_k(z);H_{\mu(j)}\|^{2-2\Lambda(r)}}h_k\right)^{n-k}\sigma.\leq.n(n+1)\kappa\left[\log^+T_f(r)\right.$$
$$\left.+(2+\epsilon)\log^+\log^+T_f(r)+2\log^+Ric_\tau(r,s)+2(n-1)\log^+r+3\log^+\log^+r+O(1)\right].$$

Next, using the Plücker formula, we have

$$N_{l_k}(r,s_0)+T_{f_{k-1}}(r,s_0)-2T_{f_k}(r,s_0)+T_{f_{k+1}}(r,s_0)=S_k(r)-S_k(s_0)+Ric_\tau(r,s_0).$$

Noticing that $T_{f_n}(r,s_0)=0$,

$$\sum_{k=0}^{n-1}(n-k)S_k(r)=N_{d_n}(r,s_0)-(n+1)T_f(r,s_0)-\frac{n(n+1)}{2}Ric_\tau(r,s_0)+O(1). \tag{2.4}$$

Finally, by the First Main Theorem,

$$\begin{aligned}&\sum_{k=0}^{n-1}\sum_{j=0}^{n}\int_{M\langle r\rangle}\max_{\mu\in T}\log\frac{1}{\|f_k(z);H_{\mu(j)}\|^{2\Lambda(r)}}\ \sigma\\
&\leq\sum_{\mu\in T}\sum_{k=0}^{n-1}\sum_{j=0}^{n}\int_{M\langle r\rangle}2\Lambda(r)\log\frac{1}{\|f_k(z);H_{\mu(j)}\|}\ \sigma+O(1)\\
&=\sum_{\mu\in T}\sum_{k=0}^{n-1}\sum_{j=0}^{n}2\Lambda(r)m_{f_k}(r,H_{\mu(j)})+O(1)\\
&\leq\sum_{k=0}^{n-1}\sum_{j=0}^{n}2q!\Lambda(r)(T_{f_k}(r,s_0)+m_{f_k}(s_0,H_{\mu(j)}))+O(1)\\
&\leq O(1).\end{aligned}\tag{2.5}$$

Combining (2.1), (2.2), (2.3), (2.4) and (2.5), we have

$$\begin{aligned}\int_{M\langle r\rangle}\max_K\sum_{j\in K}\log\frac{1}{\|f(z);H_j\|}\sigma.\leq.&(n+1)T_f(r)+\frac{n(n+1)}{2}Ric_\tau(r,s_0)-N_{d_n}(r,s_0)\\
&+\kappa\frac{n(n+1)}{2}[\log^+T_f(r)+(2+\epsilon)\log^+\log^+T_f(r)\\
&+2\log^+Ric_\tau(r,s)+2(n-1)\log^+r\\
&+3\log^+\log^+r+O(1)].\end{aligned}$$

3. Distance function, associated cycles, and Möbius inversion of cycles.

3.1. Distance function and associated cycles. Let

$$H = \{[x_0 : \cdots : x_n] \mid a_0x_0 + \cdots + a_nx_n = 0\},$$

be a hyperplane in $\mathbb{P}^n(\mathbb{C})$, with $|a_0|^2 + \cdots + |a_n|^2 = 1$. We define the Weil function for H as, for $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{C})\backslash H$,

$$\lambda_H(\mathbf{x}) := \log \frac{\max(|x_0|, \ldots, |x_n|)}{|a_0x_0 + \cdots + a_nx_n|}.$$

Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map and assume that its image is not contained in H. Choose a reduced representation $\mathbf{f} : U \to \mathbb{C}^{n+1}$ on a chart U_z for $z \in M$. We define

$$\lambda_{f,H}(z) := \lambda_H(\mathbf{f}(z)).$$

This definition is independent of the choice of the reduced representations.

Definition 3.1.1. Let C be a proper linear subspace of $\mathbb{P}^n(\mathbb{C})$, let $H_1, \ldots, H_r$ be hyperplanes such that $C = \cap H_i$. Then a *distance function* for C is a continuous function $\lambda_C : \mathbb{P}^n(\mathbb{C})\backslash C \to \mathbb{R}$ such that

$$\lambda_C = \min_{1 \le i \le r} \lambda_{H_i} + O(1).$$

Note that the definition does not depend on the choice of the H_i.

Definition 3.1.2. If $C = \sum n_iC_i$ is a cycle in $\mathbb{P}^n(\mathbb{C})$ such that all C_i are proper linear subspaces and λ_{C_i} are the distance functions for C_i for all i, then we say that a function $\lambda_C : \mathbb{P}^n(\mathbb{C})\backslash \mathrm{Supp}C \to \mathbb{R}$ is a *distance function* for C if it is continuous and if

$$\lambda_C = \sum n_i\lambda_{C_i} + O(1).$$

Here Supp$C$ denotes the *support* of C, which is $\cup_{n_i \ne 0} C_i$ (if all C_i are distinct). Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map, not lying in the support of C, and choose a reduced representation $\mathbf{f} : U \to \mathbb{C}^{n+1}$ on a chart U_z for $z \in M$, then we also define

$$\lambda_{f,C}(z) := \lambda_C(\mathbf{f}(z)).$$

We define

$$m_f(r, C) = \int_{M\langle r\rangle} \lambda_{f,C}(z)\sigma.$$

This is well-defined up to $O(1)$.

Given hyperplanes $\mathcal{H} = \{H_1, \ldots, H_q\}$ in $\mathbb{P}^n(\mathbb{C})$ (not necessarily in general position), there is one cycle of particular interest. This cycle is called the *associated cycle* of $\mathcal{H}$ which is defined as follows.

Definition 3.1.3. The *associated cycle* of $\mathcal{H}$ is the cycle $C_{\mathcal{H}} = \sum n_iC_i$ such that

(i) the set of components C_i is the set of nonempty linear subspaces of $\mathbb{P}^n(\mathbb{C})$ which can be written as an intersection of one or more of the hyperplanes H_j, and

(ii) the multiplicity n_i satisfies the equation

$$\sum_{\{j|C_j\supseteq C_i\}} n_j = \operatorname{codim} C_i$$

for every i. (In particular, $n_i = 1$ if C_i is a hyperplane and $n_i \leq 0$ otherwise.)

If $H_1, \ldots, H_q$ are in general position, then this cycle equals $\sum H_i$.

Definition 3.1.4. The set of cycles as in Definition 3.1.2 forms an abelian group under addition. We define a partial order on this group by saying that $C \geq 0$ if, writing $C = \sum n_i C_i$, we have

$$\sum_{\{i|C_i\supseteq L\}} n_i \geq 0$$

for all (nonempty) linear subspaces L of $\mathbb{P}^n(\mathbb{C})$. We also say that a cycle C is *effective* if $n_i \geq 0$ for all i; note that this is strictly stronger than saying $C \geq 0$ (unless $n < 2$).

The associated cycle of $H_1, \ldots, H_q$ then has the property (see (3.5) in [Vo 3]) that

$$C = \max_J \sum_{j\in J} H_j, \tag{3.1.1}$$

where the maximum is taken over all subsets J of $\{1, \ldots, q\}$ such that the linear forms, corresponding to $H_j, j \in J$, are linearly independent over $\mathbb{C}$.

We also need the following Lemma from [Vo3]:

Lemma 3.1.5. *Let C be a cycle as in Definition 3.1.2 and let λ_C be a distance function for C. Then $C \geq 0$ if and only if λ_C is bounded from below.*

Proof. See [Vo3] Proposition 3.6.

Combining (3.1.1) and Lemma 3.1.5, we have the following result.

Lemma 3.1.6. *Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a set of hyperplanes in $\mathbb{P}^n(\mathbb{C})$ and let $C_{\mathcal{H}}$ be its associated cycle. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map, not lying in the hyperplanes in $\mathcal{H}$. Then*

$$\lambda_{f,C_{\mathcal{H}}}(z) = \max_J \sum_{j\in J} \lambda_{f,H_j}(z) + O(1), \tag{3.1.1}$$

where the maximum is taken over all subsets J of $\{1, ..., q\}$ such that the linear forms H_j, $j \in J$, are linearly independent over $\mathbb{C}$.

3.2. Möbius inversion of cycles. We recall more definitions and results from [Vo2] and [Vo3].

Definition 3.2.1. Let $\mathcal{D}$ be a finite collection of proper linear subspaces of $\mathbb{P}^n(\mathbb{C})$ having the property that if D_1 and D_2 are in $\mathcal{D}$, then so is $D_1 \cap D_2$.

(a) Let $\mu_{\mathcal{D}}$ be the function from $\mathcal{D}$ to the group of cycles supported on $\mathcal{D}$, defined by the Möbius condition

$$\sum_{\substack{D\in\mathcal{D}\\ D\subseteq D_0}} \mu_{\mathcal{D}}(D) = D_0 \tag{3.2.1}$$

for all $D_0 \in \mathcal{D}$.

(b) Let

$$m_f^{\mathcal{D}}(D, r) = m_f(\mu_{\mathcal{D}}(D), r).$$

Write

$$\mu_{\mathcal{D}}(D_0) = \sum_{D \in \mathcal{D}} n_{D_0,D}^{\mathcal{D}} D.$$

Then (3.2.1) is equivalent to the condition

$$\sum_{\substack{D \in \mathcal{D} \\ D \subseteq D_0}} n_{D,D_1}^{\mathcal{D}} = \begin{cases} 1 & \text{if } D_1 = D_0, \\ 0 & \text{otherwise.} \end{cases}$$

This condition, in turn, is equivalent to

$$\sum_{\substack{D \in \mathcal{D} \\ D \supseteq D_1}} n_{D_0,D}^{\mathcal{D}} = \begin{cases} 1 & \text{if } D_1 = D_0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.2}$$

Indeed, the former condition says that the matrix given by the $n_{D_0,D}^{\mathcal{D}}$ is the right inverse of the matrix $(m_{D_0,D})$ given by $m_{D_0,D} = 1$ if $D_0 \supseteq D$ and $m_{D_0,D} = 0$ otherwise; the latter condition (3.2.2) says that it is also the left inverse of that matrix.

LEMMA 3.2.2. *Let $\mathcal{H}$ be a set of hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Let $\mathcal{D}$ be the set of linear subspaces that can be written as the intersection of one or more of the hyperplanes in $\mathcal{H}$. Let $C_{\mathcal{H}}$ be the associated cycle of $\mathcal{H}$. Then*

$$C_{\mathcal{H}} = \sum_{D \in \mathcal{D}} (\operatorname{codim} D) \mu_{\mathcal{D}}(D).$$

Proof. For $D_1 \in \mathcal{D}$, the coefficient of D_1 in the right hand side of the equality is $n_{D_1} = \sum_{D \in \mathcal{D}} (\operatorname{codim} D) n_{D,D_1}$. For $D_0 \in \mathcal{D}$, it follows that

$$\sum_{\substack{D_1 \in \mathcal{D} \\ D_1 \supseteq D_0}} n_{D_1} = \sum_{D \in \mathcal{D}} (\operatorname{codim} D) \sum_{\substack{D_1 \in \mathcal{D} \\ D_1 \supseteq D_0}} n_{D,D_1} = \operatorname{codim} D_0$$

by (3.2.2). Comparing with (ii) in Definition 3.1.3 then gives the lemma.

4. Projective version of the second main theorem. By Lemma 3.1.6, Theorem 2.1 can be rewritten as

$$\begin{aligned} m_f(r, \mathcal{C}_{\mathcal{H}}) .\leq .&(n+1) T_f(r, s_0) + \frac{n(n+1)}{2} Ric_\tau(r, s_0) \\ &+ \kappa \frac{n(n+1)}{2} [\log^+ T_f(r, s_0) + (2+\epsilon) \log^+ \log^+ T_f(r, s_0) \\ &+ 2 \log^+ Ric_\tau(r, s_0) + O(\log^+ r)]. \end{aligned}$$

Using Lemma 3.2.2, this also can be written as

$$\begin{aligned}\sum_{D\in\mathcal{D}}(\operatorname{codim} D)m_f^{\mathcal{D}}(D,r).\le.(n+1)T_f(r,s_0)+\frac{n(n+1)}{2}Ric_\tau(r,s_0)\\ +\kappa\frac{n(n+1)}{2}[\log^+T_f(r,s_0)+(2+\epsilon)\log^+\log^+T_f(r,s_0)\\ +2\log^+Ric_\tau(r,s_0)+O(\log^+r)],\end{aligned}\tag{4.1}$$

where $\mathcal{D}$ is the set of linear subspaces of $\mathbb{P}^n(\mathbb{C})$ that can be written as the intersection of one or more of the hyperplanes in $\mathcal{H}$.

In this section, we will extend this result to a more general case, which involves the projection of $\mathbb{P}^n(\mathbb{C})$ to E, for every linear subspace E. Let D, E be two subspaces of $\mathbb{P}^n(\mathbb{C})$, we use $< D, E >$ to denote the smallest subspace which contains D and E.

Proposition 4.1. *Let $\mathcal{H}=\{H_1,...,H_q\}$ be a set of hyperplanes in $\mathbb{P}^n(\mathbb{C})$, and let $\mathcal{D}$ be the collection of nonempty proper linear subspaces that can be written as H_i or a finite intersection of hyperplanes in $\mathcal{H}$. Let $f: M\to\mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate meromorphic map. Then, for every subspace E of $\mathbb{P}^n(\mathbb{C})$ and every $\epsilon>0$,*

$$\begin{aligned}\sum_{D\in\mathcal{D}}\operatorname{codim}<D,E>m_f^{\mathcal{D}}(D,r).\le.(\operatorname{codim} E)T_f(r)+\frac{n\operatorname{codim} E}{2}Ric_\tau(r,s_0)\\ +\kappa\frac{n\operatorname{codim} E}{2}[\log^+T_f(r,s_0)+(2+\epsilon)\log^+\log^+T_f(r,s_0)\\ +2\log^+Ric_\tau(r,s_0)+O(\log^+r)+O(1)]\end{aligned}$$

where ". ≤ ." means that the inequality holds for all $r\in[s_0,+\infty)$ outside a union of intervals of finite total length,

Proof. We first observe, by (4.1), that Proposition 4.1 holds if $E=\mathbb{P}^n(\mathbb{C})$. It also trivially holds if E is a point. Hence, we let E be a proper linear subspace of $\mathbb{P}^n(\mathbb{C})$. Without loss of generality, we assume that $E=\mathbb{P}^t(\mathbb{C})$ with $t>0$. We consider the projection $\phi:\mathbb{P}^n(\mathbb{C})\to\mathbb{P}^{n-t-1}(\mathbb{C})$ defined by $\phi[x_0:\cdots:x_n]=[x_{t+1}:\cdots:x_n]$, and consider the map $\phi\circ f: M\to\mathbb{P}^{n-t-1}(\mathbb{C})$. Then $\phi\circ f$ is still linearly non-degenerate. Let $\mathcal{D}'$ be the set of subspaces $\phi(D)$, $D\in\mathcal{D}$, together with all intersections thereof.

Then by Theorem 2.1, Lemma 3.1.6 and Lemma 3.2.2,

$$
\begin{aligned}
&\sum_{D'\in\mathcal{D}'}(\operatorname{codim} D')m^{\mathcal{D}'}_{\phi\circ f}(D',r)\\
&.\le .(\operatorname{codim} E)\cdot T_{\phi\circ f}(r,s_0)+\frac{(\operatorname{codim} E-1)\operatorname{codim} E}{2}Ric_\tau(r,s_0)\\
&+\kappa\frac{(\operatorname{codim} E-1)\operatorname{codim} E}{2}\left[\log^+ T_{\phi\circ f}(r,s_0)+(2+\epsilon)\log^+\log^+ T_{\phi\circ f}(r,s_0)\right.\\
&\left.+2\log^+ Ric_\tau(r,s_0)+O(\log^+ r)\right]\\
&.\le .(\operatorname{codim} E)\cdot T_{\phi\circ f}(r,s_0)+\frac{n\operatorname{codim} E}{2}Ric_\tau(r,s_0)\\
&+\kappa\frac{n\operatorname{codim} E}{2}\left[\log^+ T_{\phi\circ f}(r,s_0)+(2+\epsilon)\log^+\log^+ T_{\phi\circ f}(r,s_0)\right.\\
&\left.+2\log^+ Ric_\tau(r,s_0)+O(\log^+ r)\right].
\end{aligned}
\tag{4.2}
$$

We now compare the characteristic function and the proximity function of f and $\phi(f)$. Let $\{U_\lambda,\lambda\in\Lambda\}$ be an open covering of M, and let $\mathbf{f}_\lambda : U_\lambda\to\mathbb{C}^{n+1}$ be a reduced representation of f on U_λ, then there is a holomorphic function $g_{\lambda\mu}: U_\lambda\cap U_\mu\to\mathbb{C}^*$ such that

$$\mathbf{f}_\lambda = g_{\lambda\mu}\mathbf{f}_\mu \text{ on } U_\lambda\cap U_\mu.$$

It is easy to check that $\{g_{\lambda\mu}\}$ is a basic cocycle(cf. [St]). Therefore there are exists a holomorphic line bundle L_f on M with a holomorphic frame atlas $\{U_\lambda, s_\lambda\}_{\lambda\in\Lambda}$ such that

$$s_\mu = g_{\lambda\mu}s_\lambda \text{ on } U_\lambda\cap U_\mu.$$

Also define $\tilde{\mathbf{f}}_\lambda\in\Gamma(U_\lambda, M\times\mathbb{C}^{n+1})$ by $\tilde{\mathbf{f}}_\lambda(z)=(z,\mathbf{f}_\lambda(z))$ for $z\in U_\lambda$. Hence $\tilde{\mathbf{f}}_\lambda\otimes s_\lambda = g_{\lambda\mu}\tilde{\mathbf{f}}_\mu\otimes s_\lambda=\tilde{\mathbf{f}}_\mu\otimes g_{\lambda\mu}s_\lambda=\tilde{\mathbf{f}}_\mu\otimes s_\mu$ on $U_\lambda\cap U_\mu$. Therefore there exists a holomorphic section F_f of $(M\times\mathbb{C}^{n+1})\otimes L_f$ such that $F_f|_{U_\lambda}=\tilde{\mathbf{f}}_\lambda\otimes s_\lambda$. Let ℓ be the standard hermitian metric along the fibers of the trivial bundle $M\times\mathbb{C}^{n+1}$ and ρ be a hermitian metric along the fibers of L_f. Then

$$dd^c\log\|F_f\|^2_{\ell\otimes\rho}=dd^c\log\|\mathbf{f}_\lambda\|^2+dd^c\log\|s_\lambda\|^2_\rho=f^*\Omega_{FS}-c_1(L_f,\rho),$$

where Ω_{FS} is the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$. Hence, by Green's formula(cf. [St]), we have

$$
\begin{aligned}
T_f(r,s_0)=&\int_{s_0}^r\frac{dt}{t^{2m-1}}\int_{M[t]}c_1(L_f,\rho)\wedge\upsilon^{m-1}+\int_{M\langle r\rangle}\log\|F_f\|_{\ell\otimes\rho}\sigma\\
&-\int_{M<s_0>}\log\|F_f\|_{\ell\otimes\rho}\sigma.
\end{aligned}
\tag{4.3}
$$

Noticing that $L_f=L_{\phi(f)}$ since they share the same transition function $\{g_{\lambda,\mu}\}$, we also have

(4.4)
$$T_{\phi(f)}(r,s_0)=\int_{s_0}^{r}\frac{dt}{t^{2m-1}}\int_{M[t]}c_1(L_f,\rho)\wedge\upsilon^{m-1}+\int_{M\langle r\rangle}\log\|F_{\phi(f)}\|_{\ell\otimes\rho}\sigma - \int_{M<s_0>}\log\|F_{\phi(f)}\|_{\ell\otimes\rho}\sigma.$$

We also note that, on U_λ,

$$\|F_f\|_{\ell\otimes\rho}=\|\mathbf{f}_\lambda\|\cdot\|s_\lambda\|_\rho,$$

and

$$\|F_{\phi(f)}\|_{\ell\otimes\rho}=\|\phi(\mathbf{f}_\lambda)\|\cdot\|s_\lambda\|_\rho,$$

where $\|\mathbf{f}_\lambda\|=\max_{0\le j\le n}|f_{\lambda,j}|$ and $\|\phi(\mathbf{f}_\lambda)\|=\max_{t+1\le j\le n}|f_{\lambda,j}|$. Hence, by (4.3) and (4.4), we have

(4.5)
$$\begin{aligned}T_f(r,s_0)-T_{\phi(f)}(r,s_0)&=\int_{M\langle r\rangle}\log\|F_f\|_{\ell\otimes\rho}\sigma-\int_{M\langle r\rangle}\log\|F_{\phi(f)}\|_{\ell\otimes\rho}\sigma+O(1)\\&=\int_{M\langle r\rangle}\log\max_{0\le j\le n}|f_{\lambda,j}|\sigma-\int_{M\langle r\rangle}\log\max_{t+1\le j\le n}|f_{\lambda,j}|\sigma\\&=\int_{M\langle r\rangle}\log\frac{\max_{0\le j\le n}|f_{\lambda,j}|}{\max_{t+1\le j\le n}|f_{\lambda,j}|}\sigma.\end{aligned}$$

Note that the term $\frac{\max_{0\le j\le n}|f_{\lambda,j}|}{\max_{t+1\le j\le n}|f_{\lambda,j}|}$ appearing in the last expression above does not depend on λ, hence it is, in fact, a global function on M. On the other hand, by the definition, if we regard $E=\{[x_0:\cdots:x_n]\mid x_{t+1}=\cdots=x_n=0\}$, then the proximity function $m_f(E,r)$ can be written as

$$m_f(E,r)=\int_{M\langle r\rangle}\log\frac{\max_{0\le j\le n}|f_{\lambda,j}|}{\max_{t+1\le j\le n}|f_{\lambda,j}|}\sigma+O(1). \tag{4.6}$$

Comparing (4.5) and (4.6), we obtain that

$$T_{\phi(f)}(r,s_0)=T_f(r,s_0)-m_f(E,r). \tag{4.7}$$

By the same method in obtaining (4.7), for any linear subspace $D\in\mathcal{D}$ containing E, we have

$$m_{\phi(f)}(\phi(D),r)=m_f(D,r)-m_f(E,r)+O(1). \tag{4.8}$$

Combining (4.2) and (4.7), we have

(4.9)
$$\begin{aligned}&\sum_{D'\in\mathcal{D}'}(\operatorname{codim} D')m^{\mathcal{D}'}_{\phi\circ f}(D',r)\\ &. \le .(\operatorname{codim} E)\cdot T_f(r,s_0)-(\operatorname{codim} E)\cdot m_f(E,r)+\frac{n\operatorname{codim} E}{2}Ric_\tau(r,s_0)\\ &+\kappa\frac{n\operatorname{codim} E}{2}\left[\log^+ T_f(r,s_0)+(2+\epsilon)\log^+\log^+ T_f(r,s_0)\right.\\ &\left.+2\log^+ Ric_\tau(r,s_0)+O(\log^+ r)\right].\end{aligned}$$

Hence, Proposition 4.1 will be proved if the following inequality holds:

$$(4.10)\qquad \begin{aligned}&\sum_{D\in\mathcal{D}}\operatorname{codim} < D,E > m^{\mathcal{D}}_f(D,r)-(\operatorname{codim} E)m_f(E,r)\\ &\le \sum_{D'\in\mathcal{D}'}(\operatorname{codim} D')m^{\mathcal{D}'}_{\phi(f)}(D',r)+O(1).\end{aligned}$$

Thus the remaining part of the proof is to show (4.10). We first claim the following:

CLAIM 4.2. *Let $\mathcal{C}$ be a finite collection of linear spaces of $\mathbb{P}^n(\mathbb{C})$. Fix $C_0\in\mathcal{C}$ and let $\tilde{\mathcal{C}}$ be the collection of subspaces of $\mathbb{P}^n$ obtained by adding to $\mathcal{C}$ the subspace $< C_0,E >$ as well as all $< C_0,E > \cap C$, $C\in\mathcal{C}$. Thus $\tilde{\mathcal{C}}$ is closed under taking intersection. Then*

$$(4.11)\qquad \sum_{C\in\mathcal{C}}(\operatorname{codim} < C,E >)\mu_{\mathcal{C}}(C)\le\sum_{\tilde{C}\in\tilde{\mathcal{C}}}(\operatorname{codim} < \tilde{C},E >)\mu_{\tilde{\mathcal{C}}}(\tilde{C}).$$

Claim 4.2 was proved in [Vo2](see Claim 4.6 in [Vo2]). We will include a proof of Claim 4.2 later for the sake of completeness. Before proving Claim 4.2, we first show that Claim 4.2 implies (4.10). Since the inequalities of cycles implies the corresponding inequalities of proximity functions, the claim implies that the left-hand of (4.10) is not decreasing when we enlarge $\mathcal{D}$ so as to include all $< D,E >, D\in\mathcal{D}$. So we assume that $\mathcal{D}$ contains all $< D,E >, D\in\mathcal{D}$. To continue, we recall the definition of the map ϕ. Under the assumption of $E=\mathbb{P}^t(\mathbb{C})$ with $t>0$, i.e. E is given by the points $[x_0:\cdots:x_n]$ with $x_{t+1}=\cdots=x_n=0$, ϕ is the projection $\phi:\mathbb{P}^n(\mathbb{C})\to\mathbb{P}^{n-t-1}(\mathbb{C})$ defined by $\phi[x_0:\cdots:x_n]=[x_{t+1}:\cdots:x_n]$. Hence, for any subspace D with $D\not\subseteq E$, $\operatorname{codim} < D,E > = \operatorname{codim}\phi(D)$. Therefore the first term of the left hand side of (4.10) can be expressed as

$$\begin{aligned}&\sum_{D\in\mathcal{D}}\operatorname{codim} < D,E > m^{\mathcal{D}}_f(D,r)\\ &=\sum_{\substack{D\in\mathcal{D}\\ D\not\subseteq E}}\operatorname{codim} < D,E > m_f(\mu_{\mathcal{D}}(D),r)+\sum_{\substack{D\in\mathcal{D}\\ D\subseteq E}}(\operatorname{codim} E)m_f(\mu_{\mathcal{D}}(D),r)\\ &=\sum_{D'\in\mathcal{D}'}\sum_{\substack{D\in\mathcal{D}\\ \phi(D)=D'}}(\operatorname{codim} D')m_f(\mu_{\mathcal{D}}(D),r)+(\operatorname{codim} E)m_f(E,r).\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{D\in\mathcal{D}} \operatorname{codim} < D, E > m_f^{\mathcal{D}}(D,r) - (\operatorname{codim} E) m_f(E,r)\\
&= \sum_{D'\in\mathcal{D}'} \sum_{\substack{D\in\mathcal{D}\\ \phi(D)=D'}} (\operatorname{codim} D') m_f(\mu_{\mathcal{D}}(D),r)\\
&= \sum_{D'\in\mathcal{D}'} (\operatorname{codim} D') \sum_{\substack{D\in\mathcal{D}\\ \phi(D)=D'}} m_f(\mu_{\mathcal{D}}(D),r).
\end{aligned}$$

Hence (4.10) is true if we can show that for any $D' \in \mathcal{D}'$

$$m_{\phi(f)}^{\mathcal{D}'}(D',r) = \sum_{\substack{D\in\mathcal{D}\\ \phi(D)=D'}} m_f(\mu_{\mathcal{D}}(D),r). \tag{4.12}$$

To show (4.12), we first consider for $D_0 \in \mathcal{D}$ and $D_0 \supset E$. From the definition of the Mobius-type condition, we have

$$\begin{aligned}
m_f(D_0,r) &= \sum_{\substack{D\in\mathcal{D}\\ D\subseteq D_0}} m_f(\mu_{\mathcal{D}}(D),r)\\
&= \sum_{\substack{D\in\mathcal{D}\\ E\not\supseteq D\subseteq D_0}} m_f(\mu_{\mathcal{D}}(D),r) + \sum_{\substack{D\in\mathcal{D}\\ D\subseteq E}} m_f(\mu_{\mathcal{D}}(D),r)\\
&= \sum_{\substack{D'\in\mathcal{D}'\\ D'\subseteq\phi(D_0)}} \sum_{\substack{D\in\mathcal{D}\\ \phi(D)=D'}} m_f(\mu_{\mathcal{D}}(D),r) + m_f(E,r).
\end{aligned}$$

Combining this with (4.8), we have

$$m_{\phi(f)}(\phi(D_0),r) = \sum_{\substack{D'\in\mathcal{D}'\\ D'\subseteq\phi(D_0)}} \sum_{\substack{D\in\mathcal{D}\\ \phi(D)=D'}} m_f(\mu_{\mathcal{D}}(D),r). \tag{4.13}$$

We note that for each $D' \neq \emptyset$ in $\mathcal{D}$, there exists $D_0 \in \mathcal{D}$ such that $D_0 \supset E$ and $\phi(D_0) = D'$ since we have assumed that $\mathcal{D}$ contains all cycles of the form $< D, E >$, $D \in \mathcal{D}$. We will now prove (4.12) by induction on the dimension of D'. We first consider when $\dim D' = 0$. In this case,

$$m_{\phi(f)}^{\mathcal{D}'}(D',r) = m_{\phi(f)}(\mu_{\mathcal{D}'}(D'),r) = m_{\phi(f)}(D',r) = m_{\phi(f)}(\phi(D_0),r),$$

and by (4.13)

$$m_{\phi(f)}(\phi(D_0),r) = \sum_{\substack{D'\in\mathcal{D}'\\ D'\subseteq\phi(D_0)}} \sum_{\substack{D\in\mathcal{D}\\ \phi(D)=D'}} m_f(\mu_{\mathcal{D}}(D),r) = \sum_{\substack{D\in\mathcal{D}\\ \phi(D)=D'}} m_f(\mu_{\mathcal{D}}(D),r)$$

since the dimension of D' is zero. Hence (4.12) holds when $\dim D' = 0$. Assume (4.12) holds for $D' \in \mathcal{D}'$ with $\dim D' < d$. Let $\dim D' = d$. Then from (4.13), we have

(4.14)
$$\begin{aligned} m_{\phi(f)}(D', r) &= m_{\phi(f)}(\phi(D_0), r) \\ &= \sum_{\substack{D \in \mathcal{D} \\ \phi(D) = D'}} m_f(\mu_{\mathcal{D}}(D), r) + \sum_{\substack{C' \in \mathcal{D}' \\ C' \subsetneq D'}} \sum_{\substack{D \in \mathcal{D} \\ \phi(D) = C'}} m_f(\mu_{\mathcal{D}}(D), r) \\ &= \sum_{\substack{D \in \mathcal{D} \\ \phi(D) = D'}} m_f(\mu_{\mathcal{D}}(D), r) + \sum_{\substack{C' \in \mathcal{D}' \\ C' \subsetneq D'}} m_{\phi(f)}(\mu_{\mathcal{D}'}(C'), r), \end{aligned}$$

where the last equality follows from the induction hypothesis. On the other hand, from the definition of Mobius function, we have

$$m_{\phi(f)}(D', r) = m_{\phi(f)}(\mu_{\mathcal{D}'}(D'), r) + \sum_{\substack{C' \in \mathcal{D}' \\ C' \subsetneq D'}} m_{\phi(f)}(\mu_{\mathcal{D}'}(C'), r). \tag{4.15}$$

We also have, by definition

$$m^{\mathcal{D}'}_{\phi(f)}(D', r) = m_{\phi(f)}(\mu_{\mathcal{D}'}(D'), r). \tag{4.16}$$

Combining (4.14), (4.15) and (4.16) proves (4.12).

We now prove Claim 4.2. It will be convenient to assume that $\mathbb{P}^n \in \mathcal{C}$. This does not affect $\mu_{\mathcal{C}}$, nor does it affect the inequality being proved. Now, for each $\tilde{C} \in \tilde{\mathcal{C}}$ there is a minimal $C \in \mathcal{C}$ containing $\tilde{C}$; let it be denoted by $\beta(\tilde{C})$. We claim

$$\mu_{\mathcal{C}}(C) = \sum_{\tilde{C} \in \tilde{\mathcal{C}}, \beta(\tilde{C}) = C} \mu_{\tilde{\mathcal{C}}}(\tilde{C}). \tag{4.17}$$

This can be done by induction on the dimension of C. Suppose that C is a point. Then $\mu_{\mathcal{C}}(C) = C$, and the only point in $\tilde{C} \in \tilde{\mathcal{C}}$ such that $\beta(\tilde{C}) = C$ is when $\tilde{C} = C$. Therefore the assertion is clear. Assume the assertion holds for cycles in $\mathcal{C}$ with dimension less than d. Let now C be a cycle of dimension d. Then we have

$$\begin{aligned} C = \sum_{\substack{\tilde{C} \in \tilde{\mathcal{C}} \\ \tilde{C} \subseteq C}} \mu_{\tilde{\mathcal{C}}}(\tilde{C}) &= \sum_{\substack{\tilde{C} \in \tilde{\mathcal{C}} \\ \beta(\tilde{C}) = C}} \mu_{\tilde{\mathcal{C}}}(\tilde{C}) + \sum_{\substack{\tilde{C} \in \tilde{\mathcal{C}} \\ \beta(\tilde{C}) \subsetneq C}} \mu_{\tilde{\mathcal{C}}}(\tilde{C}) \\ &= \sum_{\substack{\tilde{C} \in \tilde{\mathcal{C}} \\ \beta(\tilde{C}) = C}} \mu_{\tilde{\mathcal{C}}}(\tilde{C}) + \sum_{\substack{D \in \mathcal{C} \\ D \subsetneq C}} \sum_{\substack{\tilde{C} \in \tilde{\mathcal{C}} \\ \beta(\tilde{C}) = D}} \mu_{\tilde{\mathcal{C}}}(\tilde{C}) \\ &= \sum_{\substack{\tilde{C} \in \tilde{\mathcal{C}} \\ \beta(\tilde{C}) = C}} \mu_{\tilde{\mathcal{C}}}(\tilde{C}) + \sum_{\substack{D \in \mathcal{C} \\ D \subsetneq C}} \mu_{\mathcal{C}}(D). \end{aligned}$$

Since

$$\mu_{\mathcal{C}}(C) + \sum_{\substack{D \in \mathcal{C} \\ D \subsetneq C}} \mu_{\mathcal{C}}(D) = C,$$

(4.17) follows easily. From (4.17), we see that

$$\sum_{C\in\mathcal{C}}(\text{codim} < C, E >)\mu_{\mathcal{C}}(C) = \sum_{\tilde{C}\in\tilde{\mathcal{C}}}(\text{codim} < \beta(\tilde{C}), E >)\mu_{\tilde{\mathcal{C}}}(\tilde{C})$$
$$\leq \sum_{\tilde{C}\in\tilde{\mathcal{C}}}(\text{codim} < \tilde{C}, E >)\mu_{\tilde{\mathcal{C}}}(\tilde{C}),$$

where the last step is true because codim $< \beta(\tilde{C}), E > \leq$ codim $< \tilde{C}, E >$. This proves Claim 4.2. Hence Lemma 4.1 is proved. ∎

5. An algebraic lemma. In this section, we reformulate the following theorem due to Vojta which plays an essential role in our proof. We call it *an algebraic lemma*, since it involves purely (linear) algebra.

THEOREM 5.1(AN ALGEBRAIC LEMMA). *Let $\mathcal{H} = \{H_1, ..., H_q\}$ be a set of hyperplanes in $\mathbb{P}^N(\mathbb{C})$ and let $\mathcal{D} = \{D_1, ..., D_M\}$ be the collection of cycles which can be written as H_i or a finite intersection of hyperplanes in $\mathcal{H}$. Then there exists a finite union of $\mathcal{R}$ of proper linear subspaces of $\mathbb{P}^N(\mathbb{C})$ depending only on $D_1, \ldots, D_M$, such that the following holds: Let $\mathbb{P}^n(\mathbb{C})$ be a linear subspace of $\mathbb{P}^N(\mathbb{C})$, and $\mathbb{P}^n(\mathbb{C}) \not\subset \mathcal{R}$. Then there exists a finite set $\mathcal{E}$ of linear subspaces of $\mathbb{P}^n(\mathbb{C})$ and constants $c_E \geq 0$ such that*

$$\sum_{E\in\mathcal{E}} c_E \, \text{codim}_{\mathbb{P}^n} < D_i \cap \mathbb{P}^n, E > \geq \text{codim}_{\mathbb{P}^N} D_i \tag{5.1}$$

for all i and

$$\sum_{E\in\mathcal{E}} c_E \, \text{codim}_{\mathbb{P}^n} E = N + 1. \tag{5.2}$$

To prove Theorem 5.1, we first recall the following theorem from [Vo3].

THEOREM 5.2 [VO3: THEOREM 4.6]. *Let $\mathcal{H} = \{H_1, ..., H_q\}$ be a set of hyperplanes in $\mathbb{P}^N(\mathbb{C})$ and $\mathcal{D} = \{D_1, ..., D_M\}$ be the collection of cycles which can be written as H_i or a finite intersection of hyperplanes in $\mathcal{H}$. Then there exists a finite union $\mathcal{R}$ of proper linear subspaces of $\mathbb{P}^N(\mathbb{C})$, depending only on $D_1, ..., D_M$, such that the following is true: Let $\mathbb{P}^n(\mathbb{C})$ be a linear subspace of $\mathbb{P}^N(\mathbb{C})$, with $\mathbb{P}^n \not\subseteq \mathcal{R}$ and let $(\mu_1, ..., \mu_M)$ be a M-tuple of real numbers satisfying the conditions*

(i) *$\mu_i \geq 0$ for all i, and*

(ii) *$\sum_{i=1}^M \mu_i \, \text{codim}_{\mathbb{P}^n} < D_i \cap \mathbb{P}^n, E > \leq \text{codim}_{\mathbb{P}^n} E$ for all linear subspaces $E \subset \mathbb{P}^n(\mathbb{C})$.*

Then $(\mu_1, ..., \mu_M)$ must also satisfy

$$\sum_{i=1}^M \mu_i \, \text{codim}_{\mathbb{P}^N} D_i \leq N + 1. \tag{5.3}$$

We also recall the following result from linear algebra.

LEMMA 5.3. *Let L and $L_1, ..., L_m$ be linear forms in $M+1$ variables with real coefficients. Suppose $L(\mu) \geq 0$ for all $\mu = (\mu_0, ..., \mu_M)$ satisfying the conditions*

$L_i(\mu) \geq 0$ for all i. Then there exist non-negative real numbers $c_1, ..., c_M$ such that $L = c_1L_1 + \cdots + c_ML_M$.

We now prove Theorem 5.1

Proof of Theorem 5.1. Let $\mathcal{R}$ be as in Theorem 5.2, and let $\mathbb{P}^n(\mathbb{C}) \not\subset \mathcal{R}$. Define the linear form

$$L(\mu_0, \mu_1, ..., \mu_M) := (N+1)\mu_0 - \sum_{i=1}^{M} \mu_i \operatorname{codim}_{\mathbb{P}^N} D_i, \tag{5.4}$$

and the linear forms

$$L_E(\mu_0, \mu_1, ..., \mu_M) := \mu_0 \operatorname{codim}_{\mathbb{P}^n} E - \sum_{i=1}^{M} \mu_i \operatorname{codim}_{\mathbb{P}^n} < D_i \cap \mathbb{P}^n, E > \tag{5.5}$$

for every linear subspace $E \subset \mathbb{P}^n(\mathbb{C})$. Note that, since the coefficients of such linear forms over $\mu_0, ..., \mu_M$ are integers between 0 and n, there are actually only finitely many linear equations in (5.5). In addition, we define linear forms $L_i(\mu_0, \mu_1, ..., \mu_M) := \mu_i$, for $i = 0, 1, ..., M$. By Theorem 5.2, for any M-tuple $(\mu_1, ..., \mu_M)$ satisfying condition (i) and (ii) must satisfies (5.3). This implies that $L(\mu_0, \mu_1, ..., \mu_M) \geq 0$ for all $(\mu_0, ..., \mu_M)$ satisfying the conditions $L_i(\mu_0, \mu_1, ..., \mu_M) \geq 0$ for $i = 0, 1, \ldots, M$ and $L_E(\mu_0, \mu_1, ..., \mu_M) \geq 0$ for every linear subspace $E \subset \mathbb{P}^n$. Hence Lemma 5.3 implies that there exist constants $c_i \geq 0$ for $i = 1, ..., M$, and $c_E \geq 0$ for $E \in \mathcal{E}$ such that

$$L = \sum_{i=1}^{M} c_i L_i + \sum_{E \in \mathcal{E}} c_E L_E.$$

Compare the coefficients of each μ_i for $i = 0, ..., M$, we have

$$\sum_{E \in \mathcal{E}} c_E \operatorname{codim}_{\mathbb{P}^n} < D_i \cap \mathbb{P}^n, E > \geq \operatorname{codim}_{\mathbb{P}^N} D_i$$

for all i and

$$\sum_{E \in \mathcal{E}} c_E \operatorname{codim}_{\mathbb{P}^n} E = N + 1.$$

Thus Theorem 5.1 is proved. □

6. Proof of the main theorem. Let $\mathcal{H} = \{H_1, ..., H_q\}$ be the given hyperplanes in $\mathbb{P}^N$, and let $\mathcal{D} = \{D_1, ..., D_M\}$ be the collection of cycles which can be written as H_i or a finite intersection of hyperplanes in $\mathcal{H}$. Let $\mathcal{R}$ be as in the algebraic lemma. Let $f : M \to \mathbb{P}^N$ be a holomorphic curve. If f is linearly non-degenerate, then we are done by Theorem 2.1. So we only need to consider the case when f is linearly degenerate. We then assume that $f : M \to \mathbb{P}^n$ and f is linearly non-degenerate. By the assumption of the main theorem, we have $\mathbb{P}^n(\mathbb{C}) \not\subset \mathcal{R}$. Hence, by the algebraic lemma, there exist a finite set $\mathcal{E}$ of linear subspaces of $\mathbb{P}^n(\mathbb{C})$ and constants $c_E \geq 0$ such that

$$\sum_{E \in \mathcal{E}} c_E \operatorname{codim}_{\mathbb{P}^n} < D_i \cap \mathbb{P}^n, E > \geq \operatorname{codim}_{\mathbb{P}^N} D_i \tag{6.1}$$

for all i and

$$\sum_{E\in\mathcal{E}} c_E \operatorname{codim}_{\mathbb{P}^n} E = N+1. \tag{6.2}$$

By (6.1), we have

(6.3)
$$\begin{aligned}\sum_{D\in\mathcal{D}}(\operatorname{codim}_{\mathbb{P}^N} D)m_f^{\mathcal{D}}(D,r) &\le \sum_{D\in\mathcal{D}}\sum_{E\in\mathcal{E}} c_E \operatorname{codim}_{\mathbb{P}^n} < D\cap\mathbb{P}^n, E > m_f^{\mathcal{D}}(D,r)\\ &= \sum_{E\in\mathcal{E}} c_E \sum_{D\in\mathcal{D}} \operatorname{codim}_{\mathbb{P}^n} < D\cap\mathbb{P}^n, E > m_f^{\mathcal{D}}(D,r).\end{aligned}$$

On the other hand, by Proposition 4.1, we have, for every subspace E of $\mathbb{P}^n(\mathbb{C})$,

(6.4)
$$\begin{aligned}&\sum_{D\in\mathcal{D}} \operatorname{codim}_{\mathbb{P}^n} < D\cap\mathbb{P}^n, E > m_f^{\mathcal{D}}(D_i,r)\\ &. \le .(\operatorname{codim} E)\cdot T_f(r,s_0) + \frac{n \operatorname{codim} E}{2} Ric_\tau(r,s_0)\\ &\quad + \kappa\frac{n \operatorname{codim} E}{2}\left[\log^+ T_f(r,s_0) + (2+\epsilon)\log^+\log^+ T_f(r,s_0)\right.\\ &\quad \left. +2\log^+ Ric_\tau(r,s_0) + O(\log^+ r)\right].\end{aligned}$$

Combining (6.2), (6.3) and (6.4), we have

(6.5)
$$\begin{aligned}\sum_{D\in\mathcal{D}}(\operatorname{codim}_{\mathbb{P}^N} D)m_f^{\mathcal{D}}(D,r). \le .(N+1)T_f(r,s_0) &+ \frac{N(N+1)}{2} Ric_\tau(r,s_0)\\ + \kappa\frac{N(N+1)}{2}&\left[\log^+ T_f(r,s_0) + (2+\epsilon)\log^+\log^+ T_f(r,s_0)\right.\\ &\left. +2\log^+ Ric_\tau(r,s_0) + O(\log^+ r)\right].\end{aligned}$$

On the other hand, by Lemma 3.2.2

$$m_f(C_{\mathcal{H}},r) = \sum_{D\in\mathcal{D}}(\operatorname{codim}_{\mathbb{P}^N} D)m_f^{\mathcal{D}}(D,r), \tag{6.6}$$

and by Lemma 3.1.6

$$m_f(C_{\mathcal{H}},r) = \int_{M\langle r\rangle} \max_K \sum_{j\in K} \log\frac{1}{\|f;H_j\|}\sigma + O(1), \tag{6.7}$$

where the maximum is taken over all subset K of $\{1,\dots,q\}$ such that the linear forms $H_j, j\in K$, are linearly independent. By the assumption that $H_1,\dots,H_q$ are in general position, we have

$$\sum_{j=1}^q m_f(H_j,r) \le \int_{M\langle r\rangle} \max_K \sum_{j\in K} \log\frac{1}{\|f;H_j\|}\sigma + O(1), \tag{6.8}$$

where the maximum is taken over all subset K of $\{1,\dots,q\}$ such that the linear forms $H_j, j\in K$, are linearly independent. Therefore, the theorem follows by combining (6.5), (6.6), (6.7), (6.8) and (1.1.2). □

The proof of the main theorem also gives the following more general theorem.

SECOND MAIN THEOREM FOR PARABOLIC MANIFOLDS. *Let M be a Stein parabolic manifold of complex dimension m. Let $\mathcal{H} = \{H_1, ..., H_q\}$ be a finite collection of hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. Then there exists a finite union $\mathcal{R}$ of proper linear subspaces of $\mathbb{P}^N(\mathbb{C})$ depending only on $\mathcal{H}$ such that if $f : M \to \mathbb{P}^N(\mathbb{C})$ is a meromorphic map whose image does not lie in $\mathcal{R}$, then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} m_f(H_j, r) .\leq .(N+1)T_f(r, s_0) + \frac{N(N+1)}{2} Ric_\tau(r, s_0)$$
$$+ \kappa \frac{N(N+1)}{2}[\log^+ T_f(r, s_0) + (2+\epsilon)\log^+\log^+ T_f(r, s_0)$$
$$+ \log^+ Y(r) + 2\log^+ Ric_\tau(r, s_0) + 5\log^+\log^+ r)],$$

where $\kappa = \int_{M\langle r\rangle} d^c \log\tau \wedge (dd^c \log\tau)^{m-1} > 0$ is a constant independent of r, $Ric_\tau(r, s_0)$ is the Ricci function of M (cf. [St]), and $.\leq.$ means that the inequality holds for all $r \in [s_0, +\infty)$ outside a union of intervals of finite total length.

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