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ROUGH SINGULAR INTEGRALS WITH KERNELS SUPPORTED BY SUBMANIFOLDS OF FINITE TYPE*

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1. Introduction. Let $n \geq 2$, $\mathbf{R}^n$ be the n-dimensional Euclidean space, and $\mathbf{S}^{n-1}$ denote the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. For $d \in \mathbf{N}$, let $B(0,1)$ be the unit ball centered at the origin in $\mathbf{R}^n$ and $\Phi : B(0,1) \to \mathbf{R}^d$ be a C^∞ mapping. Define the singular integral operator T_Φ and the related maximal operator $\mathcal{M}_\Phi$ by

$$T_\Phi f(x) = \text{p.v.} \int_{B(0,1)} f(x - \Phi(y)) \frac{\Omega(y)}{|y|^n} dy, \tag{1.1}$$

$$\mathcal{M}_\Phi f(x) = \sup_{0<r\leq 1} \frac{1}{r^n} \int_{|y|\leq r} |f(x - \Phi(y))| \, |\Omega(y)| \, dy \tag{1.2}$$

for $f \in \mathcal{S}(\mathbf{R}^d)$. Here Ω is a homogeneous function of degree 0, integrable over $\mathbf{S}^{n-1}$ and satisfies the vanishing condition

$$\int_{\mathbf{S}^{n-1}} \Omega(u) \, d\sigma(u) = 0. \tag{1.3}$$

The corresponding maximal truncated singular integral operator T^*_Φ is defined by

$$T^*_\Phi f(x) = \sup_{\varepsilon>0} \left| \int_{\varepsilon\leq|y|<1} f(x - \Phi(y)) \frac{\Omega(y)}{|y|^n} dy \right|. \tag{1.4}$$

When $\Phi(y) \equiv y$, T_Φ is simply the localized version of a classical Calderón-Zygmund operator and we shall denote it by T. Our point of departure is the following L^p boundedness result from [St].

THEOREM 1.1. *Let T_Φ and $\mathcal{M}_\Phi$ be given as in (1.1)-(1.3). Assume that:*
(i) Φ is of finite type at 0;
(ii) $\Omega \in C^1(\mathbf{S}^{n-1})$.
Then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that

$$\|T_\Phi f\|_{L^p(\mathbf{R}^d)} \leq C_p \|f\|_{L^p(\mathbf{R}^d)} \tag{1.5}$$

and

$$\|\mathcal{M}_\Phi f\|_{L^p(\mathbf{R}^d)} \leq C_p \|f\|_{L^p(\mathbf{R}^d)} \tag{1.6}$$

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for every $f \in L^p(\mathbf{R}^d)$.

Recently, the results in Theorem 1.1 were improved by Fan, Guo, and Pan in [FGP] who showed that the L^p boundedness of T_Φ and $\mathcal{M}_\Phi$ continues to hold if the condition $\Omega \in C^1(\mathbf{S}^{n-1})$ is replaced by the weaker condition $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Also, the authors of [FGP] were able to establish the L^p boundedness of the maximal operator T_Φ^* under the condition $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$.

The main purpose of this paper is to present further improvements of the above results in which the condition $\Omega \in L^q(\mathbf{S}^{n-1})$ is replaced by a weaker condition $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$. It is worth pointing out that the authors of this paper were able in [AqAsP] to show that the condition $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ is the best possible for the L^p boundedness of the classical operator T to hold. Namely, the L^p boundedness of T may fail for any p if it is replaced by a weaker condition $\Omega \in B_q^{0,\upsilon}(\mathbf{S}^{n-1})$ for any $-1 < \upsilon < 0$ and $q > 1$.The definition of the block spaces $B_q^{0,\upsilon}(\mathbf{S}^{n-1})$ on the sphere will be recalled in Section 2.

Our main results can be stated as follows.

THEOREM 1.2. *Let* T_Φ *and* $\mathcal{M}_\Phi$ *be given as in (1.1)-(1.3). Assume that:*
(i) Φ *is of finite type at* 0;
(ii) $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ *for some* $q > 1$.
Then

$$\|T_\Phi f\|_{L^p(\mathbf{R}^d)} \le C_p \|f\|_{L^p(\mathbf{R}^d)} \tag{1.7}$$

and

$$\|\mathcal{M}_\Phi f\|_{L^p(\mathbf{R}^d)} \le C_p \|f\|_{L^p(\mathbf{R}^d)} \tag{1.8}$$

hold for all $1 < p < \infty$ *and* $f \in L^p\left(\mathbf{R}^d\right)$.

THEOREM 1.3. *Let* Ω *and* T_Φ^* *be given as in (1.3)-(1.4). Assume that:*
(i) Φ *is of finite type at* 0;
(ii) $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ *for some* $q > 1$.
Then for $1 < p < \infty$ *there exists a constant* $C_p > 0$ *such that*

$$\|T_\Phi^* f\|_{L^p(\mathbf{R}^d)} \le C_p \|f\|_{L^p(\mathbf{R}^d)} \tag{1.9}$$

for every $f \in L^p(\mathbf{R}^d)$.

2. Preliminaries. Let us begin with the definition of block functions on $\mathbf{S}^{n-1}$.

DEFINITION 2.1. *(1) For* $x_0' \in \mathbf{S}^{n-1}$ *and* $0 < \theta_0 \le 2$, *the set*

$$B(x_0', \theta_0) = \{x' \in \mathbf{S}^{n-1} : |x' - x_0'| < \theta_0\}$$

is called a cap on $\mathbf{S}^{n-1}$.

(2) For $1 < q \le \infty$, *a measurable function b is called a* $q-$*block on* $\mathbf{S}^{n-1}$ *if b is a function supported on some cap* $I = B(x_0', \theta_0)$ *with* $\|b\|_{L^q} \le |I|^{-1/q'}$ *where* $|I| = \sigma(I)$ *and* $1/q + 1/q' = 1$.

(3) $B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) = \{\Omega \in L^1(\mathbf{S}^{n-1})$: $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ *where each* c_μ *is a complex number; each* b_μ *is a* $q-$*block supported on a cap* I_μ *on* $\mathbf{S}^{n-1}$; *and* $M_q^{\kappa,\upsilon}(\{c_k\},\{I_k\}) = \sum_{\mu=1}^{\infty} \left|c_\mu\right| (1 + \phi_{\kappa,\upsilon}(\left|I_\mu\right|)) < \infty\}$, *where*

$$\phi_{\kappa,\upsilon}(t) = \chi_{(0,1)}(t) \int_t^1 u^{-1-\kappa} \log^{\upsilon}\left(u^{-1}\right) du. \tag{2.1}$$

One observes that

$$\phi_{\kappa,\upsilon}(t) \sim t^{-\kappa} \log^{\upsilon}(t^{-1}) \text{ as } t \to 0 \text{ for } \kappa > 0, \upsilon \in \mathbf{R},$$
$$\phi_{0,\upsilon}(t) \sim \log^{\upsilon+1}\left(t^{-1}\right) \text{ as } t \to 0 \text{ for } \upsilon > -1.$$

The following properties of $B_q^{\kappa,\upsilon}$ can be found in [KS]:

$$(i)\ B_q^{\kappa,\upsilon_2} \subset B_q^{\kappa,\upsilon_1} \text{ if } \upsilon_2 > \upsilon_1 > -1 \text{ and } \kappa \geq 0; \tag{2.2}$$
$$(ii)\ B_q^{\kappa_2,\upsilon_2} \subset B_q^{\kappa_1,\upsilon_1} \text{ if } \upsilon_1, \upsilon_2 > -1 \text{ and } 0 \leq \kappa_1 < \kappa_2; \tag{2.3}$$
$$(iii)\ B_{q_2}^{\kappa,\upsilon} \subset B_{q_1}^{\kappa,\upsilon} \text{ if } 1 < q_1 < q_2; \tag{2.4}$$
$$(iv)\ L^q(\mathbf{S}^{n-1}) \subset B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) \text{ for } \upsilon > -1 \text{ and } \kappa \geq 0. \tag{2.5}$$

In their investigations of block spaces, Keitoku and Sato showed in [KS] that these spaces enjoy the following properties:

LEMMA 2.2. *(i) If* $1 < p \leq q \leq \infty$, *then for* $\kappa > \frac{1}{p'}$ *we have*

$$B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) \subseteq L^p(\mathbf{S}^{n-1}) \text{ for any } \upsilon > -1;$$

(ii)

$$B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) = L^q(\mathbf{S}^{n-1}) \text{ if and only if } \kappa \geq \frac{1}{q'} \text{ and } \upsilon \geq 0;$$

(iii) for any $\upsilon > -1$, *we have*

$$\bigcup{}_{q>1} B_q^{0,\upsilon}(\mathbf{S}^{n-1}) \nsubseteq \bigcup{}_{q>1} L^q(\mathbf{S}^{n-1}).$$

For a q-block function b on $\mathbf{S}^{n-1}$ supported in an interval with $q > 1$ and $\|b\|_q \leq |I|^{-1/q'}$, $1/q + 1/q' = 1$, we define the function $\tilde{b}$ on $\mathbf{S}^{n-1}$ by

$$\tilde{b}(x) = b(x) - \int_{\mathbf{S}^{n-1}} b(u) d\sigma(u). \tag{2.6}$$

Then one can easily see that $\tilde{b}$ enjoys the following properties:

$$\int_{\mathbf{S}^{n-1}} \tilde{b}(u)\, d\sigma(u) = 0; \tag{2.7}$$
$$\left\|\tilde{b}\right\|_q \leq 2\,|I|^{-1/q'}; \tag{2.8}$$
$$\left\|\tilde{b}\right\|_1 \leq 2. \tag{2.9}$$

To simplify matters, we shall call the function $\tilde{b}$ the blocklike function corresponding to the block function b.

We shall need the following two lemmas from [FGP].

LEMMA 2.3. *Let* $\Phi : B(0,1) \to \mathbf{R}^d$ *be a smooth mapping and* Ω *be a homogeneous function of degree* 0. *Suppose that* Φ *is of finite type at* 0 *and* $\Omega \in L^q(\mathbf{S}^{n-1})$ *for some* $q > 1$. *Then there are* $N \in \mathbf{N}$, $\delta \in (0,1]$, $C > 0$ *and* $j_0 \in \mathbf{Z}_-$ *such that*

$$\left|\int_{2^{j-1} \leq |y| < 2^j} e^{-i\xi\cdot\Phi(y)} \frac{\Omega(y)}{|y|^n} dy\right| \leq C\, \|\Omega\|_q \left(2^{Nj}\,|\xi|\right)^{-\delta} \tag{2.10}$$

for all $j \leq j_0$ *and* $\xi \in \mathbf{R}^d$.

LEMMA 2.4. *Let* $m \in \mathbf{N}$ *and* $R(\cdot)$ *be a real-valued polynomial on* $\mathbf{R}^n$ *with* $\deg(R) \leq m-1$. *Suppose that*

$$P(y) = \sum_{|\alpha|=m} a_\alpha y^\alpha + R(y),$$

Ω *is a homogeneous function of degree zero, and* $\Omega \in L^q(\mathbf{S}^{n-1})$ *for some* $q > 1$. *Then there exists a constant* $C = C(m,n) > 0$ *such that*

$$\left| \int_{2^{j-1} \leq |y| < 2^j} e^{-iP(y)} \frac{\Omega(y)}{|y|^n} dy \right| \leq C \left\| \Omega \right\|_q (2^{mj} \sum_{|\alpha|=m} |a_\alpha|)^{-\frac{1}{2q'm}}$$

holds for all $j \in \mathbf{Z}$ *and* $a_\alpha \in \mathbf{R}$.

The proofs of our results will rely heavily on the following lemma from [AqP] which is an extension of earlier results of Duoandikoetxea-Rubio de Francia in [DR] and Fan-Pan in [FP].

LEMMA 2.5. *Let* $N \in \mathbf{N}$ *and* $\left\{\sigma_k^{(l)} : k \in \mathbf{Z},\, 0 \leq l \leq N\right\}$ *be a family of Borel measures on* $\mathbf{R}^n$ *with* $\sigma_k^{(0)} = 0$ *for every* $k \in \mathbf{Z}$. *Let* $\{a_l : 1 \leq l \leq N\} \subseteq \mathbf{R}^+/(0,2)$, $\{m_l : 1 \leq l \leq N\} \subseteq \mathbf{N}$, $\{\alpha_l : 1 \leq l \leq N\} \subseteq \mathbf{R}^+$, *and let* $L_l : \mathbf{R}^n \to \mathbf{R}^{m_l}$ *be linear transformations for* $1 \leq l \leq N$. *Suppose that for all* $k \in \mathbf{Z}$, $1 \leq l \leq N$, *for all* $\xi \in \mathbf{R}^n$ *and for some* $C > 0$, $A > 1, p_0 \in (2,\infty)$ *we have the following:*

(i) $\left\|\sigma_k^{(l)}\right\| \leq CA$;

(ii) $\left|\hat{\sigma}_k^{(l)}(\xi)\right| \leq CA \left|a_l^{kA} L_l(\xi)\right|^{-\frac{\alpha_l}{A}}$;

(iii) $\left|\hat{\sigma}_k^{(l)}(\xi) - \hat{\sigma}_k^{(l-1)}(\xi)\right| \leq CA \left|a_l^{kA} L_l(\xi)\right|^{\frac{\alpha_l}{A}}$;

(iv)

$$\left\| (\sum_{k\in\mathbf{Z}} \left|\sigma_k^{(l)} * g_k\right|^2)^{\frac{1}{2}} \right\|_{p_0} \leq CA \left\| (\sum_{k\in\mathbf{Z}} |g_k|^2)^{\frac{1}{2}} \right\|_{p_0} \tag{2.11}$$

holds *for all functions* $\{g_k\}$ *on* $\mathbf{R}^n$.

Then for $p_0' < p < p_0$ *there exists a positive constant* C_p *such that*

$$\left\| \sum_{k\in\mathbf{Z}} \sigma_k^{(N)} * f \right\|_{L^p(\mathbf{R}^n)} \leq C_p A \|f\|_{L^p(\mathbf{R}^n)} \tag{2.12}$$

$$\left\| \left(\sum_{k\in\mathbf{Z}} \left|\sigma_k^{(N)} * f\right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n)} \leq C_p A \|f\|_{L^p(\mathbf{R}^n)} \tag{2.13}$$

hold for all f *in* $L^p(\mathbf{R}^n)$. *The constant* C_p *is independent of the linear transformations* $\{L_l\}_{l=1}^N$.

We shall also need the following result from [DR] (see also [AqP]):

LEMMA 2.6. *Let* $\{\lambda_j : j \in \mathbf{Z}\}$ *be a sequence of Borel measures in* $\mathbf{R}^n$ *and let*

$\lambda^{*}(f)=\sup_{j\in\mathbf{Z}}||\lambda_j|*f|$. *Assume that*

$$\|\lambda^{*}(f)\|_q\le B\|f\|_q \text{ for some } q>1 \text{ and } B>1. \tag{2.14}$$

Then, for arbitrary functions $\{g_j\}$ *on* $\mathbf{R}^n$ *and* $\left|\frac{1}{p_0}-\frac{1}{2}\right|=\frac{1}{2q}$, *the following inequality holds*

$$\left\|\left(\sum_{k\in\mathbf{Z}}|\lambda_k*g_k|^2\right)^{\frac{1}{2}}\right\|_{p_0}\le\left(B\sup_{k\in\mathbf{Z}}\|\lambda_k\|\right)^{\frac{1}{2}}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{\frac{1}{2}}\right\|_{p_0}. \tag{2.15}$$

3. L^p boundedness of certain maximal functions. For given sequences $\{\mu_k\}_{k\in\mathbf{Z}}$ and $\{\tau_k\}_{k\in\mathbf{Z}}$ of nonnegative Borel measures on $\mathbf{R}^n$ we define the maximal functions μ^* and τ^* by

$$\mu^{*}(f)=\sup_{k\in\mathbf{Z}}|\mu_k*f| \text{ and } \tau^{*}(f)=\sup_{k\in\mathbf{Z}}|\tau_k*f|.$$

We have the following lemma.

LEMMA 3.1. *Let* $\{\mu_k\}_{k\in\mathbf{Z}}$ and $\{\tau_k\}_{k\in\mathbf{Z}}$ *be sequences of nonnegative Borel measures on* $\mathbf{R}^n$. *Let* L: $\mathbf{R}^n\to\mathbf{R}^m$ *be a linear transformation. Suppose that for all* $k\in\mathbf{Z}$, $\xi\in\mathbf{R}^n$, *for some* $a\ge 2$, $\alpha, C>0$ *and for some constant* $B>1$ *we have*

(i) $\|\mu_k\|\le B$; $\|\tau_k\|\le B$;

(ii) $|\hat{\mu}_k(\xi)|\le CB(a^{kB}|L(\xi)|)^{-\frac{\alpha}{B}}$;

(iii) $|\hat{\mu}_k(\xi)-\hat{\tau}_k(\xi)|\le CB(a^{kB}|L(\xi)|)^{\frac{\alpha}{B}}$;

(iv)

$$\|\tau^{*}(f)\|_p\le B\|f\|_p \text{ for all } 1<p\le\infty \text{ and } f\in L^p(\mathbf{R}^n). \tag{3.1}$$

Then the inequality

$$\|\mu^{*}(f)\|_p\le C_pB\|f\|_p \tag{3.2}$$

holds for all $1<p\le\infty$ *and* f *in* $L^p(\mathbf{R}^n)$ *with a constant* C_p *independent of* B *and* L.

Proof. By the arguments in the proof of Lemma 6.2 in [FP], we may assume that $m\le n$ and $L\xi=\pi^n_m\xi=(\xi_1,\dots,\xi_m)$ for $\xi=(\xi_1,\dots,\xi_n)$. Now, choose and fix a $\theta\in\mathcal{S}(\mathbf{R}^m)$ such that $\hat{\theta}(\xi)=1$ for $|\xi|\le 1$ and $\hat{\theta}(\xi)=0$ for $|\xi|\ge 2$. For each $k\in\mathbf{Z}$, let $(\theta_k)\hat{}(\xi)=\hat{\theta}(a^{kB}\xi)$, and define the sequence of measures $\{\Upsilon_k\}$ by

$$\hat{\Upsilon}_k(\xi)=\hat{\mu}_k(\xi)-(\theta_k)\hat{}(\pi^n_m\xi)\hat{\tau}_k(\xi). \tag{3.3}$$

By (i)-(iii) we get

$$\left|\hat{\Upsilon}_k(\xi)\right|\le CB(a^{kB}|\pi^n_m\xi|)^{\pm\frac{\alpha}{B}} \tag{3.4}$$

for $\xi\in\mathbf{R}^n$. Let

$$S_{\Upsilon}(f)(x)=\Big(\sum_{k\in\mathbf{Z}}|\Upsilon_k*f(x)|^2\Big)^{\frac{1}{2}} \text{ and } \Upsilon^{*}(f)=\sup_{k\in\mathbf{Z}}||\Upsilon_k|*f|.$$

Then by using (3.3) we have

$$\mu^{*}(f)(x) \leq S_{\Upsilon}(f)(x) + C(\mathcal{M}_{\mathbf{R}^{m}} \otimes id_{\mathbf{R}^{n-m}})(\tau^{*}(f)(x)) \tag{3.5}$$

$$\Upsilon^{*}(f)(x) \leq S_{\Upsilon}(f)(x) + 2C[(\mathcal{M}_{\mathbf{R}^{m}} \otimes id_{\mathbf{R}^{n-m}})](\tau^{*}(f)(x)) \tag{3.6}$$

where $\mathcal{M}_{\mathbf{R}^{d}}$ is the classical Hardy-Littlewood maximal function on $\mathbf{R}^{d}$.

By (3.4) and Plancherel's theorem we obtain

$$\|S_{\Upsilon}(f)\|_{2} \leq CB\|f\|_{2} \tag{3.7}$$

which when combined with the L^{p} boundedness of $\mathcal{M}_{\mathbf{R}^{d}}$, (3.1), and (3.6)-(3.7) gives that

$$\|\Upsilon^{*}(f)\|_{2} \leq CB\|f\|_{2} \tag{3.8}$$

with C independent of B. By using the fact $\|\Upsilon_{k}\| \leq CB$ together with Lemma 2.6 (for $q = 2$) we get

$$\left\|\left(\sum_{k\in\mathbf{Z}}(|\Upsilon_{k} * g_{k}|^{2})^{\frac{1}{2}}\right\|_{p_{0}} \leq C_{p_{0}}B\left\|\left(\sum_{k\in\mathbf{Z}}|g_{k}|^{2}\right)^{\frac{1}{2}}\right\|_{p_{0}} \tag{3.9}$$

if $1/4 = |1/p_{0} - 1/2|$. Now, by (3.4), (3.9) and applying Lemma 2.5 we get

$$\|S_{\Upsilon}(f)\|_{p} \leq C_{p}B\ \|f\|_{p} \text{ for } p \in (\frac{4}{3}, 4). \tag{3.10}$$

Again, the L^{p} boundedness of $\mathcal{M}_{\mathbf{R}^{d}}$, (3.1), (3.6) and (3.10) imply that

$$\|\Upsilon^{*}(f)\|_{p} \leq CB\|f\|_{p} \text{ for } p \in (\frac{4}{3}, 4). \tag{3.11}$$

Reasoning as above, (3.4), (3.11), Lemma 2.5 and Lemma 2.6 provide

$$\|S_{\Upsilon}(f)\|_{p} \leq C_{p}B\|f\|_{p} \text{ for } p \in (\frac{8}{7}, 8). \tag{3.12}$$

By successive application of the above argument we ultimately obtain that

$$\|S_{\Upsilon}(f)\|_{p} \leq C_{p}B\|f\|_{p} \text{ for } p \in (1, \infty). \tag{3.13}$$

Therefore, by the L^{p} boundedness of $\mathcal{M}_{\mathbf{R}^{d}}$, (3.1), (3.5) and (3.13) we conclude that

$$\|\mu^{*}(f)\|_{p} \leq C_{p}B\|f\|_{p} \text{ for } p \in (1, \infty). \tag{3.14}$$

Finally, the inequality (3.2) holds trivially for $p = \infty$. This concludes the proof of our lemma.

Definition 3.2. *Let $\tilde{b}(\cdot)$ be a blocklike function defined as in (2.2) and Γ be an arbitrary function on $\mathbf{R}^{n}$. Define the measures $\{\sigma_{\Gamma,\tilde{b},j} : j \in \mathbf{Z}\}$ and the maximal operator $\sigma^{*}_{\Gamma,\tilde{b}}$ on $\mathbf{R}^{n}$ by*

$$\int_{\mathbf{R}^{d}} f\ d\sigma_{\Gamma,\tilde{b},j} = \int_{2^{j-1}\leq|u|<2^{j}} f(\Gamma(u))\frac{\tilde{b}(u)}{|u|^{n}}du; \tag{3.15}$$

$$\sigma^{*}_{\Gamma,\tilde{b}}(f) = \sup_{j\in\mathbf{Z}}\left|\left|\sigma_{\Gamma,\tilde{b},j}\right| * f\right|. \tag{3.16}$$

These measures will be useful only in the case $|I| \geq e^{-2}$ where I is the support of b. On the other hand, for the case $|I| < e^{-2}$ we need to define the following measures.

Definition 3.3. *Let $\tilde{b}(\cdot)$ be a q-blocklike function defined as in (2.2) and Γ be an arbitrary function on $\mathbf{R}^n$. We define the measures $\{\lambda_{\Gamma,\tilde{b},j} : j \in \mathbf{Z}\}$ and the maximal operators $\lambda^*_{\Gamma,\tilde{b}}$ on $\mathbf{R}^n$ by*

$$\int_{\mathbf{R}^d} f \, d\lambda_{\Gamma,\tilde{b},j} = \int_{\omega^{j-1} \leq |u| < \omega^j} f(\Gamma(u)) \frac{\tilde{b}\,(u)}{|u|^n} \, du; \tag{3.17}$$

$$\lambda^*_{\Gamma,\tilde{b}} f\,(x) = \sup_{j \in \mathbf{Z}} \left| \left| \lambda_{\Gamma,\tilde{b},j} \right| * f(x) \right| \tag{3.18}$$

where $\omega = 2^{[\log(|I|^{-1})]}$, $|I| < e^{-2}$ and $[\cdot]$ denotes the greatest integer function.

Lemma 3.4. *Let $\Phi : B(0,1) \to \mathbf{R}^d$ be a smooth mapping and for $q > 1$ let $\tilde{b}$ be a q-blocklike function defined as in (2.2). Suppose that Φ is of finite type at 0. If $|I| < e^{-2}$, then there are $N \in \mathbf{N}$, $\delta \in (0,1]$, $C > 0$ and $j_0 \in \mathbf{Z}_-$ such that*

$$\left| \hat{\lambda}_{\Phi,\tilde{b},j}(\xi) \right| \leq C[\log(|I|)](\omega^{Nj} |\xi|)^{-\frac{\delta}{[\log(|I|^{-1})]}} \tag{3.19}$$

for all $j \leq j_0$, $\xi \in \mathbf{R}^d$ with C independent of j and $[\log(|I|^{-1})]$.

Proof. By (2.4), Lemma 2.3 and the definition of $\lambda_{\Phi,\tilde{b},j}$ we get

$$\begin{aligned}
\left| \hat{\lambda}_{\Phi,\tilde{b},j}(\xi) \right| &\leq \sum_{s=0}^{[\log(|I|^{-1})]-1} \left| \int_{\omega^{(j-1)} 2^s \leq |y| < \omega^{(j-1)} 2^{(s+1)}} e^{-i\xi \cdot \Phi(y)} \frac{\tilde{b}(y)}{|y|^n} dy \right| \\
&\leq \sum_{s=0}^{[\log(|I|^{-1})]-1} C\,|I|^{-\frac{1}{q'}} \,(\omega^{N(j-1)} 2^{N(s+1)}\,|\xi|)^{-\delta} \\
&\leq C\,|I|^{-\frac{1}{q'}}\, \omega^{\delta N} (\omega^{Nj}\,|\xi|)^{-\delta} (\frac{1-\omega^{-\delta N}}{1-2^{\delta N}}) \\
&\leq C\omega^{\delta N}\,|I|^{-\frac{1}{q'}}\,(\omega^j\,|\xi|)^{-\delta}.
\end{aligned}$$

By interpolating between this estimate and the trivial estimate

$$\left| \hat{\lambda}_{\Phi,\tilde{b},j}(\xi) \right| \leq C[\log(|I|^{-1})]$$

we get the estimate in (3.19). This concludes the proof of our lemma.

By Lemma 2.4 and the argument used in the proof of Lemma 3.4 we get the following:

Lemma 3.5. *Let $m \in \mathbf{N}$, $\tilde{b}$ be a q-blocklike function (for $q > 1$) defined as in (2.2) and $R(\cdot)$ be a real-valued polynomial on $\mathbf{R}^n$ with* $\deg(R) \leq m-1$. *Suppose*

$$P(y) = \sum_{|\alpha|=m} a_\alpha y^\alpha + R(y),$$

and $|I| < e^{-2}$. *Then there exists a constant* $C = C(m,n) > 0$ *such that*

$$\left|\int_{\omega^{j-1}\le|u|<\omega^{j}} e^{-iP(y)}\frac{\tilde{b}(y)}{|y|^{n}}dy\right| \le C[\log(|I|^{-1})](\omega^{mj}\sum_{|\alpha|=m}|a_{\alpha}|)^{-\frac{1}{2qm[\log(|I|^{-1})]}}$$

holds for all $j \in \mathbf{Z}$ *and* $a_{\alpha} \in \mathbf{R}$.

By Proposition 1 on page 477 of [St] it is easy to see that the following result holds.

Lemma 3.6. *Let* $\mathcal{P} = (P_1, \ldots, P_d)$ *be a polynomial mapping from* $\mathbf{R}^n$ *into* $\mathbf{R}^d$. *Let* $\deg(\mathcal{P}) = \max_{1\le j\le d}\deg(P_j)$. *Suppose that* $\tilde{b}(\cdot)$ *is a blocklike function defined as in (2.2) and* $\sigma^{*}_{\mathcal{P},\Omega}$ *be given as in (2.16). Then for every* $1 < p \le \infty$, *there exists a constant* C_p *independent of* $\tilde{b}$ *and the coefficients of* $\mathcal{P}$ *such that*

$$\left\|\sigma^{*}_{\mathcal{P},\tilde{b}}(f)\right\|_{p} \le C_p\|f\|_p$$

for $f \in L^p(\mathbf{R}^d)$.

By the above lemma and the proof of Lemma 3.4 we obtain the following:

Lemma 3.7. *Let* $\mathcal{P} = (P_1, \ldots, P_d)$ *be a polynomial mapping from* $\mathbf{R}^n$ *into* $\mathbf{R}^d$ *and* $\tilde{b}$ *be a q-blocklike function defined as in (2.2). Let* $\deg(\mathcal{P}) = \max_{1\le j\le d}\deg(P_j)$. *Suppose that* $|I| < e^{-2}$. *Then for every* $1 < p \le \infty$, *there exists a constant* C_p *independent of* $\tilde{b}$ *and the coefficients of* $\mathcal{P}$ *such that*

$$\left\|\lambda^{*}_{\mathcal{P},\tilde{b}}(f)\right\|_{p} \le C_p[\log(|I|^{-1})]\,\|f\|_p$$

for $f \in L^p(\mathbf{R}^d)$.

Our next step is to prove the following result on maximal functions:

Theorem 3.8. *Let* $\Phi : B(0,1) \to \mathbf{R}^d$ *be a smooth mapping and for* $q > 1$ *let* $\tilde{b}$ *be a q-blocklike function defined as in (2.2). Suppose that* Φ *is of finite type at* 0. *Then for* $1 < p \le \infty$ *and* $f \in L^p(\mathbf{R}^d)$ *there exists a positive constant* C_p *which is independent of* $\tilde{b}$ *such that*

$$\left\|\lambda^{*}_{\Phi,\tilde{b}}(f)\right\|_{L^p(\mathbf{R}^d)} \le C_p[\log(|I|^{-1})]\,\|f\|_{L^p(\mathbf{R}^d)} \quad \text{if } |I| < e^{-2}; \tag{3.20}$$

$$\left\|\sigma^{*}_{\Phi,\tilde{b}}(f)\right\|_{L^p(\mathbf{R}^d)} \le C_p\|f\|_{L^p(\mathbf{R}^d)} \quad \text{if } |I| \ge e^{-2}. \tag{3.21}$$

Proof. Assume first that $|I| < e^{-2}$. Without loss of generality we may assume that $\tilde{b} \ge 0$. By Lemma 3.4, there are $N \in \mathbf{N}$, $\delta \in (0,1]$, $C > 0$ and $k_0 \in \mathbf{Z}_{-}$ such that

$$\left|\hat{\lambda}_{\Phi,\tilde{b},k}(\xi)\right| \le C[\log(|I|^{-1})](\omega^{Nk}|\xi|)^{-\frac{\delta}{[\log(|I|^{-1})]}} \tag{3.22}$$

for all $k \le k_0$, $\xi \in \mathbf{R}^d$ with C independent of k and $[\log(|I|^{-1})]$ where $\omega = 2^{[\log(|I|^{-1})]}$. For $\Phi = (\Phi_1, \dots, \Phi_d)$ we let $\mathcal{P} = (P_1, \dots, P_d)$ where

$$P_j(y) = \sum_{|\beta| \le N-1} \frac{1}{\beta!} \frac{\partial^\beta \Phi_j}{\partial y^\beta}(0) y^\beta, \quad 1 \le j \le d.$$

Then we have

$$\left| \hat{\lambda}_{\Phi,\tilde{b},k}(\xi) - \hat{\lambda}_{\mathcal{P},\tilde{b},k}(\xi) \right| \le C[\log(|I|^{-1})] \omega^{-N} \left(\omega^{Nk} |\xi| \right). \tag{3.23}$$

By (2.5) we have

$$\left| \hat{\lambda}_{\Phi,\tilde{b},k}(\xi) - \hat{\lambda}_{\mathcal{P},\tilde{b},k}(\xi) \right| \le C[\log(|I|^{-1})]. \tag{3.24}$$

By interpolating between this estimate and (3.23) we get

$$\left| \hat{\lambda}_{\Phi,\tilde{b},k}(\xi) - \hat{\lambda}_{\mathcal{P},\tilde{b},k}(\xi) \right| \le C[\log(|I|^{-1})] \left(\omega^{Nk} |\xi| \right)^{\overline{\frac{\delta}{[\log(|I|^{-1})]}}}. \tag{3.25}$$

Therefore, (3.20) follows from (3.22), (3.25), Lemma 3.1 and Lemma 3.7. The proof of the inequality (3.21) will be much easier. In fact, it follows from (2.4)-(2.5), Lemma 2.3, 3.1, and 3.6. We omit the details.

4. Proofs of the theorems. By assumption, Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where $c_\mu \in \mathbf{C}$, b_μ is a q-block with support on an ia cap I_μ on $\mathbf{S}^{n-1}$ and

$$M_q^{0,0}(\{c_k\}, \{I_k\}) = \sum_{\mu=1}^{\infty} |c_\mu| \left(1 + (\log |I_\mu|^{-1}) \right) < \infty. \tag{4.1}$$

For each $\mu = 1, 2, \dots$, let $\tilde{b}_\mu$ be the blocklike function corresponding to b_μ. By the vanishing condition on Ω we have

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu \tilde{b}_\mu \tag{4.2}$$

and hence

$$\|T_\Phi f\|_p \le \sum_{\mu=1}^{\infty} |c_\mu| \left\| T_{\Phi,\tilde{b}_\mu} f \right\|_p, \tag{4.3}$$

where

$$T_{\Phi,\tilde{b}_\mu} f(x) = \text{p.v.} \int_{B(0,1)} f(x - \Phi(u)) \frac{\tilde{b}_\mu(u')}{|u|^n} du.$$

Let δ, N, $\mathcal{P}$ be given as in the proof of Theorem 3.8. For $1 \le j \le d$, let $a_{j,\beta} = \frac{1}{\beta!} \frac{\partial^\beta \Phi_j}{\partial y^\beta}(0)$. For $0 \le l \le N-1$ we define $Q^l = (Q_1^l, \dots, Q_d^l)$ by

$$Q_j^l(y) = \sum_{|\beta| \le l} a_{j,\beta} y^\beta, \qquad j = 1, \dots, d \tag{4.4}$$

when $0 \le l \le N-1$ and $Q^N = \Phi$. For each $0 \le l \le N$, let $\lambda^{(l)}_{\tilde{b}_\mu,k} = \lambda_{Q^l,\tilde{b}_\mu,k}$ and $\sigma^{(l)}_{\tilde{b}_\mu,k} = \sigma_{Q^l,\tilde{b}_\mu,k}$. Then by (2.3)-(2.5), Lemma 2.4 we have

$$\left\|\sigma^{(l)}_{\tilde{b}_\mu,k}\right\| \le C; \tag{4.5}$$

$$\left|\hat{\sigma}^{(l)}_{\tilde{b}_\mu,k}(\xi)\right| \le C(2^{lk}\sum_{|\beta|=l}\left|\sum_{j=l}^{d} a_{j,\beta}\xi_j\right|)^{-\frac{1}{2q'l}}; \tag{4.6}$$

$$\left|\hat{\sigma}^{(N)}_{\tilde{b}_\mu,k}(\xi) - \hat{\sigma}^{(N-1)}_{\tilde{b}_\mu,k}(\xi)\right| \le C(2^{Nk}|\xi|); \tag{4.7}$$

$$\left|\hat{\sigma}^{(l)}_{\tilde{b}_\mu,k}(\xi) - \hat{\sigma}^{(l-1)}_{\tilde{b}_\mu,k}(\xi)\right| \le C(2^{lk}\sum_{|\beta|=l}\left|\sum_{j=l}^{d} a_{j,\beta}\xi_j\right|) \tag{4.8}$$

for $\left|I_\mu\right| \ge e^{-2}$, $\mu = 1, 2, \ldots, 0 \le l \le N-1$, and $k \le k_0$. Also, by (2.3)-(2.5), Lemma 3.5, and the same argument as in the proof (3.25) we have

$$\left\|\lambda^{(l)}_{\tilde{b}_\mu,k}\right\| \le CA_\mu; \tag{4.9}$$

$$\left|\hat{\lambda}^{(l)}_{\tilde{b}_\mu,k}(\xi)\right| \le CA_\mu(2^{lA_\mu k}\sum_{|\beta|=l}\left|\sum_{j=l}^{d} a_{j,\beta}\xi_j\right|)^{-\frac{1}{A_\mu 2q'l}}; \tag{4.10}$$

$$\left|\hat{\lambda}^{(l)}_{\tilde{b}_\mu,k}(\xi) - \hat{\lambda}^{(l-1)}_{\tilde{b}_\mu,k}(\xi)\right| \le CA_\mu(2^{lA_\mu k}\sum_{|\beta|=l}\left|\sum_{j=l}^{d} a_{j,\beta}\xi_j\right|)^{\frac{1}{A_\mu 2q'l}} \tag{4.11}$$

where $A_\mu = [\log(\left|I_\mu\right|^{-1})]$, $\left|I_\mu\right| < e^{-2}$, $\mu = 1, 2, \ldots$, $k \le k_0$, $0 \le l \le N-1$.

By (3.20)-(3.22), (3.25), (4.5)-(4.11), Theorem 3.8, Lemmas 2.5-2.6, and 3.6-3.7 we get

$$\left\|T_{\Phi,\tilde{b}_\mu}f\right\|_p = \left\|\sum_{j\in\mathbf{Z}_-}\lambda^{(N)}_{\tilde{b}_\mu,k} * f\right\|_p \le C_p A_\mu \|f\|_p \text{ if } \left|I_\mu\right| < e^{-2}; \tag{4.12}$$

$$\left\|T_{\Phi,\tilde{b}_\mu}f\right\|_p = \left\|\sum_{j\in\mathbf{Z}_-}\sigma^{(N)}_{\tilde{b}_\mu,k} * f\right\|_p \le C_p \|f\|_p \text{ if } \left|I_\mu\right| \ge e^{-2}, \tag{4.13}$$

for every $f \in L^p(\mathbf{R}^d)$, $\mu = 1, 2, \ldots$, and for all p, $1 < p < \infty$. Hence, (1.7) follows from (4.1), (4.3) and (4.12)-(4.13). On the other hand, (1.8) follows from (3.20)-(3.21), (4.2) and the following inequality

$$\mathcal{M}_\Phi f(x) \le 4\sum_{\mu=1}^{\infty}\left|c_\mu\right|\sigma^*_{\Phi,\tilde{b}_\mu}(|f|)(x)$$

$$\le 4\sum_{\mu=1,\left|I_\mu\right|\ge e^{-2}}^{\infty}\left|c_\mu\right|\sigma^*_{\Phi,\tilde{b}_\mu}(|f|)(x) + 8\sum_{\mu=1,\left|I_\mu\right|< e^{-2}}^{\infty}\left|c_\mu\right|\lambda^*_{\Phi,\tilde{b}_\mu}(|f|)(x). \tag{4.14}$$

This concludes the proof of Theorem 1.2.

Finally, the proof of Theorem 1.3 follows from the above estimates and the techniques in [AqP]. We omit the details.

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