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Graded Lie Agebroids of Poisson Almost Commutative Algebras

Ferdinand Ngakeu*
Department of Mathematics and Computer Science,
University of Douala, Cameroon

Abstract

We introduce and study the notion of abelian groups graded Lie algebroid structures on almost commutative algebras $\mathcal{A}$ and show that any graded Poisson bracket on $\mathcal{A}$ induces a graded Lie algebroid structure on the $\mathcal{A}$-module of 1-forms on $\mathcal{A}$ as in the classical Poisson manifolds. We also derive from our formalism the graded Poisson cohomology.

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1 Introduction

An almost commutative algebra $\mathcal{A}$ is characterized by its multiplication which is given on homogeneous elements by $fg = \rho(|f|,|g|)gf$ where ρ is a commutation factor defined on an abelian group G [1, 2, 3]. Almost commutative algebras are also found in the literature under other names such as graded ϵ-commutative algebras[4], ρ-algebras[5], Γ-graded associative algebras[6], colour algebras[7, 8, 9], etc.

The commutative algebra $C^\infty(M)$ of smooth functions on a manifold M is a particular case where $\rho = 1$. It is well known that when the latter algebra is endowed with a Poisson bracket, the cotangent bundle of M is endowed with a structure of Lie algebroid and the algebra of multivectors on M with a Poisson cohomology etc. (see [10, 11] and references therein). In a previous work [3] we introduced the notion of Poisson almost commutative algebra (PACA) with symplectic almost commutative algebras as particular cases. Hence it was natural to find which classical geometric objects on Poisson manifold have their equivalents in the context of PACA.

In this work, we start bringing answer to that question by providing Algebroid structure and Poisson cohomology for PACA.

We start in the following section 2 by a brief generality on PACA. We recall definitions of the graded Schouten-Nijenhuis structure and the graded Poisson bracket. In section 3, we

*E-mail address: fngakeu@yahoo.fr

give our formalism of graded Lie algebroid. The main result of this section is Theorem3.3 where we show that to each PACA, $\mathcal{A}$, is associated a graded Lie algebra $(\Omega^1(\mathcal{A}), [,])$. Section 4 is devoted to the so called graded Poisson cohomology. We give in addition analog of classical formulae linking interior, Lie and exterior derivatives. We compute explicitly the Poisson cohomology of the Quantum plane when the latter is endowed with its canonical graded sympletic form.

2 Poisson Almost Commutative Algebras

In this section we give a brief review on the notion of Poisson almost commutative algebra (PACA) from [3] using the same notations as in the latter reference. Let G be an abelian group and let A be a G-graded almost commutative $\mathbb{C}$-algebra, namely the product in A obeys

$$fg = \rho(|f|, |g|)gf, \tag{2.1}$$

for homogeneous elements $f, g \in A$, where $\rho : G \times G \longrightarrow \mathbb{C}$ is a two-cycle or commutation factor, i.e. $\rho(u, v) = \rho(v, u)^{-1}$, $\rho(u+v, w) = \rho(u, w)\rho(v, w)$.
The A-bimodule of graded derivations of A is denoted by $\chi^1(A)$. It is the space of graded endomorphisms X of A obeying $X.(fg) = (X.f)g + \rho(|X|, |f|)f(X.g)$. The space $(\chi^1(A), [;])$ is a graded Lie-algebra [12] with the ρ-commutator of two ρ-derivations given by $[X, Y] = XY - \rho(|X|, |Y|)YX$.
On the graded algebra $\wedge\chi(A) = \bigoplus_{q=0}^{\infty}\chi^q(A) \quad (\chi^0(A) = A)$ of graded multivectors on A, we consider the Schouten-Nijenhuis bracket given in the following definition:

Definition 2.1. [3] The graded Schouten-Nijenhuis bracket (SNB) on A is the $\mathbb{C}$-bilinear map $[.] : \wedge\chi(A) \times \wedge\chi(A) \longrightarrow \wedge\chi(A)$ which is defined by:
1) For $f, g \in A$, $[f, g] = 0$,
2) For $X \in \chi^1(A)$, $Q \in \wedge\chi(A)$, $[X, Q]$ is the Lie derivative $L_X Q$,
3) And for homogeneous $P \in \chi^p(A), Q \in \chi^q(A)$, $[P, Q]$ is given by the following formula:

$$[X_1 \wedge ... \wedge X_q, P] := \sum_{j=1}^{q}(-1)^{q-j}\rho(|X_j|, |X_{j+1}| + ... + |X_q|)X_1 \wedge ... \wedge \hat{X}_j ... \wedge X_q \wedge [X_j, P].$$

where the graded Lie derivative $L_X Q$ is defined by

$$\begin{aligned} < \alpha_1, ..., \alpha_q; L_X Q > &= \rho(\sum_{i=1}^{q}|\alpha_i|, |X|)X. < \alpha_1, ..., \alpha_q; Q > \\ &- \sum_{i=1}^{q}\rho(\sum_{k=i}^{q}|\alpha_k|, |X|) < \alpha_1, ..., L_X\alpha_i ..., \alpha_q; Q > . \end{aligned} \tag{2.2}$$

Theorem 2.2. *[3] The SNB verifies the following properties:*
For $\mathbb{Z} \times G$-homogeneous $P \in \chi^p(A)$, $Q \in \chi^q(A)$ and $R \in \chi^r(A)$.

$$(i) \qquad [Q, P\wedge R] = [Q,P]\wedge R + (-1)^{(q-1)p}\rho(|Q|,|P|)P\wedge[Q,R] \tag{2.3}$$
$$(ii) \qquad [Q,P] = -(-1)^{(q-1)(p-1)}\rho(|Q|,|P|)[P,Q] \tag{2.4}$$
$$(iii) \qquad [P\wedge Q, R] = P\wedge[Q,R] + (-1)^{(r-1)q}\rho(|Q|,|R|)[P,R]\wedge Q \tag{2.5}$$

and the following graded Jacobi identity:

$$(iv) \quad \bar{\rho}(R,P)[P,[Q,R]] + \bar{\rho}(P,Q)[Q,[R,P]] + \bar{\rho}(Q,R)[R,[P,Q]] = 0 \tag{2.6}$$

which can take the following form

$$[P,[Q,R]] = [[P,Q],R] + \bar{\rho}(P,Q)[Q,[P,R]]. \tag{2.7}$$

where $\bar{\rho}(P,Q) = (-1)^{(p-1)(q-1)}\rho(|P|,|Q|)$.

Definition 2.3 ([3])**.** A ρ-graded Poisson structure(GPS) on A is a Leibniz bracket $\{,\}$ of order 2 and of certain G-degree D, whose Jacobi anomaly vanishes identically. Namely, $\{,\} : A\times A \longrightarrow A$ is a bilinear map which in addition satisfies:

$$\{f,g\} = -\rho(|f|,|g|)\{g,f\} \tag{2.8}$$
$$\{fg,h\} = f\{g,h\} + \rho(|g|,|h|+D)\{f,h\}g \tag{2.9}$$
$$\rho(|h|,|f|)\{f,\{g,h\}\} + \rho(|f|,|g|)\{g,\{h,f\}\} + \rho(|g|,|h|)\{h,\{f,g\}\} = 0 \tag{2.10}$$

Equivalently, we have shown in [3] that a GPS on A is defined by a bivector $\pi \in \chi^2(A)$ such that its SNB vanishes, i.e $[\pi,\pi] = 0$. The relation between $\{,\}$ and its graded Poisson bivector π is given by:

$$\{f,g\} =< df, dg; \pi > \tag{2.11}$$

Hence one has $D = |\pi|$.

Definition 2.4. [3] The grade Hamiltonian derivation (or vector field) associated to $f \in A$ is defined by

$$X_f.g := \{g,f\}\rho(|f|+|\pi|,|g|) = -\{f,g\}\rho(|\pi|,|g|). \tag{2.12}$$

It is easily to check that X_f is a graded left derivation on A of G-degree $|X_f| = |f| + |\pi|$ and we have:

$$X_f = \rho(|f|,|\pi|)[\pi, f] = [f,\pi]. \tag{2.13}$$

Definition 2.5. [3] Let (A,π) be a GPS on A. A graded derivation X is called a graded ”Poisson vector field” if $[X,\pi] = 0$.

We have the following properties:

(i) All Hamiltonian vectors fields X_f are graded Poisson vectors fields.

(ii) if Y is a Poisson vector field, then $[Y, X_f] = X_{(Y.f)}$

(iii) $[X_f, X_g] = -\rho(|\pi|,|g|)X_{\{f,g\}}$.

3 Graded Lie Algebroid structures

In this section we introduce the notion of graded Lie algebroid structure and apply the formalism to almost commutative algebras endowed with graded poisson structures.
Let A be an almost commutative algebra and $\chi^1(A)$ the A-bimodule of graded derivations on A as above.

Definition 3.1. A graded Lie algebroid on A is a G-graded A-bimodule E which is also a graded Lie algebra $(E,[,])$ endowed with a module morphism $\sharp : E \longrightarrow \chi^1(A)$ of G-degree D, called anchor map, such that

$$[fu,v] = f[u,v] + \rho(|f|+|u|,|v|+D)(\sharp v.f)u, \tag{3.1}$$

$\forall u,v \in E, \quad \forall f \in A.$

Example 3.2. 1) A simple example of graded Lie algebroid on any almost commutative algebra A is $E = (\chi^1(A),[,])$ with the identity map as the anchor map. Indeed, the bracket $[X,Y] = X \circ Y - \rho(|X|,|Y|)Y \circ X$ endows $\chi^1(A)$ with a structure of graded Lie algebra and in addition, we have for all $f,a \in A, \quad X,Y \in \chi^1(A)$,

$$\begin{aligned}
[fX,Y].a &= fX(Y.a) - \rho(|f|+|X|,|Y|)Y(fX.a) \\
&= fX(Y.a) - \rho(|f|+|X|,|Y|)(Y.f)(X.a) - \rho(|f|+|X|,|Y|)fY(X.a) \\
&= f[X(Y.a) - \rho(|X|,|Y))Y(X.a)] - \rho(|f|+|X|,|Y|)(Y.f)(X.a)
\end{aligned}$$

hence

$$[fX,Y] = f[X,Y] - \rho(|f|+|X|,|Y|)(Y.f)X. \tag{3.2}$$

2) The classical Lie algebroid on a manifold M is a graded Lie algebroid on $A = C^\infty(M)$ with $G = \{0\}$ and $\rho = 1$.
3) The following example which is the main aim of our study shows that a graded poisson structure leads to an algebroid structure on A.

3.1 Graded Poisson Lie Algebroid

In this subsection we show that a graded Poisson structure on a almost commutative algebra A induces a canonical Lie algebroid structure on $\Omega^1(A)$.
Let (A,π) be a poisson almost commutative algebra (PACA) and let $\sharp : \Omega^1(A) \longrightarrow \chi^1(A)$ be the morphism of A-modules defined by $\sharp(adf) = aX_f, \quad \forall a,f \in A$.

Theorem 3.3. *There exists a unique graded Lie bracket* $[,]$ *on the* A*-bimodule* $\Omega^1(A)$ *which satisfies the following conditions:*

$$[df,dg] = d\{f,g\}, \quad \forall f,g \in A \tag{3.3}$$

$$[f\alpha,\beta] = f[\alpha,\beta] + \rho(|\alpha|+|f|,|\beta|+|\pi|)(\sharp\beta.f)\alpha, \quad \forall \alpha,\beta \in \Omega^1(A), \forall f \in A. \tag{3.4}$$

Proof. It is enough to set $\alpha = adf, \beta = bdg$. Then $|\beta| = |b| + |g|, \sharp(\beta) = bX_g$.

$$\begin{aligned} [\alpha,\beta] &= a[df,\beta] + \rho(|a|+|f|,|\beta|+|\pi|)(\sharp\beta).adf \qquad (3.5)\\ &= -\rho(|f|,|\beta|)a[\beta,df] + \rho(|a|+|f|,|\beta|+|\pi|)((\sharp\beta).a)df \\ &= -\rho(|f|,|\beta|)a[bdg,df] + \rho(|a|+|f|,|\beta|+|\pi|)((\sharp\beta).a)df \\ &= -\rho(|f|,|\beta|)a\{b[dg,df] + \rho(|b|+|g|,|f|+|\pi|)(X_f.b)dg\} \\ &+ \rho(|a|+|f|,|\beta|+|\pi|)((\sharp\beta).a)df \end{aligned}$$

hence, $[\alpha,\beta]$ is uniquely given by:

$$\begin{aligned} [\alpha,\beta] &= -\rho(|f|,|b|+|g|)ab(d\{g,f\}) - \rho(|b|+|g|,|\pi|)a(X_f.b)dg \\ &+ \rho(|a|+|f|,|b|+|g|+|\pi|)((bX_g).a)df \qquad (3.6) \end{aligned}$$

We now prove the graded Jacobi identity

$$\rho(|\gamma|,|\alpha|)[\alpha,[\beta,\gamma]] + \rho(|\alpha|,|\beta|)[\beta,[\gamma,\alpha]] + \rho(|\beta|,|\gamma|)[\gamma,[\alpha,\beta]] = 0. \qquad (3.7)$$

Let us set $\alpha = adf$, $\beta = bdg$, $\gamma = cdh$ which are all homogeneous elements in $\Omega^1(A)$. Then by direct computation we have

$$\begin{aligned} [\alpha,[\beta,\gamma]] &= [adf, -\rho(|g|,|\gamma|)bc(d\{h,g\})] \\ &+ [adf, -\rho(|\gamma|,|\pi|)b(X_g.c)dh] + [adf, \rho(|\beta|,|\gamma|+|\pi|)c(X_h.b)dg] \qquad (3.8) \end{aligned}$$

Let compute explicitly any of the three terms in Equation (3.8). We obtain

$$\begin{aligned} [adf, -\rho(|g|,|\gamma|)bc(d\{h,g\})] &= \rho(|g|,|\gamma|)\rho(|f|,|\beta|+|\gamma|)abc(d\{\{h,g\},f\}) \\ &+ \rho(|g|,|\gamma|)\rho(|\beta|+|\gamma|,|\pi|)aX_f.(bc)d\{h,g\} \qquad (3.9)\\ &- \rho(|g|,|\gamma|)\rho(|\alpha|,|\beta|+|\gamma|+2|\pi|)bc(X_{\{h,g\}}.a)df \end{aligned}$$

$$\begin{aligned} [adf, -\rho(|\gamma|,|\pi|)b(X_g.c)dh] &= \rho(|\gamma|,|\pi|)\rho(|f|,|\beta|+|\gamma|+|\pi|)ab(X_g.c)d\{h,f\} \\ &+ \rho(|\gamma|,|\pi|)\rho(|\beta|+|\gamma|,|\pi|)aX_f.(bX_g.c)dh \qquad (3.10)\\ &- \rho(|\gamma|,|\pi|)\rho(|\alpha|,|\beta|+|\gamma|+2|\pi|)b(X_g.c)(X_h.a)df \end{aligned}$$

$$\begin{aligned} [adf, \rho(|\beta|,|\gamma|+|\pi|)c(X_h.b)dg)] &= -\rho(|\beta|,|\gamma|+|\pi|)\rho(|f|,|\beta|+|\gamma|+|\pi|)ac(X_h.b)d\{g,f\} \\ &- \rho(|\beta|,|\gamma|+|\pi|)\rho(|\beta|+|\gamma|,|\pi|)aX_f.(cX_h.b)dg \qquad (3.11)\\ &+ \rho(|\beta|,|\gamma|+|\pi|)\rho(|\alpha|,|\beta|+|\gamma|+2|\pi|)c(X_h.b)(X_g.a)df \end{aligned}$$

We then compute the sum of Equations (3.9),(3.10),(3.11) which leads to the explicit expression of $\rho(|\gamma|,|\alpha|)[\alpha,[\beta,\gamma]]$ as follows:

$$\begin{aligned}
\rho(|\gamma|,|\alpha|)[\alpha,[\beta,\gamma]] &= \rho(|\gamma|,|\alpha|)\rho(|g|,|\gamma|)\rho(|f|,|\beta|+|\gamma|)abc(d\{\{h,g\},f\})\\
&+ [-\rho(|g|,|\gamma|)\rho(|\alpha|,|\beta|+2|\pi|)bcX_{\{h,g\}}.a-\rho(|\gamma|,|\pi|)\rho(|\alpha|,|\beta|+2|\pi|)b(X_g.c)(X_h.a)\\
&+ \rho(|\beta|,|\gamma|+|\pi|)\rho(|\alpha|,|\beta|+2|\pi|)c(X_h.b)(X_g.a)]df\\
&+ \rho(|\gamma|,|\alpha|)\rho(|\gamma|,|\pi|)\rho(|\beta|+|\gamma|,|\pi|)aX_f.(bX_g.c)dh\\
&- \rho(|\gamma|,|\alpha|)\rho(|\beta|,|\gamma|+|\pi|)\rho(|\beta|+|\gamma|,|\pi|)aX_f.(cX_h.b)dg\\
&+ \rho(|\gamma|,|\alpha|)\rho(|g|,|\gamma|)\rho(|\beta|+|\gamma|,|\pi|)aX_f(bc)d\{h,g\}\\
&+ \rho(|\gamma|,|\alpha|)\rho(|\gamma|,|\pi|)\rho(|f|,|\beta|+|\gamma|+|\pi|)ab(X_g.c)d\{h,f\}\\
&- \rho(|\gamma|,|\alpha|)\rho(|\beta|,|\gamma|+|\pi|)\rho(|f|,|\beta|+|\gamma|+|\pi|)ac(X_h.b)d\{g,f\}.
\end{aligned}$$

We make the cyclic permutation of the last formula and we sum the three expressions obtained of $\rho(|\gamma|,|\alpha|)[\alpha,[\beta,\gamma]]$, $\rho(|\alpha|,|\beta|)[\beta,[\gamma,\alpha]]$ and $\rho(|\beta|,|\gamma|)[\gamma,[\alpha,\beta]]$ and we finally have:

$$\begin{aligned}
\rho(|\gamma|,|\alpha|)[\alpha,[\beta,\gamma]] &+ \rho(|\alpha|,|\beta|)[\beta,[\gamma,\alpha]]+\rho(|\beta|,|\gamma|)[\gamma,[\alpha,\beta]]\\
&= \rho(|c|,|a|)\rho(|h|,|a|)\rho(|g|,|c|)\rho(|f|,|b|)abc\\
&\times (d[\rho(|h|,|f|)\{f,\{g,h\}\}+\rho(|f|,|g|)\{g,\{h,f\}\}+\rho(|g|,|h|)\{h,\{f,g\}\}])\\
&= 0
\end{aligned}$$

which proves the graded Jacobi identity. □

One then observes that the bracket [;] given in Theorem 3.3 defines a graded Lie algebroid structure on $\Omega^1(A)$ with anchor maps $\sharp$.

Definition 3.4. The graded Lie algebroid structure $(\Omega^1(A),[,],\sharp)$ from Theorem 3.3 is called graded Poisson Lie algebroid structure on $\Omega^1(A)$.

Remark 3.5. : From now on, we set $\alpha^\sharp := \sharp(\alpha)$.

Proposition 3.6. *The graded algebroid bracket from Theorem 3.3 is given by the following formula*

$$[\alpha,\beta]=\rho(|\alpha|,|\beta|+|\pi|)L^{\alpha}_{\beta^\sharp}-\rho(|\beta|,|\pi|)L^{\beta}_{\alpha^\sharp}-d(<\alpha,\beta,\pi>) \tag{3.12}$$

and the action of the anchor map $\sharp$ on this bracket gives

$$[\alpha,\beta]^\sharp=-\rho(|\beta|,|\pi|)[\alpha^\sharp,\beta^\sharp] \tag{3.13}$$

Proof. : We set $\alpha=adf$, $\beta=bdg$. Using Equation(3.6) we have

$$\begin{aligned}
[\alpha,\beta] &= -\rho(|f|,|\beta|)d(ab<dg,df,\pi>)+\rho(|f|,|\beta|)d(ab)<dg,df,\pi>\\
&- \rho(|\beta|,|\pi|)a(X_f.b)dg+\rho(|a|+|f|,|\beta|+|\pi|)(\beta^\beta.a)df\\
&= d(<\alpha,\beta,\pi>)+\rho(|f|,|\beta|)da<\beta,df,\pi>+\rho(|\alpha|,|\beta|)db<dg,\alpha,\pi>\\
&- \rho(|\beta|,|\pi|)a(X_f.b)dg+\rho(|\alpha|,|\beta|+|\pi|)(\beta^\beta.a)df
\end{aligned} \tag{3.14}$$

Let us recall that the graded Lie derivation satisfies

$$L_X^{f\omega} = (X.f)\omega + \rho(|X|,|f|)fL_X^{\omega}, \;\; L_{fX}^{\omega} = fL_X^{\omega} + df < X,\omega >, \tag{3.15}$$

$\forall f \in A, X \in \chi^1(A), \omega \in \Omega^1(A)$, from which one deduces

$$L_{\alpha^\sharp}^{\beta} = L_{\alpha^\sharp}^{bdg} = (\alpha^\sharp.b)dg + \rho(|\alpha|+|\pi|,|b|)bL_{\alpha^\sharp}^{dg} \tag{3.16}$$

$$\begin{aligned}(\alpha^\sharp.b)dg &= L_{\alpha^\sharp}^{\beta} - \rho(|\alpha|+|\pi|,|b|)bL_{\alpha^\sharp}^{dg},\\ (\beta^\sharp.a)df &= L_{\beta^\sharp}^{\alpha} - \rho(|\beta|+|\pi|,|a|)aL_{\beta^\sharp}^{df}\end{aligned} \tag{3.17}$$

and

$$L_{a\beta^\sharp}^{df} = aL_{\beta^\sharp}^{df} + da < \beta^\sharp, df >= aL_{\beta^\sharp}^{df} + \rho(|\beta|+|\pi|,|f|)da < df,\beta,\pi > . \tag{3.18}$$

Hence

$$da < \beta, df, \pi > -\rho(|f|,|\pi|)aL_{\beta^\sharp}^{df} = -\rho(|f|,|\pi|)L_{a\beta^\sharp}^{df} \tag{3.19}$$

and similarly

$$db < dg,\alpha,\pi > +\rho(|g|,|\alpha|+|\pi|)bL_{\alpha^\sharp}^{dg} = \rho(|g|,|\alpha|+|\pi|)L_{b\alpha^\sharp}^{dg} \tag{3.20}$$

We then compute (3.14) using (3.17),(3.19), (3.20)and we obtain

$$\begin{aligned}[\alpha,\beta] &= \rho(|\alpha|,|\beta|+|\pi|)L_{\beta^\sharp}^{\alpha} - \rho(|\beta|,|\pi|)L_{\alpha^\sharp}^{\beta} + d(< \alpha,\beta,\pi >)\\ &+ \rho(|f|,|\beta|)\{da < \beta, df, \pi > -\rho(|f|,|\pi|)aL_{\beta\sharp}^{df}\}\\ &+ \rho(|\alpha,|\beta|)\{db < dg,\alpha,\pi > +\rho(|g|,|\alpha|+|\pi|)bL_{\alpha^\sharp}^{dg}\}\\ &= \rho(|\alpha|,|\beta|+|\pi|)L_{\beta^\sharp}^{\alpha} - \rho(|\beta|,|\pi|)L_{\alpha^\sharp}^{\beta} + d(< \alpha,\beta,\pi >)\\ &- \rho(|f|,|\beta|+|\pi|)L_{a\beta^\sharp}^{df} + \rho(|\alpha,|\beta|)\rho(|g|,|\alpha|+|\pi|)L_{b\alpha^\sharp}^{dg}.\end{aligned} \tag{3.21}$$

Finally we compute (3.21)using

$$L_{a\beta^\sharp}^{df} = d(a\beta^\sharp.f) = \rho(|a|,|\beta|+|\pi|)d(< \beta^\sharp,\alpha >) = \rho(|\beta|+|\pi|,|f|)d(< \alpha,\beta,\pi >)$$

and

$$L_{b\alpha^\sharp}^{dg} = \rho(|\alpha|+|\pi|,|g|)d(< \beta,\alpha,\pi >)$$

and we obtain

$$[\alpha,\beta] = \rho(|\alpha|,|\beta|+|\pi|)L_{\beta^\sharp}^{\alpha} - \rho(|\beta|,|\pi|)L_{\alpha^\sharp}^{\beta} - d(< \alpha,\beta,\pi >)$$

as expected.
We now prove (3.13), namely $[\alpha,\beta]^\sharp = -\rho(|\beta|,|\pi|)[\alpha^\sharp,\beta^\sharp]$.
Applying the map $\sharp$ to (3.6) gives:

$$[\alpha,\beta]^\sharp = -\rho(|f|,|\beta|)abX_{\{g,f\}} - \rho(|\beta|,|\pi|)a(X_f.b)X_g + \rho(|\alpha|,|\beta|+|\pi|)((\beta^\sharp).a)X_f$$

On the other hand

$$\begin{aligned}
[\alpha^\sharp,\beta^\sharp] &= [aX_f, bX_g] = a(X_f.b)X_g - \rho(|\alpha|+|\pi|,|\beta|+|\pi|)b(X_g.a)X_f \\
&- \rho(|\alpha|+|\pi|,|\beta|+|\pi|)\rho(|g|+|\pi|,|a|)ba[X_g,X_f] \\
&= a(X_f.b)X_g - \rho(|\alpha|,|\beta|+|\pi|)\rho(|\pi|,|\beta|)(\beta^\sharp.a)X_f \\
&+ \rho(|\pi|,|f|)\rho(|\alpha|+|\pi|,|\beta|+|\pi|)\rho(|\beta|+|\pi|,|a|)abX_{\{g,f\}} \\
&= -\rho(|\pi|,|\beta|)[\alpha,\beta]^\sharp
\end{aligned}$$

as expected. □

4 Graded Poisson Cohomology

In this section we show that the graded Lie algebroid structure on $\Omega^1(A)$ induces on $\chi(A)$ a graded analog of the classical Poisson cohomology.

4.1 Graded contravariant exterior differentiation

Let (A,π) be a PACA with the Poisson bivector π. We define a graded exterior differentiation δ of G-degree $|\delta| = |\pi|$ on the complex $\wedge\chi(A) = \bigoplus_q^\infty \chi^q(A)$ by: $\forall f \in A,\ Q \in \chi^q(A), \forall \alpha,\alpha_i \in \Omega^1(A)$, we set:

$$< \alpha;\delta f > \quad = \alpha^\sharp.f \tag{4.1}$$

$$\begin{aligned}
< \alpha_0,\alpha_1,...,\alpha_q;\delta Q > &= \sum_{i=0}^{q}(-1)^i\rho(\sum_{k=0,k\neq i}^{q}|\alpha_k|,|\pi|)\rho(\sum_{k<i}|\alpha_k|,|\alpha_i|)\alpha_i^\sharp. < \alpha_0,...,\alpha_q;Q > \\
&+ \sum_{0\leq i<j}(-1)^{i+j+1}\rho(\sum_{0\leq k<i}|\alpha_k|,|\alpha_i|)\rho(\sum_{0\leq k<j,k\neq i}|\alpha_k|,|\alpha_j|)\rho(\sum_{k\neq i,j}|\alpha_k|,|\pi|) \\
&\times < [\alpha_i,\alpha_j],\alpha_0,...,\check{\alpha}_i,...,\check{\alpha}_j,...,\alpha_q;Q >
\end{aligned} \tag{4.2}$$

Proposition 4.1.

$$\delta Q = -[\pi,Q], \quad \mathit{Hence} \quad \delta\circ\delta = 0 \tag{4.3}$$

Proof. : First, Suppose that $\delta Q = -[\pi,Q]$. Then using Equation (2.7) one deduces:
$\delta\circ\delta Q = [\pi,[\pi,Q]] = [[\pi,\pi],Q] + \bar{\rho}(\pi,\pi)[\pi,[\pi,Q]] = -[\pi,[\pi,Q]]$ hence $\delta\circ\delta Q = 0$.
Next to show that $\delta Q = -[\pi,Q]$, we proceed by induction. We first show this equality for $Q = f \in A$ and $Q = X \in \chi^1(A)$. Let us set $\alpha = bdg$, $\quad b,g \in A$.

$$\begin{aligned}
< \alpha;-[\pi,f] > &= -\rho(|\pi|,|f|) < \alpha;X_f >= -\rho(|\pi|,|f|)b < dg;X_f > \\
&= -\rho(|\pi|,|f|)b\{g,f\} = bX_g.f = \alpha^\sharp.f =< \alpha;\delta f > .
\end{aligned}$$

$$\begin{aligned} < \alpha_1, \alpha_2; \delta X > &= \rho(|\alpha_2|, |\pi|)\alpha_1^\sharp.(< \alpha_2; X >) \\ &- \rho(|\alpha_1|, |\alpha_2| + |\pi|)\alpha_2^\sharp. < \alpha_1; X > + < [\alpha_1, \alpha_2]; X > \end{aligned} \tag{4.4}$$

To check that $< \alpha_1, \alpha_2; \delta X > = < \alpha_1, \alpha_2; -[\pi, X] >$ it is sufficient to do it with $\alpha_1 = dg,\ \alpha_2 = dh,\ g, h \in A$. Then (4.4) becomes

$$\begin{aligned} < \alpha_1, \alpha_2; \delta X > &= \rho(|h|, |\pi|).X_g < dh; X > -\rho(|g|, |h| + |\pi|)X_h. < dg; X > + < [dg, dh]; X > \\ &= \rho(|h|, |X| + |\pi|)X_g.(X.h) - \rho(|g|, |h| + |\pi| + |X|)X_h.(X.g) \\ &+ \rho(|g| + |h| + |\pi|, |X|)X.(\{g, h\}) \\ &= -\rho(|h| + |\pi|, |X|)\{g, X.h\} - \rho(|g| + |h| + |\pi|, |X|)\{X.g, h\} \\ &+ \rho(|g| + |h| + |\pi|, |X|)X.(\{g, h\}) \end{aligned} \tag{4.5}$$

On the other hand

$$\begin{aligned} & < \alpha_1, \alpha_2; -[\pi, X] > = \rho(|\pi|, |X|) < \alpha_1, \alpha_2; L_X^\pi > \\ = & \rho(|\pi|, |X|)\rho(|\alpha_1| + |\alpha_2|, |X|)X. < \alpha_1, \alpha_2; \pi > -\rho(|\pi| + |\alpha_1| + |\alpha_2|, |X|) < L_X^{\alpha_1}, \alpha_2; \pi > \\ - & \rho(|\pi| + |\alpha_2|, |X|) < \alpha_1, L_X^{\alpha_2}; \pi > \\ = & \rho(|\pi|, |X|)\rho(|g| + |h|, |X|)X. < dg, dh; \pi > -\rho(|\pi| + |g| + |h|, |X|) < L_X^{dg}, dh; \pi > \\ - & \rho(|\pi|, |X|)\rho(|h|, |X|) < dg, L_X^{dh}; \pi > \\ = & -\rho(|h| + |\pi|, |X|)\{g, X.h\} - \rho(|g| + |h| + |\pi|, |X|)\{X.g, h\} \\ + & \rho(|g| + |h| + |\pi|, |X|)X.(\{g, h\}) \end{aligned}$$

$$< \alpha_1, \alpha_2; -[\pi, X] > \ = \ < \alpha_1, \alpha_2; \delta X > . \tag{4.6}$$

Next to show that $\delta Q = -[\pi, Q]$ for $Q \in \chi^q(A), \quad q \geq 2$, we suppose by induction that this formula is true for any $Q' \in \chi^p(A)$ for $p \leq q$ and set $Q = X \wedge Q'$, then
$-[\pi, X \wedge Q'] = -[\pi, X] \wedge Q' + \rho(|\pi|, |X|)X \wedge [\pi, Q'] = \delta X \wedge Q' - \rho(|\pi|, |X|)X \wedge \delta Q'$
We then compute $< df_0, ..., df_q; \delta X \wedge Q' - \rho(|\pi|, |X|)X \wedge \delta Q' >$ using formula (4.2)and find the expression of $< df_0, ..., df_q; \delta Q >$. □

Definition 4.2. $(\chi(A), \delta)$ is called the graded Lichnerowicz-Poisson cochains Complex of the PACA (A, π).

Let us set: $Z^q(A) = \ker(\delta : \chi^q(A) \to \chi^{q+1}(A))$ and $B^q(A) = Im(\delta : \chi^{q-1}(A) \to \chi^q(A))$

$$H^q_{LP}(A) = \frac{Z^q(A)}{B^q(A)}, \quad q \geq 0. \tag{4.7}$$

Definition 4.3. The spaces ${H^q}_{LP}(A), \quad q \geq 0$ are called the Lichnerowicz-Poisson cohomology groups of the PACA (A, π).

One easily checks that $H^0(A) = \{f \in A, X_f = 0\}$ wich is the center of the PACA (A, π), $Z^1(A)$ is the space of the graded Poisson derivations and $B^1(A)$ is the space of Hamiltonian derivations.
The space $H^2(A)$ contains a particular class $[\pi]$. When this class vanishes, we say as in the classical case that (A, π) is an exact Poisson almost commutative algebra (EPACA).

4.2 Interior and Lie derivative of multivectors

We now introduce the graded analog of classical interior and Lie derivative of multivectors with respect to graded 1-forms. For $Q \in \chi^q(A), \quad \alpha, \alpha_i \in \Omega^1(A), \quad f \in A$, we set

$$\begin{aligned} i_\alpha f &= 0, \quad < \alpha_1, ..., \alpha_{q-1}; i_\alpha Q >= (-1)^{q-1} < \alpha_1, ..., \alpha_{q-1}, \alpha; Q > \\ L_\alpha f &= \alpha^\sharp . f, \\ < \alpha_1, ..., \alpha_q; L_\alpha Q > &= \rho(\sum_{k=1}^{q} |\alpha_k|, |\alpha| + |\pi|)\alpha^\sharp . < \alpha_1, ..., \alpha_q; Q > \\ &- \sum_{i=1}^{q} \rho(\sum_{k=i+1}^{q} |\alpha_k|, |\alpha| + |\pi|) < \alpha_1, ..., [\alpha_i, \alpha] ... \alpha_q; Q > \end{aligned} \tag{4.8}$$

Then we have the following properties:

Proposition 4.4.

$$L_\alpha = i_\alpha \circ \delta + \rho(|\alpha|, |\delta|)\delta \circ i_\alpha, \tag{4.9}$$

$$\delta \circ L_\alpha = \rho(|\delta|, |\alpha|) L_\alpha \circ \delta. \tag{4.10}$$

$$i_{[\alpha,\beta]} = \rho(|\alpha|, |\beta|) i_\beta \circ L_\alpha - \rho(|\beta|, |\pi|) L_\alpha \circ i_\beta \tag{4.11}$$

$$L_{[\alpha,\beta]} = \rho(|\alpha|, |\beta| + |\pi|) L_\beta \circ L_\alpha - \rho(|\beta|, |\pi|) L_\alpha \circ L_\beta \tag{4.12}$$

Remark 4.5. : To recover the usual formulae as in [10], one should take $\rho = 1$ and the opposite of the actual bracket of 1-forms.

We now give main steps in the proof of Proposition 4.4.

Proof. For all $Q \in \chi^q(A), \quad \alpha, \alpha_1, ..., \alpha_q \in \Omega^1(A)$, a direct computation gives:

$$\begin{aligned} < \alpha_1, ..., \alpha_q; \delta \circ i_\alpha Q > &= \sum_{i=1}^{q} (-1)^{q+i} \rho(\sum_{k \neq i}^{q} |\alpha_k|, |\pi|) \rho(\sum_{k<i} |\alpha_k|, |\alpha_i|) \alpha_i^\sharp . < \alpha_1, ..., \check{\alpha}_i ... \alpha_q, \alpha; Q > \\ &+ \sum_{1 \le i < j \le q} (-1)^{q+i+j} \rho(\sum_{k \neq i,j}^{q} |\alpha_k|, |\pi|) \rho(\sum_{k<j, k \neq i} |\alpha_k|, |\alpha_j|) \\ &\times \rho(\sum_{k<i} |\alpha_k|, |\alpha_i|) < [\alpha_i, \alpha_j], \alpha_1, ..., \check{\alpha}_i ... \check{\alpha}_j, ... \alpha_q, \alpha; Q > \end{aligned} \tag{4.13}$$

$$
\begin{aligned}
& <\alpha_1,...,\alpha_q; i_\alpha(\delta Q)>=(-1)^q<\alpha_1,...,\alpha_q,\alpha;\delta Q> \\
= \ & \sum_{i=1}^{q}(-1)^{q+i+1}\rho(\sum_{k=i+1}^{q}|\alpha_k|+\alpha,|\pi|)\rho(\sum_{k<i}|\alpha_k|,|\alpha_i|+|\pi|)\alpha_i^\sharp.<\alpha_1,...,\check{\alpha}_i...\alpha_q,\alpha;Q> \\
+ \ & \rho(\sum_{k=1}^{q}|\alpha_k|,|\alpha|+|\pi|)\alpha^\sharp.<\alpha_1,...,\alpha_q;Q> \\
+ \ & \sum_{1\le i<j\le q}(-1)^{q+i+j+1}\rho(\sum_{k\neq i,j}^{q}|\alpha_k|+|\alpha|,|\pi|)\rho(\sum_{k<j,k\neq i}|\alpha_k|,|\alpha_j|) \\
\times \ & \rho(\sum_{k<i}|\alpha_k|,|\alpha_i|)<[\alpha_i,\alpha_j],\alpha_1,...,\check{\alpha}_i...\check{\alpha}_j,...\alpha_q,\alpha;Q> \\
+ \ & \sum_{i=1}^{q}(-1)^i\rho(\sum_{k<i}|\alpha_k|,|\alpha_i|)\rho(\sum_{k\neq i}|\alpha_k|,|\alpha|+|\pi|)<[\alpha_i,\alpha],\alpha_1,...\alpha_q;Q>
\end{aligned}
$$

Multiplying Equation (4.13) by $\rho(|\alpha|,|\pi|)$ and summing the result with the equation above gives the expression of $<\alpha_1,...,\alpha_q;L_\alpha Q>$ and proves Equation(4.9).
The proof of Equation(4.10) is obviously deduced from (4.9). Next we have

$$
\begin{aligned}
& <\alpha_1,...,\alpha_{q-1};\rho(|\alpha|,|\beta|)(i_\beta\circ L_\alpha)(Q)>=(-1)^{q-1}\rho(|\alpha|,|\beta|)<\alpha_1,...,\alpha_{q-1},\beta;L_\alpha Q> \\
= \ & (-1)^{q-1}\rho(\sum_{k=1}^{q-1}|\alpha_k|,|\alpha|+|\pi|)\rho(|\beta|,|\pi|)\alpha^\sharp.<\alpha_1,...,\alpha_{q-1},\beta;Q> \\
- \ & (-1)^{q-1}\rho(|\alpha|,|\beta|)<\alpha_1,...,\alpha_{q-1},[\beta,\alpha];Q> \\
- \ & (-1)^{q-1}\sum_{i=1}^{q-1}\rho(\sum_{k=i+1}^{q-1}|\alpha_k|,|\alpha|+|\pi|)\rho(|\beta|,|\pi|)<\alpha_1,...,\alpha_{i-1},[\alpha_i,\alpha]...,\alpha_{q-1},\beta;Q>
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
& <\alpha_1,...,\alpha_{q-1};-\rho(|\beta|,|\pi|)(L_\alpha\circ i_\beta)(Q)> \\
= \ & -(-1)^{q-1}\rho(|\beta|,|\pi|)\rho(\sum_{k=1}^{q-1}|\alpha_k|,|\alpha|+|\pi|)\alpha^\sharp.<\alpha_1,...,\alpha_{q-1},\beta;Q> \\
+ \ & (-1)^{q-1}\sum_{i=1}^{q-1}\rho(\sum_{k=i+1}^{q-1}|\alpha_k|,|\alpha|+|\pi|)\rho(|\beta|,|\pi|)<\alpha_1,...,\alpha_{i-1},[\alpha_i,\alpha],...,\alpha_{q-1},\beta;Q>
\end{aligned}
$$

summing the two last equations above gives $-(-1)^{q-1}\rho(|\alpha|,|\beta|)<\alpha_1,...,\alpha_{q-1},[\beta,\alpha];Q>$which in turn is $<\alpha_1,...,\alpha_{q-1},i_{[\alpha,\beta]};Q>$ and that proves the property given in (4.11).
Finally the proof of Equation (4.12) is easily done using formula (4.9). □

Example 4.6. : We consider the quantum plane $A=R_q^2$ with its symplectic form $\omega=dy\wedge dx$ as in [3]. Its Poisson bivector is $\pi=\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$.
The Hamiltonian vector field associted to $f\in A$ is $X_f=[f,\pi]=q^{1-|f|_1}\frac{\partial f}{\partial y}\frac{\partial}{\partial x}-q^{|f|_2}\frac{\partial f}{\partial x}\frac{\partial}{\partial y}$. where

$|f| = (|f|_1, |f|_2)$ The poisson bracket $\{,\}$ is defined by:
$X_f.g =< X_f, dg >=< X_f, X_g, \omega >= -\rho(|g|, |\omega|)\{f, g\}$ which explicitly is
$\{f, g\} = -\rho(|\omega|, |g|)X_f.g = -q^{1-|f|_1-|g|_1+|g|_2}\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} + q^{|f|_2+|g|_2-|g|_1}\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}$.
Let $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ be a general homogeneous vector. We have:
$[X, \pi] = -qX_f \wedge \frac{\partial}{\partial x} - \frac{1}{q}X_g \wedge \frac{\partial}{\partial y}$. On the other hand, using the formula of X_f above, one finds that Poisson vectors $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ are solutions of the following graded PDE:

$$q^{1+|f|_2}\frac{\partial f}{\partial x} + q^{1-|g|_1}\frac{\partial g}{\partial y} = 0. \tag{4.14}$$

Solving this Equation (4.14) gives

$$X = k(x^m y^n \frac{\partial}{\partial x} - \frac{m}{n+1}q^n x^{m-1} y^{n+1}\frac{\partial}{\partial y}), \quad k \in \mathbb{C}, \quad m, n \in \mathbb{N}. \tag{4.15}$$

One easily checks that the latter is a Hamiltonian vector, hence $H^1(R_q^2) = \{0\}$.
on the other hand we have $\pi = \delta X_0$ where $X_0 = -y\frac{\partial}{\partial y}$, meaning that any closed bivector is exact, and this leads to $H^2(R_q^2) = \{0\}$. Finally the Poisson cohomology of the symplectic quantum plane is given by:

$$H^0(R_q^2) = \mathbb{C}, \quad H^1(R_q^2) = \{0\}, \quad H^2(R_q^2) = \{0\}. \tag{4.16}$$

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