P̲oisson S̲ummation F̲ormulae and the W̲ave E̲quation with a F̲initely S̲upported M̲easure as I̲nitial V̲elocity

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Abstract

New Poisson summation formulae have been recently discovered by Nir Lev and Alexander Olevskii since 2013. But some other examples were concealed in an old paper by Andrew Guinand dating from 1959. This was observed by the second author in 2016. In the present contribution a third approach is proposed. Guinand's work follows from some simple observations on solutions of the wave equation on the three dimensional torus. If the initial velocity is a Dirac mass at the origin, the solution is Guinand's distribution. Using this new approach one can construct a large family of initial velocities which give rise to crystalline measures generalizing Guinand's solution.

1 Poisson summation formulae

We are looking for new Poisson summation formulae. They depend on *crystalline measures* which are defined below (Definition 1.3).

Definition 1.1. The Fourier transform $\mathcal{F}(f) = \widehat{f}$ of a function f is defined by $\widehat{f}(y) = \int_{\mathbb{R}^n} \exp(-2\pi i x \cdot y) f(x) \, dx$.

This is well defined if $f \in L^1(\mathbb{R}^n)$ and extends to tempered distributions as detailed in L. Schwartz treatise [15].

Definition 1.2. A set of points $\Lambda \subset \mathbb{R}^n$ is locally finite if, for every compact set B, $\Lambda \cap B$ is finite.

A locally finite set $\Lambda \subset \mathbb{R}^n$ can be ordered as a sequence of points tending to infinity. A set of points $\Lambda \subset \mathbb{R}^n$ is uniformly discrete if

$$\inf_{\{\lambda, \lambda' \in \Lambda, \lambda' \neq \lambda\}} |\lambda' - \lambda| = \beta > 0. \tag{1}$$

Definition 1.3. A crystalline measure is an atomic measure μ on $\mathbb{R}^n$ which satisfies the conflicting but fortunately compatible properties:

(a) μ is supported by a locally finite set

(b) μ is a tempered distribution

(c) the distributional Fourier transform $\widehat{\mu}$ of μ is also an atomic measure supported by a locally finite set.

Let us comment on (a) and (b). We have $\mu = \sum_{\lambda \in \Lambda} a(\lambda)\delta_\lambda$ and Λ is a locally finite set. The linear form $\langle \mu, \phi \rangle = \sum_{\lambda \in \Lambda} a(\lambda)\phi(\lambda)$ is well defined if ϕ is a compactly supported test function. By (b) this linear form extends continuously from $\mathcal{D}(\mathbb{R}^n)$ to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. But it may happen that the series $\sum_{\lambda \in \Lambda} a(\lambda)\phi(\lambda)$ diverges when ϕ belongs to the Schwartz class. Here is an example. Let $\Lambda = \{\lambda_k, k \in \mathbb{N}\}$ be defined by $\lambda_{2k} = k$ and $\lambda_{2k+1} = k + 2^{-k}$. Let $a(\lambda_{2k}) = 2^k$ and $a(\lambda_{2k+1}) = -2^k$. Then the atomic measure $\mu = \sum_{\lambda \in \Lambda} a(\lambda)\delta_\lambda = \sum_{k \in \mathbb{N}} 2^k(\delta_{\lambda_{2k}} - \delta_{\lambda_{2k+1}})$ is a tempered distribution but the series $\sum_{\lambda \in \Lambda} a(\lambda)f(\lambda)$ may not converge when f belongs to the Schwartz class. If a series of real numbers $\sum_{-\infty}^{\infty} u_k$ converges then u_k tends to 0 as k tends to ∞. In our situation there exists a test function f such that $2^k f(2k)$ does not tend to 0 as k tends to infinity. Grouping terms by pairs suffices to restore the missing convergence. Such a pathology will never occur in what follows.

Let Λ be the support of a crystalline measure μ and let S be its spectrum, i.e. the support of $\widehat{\mu}$. We then have

$$\mu = \sum_{\lambda \in \Lambda} a(\lambda)\delta_\lambda, \quad \widehat{\mu} = \sum_{y \in S} b(y)\delta_y. \tag{2}$$

It yields the following *generalized Poisson summation formula*:

$$\sum_{\lambda \in \Lambda} a(\lambda)\widehat{f}(\lambda) = \sum_{y \in S} b(y)f(y), \quad \forall f \in \mathcal{S}(\mathbb{R}^n). \tag{3}$$

A well known example is given by the standard Poisson summation formula where Λ is a lattice. A lattice $\Gamma \subset \mathbb{R}^n$ is defined by $\Gamma = A\mathbb{Z}^n$ where $A \in GL(n, \mathbb{R})$. A *Dirac comb* is a sum $\mu = \sum_{\gamma \in \Gamma} \delta_\gamma$ of Dirac masses δ_γ on a lattice Γ. The Fourier transform of the Dirac comb on a lattice Γ is (up to a constant factor) the Dirac comb on the dual lattice Γ^*. This is the *standard Poisson summation formula* which plays a seminal role in X-ray crystallography and molecular biology. Other Poisson summation formulae, which will be called *generalized Dirac combs*, directly follow from the standard one.

Are there other crystalline measures ? This line of investigation began with the Riemann-Weil explicit formula in number theory. The Riemann-Weil explicit formula can be written

$\widehat{\mu} = \sigma + \omega$ where μ is a series of Dirac masses on the non-trivial zeros of the zeta function, σ is a series of Dirac masses on $\pm \log(p^m)$, p running over the set of prime numbers, $m = 1, 2, \ldots$, and $\omega(x) = -\log \pi + \Re \psi(1/4 + ix/2)$, ψ being the logarithmic derivative of the Γ function. Therefore an exponential decay is needed on the test function ϕ to give a meaning to $\langle \sigma + \omega, \phi \rangle$. The Selberg trace formula has a similar structure. The measures μ studied by André Weil (1952) and Atle Selberg (1956) are not crystalline measures. The same remark applies to the Poisson formulae relating the lengths of the closed geodesics and the spectrum of the Laplace operator on a compact Riemann manifold with negative curvature (Yves Colin de Verdière).

The collection of crystalline measures is a vector space. If μ is a crystalline measure and if P is a finite trigonometric sum then $P\mu$ is also a crystalline measure. These two remarks are used in the following definition:

Definition 1.4. Let σ_j be a Dirac comb supported by a coset $a_j + \Gamma_j$ of a lattice $\Gamma_j \subset \mathbb{R}^n$, $1 \leq j \leq N$. Let g_j be a finite trigonometric sum and $\mu_j = g_j \sigma_j$. Then $\mu = \mu_1 + \cdots + \mu_N$ will be called a generalized Dirac comb.

The Fourier transform of a generalized Dirac comb is a generalized Dirac comb.

Definition 1.5. A crystalline measure μ which is not a generalized Dirac comb is called an exotic crystalline measure.

It is the case if the support Λ of μ is not contained in a finite union $\bigcup_1^N (a_j + \Gamma_j)$ of co-sets of lattices.

Lemma 1.6. *If μ is a crystalline measure and if the density of the support of μ is infinite then μ is an exotic crystalline measure.*

This applies to Guinand's measure (Section 3). Our goal is the construction of exotic crystalline measures. Two methods are proposed. The first one uses Guinand's mysterious ideas [3]. The wave equation will provide us with a second construction in Section 4.

2 Quasicrystals and crystalline measures

In this section quasicrystals are defined as model sets [1], [13]. If Λ is a quasicrystal which is not a lattice, then $\mu = \sum_{\lambda \in \Lambda} \delta_\lambda$ is not a crystalline measure. Indeed the Fourier transform of μ is not even a measure. Using the cut and projection construction of quasicrystals one can fix this issue and find some weights $c(\lambda) \in [0, 1]$ such that the Fourier transform $\widehat{\mu}$ of the measure $\mu = \sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda$ is an atomic measure [13]. However in this construction the spectrum of μ is dense in $\mathbb{R}^n$. Therefore μ is not a crystalline measure. This led J. C. Lagarias [5] to propose the following conjecture:

Conjecture 2.1. *The support an exotic crystalline measure cannot be contained in a quasicrystal.*

Lev and Olevskii settled this issue in [10] and proved the following :

Theorem 2.2. *Let μ be a crystalline measure on $\mathbb{R}^n$. Let Λ be the support of μ and S its spectrum. Assume that the set $\Lambda - \Lambda$ is uniformly discrete. Then μ is a generalized Dirac comb.*

Are quasicrystals useless in the construction of crystalline measures ? It is not the case since Lev and Olevskii used a ladder of quasicrystals to circumvent the problem. They proved the existence of exotic crystalline measures in [7]. In their construction the support Λ of the crystalline measure μ is contained in the union $\bigcup_0^\infty \Lambda_j$ of an increasing sequence Λ_j of quasicrystals.

Generalized Dirac combs are "isolated points" inside the collection of crystalline measures. A crystalline measure which is too close from a generalized Dirac comb is a generalized Dirac comb. For example Lev and Olevskii proved the following theorem in [9]

Theorem 2.3. *In dimension 1, if both the support $\Lambda \subset \mathbb{R}$ and the spectrum $S \subset \mathbb{R}$ of a measure μ are uniformly discrete, then μ is a generalized Dirac comb.*

Theorem 2.2 holds in dimension $n \geq 2$ if μ is non negative [9]. The general case (uniformly discrete support and spectrum, μ signed measure) is open. Let us mention another open problem :

Conjecture 2.4. *A non negative crystalline measure is a generalized Dirac comb.*

Definition 2.5. A locally finite set Λ is an admissible single if it is the support of a non trivial crystalline measure μ. Let S be the support of $\widehat{\mu}$. We then say that (Λ, S) is an admissible pair.

Let C be the collection of admissible singles. We have not been able to characterize C by arithmetical properties. For example there exists an increasing sequence $0 < \lambda_1 < \lambda_2 < \cdots$ which is linearly independent over $\mathbb{Q}$ and such that $\Lambda = \{\pm\lambda_j, j \in \mathbb{N}\} \in C$. But there exist increasing sequences $\lambda'_j, j \in \mathbb{N}$, which are also linearly independent over $\mathbb{Q}$ such that $\Lambda' = \{\pm\lambda'_j, j \in \mathbb{N}\} \notin C$. The collection C is stable by finite unions. If $\Lambda \in C$ so is $\Lambda + F$ for every finite set F. Obviously $\mathbb{Z} \in C$ but $\mathbb{Z} \cup \{x_0\} \notin C$ if $x_0 \notin \mathbb{Z}$. Finally C is stable by translations and dilations.

3   Guinand's distribution

Let us begin with Guinand's genuine construction as it can be found in [3]. By Legendre's theorem, an integer $n \geq 0$ can be written as a sum of three squares (0^2 being admitted) if and only if n is not of the form $4^j(8k + 7)$, $j, k \in \mathbb{N}$. For instance 0, 1, 2, 3, 4, 5, 6 are sums of three squares but 7 is not. Let $r_3(n)$ be the number of decompositions of the integer $n \geq 1$ into a sum of three squares (with $r_3(n) = 0$ if n is not a sum of three squares). More precisely $r_3(n)$ is the number of points $k \in \mathbb{Z}^3$ such that $|k|^2 = n$. We have $r_3(4n) = r_3(n)$, $\forall n \in \mathbb{N}$, $r_3(0) = 1$, $r_3(1) = 6, r_3(2) = 12, \ldots$. Then $r_3(2^j) = 6$ if j is even and 12 if j is odd. The behavior of $r_3(n)$ as $n \to \infty$ is erratic. The mean behavior is more regular since E. Landau [6] (pp. 200-218), proved that

$$\sum_{0 \leq n \leq x} r_3(n) = \frac{4}{3}\pi x^{3/2} + O(x^{3/4+\epsilon}) \tag{4}$$

for every positive ϵ. See also E. Grosswald [2]. By an Abel transformation it implies

$$\sum_{0 \le n \le x} r_3(n) n^{-1/2} = 2\pi x + O(x^{1/4+\epsilon}). \tag{5}$$

In what follows this precise estimate is not needed and it suffices to know that the right hand side of (5) has a polynomial growth to conclude that the Guinand's distribution is a tempered distribution. Let B_R be the ball centered at 0 with radius R. Then (4) amounts to

$$\#(\mathbb{Z}^3 \cap B_R) = \frac{4}{3}\pi R^3 + O(R^{3/2+\epsilon}).$$

This estimate of the error term is not optimal and $3/2$ can be reduced to $21/16$ as D. R. Heath-Brown proved in [4].

Guinand began his seminal work [3] with a simple lemma

Lemma 3.1. *For every $a > 0$ we have*

$$1 + \sum_{1}^{\infty} r_3(n) \exp(-\pi n a) =$$

$$a^{-3/2} + a^{-3/2} \sum_{1}^{\infty} r_3(n) \exp(-\pi n/a). \tag{6}$$

The simplest proof consists in writing

$$1 + \sum_{1}^{\infty} r_3(n) \exp(-\pi n a) = \sum_{k \in \mathbb{Z}^3} \exp(-\pi a |k|^2)$$

and applying the standard Poisson formula to the RHS.□

Guinand continued as follows. Let $f_a(x) = x \exp(-\pi a x^2)$, $x \in \mathbb{R}$, $a > 0$. Then $f_a(x)$ is odd and its Fourier transform is

$$\widehat{f_a}(y) = -i a^{-3/2} y \exp(-\pi y^2/a).$$

Now (6) can be written

$$\frac{df_a}{dx}(0) + \sum_{1}^{\infty} r_3(n) n^{-1/2} f_a(\sqrt{n}) =$$

$$i \frac{d\widehat{f_a}}{dx}(0) + i \sum_{1}^{\infty} r_3(n) n^{-1/2} \widehat{f_a}(\sqrt{n}). \tag{7}$$

Guinand introduced the odd distribution $\sigma \in \mathcal{S}'(\mathbb{R})$ defined by

$$\sigma = -2\frac{d}{dx}\delta_0 + \sum_{1}^{\infty} r_3(n) n^{-1/2} (\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}) \tag{8}$$

which will be named Guinand's distribution. We have $\sum_{0}^{N} r_3(n) n^{-1/2} = 2\pi N + O(N^{1/4})$ by (5) which implies that σ is a tempered distribution. Guinand proved the following

Theorem 3.2. *The distributional Fourier transform of σ is $-i\sigma$.*

The proof of Theorem 3.1 is postponed after a few remarks. The oscillating behavior at infinity of Guinand's distribution follows from Theorem 3.1. Since the Fourier transform of $\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}$ is $-2i\sin(2\pi\sqrt{n}x)$, the Fourier transform of the tempered distribution

$$\sigma = -2\frac{d}{dx}\delta_0 + \sum_1^\infty r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$$

is $-i\widetilde{\sigma}$ where

$$\widetilde{\sigma} = 4\pi x + 2\sum_1^\infty r_3(n)n^{-1/2}\sin(2\pi\sqrt{n}x).$$

Then Theorem 3.1 can be written equivalently $\sigma = \widetilde{\sigma}$. This remark will play a seminal role in Section 4. The terminology of signal processing is used in the following corollary. It happens that a signal can be decomposed into the sum between a *trend* and some fluctuation around this trend. A trend indicates the large scale evolution of the signal. An obvious example is given by the stock market.

Corollary 3.3. *Guinand's distribution is the sum of the trend $4\pi x$ and a fluctuation which is an almost periodic distribution. More precisely we have*

$$\sigma(x) = 4\pi x + 2\sum_1^\infty r_3(n)n^{-1/2}\sin(2\pi\sqrt{n}x). \tag{9}$$

Let us observe that σ is not an almost periodic distribution. We recall that a tempered distribution τ is almost periodic if for every test function ϕ in the Schwartz class the convolution product $\tau * \phi$ is an almost periodic function in the sense of Bohr. This definition was proposed by L. Schwartz in [15].

Corollary 3.4. *For every positive x we have*

$$\sum_1^\infty r_3(n)n^{-1/2}\sin(2\pi\sqrt{n}\,x) = 0$$

if $x \neq \sqrt{m}, m \in \mathbb{N}$,

$$\sum_1^\infty r_3(n)n^{-1/2}\sin(2\pi\sqrt{n}\,x) = +\infty$$

if $x = \sqrt{m}, m \in \mathbb{N} \setminus \{0\}$.

An Abel summation is needed to sum these divergent series. Then Corollary 3.2 simply says that a Dirac mass at a can be viewed as a function which vanishes outside a and is $+\infty$ at a.

We return to Theorem 3.1. We need to prove $\langle \sigma, \widehat{\phi} \rangle = -i\langle \sigma, \phi \rangle$ for every test function ϕ. But (7) can be rewritten as $\langle \sigma, f_a \rangle = i\langle \sigma, \widehat{f_a} \rangle$ or $\langle \sigma, f_a \rangle = i\langle \widehat{\sigma}, f_a \rangle$. The collection f_a, $a > 0$, of odd functions is total in the subspace of odd functions of the Schwartz class and σ is a

tempered distribution. By continuity it implies $\langle \sigma, f \rangle = i \langle \widehat{\sigma}, f \rangle$ for every odd function in the Schwartz class. For even functions ϕ the identity $\langle \sigma, \widehat{\phi} \rangle = -i \langle \sigma, \phi \rangle$ is trivial since σ is odd and $\langle \sigma, \widehat{\phi} \rangle = -i \langle \sigma, \phi \rangle = 0$. Every function in the Schwartz class is the sum of an even one and of an odd one which ends the proof. $\square$

This proof was still a copy of Guinand's paper. As in [11] we now move one small step beyond Guinand's work and extract what we call Guinand's measure from Guinand's distribution. Let $\alpha \in (0, 1)$ and set

$$\tau_\alpha(x) = (\alpha^2 + \frac{1}{\alpha}) \sigma(x) - \alpha \sigma(\alpha x) - \sigma(x/\alpha). \tag{10}$$

Then the derivative of the Dirac mass at 0 disappears from this linear combination. On the Fourier transform side

$$\widehat{\tau}_\alpha(y) = (\alpha^2 + \frac{1}{\alpha}) \widehat{\sigma}(y) - \widehat{\sigma}(y/\alpha) - \alpha \widehat{\sigma}(\alpha y) = -i \tau_\alpha(y).$$

Fix $\alpha = 1/2$ in the preceding construction, let $\tau = \tau_{1/2}$ and define $\chi(n) = -1/2$ if $n \in \mathbb{N} \setminus 4\mathbb{N}$, $\chi(n) = 4$ if $n \in 4\mathbb{N} \setminus 16\mathbb{N}$, and $\chi(n) = 0$ if $n \in 16\mathbb{N}$. Then we have [11]

Theorem 3.5. *The Fourier transform of the measure*

$$\tau = \sum_1^\infty \chi(n) r_3(n) n^{-1/2} (\delta_{\sqrt{n}/2} - \delta_{-\sqrt{n}/2}) \tag{11}$$

is $-i\tau$.

A more natural proof of Theorem 3.2 will be given in Section 4. The support of τ is the set $\Lambda = \{\pm \frac{\sqrt{m}}{2}, \ m \neq 4^j(8k+7), \ j, k \in \mathbb{N}\}$. Therefore the density of Λ is infinite and τ is an exotic crystalline measure by Lemma 1.1.

A naive corollary is

Corollary 3.6. *For every positive x we have*

$$\sum_1^\infty \chi(n) r_3(n) n^{-1/2} \sin(\frac{\pi}{2} \sqrt{n} x) = 0$$

if $x \neq \sqrt{m}, m \in \mathbb{N}$, *or* $x = 4\sqrt{m}, m \in \mathbb{N}$, *while*

$$\sum_1^\infty \chi(n) r_3(n) n^{-1/2} \sin(\frac{\pi}{2} \sqrt{n} x) = \infty$$

if $x = \sqrt{m}, m \in \mathbb{N} \setminus 16\mathbb{N}$.

Here again an Abel summation is needed to sum these divergent series. By (5)

$$\sum_1^N r_3(n) n^{-1/2} = 2\pi N + O(N^{1/4}), \ N \to \infty,$$

while $\sum_1^\infty \chi(n) r_3(n) = 0$ after an Abel summation. If χ was erased from (11) τ would no longer be a crystalline measure. The cancellations provided by χ are playing a key role. The measure τ is not an almost periodic measure. A Borel measure μ is almost periodic if for every compactly supported continuous function f the convolution product $g = \mu * f$ is an almost periodic function in the sense of Bohr. An almost periodic measure is translation bounded, which is not the case for τ. Indeed $|\tau|([x, x+1]) \to \infty$, $x \to \infty$. But τ is an almost periodic distribution.

If μ is a crystalline measure and if $\widehat{\mu} = \lambda \mu$ then $\lambda \in \{1, -1, i, -i\}$. Conversely for each of these four eigenvalues there exists a crystalline measure μ such that $\widehat{\mu} = \lambda \mu$. This will be proved in a forthcoming paper.

4   Guinand's distribution and the wave equation

Since the Fourier transform of $\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}$ is $-2i \sin(2\pi \sqrt{n} t)$, the Fourier transform of the tempered distribution

$$\sigma = -2\frac{d}{dt}\delta_0 + \sum_1^\infty r_3(n) n^{-1/2} (\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$$

is $-i\widetilde{\sigma}$ where

$$\widetilde{\sigma} = 4\pi t + 2 \sum_1^\infty r_3(n) n^{-1/2} \sin(2\pi \sqrt{n} t).$$

This was already observed in Section 3. Using the variable t here is intentional. Then Theorem 3.1 can be written equivalently

$$\sigma = \widetilde{\sigma} \tag{12}$$

As it is proved below (12) becomes an obvious geometrical fact if it is translated into the language of the wave equation. Some well known properties of the wave equation on the three dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ which are needed in the proof are summarized in the following lemma [15], [16], [17], [19]:

Lemma 4.1. *Let $E = \mathcal{D}'(\mathbb{T}^3)$ denotes the space of Schwartz distributions on $\mathbb{T}^3$. Then for every $u_1(x) \in E$ there exists a unique solution $u(x,t) \in C^\infty([0,\infty), E)$ of the Cauchy problem*

(i) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta u(x,t)$

(ii) $u(x,0) = 0$, $\frac{\partial}{\partial t} u(x,0) = u_1(x)$.

Moreover $t \mapsto u(x,t)$ extended to $\mathbb{R}$ as an odd function of t belongs to $C^\infty(\mathbb{R}, E)$

Let $u_1(x) = \sum_{k \in \mathbb{Z}^3} \alpha(k) \exp(2\pi i k \cdot x)$ be the Fourier series expansion of $u_1(x)$. Then the solution $u(x,t)$ defined by Lemma 4.1 is given by

$$u(x,t) = \alpha(0)t + \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \alpha(k) \frac{\sin(2\pi t |k|)}{2\pi |k|} \exp(2\pi i k \cdot x). \tag{13}$$

A similar result holds for the wave equation on $\mathbb{R}^3$ where E is replaced by the Schwartz space $\mathcal{S}'$ of tempered distributions on $\mathbb{R}^3$. If we are given a tempered distribution $u_1(x)$ on $\mathbb{R}^3$ there exists a unique solution $u(x,t)$ of the wave equation $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta u(x,t)$ such that $u(x,0) = 0$, $\frac{\partial}{\partial t} u(x,0) = u_1(x)$. It is given by $\widehat{u}(\xi,t) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}\widehat{u_1}(\xi)$. We now introduce Guinand's distribution.

Corollary 4.2. *Let $w(x,t)$ be defined on $\mathbb{T}^3 \times \mathbb{R}$ by*

$$w(x,t) = t + \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\sin(2\pi t|k|)}{2\pi|k|} \exp(2\pi i k \cdot x). \tag{14}$$

Then $w(x,t)$ is the solution to the following Cauchy problem for the wave equation on $\mathbb{T}^3 \times \mathbb{R}$

(i) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta u(x,t)$

(ii) $u(x,0) = 0$, $\frac{\partial}{\partial t} u(x,0) = \delta_0(x)$.

But $w(x,t)$ can also be computed by periodizing the solution of the same Cauchy problem on $\mathbb{R}^3 \times \mathbb{R}$. This scheme is detailed now.

Lemma 4.3. *Let σ_t, $t \in \mathbb{R}$, be the normalized surface measure on the sphere $B_t \subset \mathbb{R}^3$ centered at 0 with radius $|t|$ (the total mass of σ_t is 1). Then $v(x,t) = t\sigma_t(x)$ belongs to $C^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^3))$ and is the solution of the Cauchy problem*

(i) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta u(x,t)$

(ii) $u(x,0) = 0$, $\frac{\partial}{\partial t} u(x,0) = \delta_0(x)$.

The solution of the wave equation with this peculiar initial data is called by some authors as "Riemann function" (see e.g. Strauss, [17] p. 322). The reason is that, at least in the one-dimensional case, the spatial primitive of this function solves the famous Riemann problem posed in gas dynamics (for many references on this problem see e.g. the monograph Toro [18]).

Corollary 4.4. *Under the assumptions of Lemma 4.2 one has that*

$$w(x,t) = \sum_{k \in \mathbb{Z}^3} t\sigma_t(x - k) \tag{15}$$

is the solution of the following Cauchy problem for the wave equation on the three dimensional torus:

(a) $w(x,0) = 0$

(b) $\frac{\partial}{\partial t} w(x,0) = \delta_0(x)$.

The two expansions of $w(x,t)$ given by (14) and (15) are equal and this is the main step to the proof of Guinand's theorem.

Lemma 4.5. *With the preceding notations we have*

$$w(x,t) = \sum_{k \in \mathbb{Z}^3} t\sigma_t(x-k) = t + \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\sin(2\pi t|k|)}{2\pi|k|} \exp(2\pi ik \cdot x). \tag{16}$$

This identity holds in $C^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^3))$. It can be proved directly. If μ is any compactly supported Borel measure on $\mathbb{R}^3$, the standard Poisson summation formula yields

$$\sum_{k \in \mathbb{Z}^3} d\mu(x-k) = \sum_{k \in \mathbb{Z}^3} \widehat{\mu}(k) \exp(2\pi ik \cdot x) \tag{17}$$

which implies (16) immediately when $\mu = \sigma_t$. The detour by the wave equation was only aimed at showing that Guinand's distribution is a natural mathematical object.

Let us compute the trace on $x = x_0$ of the LHS and RHS of (16) as a function of t. This trace is defined as follows.

Definition 4.6. A distribution $u(x,t) \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ defines a continuous mapping from $\mathbb{R}^3$ to $\mathcal{S}'(\mathbb{R})$ if for every test function $\phi \in \mathcal{S}(\mathbb{R})$, the distribution $< u(x, \cdot), \phi(\cdot) >$ is a continuous function of $x \in \mathbb{R}^3$.

The RHS of (16) fulfills this requirement since $\widehat{\phi}(|k|)$ is rapidly decreasing for $\phi \in \mathcal{S}(\mathbb{R})$. Therefore the trace $w(x_0, t)$ exists for every $x_0 \in \mathbb{R}^3$ and belongs to $\mathcal{S}'(\mathbb{R})$. For computing the trace of the LHS of (16) one uses the following observation:

Lemma 4.7. *For every $x_0 \in \mathbb{R}^3 \setminus \{0\}$, the trace on $x = x_0$ of the tempered distribution $t\sigma_t(\cdot) \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ is $\frac{1}{4\pi|x_0|}(\delta_{|x_0|} - \delta_{-|x_0|})$.*

This elementary fact follows form a simple calculation if one observes that $t\sigma_t(\cdot)$ is odd in t. Lemma 4.4 implies the following

Lemma 4.8. *If $x_0 \notin \mathbb{Z}^3$ the trace of $\sum_{k \in \mathbb{Z}^3} t\sigma_t(x-k)$ is $\sum_{k \in \mathbb{Z}^3} \frac{1}{4\pi|x_0-k|}(\delta_{|x_0-k|} - \delta_{-|x_0-k|})$.*

We can conclude:

Proposition 4.9. *Let $x_0 \notin \mathbb{Z}^3$. Then we have*

$$\sum_{k \in \mathbb{Z}^3} \frac{1}{|x_0-k|}(\delta_{|x_0-k|} - \delta_{-|x_0-k|}) = 4\pi t + 2 \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\sin(2\pi|k|t)}{|k|} \exp(2\pi ik \cdot x_0) \tag{18}$$

and theses two series converge in $\mathcal{S}'(\mathbb{R})$.

This identity does not make sense if $x_0 = 0$ which is needed for recovering Theorem 3.1. As it will be seen the divergence which occurs is responsible for the derivative of the Dirac mass in the definition of σ. To settle this problem it suffices to observe that the distribution $\sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\sin(2\pi|k|t)}{|k|} \exp(2\pi ik \cdot x)$ is continuous on $\mathbb{R}^3$. We then compute $w(0, t)$ in (16) as $\lim_{x \to 0, x \neq 0} w(x, t)$. Then $\frac{1}{|x_0|}(\delta_{|x_0|} - \delta_{-|x_0|}) \to -2\frac{d}{dt}\delta_0$ as $x_0 \to 0$ which yields a new proof of Theorem 3.1. $\square$

Using the same detour by the wave equation Theorem 3.2 can be viewed as a particular case of a more general fact. The notations are the same as above.

Theorem 4.10. *Let ν a real, finitely supported measure on $\mathbb{T}^3$ such that*

(a) 0 *does not belong to the support of ν*

(b) $\int_{\mathbb{T}^3} d\nu = 0$.

Let $u : \mathbb{T}^3 \times \mathbb{R} \mapsto \mathbb{R}$ be the solution of the Cauchy problem

(i) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta u(x,t)$

(ii) $u(x,0) = 0$, $\frac{\partial}{\partial t} u(x,0) = \nu$.

Then $t \mapsto u(0,t)$ is a crystalline measure.

The proof is identical to the one given above. Theorem 3.2 is now a direct corollary. It suffices to define ν by the following four conditions: ν is supported by $\{k/4, k \in \mathbb{Z}^3\}$, ν does not charge $\mathbb{Z}$, the mass of ν on each $k + 1/2$ is $1/2$, and the charge of ν on each $k/2 + 1/4$ is $-1/16$.

The lattice $\mathbb{Z}^3$ is now replaced by an arbitrary lattice $\Gamma \subset \mathbb{R}^3$ and the proof of Theorem 4.1 yields the following result:

Theorem 4.11. *Let $\Gamma \subset \mathbb{R}^3$ be a lattice. Let ν be a finitely supported measure on $V = \mathbb{R}^3/\Gamma$ such that $\int_V d\nu = 0$. Let us assume that 0 does not belong to the support of ν. Let $u : V \times \mathbb{R} \mapsto \mathbb{R}$ be the solution of the Cauchy problem*

(i) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta u(x,t)$

(ii) $u(x,0) = 0$, $\frac{\partial}{\partial t} u(x,0) = \nu$.

Then $t \mapsto u(0,t)$ is a crystalline measure.

Indeed

$$u(x,t) = \sum_{\gamma^* \in \Gamma^*} \widehat{\nu}(\gamma^*) \frac{\sin(2\pi t |\gamma^*|)}{2\pi |\gamma^*|} \exp(2\pi i x \cdot \gamma^*) \tag{19}$$

and we also have as above

$$u(x,t) = \sum_{\gamma \in \Gamma} (t \sigma_t * \nu)(x - \gamma). \tag{20}$$

By (20) $u(0,t)$ is an atomic measure and by (19) $u(0,t)$ is the Fourier transform of the atomic measure

$$\mu = \sum_{\gamma^* \in \Gamma^*} \frac{i \widehat{\nu}(\gamma^*)}{4\pi |\gamma^*|} (\delta_{|\gamma^*|} - \delta_{-|\gamma^*|}). \tag{21}$$

Let F be the support of ν. Then the support of the crystalline measure μ is the set $\Lambda = \{\pm |\gamma^*|, \gamma^* \in \Gamma^*, \gamma^* \neq 0\}$ and its spectrum is the set $S = \{\pm |x + \gamma|, \gamma \in \Gamma, x \in F\}$.

5   Concluding remarks

The detour by the wave equation on the three dimensional torus provided us with a remarkable understanding of Guinand's distribution and Guinand's measure. Let us observe that the crystalline measures of Theorem 4.2 are odd measures. However there exist many more odd crystalline measures than those described by Theorem 4.2. For example if α in (10) is irrational the corresponding crystalline measure τ_α cannot be described by Theorem 4.2. On the other hand Guinand proposed some examples of even crystalline measures in [3] without giving satisfactory proofs. These proofs were completed in [12]. Finally an important family of even crystalline measures was constructed by D. Radchenko and M. Viazovska in [14]. They proved the following theorem:

Theorem 5.1. *For every real number $\theta > 0$ and for a suitable choice of a_n, $n \in \mathbb{N}$, the series*

$$\sum_0^\infty a_n(\delta_{\sqrt{n}} + \delta_{-\sqrt{n}})$$

converges to a crystalline measure μ_θ whose Fourier transform is

$$\widehat{\mu_\theta} = \delta_\theta + \delta_{-\theta} + \sum_0^\infty b_n(\delta_{\sqrt{n}} + \delta_{-\sqrt{n}}).$$

The authors are indebted to Kristian Seip for pointing out this reference. Crystalline measures are still mysterious mathematical objects.

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