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Existence of Solutions of Some Nonlinear ϕ-Laplacian Equations with Neumann-Steklov Nonlinear Boundary Conditions

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Abstract

We study the existence of solutions of the quasilinear equation

$$(D(u(t))\phi(u'(t)))' = f(t,u(t),u'(t)), \qquad a.e.\ t \in [0,T],$$

submitted to nonlinear Neumann-Steklov boundary conditions on $[0,T]$, where $\phi :]-a,a[\to \mathbb{R}$, $(0 < a < +\infty)$ is an increasing homeomorphism such that $\phi(0) = 0$, $f : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ a L^1-Carathéodory function, $D : \mathbb{R} \longrightarrow]0,+\infty[$ is a continuous function. Using topological methods, we obtain existence and multiplicity results.

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1 Introduction

This work is devoted to the study of the existence of solutions of the following ϕ-Laplacian boundary value problem:

$$\begin{cases} (D(u(t))\phi(u'(t)))' = f(t,u(t),u'(t)), & a.e.\ t \in [0,T], \\ \phi(u'(0)) = g_0(u(0)), \qquad \phi(u'(T)) = g_T(u(T)), \end{cases} \tag{1.1}$$

where ϕ: $]-a,a[\to \mathbb{R}$, $(0 < a < +\infty)$ is an increasing homeomorphism such that $\phi(0) = 0$, $f : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function, $D : \mathbb{R} \longrightarrow]0,+\infty[$, g_0, g_T: $\mathbb{R} \longrightarrow \mathbb{R}$ are continuous.

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The boundary conditions in (1.1) are Neumann-Steklov nonlinear boundary conditions.
In 2011 Giovanni Cupini, Cristina Marcelli and Francesca Papalini studied in [6] the problem

$$\begin{cases} (a(u(t))\phi(u'(t)))' = f(t,u(t),u'(t)), & a.e.\ t \in \mathbb{R}, \\ u(-\infty) = \mu_1, \qquad u(+\infty) = \mu_2. \end{cases} \tag{1.2}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a positive continuous function, $f : \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function and $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$.
In 2008, Cristian Bereanu and Jean Mawhin [3] proved, for $D(x) = 1$ and f continuous, existence and multiplicity results for problem (1.1). They proved, under some conditions upon f, g_0 and g_T, an Ambrosetti-Prodi type multiplicity result.

In the following results, for $D : \mathbb{R} \longrightarrow]0,+\infty[$, we give some additional information concerning the location of the solution of (1.1) when the lower and upper-solutions are ordered or not. Moreover, in our multiplicity results, we need only one strict lower-solution and one strict upper-solution. Generally, in the lower and upper-solutions method, to show existence of at least one solution of a problem, we need existence of one lower-solution and one upper-solution, for that, we also give some ways for the construction of lower and upper-solutions of (1.1). These ways are new.

After introducing notations and preliminaries results in section 2, we study in section 3 the existence of at least one solution of (1.1) when the upper-solution and the lower-solution are in well order.
In section 4, we study the existence and location of at least one solution of (1.1) when the upper-solution and the lower-solution are not in well order.
In section 5, using results of sections 3 and 4, we prove the existence and location of at least one solution of (1.1) when the upper-solution and the lower-solution are ordered or not.
In section 6, we prove the existence of multiple solutions of (1.1) by the lower and upper-solutions method.
Finally in section 7, we give some new types of construction of lower and upper-solutions.

2 Notations and preliminaries

We denote:
∘ $C = C([0,T])$, the Banach space of continuous functions on $[0,T]$;
∘ $\|u\|_C = \|u\|_\infty = \max\{|u(t)|; t \in [0,T]\}$, the norm of C;
∘ $C^1 = C^1([0,T])$, the Banach space of continuous functions on $[0,T]$, having continuous first derivative on $[0,T]$;
∘ $\|u\|_{C^1} = \|u\|_C + \|u'\|_C$, the norm of C^1;
∘ $AC = AC([0,T])$, the set of absolutely continuous functions on $[0,T]$;
∘ $L^1 = L^1(0,T)$, the Banach space of functions Lebesgue integrable on $[0,T]$;
∘ $\|x\|_{L^1} = \int_0^T |x(t)|dt$, the norm of L^1;
∘ B_r, the corresponding open ball of C^1 of center 0 and radius r;
∘ $\overline{B_r}$, the corresponding close ball of C^1 of center 0 and radius r;
∘ d_{LS}, the Leray-Schauder degree and d_B the Brouwer degree;
∘ ∂A, the boundary of the bounded set A.

We introduce:
∘ the continuous operator $P : C^1 \to C^1$ defined by

$$P(u) = u(T) - u'(T) + \phi^{-1}(g_T(u(T)));$$

∘ the continuous operator $G : C \to C$ defined by

$$G(u) = D(u(0))g_0(u(0));$$

∘ the continuous operators $Q, H_1, H_2 : L^1 \to C$ defined by

$$Q(u) = Qu = \frac{1}{T}\int_0^T u(s)ds,\ \ H_1(u)(t) = \int_0^t u(s)ds \ \text{ and } \ H_2(u)(t) = \int_t^T u(s)ds,\ \ \forall t \in [0,T].$$

Definition 2.1. $f : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ is L^1-Carathéodory if:

(i) $f(.,x,y) : [0,T] \to \mathbb{R}$ is measurable for all $(x,y) \in \mathbb{R}^2$;

(ii) $f(t,.,.) : \mathbb{R}^2 \to \mathbb{R}$ is continuous for a.e. $t \in [0,T]$;

(iii) For each compact set $K \subset \mathbb{R}^2$, there is a function $\mu_K \in L^1$ such that $|f(t,x,y)| \le \mu_K(t)$ for a.e. $t \in [0,T]$ and all $(x,y) \in K$.

Definition 2.2. A solution of problem (1.1) is a function $u \in C^1$ such that $\|u'\|_\infty < a$, $(D \circ u).(\phi \circ u') \in AC$ and satisfies (1.1).

3 Existence of solutions with well ordered upper and lower-solutions

3.1 Equivalent Fixed point problem

Consider $N_f : C^1 \to L^1$ defined by $N_f(u) = f(.,u(.),u'(.))$, the Nemytskii operator associated to f. It is standard to show that N_f is continuous and sends bounded sets into bounded sets. We construct the associated fixed point operator, following the approach in [[3] and [4]].

Proposition 3.1. *$u \in C^1$ is a solution of the problem (1.1) if and only if u is a fixed point of the operator Θ defined on C^1 by*

$$\Theta(u) = P(u) - H_2\Big[\phi^{-1} \circ [(D_1 \circ u)[G(u) + H_1(N_f(u))]]\Big]. \tag{3.1}$$

where D_1 is given by $\forall x \in \mathbb{R}$, $D_1(x) = \frac{1}{D(x)}$.
Furthermore, $\forall u \in C^1$, $\|\,(\Theta(u))'\,\|_\infty < a$ and Θ is completely continuous on C^1 if

$$0 < m \le D(u(t)) \le M, \text{ for every } u \in C^1 \text{ and } t \in [0,T] \tag{3.2}$$

for some constants m, M.

Proof. Problem (1.1) is equivalent to

$$\begin{cases} u'(t) = \phi^{-1}[(D_1 \circ u)(t)[H_1(N_f(u))(t) + D(u(0))g_0(u(0))]], \\ \phi(u'(T)) = g_T(u(T)). \end{cases}$$

Assume that $u \in C^1$ is a solution of the problem (1.1); then

$$u'(T) = \phi^{-1}(g_T(u(T))).$$

Hence

$$\begin{aligned} u(t) &= u(T) - \int_t^T \phi^{-1}[(D_1 \circ u)(s)[D(u(0))g_0(u(0)) + \int_0^s N_f(u)(x)dx]]ds \\ &= u(T) - u'(T) + \phi^{-1}(g_T(u(T))) - \int_t^T \phi^{-1}[(D_1 \circ u)(s)[D(u(0))g_0(u(0)) + \\ &\quad \int_0^s N_f(u)(x)dx]]ds \\ &= (\Theta(u))(t). \end{aligned}$$

Assume that $u \in C^1$ is such that

$$u = \Theta(u) = P(u) - H_2\Big[\phi^{-1}[(D_1 \circ u)[G(u) + H_1(N_f(u))]]\Big], \tag{3.3}$$

then,

$$u(T) = u(T) - u'(T) + \phi^{-1}(g_T(u(T)))$$

and

$$u'(t) = \phi^{-1}[(D_1 \circ u)(t)[H_1(N_f(u))(t) + D(u(0))g_0(u(0))]].$$

Hence

$$\phi(u'(0)) = g_0(u(0)), \qquad \phi(u'(T)) = g_T(u(T)) \qquad \text{and} \qquad ((D \circ u)\phi(u'))' = N_f(u).$$

It follows that u is a solution of the problem (1.1).
Using Arzelá-Ascoli's Theorem, it is not difficult to see that Θ is completely continuous. As

$$\forall u \in C^1, \quad (\Theta(u))' = \phi^{-1} \circ [(D_1 \circ u)[G(u) + H_1(N_f(u))]],$$

we have

$$\forall u \in C^1, \quad \| (\Theta(u))' \|_\infty < a.$$

3.2 Existence of solutions

Definition 3.2. A function $\alpha \in C^1$ is a lower-solution of the problem (1.1) if $(D \circ \alpha).(\phi \circ \alpha') \in AC$, $\|\alpha'\|_\infty < a$,

$$\begin{cases} (D(\alpha(t))\phi(\alpha'(t)))' \geq f(t, \alpha(t), \alpha'(t)), & a.e.\ t \in [0, T], \\ \phi(\alpha'(0)) \geq g_0(\alpha(0)) \quad \text{and} \quad \phi(\alpha'(T)) \leq g_T(\alpha(T)). \end{cases} \tag{3.4}$$

Definition 3.3. A function $\beta \in C^1$ is an upper-solution of the problem (1.1) if $(D\circ\beta).(\phi\circ\beta') \in AC$, $\|\beta'\|_\infty < a$,

$$\begin{cases} (D(\beta(t))\phi(\beta'(t)))' \leq f(t,\beta(t),\beta'(t)), & a.e.\ t\in[0,T], \\ \phi(\beta'(0)) \leq g_0(\beta(0)) \quad \text{and} & \phi(\beta'(T)) \geq g_T(\beta(T)). \end{cases} \tag{3.5}$$

Definition 3.4. A lower-solution α of (1.1) is said to be strict if every solution u of (1.1) with $u(t)\geq\alpha(t)$ on $[0,T]$ is such that $u(t)>\alpha(t)$ on $[0,T]$.

Definition 3.5. An upper-solution β of (1.1) is said to be strict if every solution u of (1.1) with $u(t)\leq\beta(t)$ on $[0,T]$ is such that $u(t)<\beta(t)$ on $[0,T]$.

Proposition 3.6. *Let α be a lower-solution of (1.1) such that:*
(i) $\forall t_0 \in]0,T[$, there exist $\varepsilon_0>0$ and I_0 an open interval such that $t_0\in I_0$, and

$$(D(x)\phi(\alpha'(t)))' \geq f(t,x,y)\, \text{for a.e. } t\in I_0,$$

$$\forall\ (x,y)\in[\alpha(t),\alpha(t)+\varepsilon_0]\times[\alpha'(t)-\varepsilon_0,\alpha'(t)+\varepsilon_0];$$

(ii) $\phi(\alpha'(0)) > g_0(\alpha(0))$;
(iii) $\phi(\alpha'(T)) < g_T(\alpha(T))$.
Then α is a strict lower-solution.

Proof. Let u be a solution of (1.1) such that $\alpha(t)\leq u(t)$, $\forall t\in[0,T]$. As α is not a solution, α is not identical to u.
Suppose by contradiction that there exists $\widetilde{t}\in[0,T]$ such that $\alpha(\widetilde{t}) = u(\widetilde{t})$, hence

$$A = \{t\in[0,T];\ \alpha(t)=u(t)\} \neq \emptyset$$

and A is closed. Let $t_0 = \min A$, hence

$$\min_{t\in[0,T]}[u(t)-\alpha(t)] = u(t_0)-\alpha(t_0) = 0.$$

(I) Assume that $t_0\in]0,T[$, then we have $u'(t_0)-\alpha'(t_0)=0$ and there exists I_0 and $\varepsilon_0>0$ according to assumption (i). It follows that we can choose $t_1\in I_0$ such that $t_1<t_0$, $u'(t_1)<\alpha'(t_1)$, and

$$\forall t\in[t_1,t_0],\ (u(t),u'(t))\in]\alpha(t),\alpha(t)+\varepsilon_0[\times]\alpha'(t)-\varepsilon_0,\alpha'(t)+\varepsilon_0[.$$

Hence, for almost every $t\in[t_1,t_0]$,

$$(D(u(t))\phi(\alpha'(t)))' - f(t,u(t),u'(t)) \geq 0.$$

As ϕ is an increasing homeomorphism,

$$\phi(u'(t_0)) - \phi(\alpha'(t_0)) = 0$$

and

$$\phi(u'(t_1)) < \phi(\alpha'(t_1)). \tag{3.6}$$

We also have

$$\begin{aligned}
&-(D(u(t_1))\phi(u'(t_1)) - D(u(t_1))\phi(\alpha'(t_1))) \\
&= -\int_{t_1}^{t_0}[(D(u(s))\phi(\alpha'(s)))' - (D(u(s))\phi(u'(s)))']ds \\
&= -\int_{t_1}^{t_0}[(D(u(s))\phi(\alpha'(s)))' - f(s,u(s),u'(s))]ds \le 0,
\end{aligned}$$

which contradicts (3.6).

(II) Assume that $t_0 = 0$; then $\alpha'(0) \le u'(0)$, so that;

$$\phi(u'(0)) - \phi(\alpha'(0)) \ge 0. \tag{3.7}$$

As $\phi(\alpha'(0)) > g_0(\alpha(0))$, using (3.7) we have the contradiction

$$0 \le \phi(u'(0)) - \phi(\alpha'(0)) = g_0(u(0)) - \phi(\alpha'(0)) = g_0(\alpha(0)) - \phi(\alpha'(0)) < 0.$$

(III) Assume that $t_0 = T$; then $(u-\alpha)'(T) \le 0$, so that;

$$\phi(u'(T)) - \phi(\alpha'(T)) \le 0. \tag{3.8}$$

As $\phi(\alpha'(T)) < g_T(\alpha(T))$, using (3.8) we have the contradiction

$$0 \ge \phi(u'(T)) - \phi(\alpha'(T)) = g_T(u(T)) - \phi(\alpha'(T)) = g_T(\alpha(T)) - \phi(\alpha'(T)) > 0.$$

Proposition 3.7. *Let β be an upper-solution of (1.1) such that:*
(i) $\forall t_0 \in]0,T[$, there exist $\varepsilon_0 > 0$ and I_0 an open interval such that $t_0 \in I_0$, and

$$(D(x)\phi(\beta'(t)))' \le f(t,x,y) \text{ for a.e. } t \in I_0,$$

$$\forall\ (x,y) \in [\beta(t)-\varepsilon_0, \beta(t)] \times [\beta'(t)-\varepsilon_0, \beta'(t)+\varepsilon_0];$$

(ii) $\phi(\beta'(0)) < g_0(\beta(0))$;
(iii) $\phi(\beta'(T)) > g_T(\beta(T))$.
Then β is a strict upper-solution.

Proof. The proof is similar to the proof of Proposition 3.6.

We introduce the following Lemma (See [7], Lemma 6.3 and Corollary 6.4). It is fundamental for the proof of the following theorem. For $\alpha \in C^1$ and $\beta \in C^1$ such that

$$\alpha(t) \le \beta(t), \qquad \forall t \in [0,T],$$

we can define a function $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ by

$$\gamma(t,x) = \max\{\alpha(t), \min\{x, \beta(t)\}\}.$$

Lemma 3.8. *For $u \in C^1$ the three following properties are true.*

a) *$\frac{d}{dt}\gamma(t,u(t))$ exists for a.e. $t \in [0,T]$.*

b) $\frac{d}{dt}\gamma(t,u(t)) = \begin{cases} \alpha'(t) & if\ u(t) < \alpha(t) \\ u'(t) & if\ \alpha(t) \leq u(t) \leq \beta(t) \\ \beta'(t) & if\ \beta(t) < u(t) \end{cases}$.

c) *if* $(u_n)_n \subset C^1$ *is such that* $u_n \to u$ *in* C^1*. Then* $\gamma(.,u_n) \to \gamma(.,u)$ *in* C *and for almost every* $t \in [0,T]$, $\lim_{n\to+\infty} \frac{d}{dt}\gamma(t,u_n(t)) = \frac{d}{dt}\gamma(t,u(t))$.

Theorem 3.9. *Assume that there exist a lower-solution* α *and an upper-solution* β *of (1.1) such that* $\forall t \in [0,T]$, $\alpha(t) \leq \beta(t)$*. Then the problem (1.1) admits at least one solution u, such that*

$$\alpha(t) \leq u(t) \leq \beta(t), \qquad \forall t \in [0,T].$$

Moreover, if α *and* β *are strict, then*

$$\alpha(t) < u(t) < \beta(t),\ \forall t \in [0,T], \quad and \quad d_{LS}[I-\Theta, \Omega_{\alpha,\beta}, 0] = 1,$$

where

$$\Omega_{\alpha,\beta} = \{u \in C^1;\ \forall t \in [0,T],\ \alpha(t) < u(t) < \beta(t),\ \| u' \|_\infty < a\},$$

Θ *is the fixed point operator associated to (1.1).*

In order to prove the Theorem 3.9, we consider the auxiliary boundary value problem

$$\begin{cases} (D(\gamma(t,u(t)))\phi(u'(t)))' = F(u)(t), & a.e.\ \ t \in [0,T], \\ \phi(u'(0)) = g_0(\gamma(0,u(0))), \\ u(T) = \gamma(T,u(T)) - \delta(u'(T)) + \phi^{-1}(g_T(\gamma(T,u(T)))), \end{cases} \tag{3.9}$$

where $\delta : \mathbb{R} \to \mathbb{R}$ is defined by

$$\delta(x) = \max\{-a, \min\{x,a\}\}.$$

and $F : C^1 \to L^1$ given by, $\forall u \in C^1$ and for a.e. $t \in [0,T]$,

$$F(u)(t) = f(t, \gamma(t,u(t)), \delta(\frac{d}{dt}\gamma(t,u(t)))) + \arctan(u(t) - \gamma(t,u(t))).$$

A solution of problem (3.9) is a function $u \in C^1$ such that $\|u'\|_\infty < a$, $(D \circ u).(\phi \circ u') \in AC$ and satisfies (3.9).

Consider the operators $P_1 : C^1 \to C^1$ and $G_1 : C^1 \to C^1$ given by, $\forall u \in C^1$,

$$P_1(u) = \gamma(T,u(T)) - \delta(u'(T)) + \phi^{-1}(g_T(\gamma(T,u(T))))$$

and

$$G_1(u) = D(\gamma(0,u(0)))g_0(\gamma(0,u(0))).$$

We show that the problem (3.9) is equivalent to the fixed point problem $u = \widetilde{\Theta}(u)$ where $\widetilde{\Theta} : C^1 \to C^1$ is defined by

$$\widetilde{\Theta}(u)(t) = P_1(u) - \int_t^T \phi^{-1}\Big[D_1(\gamma(s,u(s)))[G_1(u) + \int_0^s F(u)(x)dx]\Big]ds, \quad \forall t \in [0,T].$$

By Arzelá-Ascoli Theorem, $\widetilde{\Theta}$ is completely continuous. We can see that for all u in C^1, we have

$$||\widetilde{\Theta}(u)||_{C^1} < \max\{|\alpha_L|, |\beta_M|\} + 3a + aT.$$

Lemma 3.10. *Any solution u of (3.9) is such that $\alpha(t) \le u(t) \le \beta(t)$, $\quad \forall t \in [0,T]$*

Proof. We limit ourselves to prove that $\alpha(t) \le u(t)$ for every $t \in [0,T]$; the proof of the other inequality $u(t) \le \beta(t)$ for every $t \in [0,T]$ is similar. Let us assume on the contrary that, for some $t_0 \in [0,T]$, $\max\limits_{t\in[0,T]}[\alpha(t) - u(t)] = \alpha(t_0) - u(t_0) > 0$.

If $t_0 \in]0,T[$ then $\alpha'(t_0) = u'(t_0)$. Thus $\phi(\alpha'(t_0)) = \phi(u'(t_0))$. We can find $\omega > 0$ such that $\forall t \in]t_0, t_0+\omega[$, $\alpha(t) > u(t)$. We have, $\forall t \in]t_0, t_0+\omega[$, $\gamma(t,u(t)) = \alpha(t)$ and $\delta(\frac{d}{dt}\gamma(t,u(t))) = \alpha'(t)$ for a.e. $t \in]t_0, t_0+\omega[$. It follows that $\forall t \in]t_0, t_0+\omega[$,

$$\begin{aligned} D(\alpha(t))(\phi(\alpha'(t)) - \phi(u'(t))) &= \int_{t_0}^{t} [(D(\alpha(s))[\phi(\alpha'(s)) - \phi(u'(s))]]' ds \\ &= \int_{t_0}^{t} [(D(\alpha(s))\phi(\alpha'(s)))' - (D(\gamma(s,u(s)))\phi(u'(s)))']ds \\ &= \int_{t_0}^{t} [(D(\alpha(s))\phi(\alpha'(s)))' - f(s,\alpha(s),\alpha'(s)) + \\ &\quad \arctan(\alpha(s) - u(s))]ds \\ &\ge \int_{t_0}^{t} \arctan(\alpha(s) - u(s))ds > 0. \end{aligned}$$

As $\forall x \in \mathbb{R}$, $D(x) > 0$ and ϕ is an increasing homeomorphism, we obtain

$$D(\alpha(t))(\phi(\alpha'(t)) - \phi(u'(t))) > 0 \Rightarrow \phi(\alpha'(t)) - \phi(u'(t)) > 0 \Rightarrow \alpha'(t) - u'(t) > 0,$$

a contradiction.

If $t_0 = 0$ then $\alpha'(0) \le u'(0)$, hence $\phi(\alpha'(0)) \le \phi(u'(0))$. Moreover, as $\gamma(0,u(0)) = \alpha(0)$ we deduce, using (3.4), that

$$\phi(\alpha'(0)) - \phi(u'(0)) = \phi(\alpha'(0)) - g_0(\alpha(0)) \ge 0.$$

We have $\phi(\alpha'(0)) \le \phi(u'(0))$ and $\phi(\alpha'(0)) - \phi(u'(0)) \ge 0$ hence

$$\phi(\alpha'(0)) - \phi(u'(0)) = 0.$$

We can find $\omega > 0$ such that $\forall t \in]0,\omega[$, $\alpha(t) > u(t)$. We have $\forall t \in]0,\omega[$, $\gamma(t,u(t)) = \alpha(t)$ and $\delta(\frac{d}{dt}\gamma(t,u(t))) = \alpha'(t)$ for a.e. $t \in]0,\omega[$. It follows that $\forall t \in]0,\omega[$,

$$\begin{aligned} D(\alpha(t))(\phi(\alpha'(t)) - \phi(u'(t))) &= \int_{0}^{t} [(D(\alpha(s))[\phi(\alpha'(s)) - \phi(u'(s))]]' ds \\ &= \int_{0}^{t} [(D(\alpha(s))\phi(\alpha'(s)))' - (D(\gamma(s,u(s)))\phi(u'(s)))']ds \\ &= \int_{0}^{t} [(D(\alpha(s))\phi(\alpha'(s)))' - f(s,\alpha(s),\alpha'(s)) + \\ &\quad \arctan(\alpha(s) - u(s))]ds \\ &\ge \int_{0}^{t} \arctan(\alpha(s) - u(s))ds > 0. \end{aligned}$$

As $\forall x \in \mathbb{R}$, $D(x) > 0$ and ϕ is an increasing homeomorphism, we obtain

$$D(\alpha(t))(\phi(\alpha'(t)) - \phi(u'(t))) > 0 \Rightarrow \phi(\alpha'(t)) - \phi(u'(t)) > 0 \Rightarrow \alpha'(t) - u'(t) > 0,$$

a contradiction.

If $t_0 = T$ then $\alpha(T) - u(T) > 0$ and $\alpha'(T) \geq u'(T)$.
As $\alpha(T) - u(T) > 0$, we have $\gamma(T, u(T)) = \alpha(T)$. As $\alpha'(T) \geq u'(T)$, we have $\delta(\alpha'(T)) \geq \delta(u'(T))$. Therefore we obtain the contradiction

$$\begin{aligned} 0 < \alpha(T) - u(T) &= \alpha(T) - \alpha(T) + \delta(u'(T)) - \phi^{-1}(g_T(\alpha(T))); \\ &= \delta(u'(T)) - \phi^{-1}(g_T(\alpha(T))); \\ &\leq \delta(\alpha'(T)) - \phi^{-1}(g_T(\alpha(T))); \\ &\leq \alpha'(T) - \phi^{-1}(g_T(\alpha(T))) \leq 0. \end{aligned}$$

Proof of Theorem 3.9. $\widetilde{\Theta}$ is completely continuous and $\widetilde{\Theta}(C^1) \subset B_\lambda$ for $\lambda > \max\{|\alpha_L|, |\beta_M|\} + 3a + aT$. Hence we have, by a straightforward application of Schauder Theorem, $\widetilde{\Theta}$ has a fixed point U which is a solution of (3.9). Therefore, using Lemma 3.10, U is also a solution of (1.1).
Assume that α and β are strict lower and upper-solutions of (1.1).
Let

$$\lambda > \max\{|\alpha_L|, |\beta_M|\} + 3a + aT$$

large enough such that

$$\widetilde{\Theta}(v) \neq v \quad \text{for any} \quad v \in \partial\overline{B_\lambda}.$$

Since $\widetilde{\Theta}$ is completely continuous, we can calculate the topological degree of $I - \widetilde{\Theta}$. The function H define by $H(t, v) = t\widetilde{\Theta}(v)$ is continuous and compact on $[0,1] \times \overline{B_\lambda}$. If for some $t \in [0,1]$ and $v \in \partial\overline{B_\lambda}$ we have $v - H(t, v) = 0$, then $t\widetilde{\Theta}(v) = v$. As $\|v\|_{C^1} = \lambda$ and $\|\widetilde{\Theta}(v)\|_{C^1} \leq \lambda$ for any $v \in \partial\overline{B_\lambda}$, this imposes $t = 1$ and $v = \widetilde{\Theta}(v)$, therefore the presence of a fixed point of $\widetilde{\Theta}$ on $\partial\overline{B_\lambda}$, situation which we excluded. We can thus apply the homotopy invariance properties of Leray-Schauder degree, to obtain

$$d_{LS}(I - \widetilde{\Theta}, B_\lambda, 0) = d_{LS}(I, B_\lambda, 0) = 1.$$

By the definition of strict lower or upper-solution, neither α nor β can be a solution of (3.9). Hence (3.9) has no solution on the boundary of $\Omega_{\alpha,\beta}$. Moreover, using the additivity-excision property of the Leray-Schauder degree (see [10]), we have

$$d_{LS}[I - \widetilde{\Theta}, \Omega_{\alpha,\beta}, 0] = d_{LS}[I - \widetilde{\Theta}, B_\lambda, 0] = 1.$$

On the other hand, as the completely continuous operator Θ associated to (1.1) is equal to $\widetilde{\Theta}$ on $\Omega_{\alpha,\beta}$, we deduce that

$$d_{LS}[I - \Theta, \Omega_{\alpha,\beta}, 0] = 1.$$

4 Existence of solutions with non ordered upper and lower-solutions

In the following theorem, we give some additional information concerning the location of the solution when the lower and upper-solutions are not ordered. The proof of the following theorem is similar to the proof of [Theorem 8.10 in [11]] and [Theorem 1 in [2]].

Theorem 4.1. *Assume that there exist a lower-solution α and an upper-solution β of (1.1) such that*

$$\exists \widetilde{t} \in [0,T] \quad \text{such that} \quad \alpha(\widetilde{t}) > \beta(\widetilde{t}); \tag{4.1}$$

Then the problem (1.1) admits at least one solution u, such that

$$\min\{\alpha(t_u),\beta(t_u)\} \leq u(t_u) \leq \max\{\alpha(t_u),\beta(t_u)\} \tag{4.2}$$

for some $t_u \in [0,T]$ and

$$\|u\|_\infty \leq \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} + aT. \tag{4.3}$$

Proof. Let $R = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} + aT$. Consider the functions $f^* : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f^*(t,u,v) = \begin{cases} 2 & \text{if} \quad u > R+1 \\ (1+R-u)f(t,u,v) + 2(u-R) & \text{if} \quad R < u \leq R+1 \\ f(t,u,v) & \text{if} \quad -R \leq u \leq R \\ (1+R+u)f(t,u,v) + 2(u+R) & \text{if} \quad -R-1 \leq u < -R \\ -2 & \text{if} \quad u < -R-1, \end{cases} \tag{4.4}$$

$g_T^* : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$g_T^*(u) = \begin{cases} -2 & \text{if} \quad u > R+1 \\ (1+R-u)g_T(u) - 2(u-R) & \text{if} \quad R < u \leq R+1 \\ g_T(u) & \text{if} \quad -R \leq u \leq R \\ (1+R+u)g_T(u) - 2(u+R) & \text{if} \quad -R-1 \leq u < -R \\ 2 & \text{if} \quad u < -R-1 \end{cases} \tag{4.5}$$

and $g_0^* : \mathbb{R} \longrightarrow \mathbb{R}$ given by:

$$g_0^*(u) = \begin{cases} 2 & \text{if} \quad u > R+1 \\ (1+R-u)g_0(u) + 2(u-R) & \text{if} \quad R < u \leq R+1 \\ g_0(u) & \text{if} \quad -R \leq u \leq R \\ (1+R+u)g_0(u) + 2(u+R) & \text{if} \quad -R-1 \leq u < -R \\ -2 & \text{if} \quad u < -R-1. \end{cases} \tag{4.6}$$

f^* is a L^1-Carathéodory function, g_0^* and g_T^* are continuous. Consider the modified problem

$$\begin{cases} (D(u(t))\phi(u'(t)))' = f^*(t,u(t),u'(t)) \ a.e. \ t \in [0,T], \\ \phi(u'(0)) = g_0^*(u(0)), \qquad \phi(u'(T)) = g_T^*(u(T)). \end{cases} \tag{4.7}$$

α is a lower-solution, and β is an upper-solution of (4.7).
Let $\eta : \mathbb{R} \longrightarrow \mathbb{R}$ given by: $\eta(t) = R+2$. We have

$$(D(\eta(t))\phi(\eta'(t)))' = 0 < 2 = f^*(t,\eta(t),\eta'(t)), \quad a.e.\ t \in [0,T],$$
$$\phi(\eta'(0)) = 0 < 2 = g_0^*(\eta(0)), \qquad \phi(\eta'(T)) = 0 > -2 = g_T^*(\eta(T)).$$

Hence, η is an upper-solution of (4.7).
Let $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ given by: $\sigma(t) = -R-2$. We have

$$(D(\sigma(t))\phi(\sigma'(t)))' = 0 > -2 = f^\star(t,\sigma(t),\sigma'(t)), \quad a.e.\ t \in [0,T],$$
$$\phi(\sigma'(0)) = 0 > -2 = g_0^\star(\sigma(0)), \qquad \phi(\sigma'(T)) = 0 < 2 = g_T^\star(\sigma(T)).$$

Hence, σ is a lower-solution of (4.7). We have

$$\forall t \in [0,T], \qquad \sigma(t) < \min\{\beta(t),\alpha(t)\} \leq \max\{\beta(t),\alpha(t)\} < \eta(t).$$

Let

$$\Omega_{\sigma,\beta} = \{u \in C^1; \quad \forall t \in [0,T], \quad \sigma(t) < u(t) < \beta(t), \quad \| u' \|_\infty < a\},$$
$$\Omega_{\alpha,\eta} = \{u \in C^1; \quad \forall t \in [0,T], \quad \alpha(t) < u(t) < \eta(t), \quad \| u' \|_\infty < a\},$$

and

$$\Omega_{\sigma,\eta} = \{u \in C^1; \quad \forall t \in [0,T], \quad \sigma(t) < u(t) < \eta(t), \quad \| u' \|_\infty < a\}.$$

Using (4.1), we have

$$\Omega_{\sigma,\beta} \cap \Omega_{\alpha,\eta} = \emptyset.$$

We also have

$$(\Omega_{\sigma,\beta} \cup \Omega_{\alpha,\eta}) \subset \Omega_{\sigma,\eta}.$$

Consider

$$\Omega = \Omega_{\sigma,\eta} \setminus (\overline{\Omega}_{\sigma,\beta} \cup \overline{\Omega}_{\alpha,\eta}).$$

It follows that

$$\Omega = \{u \in \Omega_{\sigma,\eta};\ \exists\, (t_1,t_2) \in [0,T]^2 \text{ such that } \beta(t_1) < u(t_1) \quad \text{and} \quad u(t_2) < \alpha(t_2)\}$$

and

$$\partial\Omega = \partial\Omega_{\sigma,\eta} \cup \partial\Omega_{\sigma,\beta} \cup \partial\Omega_{\alpha,\eta}.$$

As any constant function between $\beta(\widetilde{t})$ and $\alpha(\widetilde{t})$ is contained in Ω, Ω is a non-empty set.
Let Γ_1 be the fixed point operator associated to (4.7). Next, let us consider $u \in \Omega$ such that $\Gamma_1(u) = u$ and $\|u\|_\infty = R+2$. Notice that we have $\|u'\|_\infty < a$. Hence there exists $t_0 \in [0,T]$ such that $u(t_0) = \max\limits_{[0,T]} u = R+2$ or $u(t_0) = \min\limits_{[0,T]} u = -R-2$. Consider the case $u(t_0) = \max\limits_{[0,T]} u = R+2$.
If $t_0 \in]0,T[$, then $u'(t_0) = 0$ and there exists $\varepsilon > 0$ such that $u(t) > R+1$ for all $t \in [t_0, t_0+\varepsilon]$.
Moreover,

$$(D(u(t))\phi(u'(t)))' = 2 > 0,$$

hence

$$D(u(t))\phi(u'(t)) = \int_{t_0}^{t} (D(u(s))\phi(u'(s)))' ds > 0, \quad \text{for all } t \in [t_0, t_0+\varepsilon].$$

As $(D(u(t)) > 0$, we have $\phi(u'(t)) > 0$ for all $t \in [t_0, t_0 + \varepsilon]$.
This implying that u is strictly increasing on $[t_0, t_0 + \varepsilon]$, which is a contradiction.

If $t_0 = 0$, then $u'(0) \leq 0$ and we obtain the contradiction

$$0 \geq \phi(u'(0)) = g_0^*(R+2) = 2 > 0.$$

If $t_0 = T$, then $u'(T) \geq 0$ and we obtain the contradiction

$$0 \leq \phi(u'(T)) = g_T^*(R+2) = -2 < 0.$$

In the same way, we obtain a contradiction if $u(t_0) = \min_{[0,T]} u = -R-2$.
Therefore

$$[u \in \partial\Omega,\ \Gamma_1(u) = u] \Rightarrow \|u\|_\infty < R+2. \tag{4.8}$$

Now, let $u \in \partial\Omega$ be such that $\Gamma_1(u) = u$. It follows from (4.8) that $\|u\|_\infty < R+2$, $\|u'\|_\infty < a$, and $u \in \partial\Omega_{\sigma,\beta} \cup \partial\Omega_{\alpha,\eta}$. It follows that there exists $t_0 \in [0,T]$ such that $u(t_0) = \alpha(t_0)$ or $u(t_0) = \beta(t_0)$, implying that

$$|u(t_0)| \leq \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}.$$

Then,

$$|u(t)| \leq |u(t_0)| + \int_0^T |u'(t)|dt < R \qquad \text{for all } t \in [0,T],$$

therefore,

$$[u \in \partial\Omega,\ \Gamma_1(u) = u] \Rightarrow \|u\|_\infty < R. \tag{4.9}$$

We have two cases.
Case 1. Assume that there exists $u \in \partial\Omega$ be such that $\Gamma_1(u) = u$. Using (4.9), we deduce that $\|u\|_\infty < R$, implying that u is a solution of (1.1), and (4.2) and (4.3) are satisfied. In this case there exists $\tau \in [0,T]$ such that $u(\tau) = \alpha(\tau)$ or $u(\tau) = \beta(\tau)$.
Case 2. Assume that $\Gamma_1(u) \neq u$ for all $u \in \partial\Omega$. Then, like in the proof of Theorem 3.9, we have

$$\begin{aligned} d_{LS}[I - \Gamma_1, \Omega_{\sigma,\eta}, 0] &= d_{LS}[I - \Gamma_1, \Omega_{\sigma,\beta}, 0] \\ &= d_{LS}[I - \Gamma_1, \Omega_{\alpha,\eta}, 0] \\ &= 1 \end{aligned}$$

By the additivity property of the Leray-Schauder degree, we have

$$d_{LS}[I - \Gamma_1, \Omega, 0] = -1.$$

By the existence property of the Leray-Schauder degree, there exists $u \in \Omega$ such that $\Gamma_1(u) = u$. It follows that there exists $(t_1, t_2) \in [0,T]^2$ such that $u(t_1) < \alpha(t_1)$ and $u(t_2) > \beta(t_2)$. Then, using once again that $\|u'\|_\infty < a$, it follows that $\|u\|_\infty < R$, hence u is a solution of (1.1) and (4.3) is satisfied. Moreover, from $u \in \Omega$ it follows that (4.2) holds true.

5 Existence of solutions with upper and lower-solutions ordered or not

Theorem 5.1. *Assume that there exist a lower-solution α and an upper-solution β of (1.1). Then the problem (1.1) admits at least one solution u, such that*

$$||u||_\infty \le \max\{||\alpha||_\infty, ||\beta||_\infty\} + aT. \tag{5.1}$$

Proof. If $\forall t \in [0,T]$, $\alpha(t) \le \beta(t)$, by Theorem 3.9, the problem (1.1) admits at least one solution u, such that

$$\alpha(t) \le u(t) \le \beta(t), \qquad \forall t \in [0,T].$$

Moreover, (5.1) is satisfied.
If

$$\exists \widetilde{t} \in [0,T] \quad \text{such that} \quad \alpha(\widetilde{t}) > \beta(\widetilde{t}), \tag{5.2}$$

by Theorem 4.1, the problem (1.1) admits at least one solution u, such that

$$||u||_\infty \le \max\{||\alpha||_\infty, ||\beta||_\infty\} + aT.$$

Example 5.2. Consider the problem

$$\begin{array}{l} \left(\frac{e^{u(t)}u'(t)}{\sqrt{a^2-(u'(t))^2}}\right)' = -c|u'(t)|^q - \frac{b\max\{0,u(t)\}}{\sqrt{t}} + t \quad \text{for a.e.} \quad t \in [0,T], \\ \frac{u'(0)}{\sqrt{a^2-(u'(0))^2}} = -(u(0))^2 \qquad \text{and} \qquad \frac{u'(T)}{\sqrt{a^2-(u'(T))^2}} = (u(T))^2 \end{array}$$

where $a > 0$, $b > 0$, $c > 0$, $T > 0$ and $q > 0$. Then $\alpha(t) = \frac{T\sqrt{T}}{b}$ and $\beta(t) = 0$ are lower and upper-solutions. Using Theorem 5.1, we deduce the existence of at least one solution.

6 Existence of multiple solutions

In this section we use Theorem 3.9 and Theorem 4.1 to prove existence of multiple solutions for the problem (1.1). In the following theorem we prove existence of at least three solutions of problem (1.1).

Theorem 6.1. *Assume that there exist α a lower-solution and σ a strict lower-solution of problem (1.1), η an upper-solution and β a strict upper-solution of the problem (1.1), such that*

$$\forall t \in [0,T], \quad \alpha(t) \le \beta(t) < \sigma(t) \le \eta(t). \tag{6.1}$$

Then the boundary value problem (1.1) admits at least three solutions u, v and w, with

$$\alpha(t) \le u(t) < \beta(t),\ \forall t \in [0,T]; \qquad \sigma(t) < v(t) \le \eta(t),\ \forall t \in [0,T];$$

$$\beta(t_w) \le w(t_w) \le \sigma(t_w), \quad \textit{for some} \quad t_w \in [0,T].$$

Proof. Using Theorem 3.9 and the fact that β and η are strict, the problem (1.1) admits at least two solutions u, v such that

$$\alpha(t) \le u(t) < \beta(t), \quad \forall t \in [0,T] \quad \text{and} \quad \sigma(t) < v(t) \le \eta(t), \quad \forall t \in [0,T]. \tag{6.2}$$

Using Theorem 4.1, the problem (1.1) admits at least one solution w such that

$$\beta(t_w) = \min\{\sigma(t_w), \beta(t_w)\} \le w(t_w) \le \max\{\sigma(t_w), \beta(t_w)\} = \sigma(t_w) \tag{6.3}$$

for some $t_w \in [0,T]$. Using (6.2) and (6.3), we have $u \ne w$ and $w \ne v$.

In the following Theorems we prove existence of at least two solutions of the problem (1.1).

Theorem 6.2. *With the previous notations, assume that there exist α and σ two lower-solutions of the problem (1.1), and β a strict upper-solution of the problem (1.1), such that*

$$\forall t \in [0,T], \quad \alpha(t) \le \beta(t) < \sigma(t);$$

Then the boundary value problem (1.1) admits at least two solutions u and w, with

$$\alpha(t) \le u(t) < \beta(t), \qquad \forall t \in [0,T],$$

and

$$\exists\, t_w \in [0,T] \text{ such that } \beta(t_w) \le w(t_w) \le \sigma(t_w). \tag{6.4}$$

Proof. Using Theorem 3.9, and the fact that β is strict, the problem (1.1) admits at least one solution u, such that

$$\alpha(t) \le u(t) < \beta(t). \tag{6.5}$$

Using Theorem 4.1, the problem (1.1) admits at least one solution w such that

$$\beta(t_w) = \min\{\sigma(t_w), \beta(t_w)\} \le w(t_w) \le \max\{\sigma(t_w), \beta(t_w)\} = \sigma(t_w) \tag{6.6}$$

for some $t_w \in [0,T]$. Using (6.5) and (6.6), we have $u \ne w$.

Theorem 6.3. *With the previous notations, assume that there exist a strict lower-solution α, and two upper-solutions β and η of (1.1) such that*

$$\forall t \in [0,T], \qquad \beta(t) < \alpha(t) \le \eta(t);$$

Then the boundary value problem (1.1) admits at least two solutions u and w, with

$$\alpha(t) < u(t) \le \eta(t), \quad \forall t \in [0,T],$$

and

$$\exists\, t_w \in [0,T] \text{ such that } \beta(t_w) \le w(t_w) \le \alpha(t_w). \tag{6.7}$$

Proof. The proof is similar to the proof of Theorem 6.2.

7 Construction of lower and upper-solutions

Theorem 7.1. *If there exists a constant ω such that*

$$f(t,\omega,0) \leq 0, \quad a.e.\ t \in [0,T], \quad g_0(\omega) \leq 0 \quad and \quad g_T(\omega) \geq 0, \tag{7.1}$$

then, the function α given by $\alpha(t) = \omega$, $\forall t \in [0,T]$, is a lower-solution of problem (1.1).
If there exists a constant ω_1 such that

$$f(t,\omega_1,0) \geq 0, \quad a.e.\ t \in [0,T], \quad g_0(\omega_1) \geq 0 \quad and \quad g_T(\omega_1) \leq 0, \tag{7.2}$$

then, the function β given by $\beta(t) = \omega_1$, $\forall t \in [0,T]$, is an upper-solution of problem (1.1).

Proof. If (7.1) holds, for α given by $\alpha(t) = \omega$, $\forall t \in [0,T]$, we have

$$\begin{aligned}
&(D(\alpha(t))\phi(\alpha'(t)))' = 0 \geq f(t,\omega,0) = f(t,\alpha(t),\alpha'(t)), \quad a.e.\ t \in [0,T],\\
&\phi(\alpha'(0)) = 0 \geq g_0(\omega) = g_0(\alpha(0)),\\
&\phi(\alpha'(T)) = 0 \leq g_T(\omega) = g_T(\alpha(T)).
\end{aligned}$$

If (7.2) holds, for β given by $\beta(t) = \omega_1$, $\forall t \in [0,T]$, we have

$$\begin{aligned}
&(D(\beta(t))\phi(\beta'(t)))' = 0 \leq f(t,\omega_1,0) = f(t,\beta(t),\beta'(t)), \quad a.e.\ t \in [0,T],\\
&\phi(\beta'(0)) = 0 \leq g_0(\omega_1) = g_0(\beta(0)),\\
&\phi(\beta'(T)) = 0 \geq g_T(\omega_1) = g_T(\beta(T)).
\end{aligned}$$

α is a lower-solution and β an upper-solution of problem (1.1).

Lemma 7.2. *Assume that there exist $(A,B) \in \mathbb{R}^2$ and $h \in L^1$ such that*

$$\int_0^T h(t)dt - B + A = 0. \tag{7.3}$$

Assume that $0 < m \leq D(x) \leq M$, $\forall x \in \mathbb{R}$, for some constants m, M.
Then the problem

$$\begin{cases} (D(u(t))\phi(u'(t)))' = h(t), & a.e.\ t \in [0,T],\\ D(u(0))\phi(u'(0)) = A, & D(u(T))\phi(u'(T)) = B, \end{cases} \tag{7.4}$$

has at least one solution.

Proof. We use an argument used by Bereanu and Mawhin in [3]. Let us decompose any $u \in C^1$ in the form $u = \underline{u} + \widetilde{u}$, ($\underline{u} = u(0)$, $\widetilde{u}(0) = 0$) and let $\widetilde{C}^1 = \{u \in C^1 : u(0) = 0\}$.
Consider the family of problem

$$\begin{cases} (D(\widetilde{u}(t))\phi(\widetilde{u}'(t)))' = \lambda h(t), & a.e.\ t \in [0,T],\\ D(\widetilde{u}(0))\phi(\widetilde{u}'(0)) = \lambda A, & D(\widetilde{u}(T))\phi(\widetilde{u}'(T)) = \lambda B \qquad \lambda \in [0,1]. \end{cases} \tag{7.5}$$

For each $\lambda \in [0,1]$, the problem (7.5) is equivalent to the fixed point problem in $\widetilde{C}^1$,

$$\widetilde{u} = H_1 \circ \left[\phi^{-1} \circ [(D_1 \circ \widetilde{u})[\lambda A + \lambda H_1(h)]]\right] = N(\lambda,\widetilde{u}). \tag{7.6}$$

Using Arzelá-Ascoli's Theorem, we see that, $N : [0,1]\times\widetilde{C}^1 \to \widetilde{C}^1$ is completely continuous. For each $\lambda \in [0,1]$, any possible fixed point $\widetilde{u}$ of $N(\lambda,.)$ is such that $\|\widetilde{u}\|_{C^1} < a(T+1)$. Therefore,

$$\begin{aligned} d_{LS}[I-N(1,.),B_{a(T+1)},0] &= d_{LS}[I-N(0,.),B_{a(T+1)},0] \\ &= d_{LS}[I,B_{a(T+1)},0] = 1. \end{aligned}$$

Then, from the existence property of the Leray-Schauder degree, there exists $\widetilde{u} \in \widetilde{C}^1$ such that $\widetilde{u} = N(1,\widetilde{u})$, which is a solution of the problem (7.4).

Theorem 7.3. *Assume that:*

(1) $\exists\theta > 0$ *such that* $D(x-\theta) = D(x) = D(x+\theta)$, $\forall x \in \mathbb{R}$*;*

(2) *There exists* $(d,e) \in \mathbb{R}^2$ *such that, for a.e.* $t \in [0,T]$ *and for all* $(x,y) \in]-\infty,d]\times]-a,a[$,

$$f(t,x,y) \le f(t,d,e);$$

(3) g_0 *is an increasing homeomorphism;*

(4) g_T *is a decreasing homeomorphism.*

Then the problem (1.1) has at least one lower-solution.

Proof. As D is positive, continuous and periodic, there exist two constants m and M such that $0 < m \le D(x) \le M$, $\forall x \in \mathbb{R}$.
Let $(i,j) \in \mathbb{R}^2$ be such that

$$g_0(i) > 0, \quad g_T(j) > 0 \quad \text{and} \quad \int_0^T f(t,d,e)dt - mg_T(j) + Mg_0(i) = 0. \tag{7.7}$$

Using Lemma 7.2, we have that the problem

$$\begin{cases} (D(u(t))\phi(u'(t)))' = f(t,d,e), & a.e.\ t \in [0,T], \\ D(u(0))\phi(u'(0)) = Mg_0(i), & D(u(T))\phi(u'(T)) = mg_T(j), \end{cases} \tag{7.8}$$

admits at least one solution.
Let w be a solution of problem (7.8). Consider the function

$$\alpha = w - A\theta, \text{ with } A \in \mathbb{N}^*, \text{ such that } w(t) - A\theta < \min\{d,i,j\},\ \forall t \in [0,T].$$

$\forall t \in [0,T]$, we have $\alpha'(t) \in]-a,a[$, $\alpha'(t) = w'(t)$, $\alpha(t) \in]-\infty,d[$ and $D(\alpha(t)) = D(w(t))$. Therefore

$$\begin{aligned} &(D(\alpha(t))\phi(\alpha'(t)))' = (D(w(t))\phi(w'(t)))' = f(t,d,e) \ge f(t,\alpha(t),\alpha'(t)), \quad a.e.\ t \in [0,T], \\ &D(\alpha(0))\phi(\alpha'(0)) = D(w(0))\phi(w'(0)) = Mg_0(i) \ge D(\alpha(0))g_0(i) \ge D(\alpha(0))g_0(\alpha(0)), \\ &D(\alpha(T))\phi(\alpha'(T)) = D(w(T))\phi(w'(T)) = mg_T(j) \le D(\alpha(T))g_T(j) \le D(\alpha(T))g_T(\alpha(T)). \end{aligned}$$

Consequently, α is a lower-solution of problem (1.1).

Theorem 7.4. *Assume that:*

(1) $\exists\theta > 0$ *such that* $D(x-\theta) = D(x) = D(x+\theta)$, $\forall x \in \mathbb{R}$;

(2) *There exists* $(d_1, e_1) \in \mathbb{R}^2$ *such that, for a.e.* $t \in [0,T]$ *and for all* $(x,y) \in [d_1,+\infty[\times]-a,a[$,
$$f(t,x,y) \geq f(t,d_1,e_1);$$

(3) g_0 *is an increasing homeomorphism;*

(4) g_T *is a decreasing homeomorphism.*

Then the problem (1.1) has at least one upper-solution.

Proof. As D is positive, continuous and periodic, there exist two constants m and M such that $0 < m \leq D(x) \leq M$, $\forall x \in \mathbb{R}$.
Let $(i_1, j_1) \in \mathbb{R}^2$ be such that

$$g_0(i_1) < 0, \quad g_T(j_1) < 0 \quad \text{and} \quad \int_0^T f(t,d_1,e_1)dt - mg_T(j_1) + Mg_0(i_1) = 0. \tag{7.9}$$

Using Lemma 7.2, we have that the problem

$$\begin{cases} (D(u(t))\phi(u'(t)))' = f(t,d_1,e_1), & a.e.\ t \in [0,T], \\ D(u(0))\phi(u'(0)) = Mg_0(i_1), & D(u(T))\phi(u'(T)) = mg_T(j_1), \end{cases} \tag{7.10}$$

admits at least one solution.
Let w_1 be a solution of problem (7.10). Consider the function

$$\beta = w_1 + B\theta, \text{ with } B \in \mathbb{N}^*, \text{ such that } w_1(t) + B\theta > \max\{d_1, i_1, j_1\},\ \forall t \in [0,T].$$

$\forall t \in [0,T]$, we have $\beta'(t) \in]-a,a[$, $\beta'(t) = w_1'(t)$, $\beta(t) \in]d_1,+\infty[$ and $D(\beta(t)) = D(w_1(t))$. Therefore

$$\begin{aligned} &(D(\beta(t))\phi(\beta'(t)))' = (D(w_1(t))\phi(w_1'(t)))' = f(t,d_1,e_1) \leq f(t,\beta(t),\beta'(t)), \quad a.e.\ t \in [0,T], \\ &D(\beta(0))\phi(\beta'(0)) = D(w_1(0))\phi(w_1'(0)) = Mg_0(i_1) \leq D(\beta(0))g_0(i_1) \leq D(\beta(0))g_0(\beta(0)), \\ &D(\beta(T))\phi(\beta'(T)) = D(w_1(t))\phi(w_1'(T)) = mg_T(j_1) \geq D(\beta(T))g_T(j_1) \geq D(\beta(T))g_T(\beta(T)). \end{aligned}$$

Consequently, β is an upper-solution of problem (1.1).

Theorem 7.5. *Assume that:*

(1) $\exists\theta > 0$ *such that* $D(x-\theta) = D(x) = D(x+\theta)$, $\forall x \in \mathbb{R}$;

(2) *There exist* $(d,e) \in \mathbb{R}^2$ *and* $(d_1,e_1) \in \mathbb{R}^2$ *such that,*
for a.e. $t \in [0,T]$ *and for all* $(x,y) \in]-\infty,d]\times]-a,a[$, $f(t,x,y) \leq f(t,d,e)$,
for a.e. $t \in [0,T]$ *and for all* $(x,y) \in [d_1,+\infty[\times]-a,a[$, $f(t,x,y) \geq f(t,d_1,e_1)$;

(3) g_0 *is an increasing homeomorphism;*

(4) g_T *is a decreasing homeomorphism.*

Then the problem (1.1) has at least one lower-solution and at least one upper-solution. Therefore the problem (1.1) admits at least one solution.

Proof. By Theorem 7.3, the problem (1.1) has at least one lower-solution and by the Theorem 7.4, the problem (1.1) has at least one upper-solution. Therefore, by Theorem 5.1, the problem (1.1) admits at least one solution.

Corollary 7.6. *Assume that:*

(a) $\exists\theta > 0$ *such that* $D(x-\theta) = D(x) = D(x+\theta)$, $\forall x \in \mathbb{R}$;

(b) *for a.e.* $t \in [0,T]$ *and for all* $y \in [-a,a]$, $f(t,.,y)$ *is increasing*
for a.e. $t \in [0,T]$ *and for all* $x \in]-\infty,+\infty[$, $f(t,x,.)$ *is increasing in* $[-a,a]$;

(c) g_0 *is an increasing homeomorphism;*

(d) g_T *is a decreasing homeomorphism.*

Then the problem (1.1) has at least one lower-solution and at least one upper-solution.

Proof. By (b), we have:
There exists $d \in \mathbb{R}$ such that for a.e. $t \in [0,T]$ and for all $(x,y) \in]-\infty,d]\times]-a,a]$, $f(t,x,y) \leq f(t,d,a)$.
There exists $d_1 \in \mathbb{R}$ such that for a.e. $t \in [0,T]$ and for all $(x,y) \in [d_1,+\infty[\times[-a,a[$, $f(t,x,y) \geq f(t,d_1,-a)$.
It follows that, conditions (1), (2), (3) and (4) of Theorem 7.5 hold.

Theorem 7.7. *Assume that:*

(1) *D is a function having continuous first derivative on $\mathbb{R}$ and,*
$\exists(b,c) \in]-\infty,+\infty[\times]0,+\infty[$ *such that* $D'(x) \geq 0$, $\forall x \in]-\infty,b]$ *and* $D(x) \geq c$, $\forall x \in \mathbb{R}$;

(2) *There exists* $(d,e) \in \mathbb{R}\times]-a,a]$ *such that for a.e.* $t \in [0,T]$, $f(t,d,e) \geq 0$ *and, for a.e.* $t \in [0,T]$ *and for all* $(x,y) \in]-\infty,d]\times]-a,e]$,

$$f(t,x,y) \leq f(t,d,e);$$

(3) g_0 *is an increasing function;*

(4) g_T *is a decreasing function;*

(5) *There exists* $(i,j) \in \mathbb{R}^2$ *such that*

$$\phi^{-1}\Big(g_0(i) + \frac{1}{c}\max_{t\in[0,T]}\int_0^t f(s,d,e)ds\Big) < e \quad \text{and} \quad \frac{1}{c}\int_0^T f(t,d,e)dt - g_T(j) + g_0(i) = 0. \tag{7.11}$$

Then the problem (1.1) has at least one lower-solution.

Proof. Using Lemma 7.2, the problem

$$\begin{cases} (c\phi(u'(t)))' = f(t,d,e), & a.e.\ t\in[0,T], \\ \phi(u'(0)) = g_0(i), & \phi(u'(T)) = g_T(j), \end{cases} \tag{7.12}$$

admits at least one solution.
Let w be a solution of problem (7.12). Consider the function

$$\alpha = w - A, \text{ with } A, \text{ such that } w(t) - A < \min\{d,i,j,b\},\ \forall t\in[0,T].$$

We have

$$\phi(\alpha'(t)) = \phi(w'(t)) = g_0(i) + \frac{1}{c}\int_0^t f(s,d,e)ds, \quad \forall t\in[0,T],$$

hence $\alpha'(t) < e$, $\forall t\in[0,T]$, Therefore, for a.e. $t\in[0,T]$,

$$\begin{aligned}(D(\alpha(t))\phi(\alpha'(t)))' &= (D(\alpha(t)))'\phi(\alpha'(t)) + D(\alpha(t))(\phi(\alpha'(t)))', \\ &= D'(\alpha(t))[\alpha'(t)\phi(\alpha'(t))] + D(\alpha(t))(\phi(w'(t)))', \\ &\geq D(\alpha(t))(\frac{1}{c}f(t,d,e)), \\ &\geq c(\frac{1}{c}f(t,d,e)), \\ &\geq f(t,d,e), \\ &\geq f(t,\alpha(t),\alpha'(t)),\end{aligned}$$

and

$$\begin{aligned}\phi(\alpha'(0)) = \phi(w'(0)) = g_0(i) \geq g_0(\alpha(0)), \\ \phi(\alpha'(T)) = \phi(w'(T)) = g_T(j) \leq g_T(\alpha(T)).\end{aligned}$$

Consequently, α is a lower-solution of problem (1.1).

Theorem 7.8. *Assume that:*

(1) *D is a function having continuous first derivative on $\mathbb{R}$ and,* $\exists(b_1,c_1)\in]-\infty,+\infty[\times]0,+\infty[$ *such that* $D'(x)\leq 0$, $\forall x\in[b_1,+\infty[$ *and* $D(x)\geq c_1$, $\forall x\in\mathbb{R}$*;*

(2) *There exists* $(d_1,e_1)\in\mathbb{R}\times[-a,a[$ *such that for a.e.* $t\in[0,T]$, $f(t,d_1,e_1)\leq 0$ *and, for a.e.* $t\in[0,T]$ *and for all* $(x,y)\in[d_1,+\infty[\times[e_1,a[$,

$$f(t,x,y)\geq f(t,d_1,e_1)$$

(3) *g_0 is an increasing function;*

(4) *g_T is a decreasing function;*

(5) *There exists* $(i_1,j_1)\in\mathbb{R}^2$ *such that*

$$\phi^{-1}\Big(g_0(i_1) + \frac{1}{c_1}\min_{t\in[0,T]}\int_0^t f(s,d_1,e_1)ds\Big) > e_1 \quad and \quad \frac{1}{c_1}\int_0^T f(t,d_1,e_1)dt - g_T(j_1) + g_0(i_1) = 0. \tag{7.13}$$

Then the problem (1.1) has at least one upper-solution.

Proof. Using Lemma 7.2, the problem

$$\begin{cases} (c_1\phi(u'(t)))' = f(t,d_1,e_1), & a.e.\ t \in [0,T], \\ \phi(u'(0)) = g_0(i_1), & \phi(u'(T)) = g_T(j_1), \end{cases} \tag{7.14}$$

admits at least one solution.
Let w_1 be a solution of problem (7.14). Consider the function

$$\beta = w_1 + B, \text{ with } B, \text{ such that } w_1(t) + B > \max\{d_1, i_1, j_1, b_1\},\ \forall t \in [0,T].$$

We have

$$\phi(\beta'(t)) = \phi(w_1'(t)) = g_0(i_1) + \frac{1}{c_1}\int_0^t f(s,d_1,e_1)ds, \quad \forall t \in [0,T],$$

hence $\beta'(t) > e_1$, $\forall t \in [0,T]$, Therefore

$$\begin{aligned}
(D(\beta(t))\phi(\beta'(t)))' &= (D(\beta(t)))'\phi(\beta'(t)) + D(\beta(t))(\phi(\beta'(t)))', \\
&= D'(\beta(t))[\beta'(t)\phi(\beta'(t))] + D(\beta(t))(\phi(w_1'(t)))', \\
&\le D(\beta(t))(\frac{1}{c_1}f(t,d_1,e_1)), \\
&\le c_1(\frac{1}{c_1}f(t,d_1,e_1)), \\
&\le f(t,d_1,e_1), \\
&\le f(t,\beta(t),\beta'(t)),
\end{aligned}$$

and

$$\begin{aligned}
\phi(\beta'(0)) &= \phi(w'(0)) = g_0(i_1) \le g_0(\beta(0)), \\
\phi(\beta'(T)) &= \phi(w'(T)) = g_T(j_1) \ge g_T(\beta(T)).
\end{aligned}$$

Consequently, β is an upper-solution of the problem (1.1).

Theorem 7.9. *Assume that:*

(1) *D is a function having continuous first derivative on $\mathbb{R}$ and,*
$\exists(b,c,b_1) \in]-\infty,+\infty[\times]0,+\infty[\times]-\infty,+\infty[$ *such that* $D'(x) \ge 0,\ \forall x \in]-\infty,b]$, $D'(x) \le 0,\ \forall x \in [b_1,+\infty[$ *and* $D(x) \ge c,\ \forall x \in \mathbb{R}$*;*

(2) *There exists* $(d,e) \in \mathbb{R}\times]-a,a]$ *such that for a.e.* $t \in [0,T]$, $f(t,d,e) \ge 0$ *and, for a.e.* $t \in [0,T]$ *and for all* $(x,y) \in]-\infty,d]\times]-a,e]$, $f(t,x,y) \le f(t,d,e)$
there exists $(d_1,e_1) \in \mathbb{R}\times[-a,a[$ *such that for a.e.* $t \in [0,T]$, $f(t,d_1,e_1) \le 0$ *and, for a.e.* $t \in [0,T]$ *and for all* $(x,y) \in [d_1,+\infty[\times[e_1,a[$, $f(t,x,y) \ge f(t,d_1,e_1)$*;*

(3) *g_0 is an increasing function;*

(4) *g_T is a decreasing function;*

(5) *There exist* $(i,j)\in\mathbb{R}^2$ *and* $(i_1,j_1)\in\mathbb{R}^2$ *such that*

$$\phi^{-1}\Big(g_0(i)+\frac{1}{c}\max_{t\in[0,T]}\int_0^t f(s,d,e)ds\Big)<e \quad \textit{and} \quad \frac{1}{c}\int_0^T f(t,d,e)dt-g_T(j)+g_0(i)=0$$

and

$$\phi^{-1}\Big(g_0(i_1)+\frac{1}{c}\min_{t\in[0,T]}\int_0^t f(s,d_1,e_1)ds\Big)>e_1 \quad \textit{and} \quad \frac{1}{c}\int_0^T f(t,d_1,e_1)dt-g_T(j_1)+g_0(i_1)=0.$$

Then the problem (1.1) has at least one lower-solution and at least one upper-solution. Therefore the problem (1.1) admits at least one solution.

Proof. By Theorem 7.7, the problem (1.1) has at least one lower-solution and by Theorem 7.8, the problem (1.1) has at least one upper-solution. Therefore, by Theorem 5.1, the problem (1.1) admits at least one solution.

Corollary 7.10. *Assume that:*

(a) *D is a function having continuous first derivative on $\mathbb{R}$ and,* $\exists(b,c,b_1)\in]-\infty,+\infty[\times]0,+\infty[\times]-\infty,+\infty[$ *such that* $D'(x)\geq 0$, $\forall x\in]-\infty,b]$, $D'(x)\leq 0$, $\forall x\in[b_1,+\infty[$ *and* $D(x)\geq c$, $\forall x\in\mathbb{R}$;

(b) *There exists $d\in\mathbb{R}$ such that for a.e. $t\in[0,T]$, $f(t,d,a)\geq 0$, there exists $d_1\in\mathbb{R}$ such that for a.e. $t\in[0,T]$, $f(t,d_1,-a)\leq 0$ and,*
for a.e. $t\in[0,T]$ and for all $y\in[-a,a]$, $f(t,.,y)$ is increasing
for a.e. $t\in[0,T]$ and for all $x\in]-\infty,+\infty[$, $f(t,x,.)$ is increasing in $[-a,a]$;

(c) *g_0 is an increasing homeomorphism;*

(d) *g_T is a decreasing homeomorphism.*

Then the problem (1.1) has at least one lower-solution and at least one upper-solution. Therefore the problem (1.1) admits at least one solution.

Proof. By (b), (c) and (d), we have:
There exists $d\in\mathbb{R}$ such that for a.e. $t\in[0,T]$, $f(t,d,a)\geq 0$ and, for a.e. $t\in[0,T]$ and for all $(x,y)\in]-\infty,d]\times]-a,a]$,

$$f(t,x,y)\leq f(t,d,a);$$

$\exists(i,j)\in\mathbb{R}^2$ such that

$$\phi^{-1}\Big(g_0(i)+\frac{1}{c}\max_{t\in[0,T]}\int_0^t f(s,d,a)ds\Big)<a \quad \text{and} \quad \frac{1}{c}\int_0^T f(t,d,a)dt-g_T(j)+g_0(i)=0,$$

There exists $d_1\in\mathbb{R}$ such that for a.e. $t\in[0,T]$, $f(t,d_1,-a)\leq 0$ and, for a.e. $t\in[0,T]$ and for all $(x,y)\in[d_1,+\infty[\times[-a,a[$,

$$f(t,x,y)\geq f(t,d_1,-a);$$

$\exists(i_1,j_1)\in\mathbb{R}^2$ such that

$$\phi^{-1}\Big(g_0(i_1)+\frac{1}{c}\max_{t\in[0,T]}\int_0^t f(s,d_1,-a)ds\Big)>-a \quad \text{and} \quad \frac{1}{c}\int_0^T f(t,d_1,-a)dt-g_T(j_1)+g_0(i_1)=0.$$

Therefore, conditions (1), (2), (3), (4) and (5) of Theorem 7.9 hold.

Example 7.11. Consider the problem

$$\begin{cases} \left(\frac{(u(t))^4+(u(t))^2+1}{(u(t))^4+1}\cdot\frac{u'(t)}{\sqrt{1-(u'(t))^2}}\right)' = \frac{(u'(t))^3}{\sqrt{t}} + \frac{-u(t)+1}{(u(t))^2-2u(t)+2} \quad \text{for a.e.} \quad t\in[0,1], \\ \frac{u'(0)}{\sqrt{1-(u'(0))^2}} = u(0) \qquad \text{and} \qquad \frac{u'(T)}{\sqrt{1-(u'(T))^2}} = -u(1) \end{cases} \tag{7.15}$$

We have $a=1$, $D(x)=\frac{x^4+x^2+1}{x^4+1}$, $g_0(x)=x$ and $g_1(x)=-x$, $\forall x\in\mathbb{R}$;
for a.e. $t\in[0,1]$ and for all $(x,y)\in\mathbb{R}^2$, $f(t,x,y)=\frac{y^3}{\sqrt{t}}+\frac{-x+1}{x^2-2x+2}$.
We also have: $D'(x)\geq 0$, $\forall x\in]-\infty,-1]$, $D'(x)\leq 0$, $\forall x\in[1,+\infty[$ and $D(x)\geq 1$, $\forall x\in\mathbb{R}$.
For a.e. $t\in[0,1]$, $f(t,0,1)=\frac{1}{\sqrt{t}}+\frac{1}{2}>0$
and, for a.e. $t\in[0,1]$ and for all $(x,y)\in]-\infty,0]\times]-1,1]$, $f(t,x,y)\leq f(t,0,1)$;
For a.e. $t\in[0,1]$, $f(t,2,-1)=\frac{-1}{\sqrt{t}}-\frac{1}{2}<0$
and, for a.e. $t\in[0,1]$ and for all $(x,y)\in[2,+\infty[\times[-1,1[$, $f(t,x,y)\geq f(t,2,-1)$;

$$\phi^{-1}\Big(g_0(0)+\max_{t\in[0,1]}\int_0^t f(s,0,1)ds\Big)<1 \quad \text{and} \quad \int_0^1 f(t,0,1)dt-g_1(-\frac{5}{2})+g_0(0)=0.$$

$$\phi^{-1}\Big(g_0(0)+\max_{t\in[0,1]}\int_0^t f(s,2,-1)ds\Big)>-1 \quad \text{and} \quad \int_0^1 f(t,2,-1)dt-g_1(\frac{5}{2})+g_0(0)=0.$$

g_0 is an increasing function and g_1 a decreasing function.
Taking $b=-1$, $b_1=1$, $c=1$, $d=0$, $e=1$, $d_1=2$, $e_1=-1$, $i=i_1=0$, $j=-\frac{5}{2}$ and $j_1=\frac{5}{2}$, and the fact that g_0 is an increasing function and g_1 a decreasing function, by Theorem 7.9, we deduce the existence of at least one lower-solution, at least one upper-solution and at least one solution of the problem (7.15).

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