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Periodic Homogenization of Schrödinger Type Equations with Rapidly Oscillating Potential

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Abstract

This paper is devoted to the homogenization of Shrödinger type equations with periodically oscillating coefficients of the diffusion term, and a rapidly oscillating periodic potential. One convergence theorem is proved and we derive the macroscopic homogenized model. Our approach is the well known two-scale convergence method.

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1 Introduction

Let us consider a (non-empty) smooth bounded open subset Ω of $\mathbb{R}^N_x$ (the N-numerical space $\mathbb{R}^N$ of variables $x = (x_1, ..., x_N)$), where N is a given positive integer, and let T and ε be real numbers with $T > 0$ and $0 < \varepsilon < 1$. We consider the partial differential operator

$$\mathcal{A}^\varepsilon = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a^\varepsilon_{ij}\frac{\partial}{\partial x_j}\right)$$

in Ω, where $a^\varepsilon_{ij}(x) = a_{ij}\left(\frac{x}{\varepsilon}\right) \quad (x \in \Omega)$, $a_{ij} \in L^\infty\left(\mathbb{R}^N_y;\mathbb{R}\right)$ $(1 \leq i, j \leq N)$ with

$$a_{ij} = a_{ji}, \tag{1.1}$$

and the assumption that there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(y)\zeta_j\overline{\zeta_i} \geq \alpha|\zeta|^2 \text{ for all } \zeta = \left(\zeta_j\right) \in \mathbb{C}^N \text{ and} \tag{1.2}$$

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for almost all $y \in \mathbb{R}^N$, where $\mathbb{R}^N_y$ is the N-numerical space $\mathbb{R}^N$ of variables $y = (y_1, ..., y_N)$, and where $|\cdot|$ denotes the Euclidean norm in $\mathbb{C}^N$. Let us consider for fixed $0 < \varepsilon < 1$, the following initial boundary value problem:

$$\mathbf{i}\frac{\partial u_\varepsilon}{\partial t} + \mathcal{A}^\varepsilon u_\varepsilon + \frac{1}{\varepsilon}\mathcal{V}^\varepsilon u_\varepsilon = f \text{ in } \Omega\times]0,T[\tag{1.3}$$

$$u_\varepsilon = 0 \text{ on } \partial\Omega\times]0,T[\tag{1.4}$$

$$u_\varepsilon(0) = 0 \text{ in } \Omega, \tag{1.5}$$

where $\mathcal{V}^\varepsilon(x) = \mathcal{V}\left(\frac{x}{\varepsilon}\right)$ is a real potential with $\mathcal{V} \in L^\infty\left(\mathbb{R}^N_y; \mathbb{R}\right)$, and where $f \in L^2\left(0, T; L^2(\Omega)\right)$. In view of (1.1)-(1.2), we will show later that the initial boundary value problem (1.3)-(1.5) admits a unique solution in $C\left([0,T]; H^1_0(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$, provided some regularity assumptions on f, and some hypothesis on $\mathcal{V}$.

The aim here is to investigate the limiting behaviour of u_ε solution of (1.3)-(1.5) when ε goes to zero, under the periodicity hypotheses on the coefficients a_{ij} and the potential $\mathcal{V}$, and the assumption that the mean value of $\mathcal{V}$ is null.

The asymptotic analysis of boundary value problems with rapidly oscillating potential has been studied for the first time in the book of Bensoussan, Lions and Papanicolaou [4] using the asymptotic expansions. Indeed, they considered the following Dirichlet's boundary value problem:

$$\begin{cases} \mathcal{A}^\varepsilon u_\varepsilon + \frac{1}{\varepsilon}\mathcal{V}^\varepsilon u_\varepsilon = f \text{ in } \Omega \\ \qquad\qquad u_\varepsilon = 0 \text{ on } \Omega, \end{cases}$$

which is the stationary case of (1.3)-(1.5). They also considered the Schrödinger model

$$\mathbf{i}\frac{1}{\varepsilon}\frac{\partial u_\varepsilon}{\partial t} + \mathcal{A}^\varepsilon u_\varepsilon + \frac{1}{\varepsilon^2}\mathcal{V}^\varepsilon u_\varepsilon = 0 \text{ in } \mathbb{R}^N \times \mathbb{R}^*_+$$

with initial condition, which is scaled differently from (1.3)-(1.5). Recently, Allaire and Piatnitski in [2] have investigated the homogenization of the Schrödinger type equation

$$\mathbf{i}\frac{\partial u_\varepsilon}{\partial t} + \mathcal{A}^\varepsilon u_\varepsilon + \frac{1}{\varepsilon^2}\mathcal{V}^\varepsilon u_\varepsilon = 0 \text{ in } \mathbb{R}^N \times \mathbb{R}^*_+$$

with initial data, using the two-scale convergence method combined with the bloch waves decomposition. Let us recall that, the scaling of the model under investigation is different from the semi-classical scaling which is

$$\mathbf{i}\frac{\partial u_\varepsilon}{\partial t} + \varepsilon\mathcal{A}^\varepsilon u_\varepsilon + \frac{1}{\varepsilon}\mathcal{V}^\varepsilon u_\varepsilon = \varepsilon f \text{ in } \Omega\times]0,T[.$$

Further, as the oscillatory potential $\mathcal{V}^\varepsilon$ admits a "penalty" factor $\frac{1}{\varepsilon}$, we can also think of the homogenization process for (1.3)-(1.5) as results of the singular perturbations type for

$$\begin{cases} \varepsilon\left(\mathbf{i}\frac{\partial u_\varepsilon}{\partial t} + \mathcal{A}^\varepsilon u_\varepsilon\right) + \mathcal{V}^\varepsilon u_\varepsilon = \varepsilon f \text{ in } \Omega\times]0,T[\\ \qquad\qquad u_\varepsilon = 0 \text{ on } \partial\Omega\times]0,T[\\ \qquad\qquad u_\varepsilon(0) = 0 \text{ in } \Omega. \end{cases}$$

Clearly, in our study we present an other point of view concerning the asymptotic analysis of the Schrödinger model, when the potential is scaled as ε^{-1}. The main result of this paper is stated as follows: Suppose that the conditions (3.1)-(3.2) and (3.4)-(3.5) are satisfied. Suppose also that (3.16)-(3.17) are verified. Let $u_\varepsilon \in \mathcal{C}\left([0,T];H_0^1(\Omega)\right)\cap\mathcal{C}^1\left([0,T];L^2(\Omega)\right)$ be the solution to (1.3)-(1.5) for $\varepsilon > 0$. Then there exists some $u_0 \in L^2\left(0,T;H_0^1(\Omega)\right)$ and some $u_1 \in L^2\left(Q;L^2_{per}\left(Z;H^1_\#(Y)\right)\right)$ such that u_ε converges in $L^2(Q)$-strong to u_0 and ∇u_ε weakly two-scale converges in $L^2(Q)$ to $\nabla u_0 + \nabla_y u_1$ as ε tends to zero. Further, the couple $\mathbf{u} = (u_0, u_1)$ is the unique solution to (3.13). This result is proved in Theorem 3.6 and Theorem 3.9. The derived macroscopic homogenized model given by (3.31)-(3.33) is of Schrödinger type with an additional advection term, while the equations at the microscopic scale are given by (3.27)-(3.28) and the global equation (including the macroscopic and the microscopic scales) by (3.13).

This study is motivated by the fact that the asymptotic analysis of (1.3)-(1.5) is connected with the modelling of the wave function for a particle submitted to a potential. Let us note that the classical Schrödinger equation corresponds to the choice $\mathcal{A}^\varepsilon = -\Delta$.

Unless otherwise specified, vector spaces throughout are considered over the complex field, $\mathbb{C}$, and scalar functions are assumed to take complex values. Let us recall some basic notation. If X and F denote a locally compact space and a Banach space, respectively, then we write $\mathcal{C}(X;F)$ for continuous mappings of X into F, and $\mathcal{B}(X;F)$ for those mappings in $\mathcal{C}(X;F)$ that are bounded. We shall assume $\mathcal{B}(X;F)$ to be equipped with the supremum norm $\|u\|_\infty = \sup_{x\in X}\|u(x)\|$ ($\|\cdot\|$ denotes the norm in F). For shortness we will write $\mathcal{C}(X) = \mathcal{C}(X;\mathbb{C})$ and $\mathcal{B}(X) = \mathcal{B}(X;\mathbb{C})$. Likewise in the case when $F = \mathbb{C}$, the usual spaces $L^p(X;F)$ and $L^p_{loc}(X;F)$ (X provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{loc}(X)$, respectively. Finally, the numerical space $\mathbb{R}^N$ and its open sets are each provided with Lebesgue measure denoted by $dx = dx_1...dx_N$.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results on the two-scale convergence, whereas in Section 3 one convergence theorem is established for (1.3)-(1.5).

2 Preliminaries

We set $Y = \left(-\frac{1}{2},\frac{1}{2}\right)^N$, Y considered as a subset of $\mathbb{R}^N_y$ (the space $\mathbb{R}^N$ of variables $y = (y_1,...,y_N)$). We set also $Z = \left(-\frac{1}{2},\frac{1}{2}\right)$, Z considered as a subset of $\mathbb{R}_\tau$ (the space $\mathbb{R}$ of variables τ).

Let us first recall that a function $u \in L^1_{loc}\left(\mathbb{R}^N_y \times \mathbb{R}_\tau\right)$ is said to be $Y\times Z$-periodic if for each $(k,l) \in \mathbb{Z}^N \times \mathbb{Z}$ ($\mathbb{Z}$ denotes the integers), we have $u(y+k,\tau+l) = u(y,\tau)$ almost everywhere (a.e.) in $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$. If in addition u is continuous, then the preceding equality holds for every $(y,\tau) \in \mathbb{R}^N\times\mathbb{R}$, of course. The space of all $Y\times Z$-periodic continuous complex functions on $\mathbb{R}^N_y \times \mathbb{R}_\tau$ is denoted by $\mathcal{C}_{per}(Y\times Z)$; that of all $Y\times Z$-periodic functions in $L^p_{loc}\left(\mathbb{R}^N_y\times\mathbb{R}_\tau\right)$ ($1 \le p \le \infty$) is denoted by $L^p_{per}(Y\times Z)$. $\mathcal{C}_{per}(Y\times Z)$ is a Banach space under the supremum norm on $\mathbb{R}^N\times\mathbb{R}$, whereas $L^p_{per}(Y\times Z)$ is a Banach space under the norm

$$\|u\|_{L^p(Y\times Z)} = \left(\int_Z\int_Y |u(y,\tau)|^p\,dyd\tau\right)^{\frac{1}{p}} \quad \left(u \in L^p_{per}(Y\times Z)\right).$$

The space $H^1_\#(Y)$ of Y-periodic functions $u \in H^1_{loc}\left(\mathbb{R}^N_y\right) = W^{1,2}_{loc}\left(\mathbb{R}^N_y\right)$ such that $\int_Y u(y)\,dy = 0$ will be of our interest in this study. Provided with the gradient norm,

$$\|u\|_{H^1_\#(Y)} = \left(\int_Y \left|\nabla_y u\right|^2 dy\right)^{\frac{1}{2}} \quad \left(u \in H^1_\#(Y)\right),$$

where $\nabla_y u = \left(\frac{\partial u}{\partial y_1}, ..., \frac{\partial u}{\partial y_N}\right)$, $H^1_\#(Y)$ is a Hilbert space. We will also need the space $L^2_{per}\left(Z; H^1_\#(Y)\right)$ with the norm

$$\|u\|_{L^2_{per}\left(Z;H^1_\#(Y)\right)} = \left(\int_Z \int_Y \left|\nabla_y u(y,\tau)\right|^2 dy d\tau\right)^{\frac{1}{2}} \quad \left(u \in L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$$

which is a Hilbert space.

Before we can recall the concept of two-scale convergence, let us introduce one further notation. The letter E throughout will denote a family of real numbers $0 < \varepsilon < 1$ admitting 0 as an accumulation point. For example, E may be the whole interval $(0,1)$; E may also be an ordinary sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $0 < \varepsilon_n < 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. In the latter case E will be referred to as a *fundamental sequence*.

Let Ω be a bounded open set in $\mathbb{R}^N_x$ and $Q = \Omega\times]0,T[$ with $T \in \mathbb{R}^*_+$, and let $1 \leq p < \infty$.

Definition 2.1. A sequence $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(Q)$ is said to:

(i) weakly two-scale converge in $L^p(Q)$ to some $u_0 \in L^p\left(Q; L^p_{per}(Y\times Z)\right)$ if as $E \ni \varepsilon \to 0$,

$$\int_Q u_\varepsilon(x,t)\psi^\varepsilon(x,t)\,dxdt \to \int\int\int_{Q\times Y\times Z} u_0(x,t,y,\tau)\psi(x,t,y,\tau)\,dxdtdyd\tau \tag{2.1}$$

for all $\psi \in L^{p'}\left(Q; C_{per}(Y\times Z)\right)$ $\left(\frac{1}{p'} = 1 - \frac{1}{p}\right)$, where $\psi^\varepsilon(x,t) = \psi\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)$ $((x,t) \in Q)$;

(ii) strongly two-scale converge in $L^p(Q)$ to some $u_0 \in L^p\left(Q; L^p_{per}(Y\times Z)\right)$ if the following property is verified:

$$\begin{cases} \text{Given } \eta > 0 \text{ and } v \in L^p\left(Q; C_{per}(Y\times Z)\right) \text{ with} \\ \|u_0 - v\|_{L^p(Q\times Y\times Z)} \leq \frac{\eta}{2}, \text{ there is some } \alpha > 0 \text{ such} \\ \text{that } \|u_\varepsilon - v^\varepsilon\|_{L^p(Q)} \leq \eta \text{ provided } E \ni \varepsilon \leq \alpha, \end{cases}$$

where $v^\varepsilon(x,t) = v\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)$ $((x,t) \in Q)$.

We will briefly express weak and strong two-scale convergence by writing $u_\varepsilon \to u_0$ in $L^p(Q)$-weak 2-s and $u_\varepsilon \to u_0$ in $L^p(Q)$-strong 2-s, respectively.

Remark 2.2. It is of interest to know that if $u_\varepsilon \to u_0$ in $L^p(Q)$-weak 2-s, then (2.1) holds for $\psi \in C\left(\overline{Q}; L^\infty_{per}(Y\times Z)\right)$. See [12, Proposition 10] for the proof.

For more details about the two-scale convergence the reader can refer to [8].

However, we recall below two fundamental results. First of all, let

$$\mathcal{Y}(0,T) = \left\{v \in L^2\left(0,T; H^1_0(\Omega)\right) : v' \in L^2\left(0,T; H^{-1}(\Omega)\right)\right\}.$$

$\mathcal{Y}(0,T)$ is provided with the norm

$$\|v\|_{\mathcal{Y}(0,T)} = \left(\|v\|^2_{L^2\left(0,T;H^1_0(\Omega)\right)} + \left\|v'\right\|^2_{L^2\left(0,T;H^{-1}(\Omega)\right)}\right)^{\frac{1}{2}} \qquad (v \in \mathcal{Y}(0,T))$$

which makes it a Hilbert space.

Theorem 2.3. *Assume that $1 < p < \infty$ and further E is a fundamental sequence. Let a sequence $(u_\varepsilon)_{\varepsilon\in E}$ be bounded in $L^p(Q)$. Then, a subsequence E' can be extracted from E such that $(u_\varepsilon)_{\varepsilon\in E'}$ weakly two-scale converges in $L^p(Q)$.*

Theorem 2.4. *Let E be a fundamental sequence. Suppose a sequence $(u_\varepsilon)_{\varepsilon\in E}$ is bounded in $\mathcal{Y}(0,T)$. Then, a subsequence E' can be extracted from E such that, as $E' \ni \varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \text{ in } \mathcal{Y}(0,T)\text{-weak},$$

$$u_\varepsilon \to u_0 \text{ in } L^2(Q)\text{-weak } 2\text{-}s,$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^2(Q)\text{-weak } 2\text{-}s \ (1 \le j \le N),$$

where $u_0 \in \mathcal{Y}(0,T)$, $u_1 \in L^2\left(Q; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$.

The proof of Theorem 2.3 can be found in, e.g., [8], [10], whereas Theorem 2.4 has its proof in, e.g., [12].

Let us prove the following lemma.

Lemma 2.5. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $\mathcal{Y}(0,T)$, where E is a fundamental sequence. There exists a subsequence E' extracted from E such that*

$$\int_Q \frac{1}{\varepsilon} u_\varepsilon \psi^\varepsilon dxdt \to \int_Q \int\int_{Y\times Z} u_1(x,t,y,\tau)\psi(x,t,y,\tau)\,dxdtdyd\tau \tag{2.2}$$

for all $\psi \in \mathcal{D}(Q)\otimes\left(C_{per}(Y)/\mathbb{C}\right)\otimes C_{per}(Z)$ as $E' \ni \varepsilon \to 0$, where $u_1 \in L^2\left(Q; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$.

Proof. As $(u_\varepsilon)_{\varepsilon\in E}$ is a bounded sequence in $\mathcal{Y}(0,T)$, thanks to Theorem 2.4, there exists a subsequence E' extracted from E and functions $u_0 \in \mathcal{Y}(0,T)$, $u_1 \in L^2\left(Q; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$ such that

$$u_\varepsilon \to u_0 \text{ in } \mathcal{Y}(0,T)\text{-weak},$$

$$u_\varepsilon \to u_0 \text{ in } L^2(Q)\text{-weak } 2\text{-}s, \tag{2.3}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^2(Q)\text{-weak } 2\text{-}s \ (1 \le j \le N), \tag{2.4}$$

as $E' \ni \varepsilon \to 0$. Let $\theta \in \mathcal{D}(Q)\otimes C^\infty_{per}(Y)\otimes C_{per}(Z)$. We have

$$\frac{1}{\varepsilon}\left(\Delta_y\theta\right)^\varepsilon = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\frac{\partial\theta}{\partial y_i}\right)^\varepsilon - \sum_{i=1}^N \left(\frac{\partial^2\theta}{\partial x_i \partial y_i}\right)^\varepsilon,$$

as is easily seen by observing that

$$\frac{\partial \Phi^{\varepsilon}}{\partial x_i} = \left(\frac{\partial \Phi}{\partial x_i}\right)^{\varepsilon} + \frac{1}{\varepsilon}\left(\frac{\partial \Phi}{\partial y_i}\right)^{\varepsilon}, \qquad \Phi \in \mathcal{C}^1\left(Q \times \mathbb{R}_y^N \times \mathbb{R}_\tau\right).$$

Hence,

$$\int_Q \frac{1}{\varepsilon} u_\varepsilon \left(\Delta_y \theta\right)^{\varepsilon} dxdt = -\int_Q \nabla_x u_\varepsilon \cdot \left(\nabla_y \theta\right)^{\varepsilon} dxdt - \int_Q u_\varepsilon \sum_{i=1}^{N} \left(\frac{\partial^2 \theta}{\partial x_i \partial y_i}\right)^{\varepsilon} dxdt, \tag{2.5}$$

where the dot denotes the Euclidean inner product. On the other hand, according to (2.3) and (2.4) we have

$$\int_Q u_\varepsilon \left(\frac{\partial^2 \theta}{\partial x_i \partial y_i}\right)^{\varepsilon} dxdt \to \int_Q u_0 \left(\int \int_{Y \times Z} \frac{\partial^2 \theta}{\partial x_i \partial y_i} dy d\tau\right) dxdt = 0$$

and

$$\int_Q \nabla_x u_\varepsilon \cdot \left(\nabla_y \theta\right)^{\varepsilon} dxdt \to \int \int \int_{Q \times Y \times Z} \left(\nabla_x u_0 + \nabla_y u_1\right) \cdot \nabla_y \theta dxdtdyd\tau$$

as $E' \ni \varepsilon \to 0$. Therefore, on letting $E' \ni \varepsilon \to 0$ in (2.5), one has

$$\int_Q \frac{1}{\varepsilon} u_\varepsilon \left(\Delta_y \theta\right)^{\varepsilon} dxdt \to \int \int \int_{Q \times Y \times Z} u_1 \Delta_y \theta dxdtdyd\tau.$$

With this in mind, let $\psi \in \mathcal{D}(Q) \otimes \left(\mathcal{C}_{per}^{\infty}(Y)/\mathbb{C}\right) \otimes \mathcal{C}_{per}(Z)$, i.e.,

$$\psi = \sum_{i \in I} \varphi_i \otimes \psi_i \otimes \chi_i$$

with $\varphi_i \in \mathcal{D}(Q)$, $\psi_i \in \mathcal{C}_{per}^{\infty}(Y)/\mathbb{C}$ and $\chi_i \in \mathcal{C}_{per}(Z)$, where I is a finite set (depending on ψ). For any $i \in I$, let $\theta_i \in H^1(Y)$ such that $\Delta_y \theta_i = \psi_i$. In view of the hypoellipticity of the Laplace operator Δ_y, the function θ_i is of class $\mathcal{C}^{\infty}$, thus, it belongs to $\mathcal{C}_{per}^{\infty}(Y)$. Let

$$\theta = \sum_{i \in I} \varphi_i \otimes \theta_i \otimes \chi_i.$$

We have $\theta \in \mathcal{D}(Q) \otimes \mathcal{C}_{per}^{\infty}(Y) \otimes \mathcal{C}_{per}(Z)$ and $\Delta_y \theta = \psi$. Hence, (2.2) follows and the lemma is proved. □

3 Convergence of the homogenization process

3.1 Preliminary results

Let B^{ε} be the linear operator in $L^2(\Omega)$ with domain

$$D(B^{\varepsilon}) = \left\{ v \in H_0^1(\Omega) : \mathcal{A}^{\varepsilon} v + \frac{1}{\varepsilon} \mathcal{V}^{\varepsilon} v \in L^2(\Omega) \right\},$$

defined by

$$B^{\varepsilon}u = \mathbf{i}\mathcal{A}^{\varepsilon}u + \frac{\mathbf{i}}{\varepsilon}\mathcal{V}^{\varepsilon}u \quad \text{ for all } u \in D(B^{\varepsilon}).$$

In the sequel, we suppose that the coefficients $\left(a_{ij}\right)_{1\leq i,j\leq N}$ verify

$$a_{ij} \in W^{1,\infty}\left(\mathbb{R}^N_y;\mathbb{R}\right) \quad (1 \leq i, j \leq N), \tag{3.1}$$

where $W^{1,\infty}\left(\mathbb{R}^N_y;\mathbb{R}\right)$ is the Sobolev space of functions in $L^{\infty}\left(\mathbb{R}^N_y;\mathbb{R}\right)$ with their derivatives of order 1. Then B^{ε} is of dense domain, and skew-adjoint since $\mathcal{A}^{\varepsilon} + \frac{1}{\varepsilon}\mathcal{V}^{\varepsilon}$ is self-adjoint (see [6] for more details). Consequently, B^{ε} is a m-dissipative operator in $L^2(\Omega)$ by virtue of [6, Corollary 2.4.11]. It follows by the Hille-Yosida-Philips theorem that B^{ε} is the generator of a contraction semi-group. Thus, according to [6, Chapter 4] (1.3)-(1.5) admits a unique solution $u_{\varepsilon} \in C([0,T];D(B^{\varepsilon})) \cap C^1\left([0,T];L^2(\Omega)\right)$, provided $f \in C\left([0,T];L^2(\Omega)\right)$. Further, in the sequel the potential $\mathcal{V}$ is assumed to satisfy

$$\left\|\frac{1}{\varepsilon}\mathcal{V}^{\varepsilon}\right\|_{\mathcal{L}\left(H^1_0(\Omega),H^{-1}(\Omega)\right)} \leq \beta \qquad (\varepsilon > 0), \tag{3.2}$$

where $\beta > 0$ is a constant independent of ε and where $\mathcal{L}\left(H^1_0(\Omega), H^{-1}(\Omega)\right)$ is the space of linear continuous mappings of $H^1_0(\Omega)$ into $H^{-1}(\Omega)$ ($\frac{1}{\varepsilon}\mathcal{V}^{\varepsilon}$ is the linear operator of $H^1_0(\Omega)$ into $H^{-1}(\Omega)$ defined by $u \mapsto \frac{1}{\varepsilon}\mathcal{V}^{\varepsilon}u$). For an illustrative example, if the potential $\mathcal{V}$ belongs to $C_{per}(Y) \cap C^2\left(\mathbb{R}^N_y;\mathbb{R}\right)$ ($C_{per}(Y)$ is the space of Y-periodic continuous complex functions on $\mathbb{R}^N_y$) and verifies

$$\int_Y \mathcal{V}(y)\,dy = 0. \tag{3.3}$$

Then, the linear operator $\frac{1}{\varepsilon}\mathcal{V}^{\varepsilon}$ of $H^1_0(\Omega)$ into $H^{-1}(\Omega)$ verifies (3.2). Indeed, since $\mathcal{V} \in C_{per}(Y)$ and verifies (3.3), the equation

$$-\Delta_y\chi + \mathcal{V} = 0$$

admits a unique solution χ in $H^1_{\#}(Y)$ which is sufficiently smooth. Moreover, for all $\varepsilon > 0$, we have

$$-\varepsilon\Delta\chi^{\varepsilon} + \frac{1}{\varepsilon}\mathcal{V}^{\varepsilon} = 0.$$

Thus, for any $u \in H^1_0(\Omega)$, we have

$$\left(\frac{1}{\varepsilon}\mathcal{V}^{\varepsilon}u, v\right) = -\varepsilon\int_{\Omega}\nabla\chi^{\varepsilon}\cdot\nabla(uv)\,dx = -\int_{\Omega}\left(\nabla_y\chi\right)^{\varepsilon}\cdot\nabla(uv)\,dx$$

for all $v \in \mathcal{D}(\Omega)$ ($\mathcal{D}(\Omega)$ is the space of functions in $C^{\infty}(\Omega)$ with compact supports), and this implies

$$\left|\left(\frac{1}{\varepsilon}\mathcal{V}^{\varepsilon}u, v\right)\right| \leq c_0\left(\|u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\right)$$

where $c_0 = \max_{1\leq j\leq N}\left\|\frac{\partial\chi}{\partial y_j}\right\|_{\infty}$. Accordingly,

$$\left|\left(\frac{1}{\varepsilon}\mathcal{V}^{\varepsilon}u, v\right)\right| \leq 2c_0c_1\|u\|_{H^1_0(\Omega)}\|v\|_{H^1_0(\Omega)}, \quad \forall v \in \mathcal{D}(\Omega),$$

c_1 being the constant in the Poincaré inequality. Thus, by the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$, the precedent inequality holds for all $v \in H_0^1(\Omega)$. Hence, (3.2) follows with $\beta = 2c_0c_1$.

Now, let us prove some estimates for (1.3)-(1.5).

Lemma 3.1. *Suppose that*

$$\alpha > \beta \tag{3.4}$$

(α being the constant in (1.2) and β the one in (3.2)) and

$$f \in C^1\left(0,T;L^2(\Omega)\right). \tag{3.5}$$

Then, there exists a constant $c > 0$ independent of ε such that the solution u_ε of (1.3)-(1.5) verifies:

$$\|u_\varepsilon\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \leq c \tag{3.6}$$

and

$$\left\|u'_\varepsilon\right\|_{L^2\left(0,T;H^{-1}(\Omega)\right)} \leq c. \tag{3.7}$$

Before the proof of this lemma, let us make some useful remarks. Let us put

$$a^\varepsilon(u,v) = \sum_{i,j=1}^N \int_\Omega a_{ij}^\varepsilon \frac{\partial u}{\partial x_j}\frac{\overline{\partial v}}{\partial x_i}dx \quad \text{for all } u,v \in H^1(\Omega).$$

Remark 3.2. As $u_\varepsilon \in C([0,T];D(B^\varepsilon)) \cap C^1\left([0,T];L^2(\Omega)\right)$, the function $t \mapsto a^\varepsilon(u_\varepsilon(t),u_\varepsilon(t))$ belongs to $C^1([0,T])$ and

$$\frac{d}{dt}a^\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) = 2\,\mathrm{Re}\left(\mathcal{A}^\varepsilon u_\varepsilon(t),u'_\varepsilon(t)\right) \text{ for all } t \in [0,T].$$

On the other hand, we have

$$\frac{d}{dt}\left(\mathcal{V}^\varepsilon u_\varepsilon(t),u_\varepsilon(t)\right) = 2\,\mathrm{Re}\left(\mathcal{V}^\varepsilon u_\varepsilon(t),u'_\varepsilon(t)\right) \quad (t \in [0,T]),$$

where $(,)$ denotes the scalar product in $L^2(\Omega)$. Further, by (3.5) we have

$$\frac{d}{dt}\left(f(t),u_\varepsilon(t)\right) = \left(f'(t),u_\varepsilon(t)\right) + \left(f(t),u'_\varepsilon(t)\right) \quad (t \in [0,T]).$$

Proof of Lemma 3.1. Taking the scalar product in $L^2(\Omega)$ of (1.3) with u_ε yields

$$\mathbf{i}\left(u'_\varepsilon(t),u_\varepsilon(t)\right) + a^\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) + \frac{1}{\varepsilon}\left(\mathcal{V}^\varepsilon u_\varepsilon(t),u_\varepsilon(t)\right) = \left(f(t),u_\varepsilon(t)\right) \qquad (t \in [0,T]).$$

Using (1.1) and the fact that a_{ij} is real, we see that $t \longmapsto a^\varepsilon(u_\varepsilon(t),u_\varepsilon(t))$ is a real valued function. Thus, by the preceding equality we have

$$\mathrm{Re}\left(u'_\varepsilon(t),u_\varepsilon(t)\right) = -\mathrm{Re}\left(\mathbf{i}f(t),u_\varepsilon(t)\right) \qquad (t \in [0,T]),$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(t)\|_{L^2(\Omega)}^2 = -\mathrm{Re}\left(\mathbf{i}f(t),u_\varepsilon(t)\right) \qquad (t \in [0,T]).$$

Integrating the preceding equality in $[0,t]$ with $t \in [0,T]$ leads to

$$\|u_\varepsilon(t)\|^2_{L^2(\Omega)} \leq 2\int_0^T \int_\Omega |f|\,|u_\varepsilon|\,dxdt. \tag{3.8}$$

Moreover,

$$2\int_0^T \int_\Omega |f|\,|u_\varepsilon|\,dxdt \leq \int_0^T \int_\Omega \left(2T\,|f|^2 + \frac{1}{2T}\,|u_\varepsilon|^2\right)dxdt.$$

Consequently, an integration on $[0,T]$ of (3.8) leads to

$$\frac{1}{2}\,\|u_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} \leq 2T^2\,\|f\|^2_{L^2(0,T;L^2(\Omega))}. \tag{3.9}$$

It follows from the preceding inequality that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2\left(0,T;L^2(\Omega)\right)$. Now, let us prove (3.6). Taking the scalar product in $L^2(\Omega)$ of (1.3) with u'_ε, one as

$$\mathbf{i}\left\|u'_\varepsilon(t)\right\|^2_{L^2(\Omega)} + (\mathcal{A}^\varepsilon u_\varepsilon(t), u'_\varepsilon(t)) + \frac{1}{\varepsilon}(\mathcal{V}^\varepsilon u_\varepsilon(t), u'_\varepsilon(t)) = (f(t), u'_\varepsilon(t)) \qquad (t \in [0,T]).$$

By the preceding equality we have,

$$\mathrm{Re}\,(\mathcal{A}^\varepsilon u_\varepsilon(t), u'_\varepsilon(t)) + \frac{1}{\varepsilon}\mathrm{Re}\,(\mathcal{V}^\varepsilon u_\varepsilon(t), u'_\varepsilon(t)) = \mathrm{Re}\,(f(t), u'_\varepsilon(t)) \qquad (t \in [0,T]).$$

Thus, using Remark 3.2 leads to

$$\frac{1}{2}\frac{d}{dt}a^\varepsilon(u_\varepsilon(t), u_\varepsilon(t)) + \frac{1}{2\varepsilon}\frac{d}{dt}(\mathcal{V}^\varepsilon u_\varepsilon(t), u_\varepsilon(t)) = \mathrm{Re}\,\frac{d}{dt}(f(t), u_\varepsilon(t)) - \mathrm{Re}\,(f'(t), u_\varepsilon(t)). \tag{3.10}$$

An integration on $[0,t]$ of (3.10) yields,

$$\frac{1}{2}a^\varepsilon(u_\varepsilon(t), u_\varepsilon(t)) + \frac{1}{2\varepsilon}(\mathcal{V}^\varepsilon u_\varepsilon(t), u_\varepsilon(t)) = \mathrm{Re}\,(f(t), u_\varepsilon(t)) - \mathrm{Re}\int_0^t (f'(s), u_\varepsilon(s))\,ds. \tag{3.11}$$

It follows from (1.2) and (3.11) that, by (3.2) we have

$$\alpha\,\|u_\varepsilon(t)\|^2_{H^1_0(\Omega)} \leq \beta\,\|u_\varepsilon(t)\|^2_{H^1_0(\Omega)} + 2\,\|f(t)\|_{L^2(\Omega)}\,\|u_\varepsilon(t)\|_{L^2(\Omega)} + 2\left\|f'\right\|_{L^2(0,T;L^2(\Omega))}\|u_\varepsilon\|_{L^2(0,T;L^2(\Omega))}.$$

Integrating on $[0,T]$ the preceding inequality and using (3.9) and (3.4), we see that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2\left(0,T;H^1_0(\Omega)\right)$, and (3.6) follows. Now, we can prove (3.7). By (1.3), we have

$$\mathbf{i}\int_0^T (u'_\varepsilon(t), v(t))\,dt + \int_0^T a^\varepsilon(u_\varepsilon(t), v(t))\,dt + \frac{1}{\varepsilon}\int_0^T (\mathcal{V}^\varepsilon u_\varepsilon(t), v(t))\,dt = \int_0^T (f(t), v(t))\,dt$$

for all $v \in L^2\left(0,T;H^1_0(\Omega)\right)$. Hence,

$$\left|\int_0^T (u'_\varepsilon(t), v(t))\,dt\right| \leq c_2\,\|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))}\,\|v\|_{L^2(0,T;H^1_0(\Omega))} + \\ \beta\,\|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))}\,\|v\|_{L^2(0,T;H^1_0(\Omega))} + c_0\,\|f\|_{L^2(0,T;L^2(\Omega))}\,\|v\|_{L^2(0,T;H^1_0(\Omega))},$$

where $c_2 = \max_{1\leq i,j\leq N}\left\|a_{ij}\right\|_\infty$, β is given by (3.2) and c_0 is the constant in the Poincaré inequality. It follows from the preceding inequality that

$$\left\|u'_\varepsilon\right\|_{L^2(0,T;H^{-1}(\Omega))} \leq (c_2 + \beta)\,\|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))} + c_0\,\|f\|_{L^2(0,T;L^2(\Omega))}.$$

Then, by (3.6) we conclude that the sequence $(u'_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2\left(0,T;H^{-1}(\Omega)\right)$. The lemma is proved. □

3.2 A convergence theorem

Let us first introduce some functions spaces.
We consider the space

$$\mathbb{F}_0^1 = \mathcal{Y}(0,T) \times L^2\left(Q; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$$

provided with the norm

$$\|\mathbf{u}\|_{\mathbb{F}_0^1} = \left(\|u_0\|^2_{\mathcal{Y}(0,T)} + \|u_1\|^2_{L^2\left(Q;L^2_{per}\left(Z;H^1_\#(Y)\right)\right)}\right)^{\frac{1}{2}} \qquad \left(\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^1\right),$$

which makes it Hilbert space. We consider also the space

$$\mathcal{F}_0^\infty = \mathcal{D}(Q) \times \left[\mathcal{D}(Q) \otimes \left[\left(C_{per}(Y)/\mathbb{C}\right) \otimes C_{per}(Z)\right]\right]$$

which is a dense subspace of $\mathbb{F}_0^1$. For $\mathbf{u} = (u_0, u_1)$ and $\mathbf{v} = (v_0, v_1) \in H^1_0(\Omega) \times L^2\left(\Omega; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$, we set

$$\mathfrak{a}(\mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^N \int\int\int_{\Omega\times Y\times Z} a_{ij}(y) \left(\frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right) \left(\overline{\frac{\partial v_0}{\partial x_i}} + \overline{\frac{\partial v_1}{\partial y_i}}\right) dxdyd\tau.$$

This defines a sesquilinear hermitian form on $\left[H^1_0(\Omega) \times L^2\left(\Omega; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)\right]^2$ which is continuous and verifies

$$\mathfrak{a}(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|^2_{H^1_0(\Omega)\times L^2\left(\Omega;L^2_{per}\left(Z;H^1_\#(Y)\right)\right)} \qquad \left(\mathbf{v} \in H^1_0(\Omega) \times L^2\left(\Omega; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)\right), \quad (3.12)$$

according to (1.1)-(1.2). Further, we have the following lemma.

Lemma 3.3. *Let $f \in L^2\left(0,T; L^2(\Omega)\right)$ and $\mathcal{V} \in L^\infty\left(\mathbb{R}^N_y; \mathbb{R}\right)$. Then the variational problem*

$$\begin{cases} \mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^1 \text{ with } u_0(0) = 0 : \\ \boldsymbol{i} \int_0^T \left\langle u_0'(t), \overline{v_0}(t)\right\rangle dt + \int_0^T \mathfrak{a}(\mathbf{u}(t), \mathbf{v}(t))\, dt + \int\int\int_{Q\times Y\times Z} (u_1\overline{v_0} + u_0\overline{v_1}) \mathcal{V} dxdtdyd\tau \\ \qquad\qquad = \int_0^T (f(t), v_0(t))\, dt \\ \text{for all } \mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^1, \end{cases} \quad (3.13)$$

admits at most one solution ($\langle , \rangle$ is the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$).

Proof. Suppose $\mathbf{u} = (u_0, u_1)$ and $\mathbf{w} = (w_0, w_1)$ are solutions of (3.13). We set $\mathbf{z} = \mathbf{u} - \mathbf{w}$ ($\mathbf{z} = (z_0, z_1)$ with $z_0 = u_0 - w_0$ and $z_1 = u_1 - w_1$). By (3.13), we see that $\mathbf{z}$ verifies

$$\mathbf{i} \int_0^T \left\langle z_0'(t), \overline{v_0}(t)\right\rangle dt + \int_0^T \mathfrak{a}(\mathbf{z}(t), \mathbf{v}(t))\, dt + \int\int\int_{Q\times Y\times Z} (z_1\overline{v_0} + z_0\overline{v_1}) \mathcal{V} dxdtdyd\tau = 0 \quad (3.14)$$

for all $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^1$. Taking in particular $\mathbf{v} = \varphi \otimes \mathbf{v}_*$ with $\varphi \in \mathcal{D}(]0,T[)$ and $\mathbf{v}_* = (v_0, v_1) \in H^1_0(\Omega) \times L^2\left(\Omega; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$ in (3.14), we obtain

$$\mathbf{i}\left\langle z_0'(t), \overline{v_0}\right\rangle + \mathfrak{a}(\mathbf{z}(t), \mathbf{v}_*) + \int\int\int_{\Omega\times Y\times Z} (z_1(t)\overline{v_0} + z_0(t)\overline{v_1}) \mathcal{V} dxdyd\tau = 0 \quad (t \in [0,T])$$

for all $\mathbf{v}_* = (v_0, v_1) \in H_0^1(\Omega) \times L^2\left(\Omega; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$. Thus, choosing $\mathbf{v}_* = \mathbf{z}(t)$ for $t \in [0,T]$ in the preceding equality yields,

$$\mathbf{i}\left\langle z_0'(t), \overline{z_0}(t)\right\rangle + \mathfrak{a}(\mathbf{z}(t), \mathbf{z}(t)) + \int\int\int_{\Omega\times Y\times Z} (z_1(t)\overline{z_0}(t) + z_0(t)\overline{z_1}(t))\mathcal{V} dxdyd\tau = 0 \quad (t \in [0,T]). \tag{3.15}$$

But, according to (1.1) and by the fact that a_{ij} is real, $t \mapsto \mathfrak{a}(\mathbf{z}(t), \mathbf{z}(t))$ is a real valued function. Consequently, by the preceding equality we have

$$\mathrm{Re}\left\langle z_0'(t), \overline{z_0}(t)\right\rangle = 0 \qquad (t \in [0,T]),$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\|z_0(t)\|^2_{L^2(\Omega)} = 0 \qquad (t \in [0,T]).$$

Hence $z_0(t) = 0$ for all $t \in [0,T]$. Then, by (3.12) and (3.15) we see that $\mathbf{z}(t) = 0$ for all $t \in [0,T]$, and the lemma follows. □

In the sequel the coefficients a_{ij} $(1 \le i, j \le N)$ are assumed to verify the periodicity hypothesis

$$a_{ij}(y+k) = a_{ij}(y) \quad \text{a.e. in } \mathbb{R}^N \ (1 \le i, j \le N) \tag{3.16}$$

for all $k \in \mathbb{Z}^N$. Moreover, the potential $\mathcal{V}$ is supposed to satisfy

$$\mathcal{V}(y+k) = \mathcal{V}(y) \quad \text{a.e. in } \mathbb{R}^N \tag{3.17}$$

for all $k \in \mathbb{Z}^N$, and (3.3). Therefore the functions a_{ij} $(1 \le i, j \le N)$ and $\mathcal{V}$ are Y-periodic, where $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^N$. Further, $\mathcal{V}$ is of zero mean value.

Before we can prove our convergence theorem, let us state the following useful lemma whose proof is left to the reader.

Lemma 3.4. *There exist a constant $c_0 = c_0(\Omega, Y, T) > 0$ and some real number $\varepsilon_0 > 0$ such that for all $w \in \left(\mathcal{C}_{per}(Y)/\mathbb{C}\right) \otimes \mathcal{C}_{per}(Z)$,*

$$\left|\int_Q w^\varepsilon \varphi dxdt\right| \le c_0 \varepsilon \|w\|_{L^2_{per}(Y\times Z)} \|\varphi\|_{L^2\left(0,T;H_0^1(\Omega)\right)}$$

for all $\varepsilon \in E$, $\varepsilon \le \varepsilon_0$ and for all $\varphi \in \mathcal{D}^1(Q)$, where E is a fundamental sequence.

Now, let us make this useful remark.

Remark 3.5. In view of the density of $\left(\mathcal{C}_{per}(Y)/\mathbb{C}\right) \otimes \mathcal{C}_{per}(Z)$ in $L^2_{per}\left(Z; L^2_{per}(Y)/\mathbb{C}\right)$, if (2.2) holds for any $w \in \left(\mathcal{C}_{per}(Y)/\mathbb{C}\right) \otimes \mathcal{C}_{per}(Z)$, then (2.2) holds for any $w \in L^2_{per}\left(Z; L^2_{per}(Y)/\mathbb{C}\right)$. Indeed, by virtue of Lemma 3.4, we have

$$\left|\int_Q w^\varepsilon \varphi dxdt\right| \le c_0 \varepsilon \|w\|_{L^2_{per}(Y\times Z)} \|\varphi\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \tag{3.18}$$

for all $w \in \left(\mathcal{C}_{per}(Y)/\mathbb{C}\right) \otimes \mathcal{C}_{per}(Z)$, $\varphi \in \mathcal{D}^1(Q)$ and all $\varepsilon \in E$, $\varepsilon < \varepsilon_0$. By density, (3.18) holds for all $w \in L^2_{per}\left(Z; L^2_{per}(Y)/\mathbb{C}\right)$ and all $\varphi \in L^2\left(0, T; H_0^1(\Omega)\right)$. In particular, we have

$$\left|\int_Q \frac{1}{\varepsilon} u_\varepsilon w^\varepsilon \varphi dxdt\right| \le c_0 \|w\|_{L^2(Y\times Z)} \|\varphi\|_\infty \|u_\varepsilon\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \quad (\varepsilon < \varepsilon_0) \tag{3.19}$$

for all $w \in L^2_{per}\left(Z; L^2_{per}(Y)/\mathbb{C}\right)$. Now, let $c_1 > 0$ be a constant such that

$\|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))} \leq c_1$ (for all $\varepsilon \in E$) and $\|u_1\|_{L^2(Q\times Y\times Z)} \leq c_1$. Further, fix $w \in L^2_{per}\left(Z; L^2_{per}(Y)/\mathbb{C}\right)$, $\varphi \in \mathcal{D}(Q)$ and let $c_2 > \max\{c_0 c_1 \|\varphi\|_\infty, \ c_1 \|\varphi\|_\infty\}$. Consider an arbitrary real $\eta > 0$. By density, choose $\psi \in \left(\mathcal{C}_{per}(Y)/\mathbb{C}\right) \otimes \mathcal{C}_{per}(Z)$ such that $\|w - \psi\|_{L^2(Y\times Z)} \leq \frac{\eta}{3c_2}$. Now writing

$$\int_Q \frac{1}{\varepsilon} u_\varepsilon w^\varepsilon \varphi dxdt - \int_Q \int \int_{Y\times Z} w u_1 \varphi dxdtdyd\tau =$$

$$\int_Q \frac{1}{\varepsilon} u_\varepsilon (w^\varepsilon - \psi^\varepsilon) \varphi dxdt + \int_Q \frac{1}{\varepsilon} u_\varepsilon \psi^\varepsilon \varphi dxdt - \int_Q \int \int_{Y\times Z} \psi u_1 \varphi dxdtdyd\tau$$

$$+ \int_Q \int \int_{Y\times Z} (\psi - w) u_1 \varphi dxdtdyd\tau,$$

we estimate the first integral on the right-hand side by using (3.19). We obtain

$$\left| \int_Q \frac{1}{\varepsilon} u_\varepsilon w^\varepsilon \varphi dxdt - \int_Q \int \int_{Y\times Z} w u_1 \varphi dxdtdyd\tau \right|$$

$$\leq 2c_2 \|w - \psi\|_{L^2(Y\times Z)} + \left| \int_Q \frac{1}{\varepsilon} u_\varepsilon \psi^\varepsilon \varphi dxdt - \int_Q \int \int_{Y\times Z} \psi u_1 \varphi dxdtdyd\tau \right| \tag{3.20}$$

for all $\varepsilon < \varepsilon_0$. Finally, using (2.2) we see that there exists $\alpha > 0$ such that

$$\left| \int_Q \frac{1}{\varepsilon} u_\varepsilon \psi^\varepsilon \varphi dxdt - \int_Q \int \int_{Y\times Z} \psi u_1 \varphi dxdtdyd\tau \right| \leq \frac{\eta}{3}$$

for all $\varepsilon < \alpha$. Thus, by (3.20) we have

$$\left| \int_Q \frac{1}{\varepsilon} u_\varepsilon w^\varepsilon \varphi dxdt - \int_Q \int \int_{Y\times Z} w u_1 \varphi dxdtdyd\tau \right| \leq \frac{2\eta}{3} + \frac{\eta}{3} = \eta$$

for all $\varepsilon < \alpha$.

We are now in position to prove our convergence theorem.

Theorem 3.6. *Suppose the hypotheses of Lemma 3.1 are satisfied. For fixed $\varepsilon > 0$, let u_ε be the solution of (1.3)-(1.5). Then, as $\varepsilon \to 0$, we have:*

$$u_\varepsilon \to u_0 \text{ in } \mathcal{Y}(0,T)\text{-weak,} \tag{3.21}$$

$$u_\varepsilon \to u_0 \text{ in } L^2(Q)\text{-strong} \tag{3.22}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^2(Q)\text{-weak 2-s} \quad (1 \leq i, j \leq N), \tag{3.23}$$

where $\mathbf{u} = (u_0, u_1) \in \mathbb{F}^1_0$ is the unique solution of (3.13).

Proof. According to Lemma 3.1, the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $\mathcal{Y}(0,T)$. Hence, if E is a fundamental sequence, by virtue of Theorem 2.4 there are some subsequence E' extracted from E and some vector function $\mathbf{u}=(u_0,u_1)\in\mathbb{F}_0^1$ such that (3.21)-(3.23) hold when $E'\ni\varepsilon\to 0$. Thus, thanks to Lemma 3.3, the theorem is certainly proved if we can show that $\mathbf{u}$ verifies (3.13). Indeed, we begin by verifying that $u_0(0)=0$ (it is worth recalling that u_0 may be viewed as a continuous mapping of $[0,T]$ into $L^2(\Omega)$).

Let $v\in H_0^1(\Omega)$, and let $\varphi\in C^1([0,T])$ with $\varphi(T)=0$. By integration by parts, we have,

$$\int_0^T \langle u'_\varepsilon(t),v\rangle\varphi(t)\,dt+\int_0^T \langle u_\varepsilon(t),v\rangle\varphi'(t)\,dt=-\langle u_\varepsilon(0),v\rangle\varphi(0)=0,$$

since $u_\varepsilon(0)=0$. In view of (3.21)-(3.22), we pass to the limit in the preceding equality as $E'\ni\varepsilon\to 0$. We obtain

$$\int_0^T \left\langle u'_0(t),v\right\rangle\varphi(t)\,dt+\int_0^T \langle u_0(t),v\rangle\varphi'(t)\,dt=0.$$

Since φ and v are arbitrary, we see that $u_0(0)=0$.
Finally, let us prove the variational equality of (3.13). Fix any arbitrary two functions

$$\psi_0\in\mathcal{D}(Q)\ \text{ and }\psi_1\in\mathcal{D}(Q)\otimes\left[\left(\mathcal{C}_{per}(Y)/\mathbb{C}\right)\otimes\mathcal{C}_{per}(Z)\right],$$

and let

$$\psi_\varepsilon=\psi_0+\varepsilon\psi_1^\varepsilon,\ \text{i.e.},\ \psi_\varepsilon(x,t)=\psi_0(x,t)+\varepsilon\psi_1\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)\ \text{for all }\ (x,t)\in Q,$$

where $\varepsilon>0$ is arbitrary. By (1.3), one as

$$\mathbf{i}\int_0^T \left\langle u'_\varepsilon(t),\overline{\psi}_\varepsilon(t)\right\rangle dt+\int_0^T a^\varepsilon(u_\varepsilon(t),\psi_\varepsilon(t))\,dt+\frac{1}{\varepsilon}\int_0^T (\mathcal{V}^\varepsilon u_\varepsilon(t),\psi_\varepsilon(t))\,dt=\int_0^T (f(t),\psi_\varepsilon(t))\,dt. \tag{3.24}$$

The aim is to pass to the limit in (3.24) as $E'\ni\varepsilon\to 0$. First, we have

$$\int_0^T \left\langle u'_\varepsilon(t),\overline{\psi}_\varepsilon(t)\right\rangle dt=-\int_Q u_\varepsilon\frac{\partial\overline{\psi}_\varepsilon}{\partial t}dxdt=-\int_Q u_\varepsilon\left(\frac{\partial\overline{\psi}_0}{\partial t}+\varepsilon\left(\frac{\partial\overline{\psi}_1}{\partial t}\right)^\varepsilon+\left(\frac{\partial\overline{\psi}_1}{\partial\tau}\right)^\varepsilon\right)dxdt.$$

Thus, in view of (3.22) (and using Definition 2.1), we have,

$$\int_0^T \left\langle u'_\varepsilon(t),\overline{\psi}_\varepsilon(t)\right\rangle dt\to-\int_Q u_0\frac{\partial\overline{\psi}_0}{\partial t}dxdt=\int_0^T \left\langle u'_0(t),\overline{\psi}_0(t)\right\rangle dt$$

as $E'\ni\varepsilon\to 0$, since

$$\int_Q\left(\int\int_{Y\times Z}\frac{\partial\overline{\psi}_1}{\partial\tau}dyd\tau\right)u_0dxdt=0$$

by virtue of the $Y\times Z$-periodicity of ψ_1.
Next, we have

$$\int_0^T a^\varepsilon(u_\varepsilon(t),\psi_\varepsilon(t))\,dt\to\int_0^T \mathfrak{a}(\mathbf{u}(t),\phi(t))\,dt$$

as $E' \ni \varepsilon \to 0$, where $\phi = (\psi_0, \psi_1)$ (proceed as in the proof of the similar result in [11, p.179]). On the other hand,

$$\frac{1}{\varepsilon}\int_0^T (\mathcal{V}^\varepsilon u_\varepsilon(t), \psi_\varepsilon(t))\,dt = \frac{1}{\varepsilon}\int_Q \mathcal{V}^\varepsilon u_\varepsilon \overline{\psi}_0 dxdt + \int_Q \mathcal{V}^\varepsilon u_\varepsilon \overline{\psi}_1^\varepsilon dxdt. \tag{3.25}$$

In view of Lemma 2.5 and Remark 3.5, and by the fact that $\mathcal{V}$ belongs to $L^2_{per}\left(Z; L^2_{per}(Y)/\mathbb{C}\right)$ (*by virtue of (3.3) and (3.17)*), we pass to the limit in (3.25). This yields,

$$\frac{1}{\varepsilon}\int_0^T (\mathcal{V}^\varepsilon u_\varepsilon(t), \psi_\varepsilon(t))\,dt \to \int\int\int_{Q\times Y\times Z} \left(u_1\overline{\psi}_0 + u_0\overline{\psi}_1\right)\mathcal{V}dxdtdyd\tau$$

as $E' \ni \varepsilon \to 0$. Hence, passing to the limit in (3.24) as $E' \ni \varepsilon \to 0$ leads to

$$\begin{array}{c}\mathbf{i}\int_0^T \left\langle u_0'(t), \overline{\psi}_0(t)\right\rangle dt + \int_0^T \mathfrak{a}(\mathbf{u}(t), \phi(t))\,dt + \int\int\int_{Q\times Y\times Z}\left(u_1\overline{\psi}_0 + u_0\overline{\psi}_1\right)\mathcal{V}dxdtdyd\tau \\ = \int_0^T (f(t), \psi_0(t))\,dt\end{array} \tag{3.26}$$

for all $\phi = (\psi_0, \psi_1) \in \mathcal{F}_0^\infty$. Moreover, since $\mathcal{F}_0^\infty$ is a dense subspace of $\mathbb{F}_0^1$, by (3.26) we see that $\mathbf{u} = (u_0, u_1)$ verifies (3.13). Thanks to the uniqueness of the solution for (3.13) and the fact that the sequence E is arbitrary, we have (3.21)-(3.23) as $\varepsilon \to 0$. The theorem is proved. $\square$

For further needs, we wish to give a simple representation of the function u_1 in Theorem 3.6. For this purpose, let us introduce the form $\widehat{\mathfrak{a}}$ on $L^2_{per}\left(Z; H^1_\#(Y)\right) \times L^2_{per}\left(Z; H^1_\#(Y)\right)$ defined by

$$\widehat{\mathfrak{a}}(w, v) = \sum_{i,j=}^N \int\int_{Y\times Z} a_{ij}\frac{\partial w}{\partial y_j}\overline{\frac{\partial v}{\partial y_i}}dyd\tau$$

for all w, $v \in L^2_{per}\left(Z; H^1_\#(Y)\right)$. By virtue of (1.1)-(1.2), the sesquilinear form $\widehat{\mathfrak{a}}$ is continuous, hermitian and coercive with,

$$\widehat{\mathfrak{a}}(v, v) \geq \alpha \|v\|^2_{L^2_{per}(Z;H^1_\#(Y))} \text{ for all } v \in L^2_{per}\left(Z; H^1_\#(Y)\right).$$

Next, for any indice l with $1 \leq l \leq N$, we consider the variational problem

$$\left\{\begin{array}{l}\chi^l \in L^2_{per}\left(Z; H^1_\#(Y)\right) \\ \widehat{\mathfrak{a}}\left(\chi^l, v\right) = \sum_{i=1}^N \int\int_{Y\times Z} a_{il}\overline{\frac{\partial v}{\partial y_i}}dyd\tau \\ \text{for all } v \in L^2_{per}\left(Z; H^1_\#(Y)\right),\end{array}\right. \tag{3.27}$$

which determines χ^l in a unique manner. Further, let $\eta \in L^2_{per}\left(Z; H^1_\#(Y)\right)$ be the unique function defined by

$$\widehat{\mathfrak{a}}(\eta, v) = \int\int_{Y\times Z} \mathcal{V}\overline{v}dyd\tau \quad \text{for all } v \in L^2_{per}\left(Z; H^1_\#(Y)\right). \tag{3.28}$$

Lemma 3.7. *Under the hypotheses of Theorem 3.6, we have*

$$u_1(x,t,y,\tau) = -\sum_{j=1}^{N} \frac{\partial u_0}{\partial x_j}(x,t)\chi^j(y,\tau) + \eta(y,\tau)u_0(x,t) \tag{3.29}$$

for almost all $(x,t,y,\tau) \in Q\times Y\times Z$.

Proof. In (3.13) choose the particular test function $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^1$ with $v_0 = 0$ and $v_1 = \varphi\otimes v$, where $\varphi \in \mathcal{D}(Q)$ and $v \in L^2_{per}\left(Z; H^1_\#(Y)\right)$. This yields

$$\widehat{\mathfrak{a}}(u_1(x,t), v) = -\sum_{i,j=1}^{N} \frac{\partial u_0}{\partial x_j}(x,t)\int\int_{Y\times Z} a_{ij}\frac{\overline{\partial v}}{\partial y_i}dyd\tau + u_0(x,t)\int\int_{Y\times Z}\mathcal{V}\overline{v}dyd\tau \tag{3.30}$$

almost everywhere in $(x,t)\in Q$ and for all $v \in L^2_{per}\left(Z; H^1_\#(Y)\right)$. But it is clear that $u_1(x,t)$ (for fixed $(x,t)\in Q$) is the sole function in $L^2_{per}\left(Z; H^1_\#(Y)\right)$ solving the variational equation (3.30). On the other hand, in view of (3.27)-(3.28) it is an easy matter to check that the right hand side of (3.29) solves the same variational equation. Hence the lemma follows immediatly. □

3.3 The macroscopic homogenized equation

Our aim here is to derive the initial boundary value problem for u_0. To begin, for $1 \le i, j \le N$, let

$$q_{ij} = \int_Y a_{ij}dy - \sum_{l=1}^{N}\int\int_{Y\times Z} a_{il}\frac{\partial \chi^j}{\partial y_l}dyd\tau,$$

$$b_i = -\int\int_{Y\times Z}\chi^i\mathcal{V}dyd\tau - \sum_{j=1}^{N}\int\int_{Y\times Z} a_{ij}\frac{\partial\eta}{\partial y_j}dyd\tau.$$

Fruther, let

$$\mu = \int\int_{Y\times Z}\eta\mathcal{V}dyd\tau.$$

To the coefficients q_{ij} we attach the differential operator $\mathcal{Q}$ on Q mapping $\mathcal{D}'(Q)$ into $\mathcal{D}'(Q)$ ($\mathcal{D}'(Q)$ being the usual space of complex distributions on Q) as

$$\mathcal{Q}u = -\sum_{i,j=}^{N} q_{ij}\frac{\partial^2 u}{\partial x_j\partial x_i} \text{ for all } u \in \mathcal{D}'(Q).$$

Let

$$b = (b_i)_{i=1,\dots,N}.$$

We consider the following initial boundary value problem:

$$\mathbf{i}\frac{\partial u_0}{\partial t} + \mathcal{Q}u_0 + b\cdot\nabla u_0 + \mu u_0 = f \text{ in } Q = \Omega\times]0,T[\tag{3.31}$$

$$u_0 = 0 \text{ on } \partial\Omega\times]0,T[\tag{3.32}$$

$$u_0(0) = 0 \text{ in } \Omega. \tag{3.33}$$

The initial boundary value problem (3.31)-(3.33) is the so-called macroscopic homogenized equation.

Lemma 3.8. *Suppose the hypotheses of Lemma 3.1 are satisfied. Then, the initial boundary value problem (3.31)-(3.33) admits at most one weak solution u_0 in $\mathcal{Y}(0,T)$.*

Proof. It is an easy exercise to show that if $u_0 \in \mathcal{Y}(0,T)$ verifies (3.31)-(3.33) then $\mathbf{u} = (u_0, u_1)$ [with u_1 given by (3.29)] satisfies (3.13). Hence, the unicity in (3.31)-(3.33) follows by Lemma 3.3. □

Theorem 3.9. *Suppose the hypotheses of Lemma 3.1 are satisfied. For $\varepsilon > 0$, let $u_\varepsilon \in \mathcal{Y}(0,T)$ be defined by (1.3)-(1.5). Then, as $\varepsilon \to 0$, we have $u_\varepsilon \to u_0$ in $\mathcal{Y}(0,T)$-weak, where u_0 is the unique weak solution of (3.31)-(3.33) in $\mathcal{Y}(0,T)$.*

Proof. As in the proof of Theorem 3.6, from any fundamental sequence E one can extract a subsequence E' such that as $E' \ni \varepsilon \to 0$, we have (3.21)-(3.23), and further (3.26) holds for all $\phi = (\psi_0, \psi_1) \in \mathcal{F}_0^\infty$, where $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^1$. Now, substituting (3.29) in (3.26) and then choosing therein the ϕ's such that $\psi_1 = 0$, a simple computation yields (3.31) with (3.32)-(3.33), of course. Hence the theorem follows by Lemma 3.8 and using of an obvious argument. □

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