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Pseudo-Almost Periodic and Pseudo-Almost Automorphic Solutions of Class r Under the Light of Measure Theory

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Abstract

The aim of this work is to present new approach to study weighted pseudo almost periodic and automorphic functions using the measure theory. We present a new concept of weighted ergodic functions which is more general than the classical one. Then we establish many interesting results on the functional space of such functions. We study the existence and uniqueness of (μ, ν)-pseudo almost periodic and automorphic solutions of class r for some neutral partial functional differential equations in a Banach space when the delay is distributed using the spectral decomposition of the phase space developed in Adimy and co-authors. Here we assume that the undelayed part is not necessarily densely defined and satisfies the well-known Hille-Yosida condition, the delayed part are assumed to be pseudo almost periodic with respect to the first argument and Lipschitz continuous with respect to the second argument.

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1 Introduction

In this work, we present a new approach dealing with weighted pseudo almost periodic functions and their applications in evolution equations and partial functional differential equations. Here we use the measure theory to define an ergodic function and we investigate many interesting properties of such functions. Weighted pseudo almost periodic functions

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started recently and becomes an interesting field in dynamical systems. The study of existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic and pseudo almost periodic solutions is one of the most attractive topics in the qualitative theory of differential equations due both to its mathematical interest and applications in physics, mathematical biology, and control theory, among other areas. Most of these problems need to be studied in abstract spaces and the operators are defined over non-dense domains. In this context the literature is very scarce (see [1] ,[2], [4] and the bibliography therein).

In this work, we study the existence and uniqueness of (μ,ν)-pseudo almost periodic and automorphic solutions of class r for the following neutral partial functional differential equation

$$u'(t) = Au(t) + L(u_t) + f(t) \text{ for } t \in \mathbb{R}, \tag{1.1}$$

where A is a linear operator on a Banach space X satisfying the Hille-Yosida condition, that is, there exist $M_0 \geq 1$ and $\omega \in \mathbb{R}$ such that $]\omega,+\infty[\subset \rho(A)$ and

$$|R(\lambda,A)^n| \leq \frac{M_0}{(\lambda-\omega)^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda,A) = (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$. In sequel, without lost of generality, we suppose that $M_0 = 1$. $C = C([-r,0];X)$ denotes the space of continuous functions from $[-r,0]$ to X endowed with the uniform topology norm. For every $t \geq 0$, u_t denotes the history function of C defined by

$$u_t(\theta) = u(t+\theta) \text{ for } -r \leq \theta \leq 0.$$

L is a bounded linear operator from C into X and $f : \mathbb{R} \to X$ is a continuous function.
Some recent contributions concerning pseudo almost periodic solutions for abstract differential equations similar to equation (1.1) have been made. For example in [2] the authors have shown that if the inhomogeneous term f depends only on variable t and it is a pseudo almost periodic function, then equation (1.1) has a unique pseudo almost periodic solution. In [4] the authors have proven that if $f : \mathbb{R}\times X_0 \to X$ is a suitable continuous function, where $X_0 = \overline{D(A)}$, the problem

$$x'(t) = Ax(t) + f(t,x(t)), t \in \mathbb{R}$$

has a unique pseudo almost periodic solution, while in [1] the authors have treated the existence of almost periodic solutions for a class of partial neutral functional differential equations defined by a linear operator of Hille-Yosida type with non-dense domain. In [3], the authors studied the existence and uniqueness of pseudo almost periodic solutions for a first-order abstract functional differential equation with a linear part dominated by a Hille-Yosida type operator with a non-dense domain.
In [9], the authors introduce some new classes of functions called weighted pseudo-almost periodic functions, which implement in a natural fashion the classical pseudo-almost periodic functions due to Zhang ([19, 20, 21]). Properties of these weighted pseudo-almost periodic functions are discussed, including a composition result for weighted pseudo-almost periodic functions. The results obtained are subsequently utilized to study the existence and

uniqueness of a weighted pseudo-almost periodic solution to the heat equation with Dirichlet conditions.
In [6], the authors present new approach to study weighted pseudo almost periodic functions using the measure theory. They present a new concept of weighted ergodic functions which is more general than the classical one. Then they establish many interesting results on the functional space of such functions like completeness and composition theorems. The theory of their work generalizes the classical results on weighted pseudo almost periodic functions.

The aim of this work is to prove the existence of (μ,ν)-pseudo almost periodic and automorphic solutions of equation (1.1) when the delay is distributed on $[-r,0]$. Our approach is based on the spectral decomposition of the phase space developed in [3] and a new approach developped in [6].
This work is organised as follow, in section 2 we recall some prelimary results on spectral decomposition. In section 3, we recall some prelimary results on (μ,ν)-pseudo almost periodic functions and neutral partial functional differential equations that will be used in this work. In section 4, we give some properties of (μ,ν)-pseudo almost periodic functions of class r. In section 5 and 7, we discuss the main result of this paper. Using the strict contraction principle we show the existence and uniqueness of (μ,ν)-pseudo almost periodic solution of class r for equation (1.1). Sections 6 and 8 are devoted to some applications arising in population dynamics.

2 Spectral decomposition

To equation (1.1), we associate the following initial value problem

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(u_t) + f(t) \text{ for } t \geq 0 \\ u_0 = \varphi \in C = C([-r,0];X), \end{cases} \tag{2.1}$$

where $f : \mathbb{R}^+ \to X$ is a continuous function.

Definition 2.1. We say that a continuous function u from $[-r,+\infty[$ into X is an integral solution of equation (2.1), if the following conditions hold:
i) $\displaystyle\int_0^t u(s)ds \in D(A)$ for $t \geq 0$,
ii) $u(t) = \varphi(0) + A\displaystyle\int_0^t u(s)ds + \int_0^t (L(u_s) + f(s))ds$ for $t \geq 0$,
iii) $u_0 = \varphi$.

If $\overline{D(A)} = X$, the integral solutions coincide with the known mild solutions.
One can see that if $u(t)$ is an integral solution of equation (2.1), then $u(t) \in \overline{D(A)}$ for all $t \geq 0$, in particular $\varphi(0) \in \overline{D(A)}$.
Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ which defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : \ Ax \in \overline{D(A)}\} \\ A_0 x = Ax \text{ ;for } \ x \in D(A_0) \end{cases}$$

We make the following assertion:
($\mathbf{H_0}$) A satisfies the Hille-Yosida condition.

Lemma 2.2. *[1] A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

Proposition 2.3. *[2] Assume that ($\boldsymbol{H_0}$) holds, then for all $\varphi \in C$ such that $\varphi(0) \in \overline{D(A)}$, equation (2.1) has a unique integral solution u on $[-r, +\infty[$. Moreover, u is given by*

$$u(t) = T_0(t)\varphi(0) + \lim_{\lambda \to +\infty} \int_0^t T_0(t-s)B_\lambda(L(u_s) + f(s))ds, \ \ for \ \ t \geq 0,$$

where $B_\lambda = \lambda R(\lambda, A)$, for $\lambda > \omega$.

The phase space C_0 of equation (2.1) is defined by

$$C_0 = \{\varphi \in C : \ \varphi(0) \in \overline{D(A)}\}.$$

For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_0 by

$$\mathcal{U}(t)\varphi = v_t(., \varphi)$$

where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt}v(t) = Av(t) + L(v_t) \text{ for } t \geq 0 \\ v_0 = \varphi \in C. \end{cases}$$

Proposition 2.4. *[3] $(\mathcal{U}(t))_{t\geq 0}$ is a strongly continuous semigroup of linear operators on C_0. Moreover, $(\mathcal{U}(t))_{t\geq 0}$ satisfies, for $t \geq 0$ and $\theta \in [-r, 0]$, the following translation property*

$$(\mathcal{U}(t)\varphi)(\theta) = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0) \, for \, t+\theta \geq 0 \\ \varphi(t+\theta) \, for \, t+\theta \leq 0. \end{cases}$$

Proposition 2.5. *[3] Let $\mathcal{A}_{\mathcal{U}}$ defined on C_0 by*

$$\begin{cases} D(\mathcal{A}_{\mathcal{U}}) = \left\{\varphi \in C^1([-r, 0]; X); \ \varphi(0) \in D(A), \ \varphi(0)' \in \overline{D(A)} \ \ and \ \ \varphi(0)' = A\varphi(0) + L(\varphi)\right\} \\ \mathcal{A}_{\mathcal{U}}\varphi = \varphi' \ \ for \ \ \varphi \in D(\mathcal{A}_{\mathcal{U}}). \end{cases}$$

Then $\mathcal{A}_{\mathcal{U}}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ on C_0.

Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{X_0 c : \ c \in X\}$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 \ \text{ if } \ \theta \in [-r, 0[, \\ c \ \text{ if } \ \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0 \rangle$ equipped with the norm $|\phi + X_0 c| = |\phi|_C + |c|$ for $(\phi, c) \in C_0 \times X$ is a Banach space and consider the extension $\mathcal{A}_{\mathcal{U}}$ defined on $C_0 \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_{\mathcal{U}}}) = \left\{\varphi \in C^1([-r, 0]; X) : \ \varphi(0) \in D(A) \ \text{ and } \varphi(0)' \in \overline{D(A)}\right\} \\ \widetilde{\mathcal{A}_{\mathcal{U}}}\varphi = \varphi' + X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

Lemma 2.6. *[3] Assume that ($\boldsymbol{H_0}$) holds. Then, $\widetilde{\mathcal{A}_{\mathcal{U}}}$ satisfies the Hille-Yosida condition on $C_0 \oplus \langle X_0 \rangle$ there exist $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}_{\mathcal{U}}})$ and*

$$|(\lambda I - \widetilde{\mathcal{A}_{\mathcal{U}}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

Moreover, the part of $\widetilde{\mathcal{A}_{\mathcal{U}}}$ on $D(\widetilde{\mathcal{A}_{\mathcal{U}}}) = C_0$ is exactly the operator $\mathcal{A}_{\mathcal{U}}$.

Now, we can state the variation of constants formula associated to equation (2.1).

Proposition 2.7. *[3] Assume that ($\boldsymbol{H_0}$) holds. Then for all $\varphi \in C_0$, the solution u of equation* (2.1) *is given by the following formula*

$$u_t = \mathcal{U}(t)\varphi + \lim_{\lambda \to +\infty} \int_0^t \mathcal{U}(t-s)\widetilde{B}_\lambda(X_0 f(s))ds \text{ for } t \geq 0,$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}}_{\mathcal{U}})^{-1}$ for $\lambda > \widetilde{\omega}$.

Definition 2.8. We say a semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if

$$\sigma(\mathcal{A}_{\mathcal{U}}) \cap i\mathbb{R} = \emptyset$$

For the sequel, we make the following assumption:
($\mathbf{H_1}$) $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

Proposition 2.9. *[3] Assume that ($\boldsymbol{H_0}$) and ($\boldsymbol{H_1}$). Then the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is compact for $t > r$.*

From the compactness of the semigroup $(\mathcal{U}(t))_{t\geq 0}$, we get the following result on the spectral decomposition of the phase space C_0.

Proposition 2.10. *[14] Assume that ($\boldsymbol{H_1}$) holds. If the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic, then the space C_0 is decomposed as a direct sum*

$$C_0 = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restricted semigroup on $\mathcal{U}$ is a group and there exist positive constants $\overline{M}$ and ω such that

$$\begin{aligned} |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{-\omega t}|\varphi| \text{ for } t \geq 0 \text{ and } \varphi \in S \\ |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{\omega t}|\varphi| \text{ for } t \leq 0 \text{ and } \varphi \in U, \end{aligned}$$

where S and U are called respectively the stable and unstable space, Π^s and Π^u denote respectively the projection operator on S and U.

3 (μ,ν)-Pseudo almost periodic functions

In this section, we recall some properties about μ-pseudo almost periodic functions. The notion of μ-pseudo almost periodicity is a generalization of the pseudo almost periodicity introduced by Zhang [19, 20, 21]; it is also a generalization of weighted pseudo almost periodicity given by Diagana [9]. Let $BC(\mathbb{R};X)$ be the space of all bounded and continuous function from $\mathbb{R}$ to X equipped with the uniform topology norm.
We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < \infty$, for all $a,b \in \mathbb{R}$ ($a \leq b$).

Definition 3.1. A bounded continuous function $\phi : \mathbb{R} \to X$ is called almost periodic if for each $\varepsilon > 0$, there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{K}(\varepsilon,\phi,X)$ such that $|\phi(t+\tau)-\phi(t)| < \varepsilon$ for all $(t,\tau) \in \mathbb{R}\times\mathcal{K}(\varepsilon,\phi,X)$.

We denote by $AP(\mathbb{R};X)$, the space of all such functions.

Definition 3.2. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R}\times X_1 \to X_2$ is called almost periodic in $t \in \mathbb{R}$ uniformly in $x \in X_1$ if for each $\varepsilon > 0$ and all compact $K \subset X_1$, there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{K}(\varepsilon,\phi,K)$ such that $|\phi(t+\tau,x)-\phi(t,x)| < \varepsilon$ for all $t \in \mathbb{R},\ x \in K,\ \tau \in \mathcal{K}(\varepsilon,\phi,K)$.

We denote by $AP(\mathbb{R}\times X_1;X_2)$, the space of all such functions.
The next lemma is also a characterization of almost periodic functions.

Lemma 3.3. *A function $\phi \in C(\mathbb{R},X)$ is almost periodic if and only if the space of functions $\{\phi_\tau :\ \tau \in \mathbb{R}\}$, where $(\phi_\tau)(t) = \phi(t+\tau)$, is relatively compact in $BC(\mathbb{R};X)$.*

In the sequel, we recall some preliminary results concerning the (μ,ν)-Pseudo almost periodic functions.
$\mathcal{E}(\mathbb{R};X,\mu,\nu)$ stands for the space of functions

$$\mathcal{E}(\mathbb{R};X,\mu,\nu) = \Big\{u \in BC(\mathbb{R};X) :\ \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} |u(t)| d\mu(t) = 0\Big\}.$$

To study delayed differential equations for which the history belong to $C([-r,0];X)$, we need to introduce the space

$$\mathcal{E}(\mathbb{R};X,\mu,\nu,r) = \Big\{u \in BC(\mathbb{R};X) :\ \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |u(\theta)|\Big) d\mu(t) = 0\Big\}.$$

In addition to above-mentioned space, we consider the following spaces

$$\mathcal{E}(\mathbb{R}\times X_1,X_2,\mu,\nu) = \Big\{u \in BC(\mathbb{R}\times X_1;X_2) :\ \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} |u(t,x)|_{X_2} d\mu(t) = 0\Big\},$$

$$\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu,r) = \Big\{u \in BC(\mathbb{R}\times X_1;X_2) :\ \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |u(\theta,x)|_{X_2}\Big) d\mu(t) = 0\Big\},$$

where in both cases the limit (as $\tau \to +\infty$) is uniform in compact subset of X_1.
In view of previous definitions, it is clear that the spaces $\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$ and $\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu,r)$

are continuously embedded in $\mathcal{E}(\mathbb{R};X,\mu,\nu)$ and $\mathcal{E}(\mathbb{R}\times X_1,X_2,\mu,\nu)$, respectively.
On the other hand, one can observe that a ρ-weighted pseudo almost periodic functions is μ-pseudo almost periodic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ:

$$d\mu(t) = \rho(t)dt$$

and ν is the usual Lebesgue measure on $\mathbb{R}$, i.e $\nu([-\tau,\tau] = 2\tau$ for all $\tau \geq 0$.

Example 3.4. [6] Let ρ be a nonnegative $\mathcal{B}$-measurable function. Denote by μ the positive measure defined by

$$\mu(A) = \int_A \rho(t)dt, \text{ for } A \in \mathcal{B}, \tag{3.1}$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ which occurs in equation (3.1) is called the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$.

From $\mu,\nu \in \mathcal{M}$, we formulate the following hypothese.
($\mathbf{H_2}$) Let $\mu,\nu \in \mathcal{M}$ be such that $\limsup\limits_{\tau\to+\infty} \dfrac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \alpha < \infty$.
We have the following result.

Lemma 3.5. *Asumme ($\boldsymbol{H_2}$) holds and let $f \in BC(\mathbb{R};X)$. Then $f \in \mathcal{E}(\mathbb{R};X,\mu,\nu)$ if and only if for any $\varepsilon > 0$,*

$$\lim_{\tau\to+\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau]} = 0$$

where

$$M_{\tau,\varepsilon}(f) = \{t \in [-\tau,\tau] : \ |f(t)| \geq \varepsilon\}.$$

Proof. Suppose that $f \in \mathcal{E}(\mathbb{R};X,\mu,\nu)$. Then

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} |f(t)|d\mu(t) &= \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)} |f(t)|d\mu(t) + \frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f)} |f(t)|d\mu(t)\\
&\geq \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)} |f(t)|d\mu(t)\\
&\geq \frac{\varepsilon}{\nu([-\tau,\tau])} M_{\tau,\varepsilon}(f).
\end{aligned}$$

Consequently

$$\lim_{\tau\to+\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau]} = 0.$$

Suppose that $f \in BC(\mathbb{R};X)$ such that for any $\varepsilon > 0$,

$$\lim_{\tau\to+\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau]} = 0$$

We can assume $|f(t)| \leq N$ for all $t \in \mathbb{R}$. Using ($\mathbf{H_2}$), we have

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|f(t)|d\mu(t) &= \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)}|f(t)|d\mu(t)+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f)}|f(t)|d\mu(t)\\
&\leq \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)}d\mu(t)+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f)}|f(t)|d\mu(t)\\
&\leq \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)}d\mu(t)+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}d\mu(t)\\
&\leq \frac{N}{\nu([-\tau,\tau])}M_{\tau,\varepsilon}(f)+\frac{\varepsilon\mu([-\tau,\tau])}{\nu([-\tau,\tau])}.
\end{aligned}$$

Which implies that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|f(t)|d\mu(t)\leq \alpha\varepsilon \text{ for any } \varepsilon>0.$$

Therefore $f \in \mathcal{E}(\mathbb{R};X,\mu,\nu)$.■

Definition 3.6. Let $\mu,\nu \in \mathcal{M}$. A bounded continuous function $\phi : \mathbb{R} \to X$ is called (μ,ν)-pseudo almost periodic if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R},X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R};X,\mu,\nu)$.

We denote by $PAP(\mathbb{R};X,\mu,\nu)$ the space of all such functions.

Definition 3.7. Let $\mu,\nu \in \mathcal{M}$ and X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R}\times X_1 \to X_2$ is called uniformly (μ,ν)-pseudo almost periodic if $\phi = \phi_1 + \phi_2$, where
$\phi_1 \in AP(\mathbb{R}\times X_1;X_2)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}\times X_1,X_2,\mu,\nu)$.

We denote by $PAP(\mathbb{R}\times X_1;X_2,\mu,\nu)$, the space of all such functions.

Definition 3.8. $\mu,\nu \in \mathcal{M}$. A bounded continuous function $\phi : \mathbb{R} \to X$ is called (μ,ν)-pseudo almost periodic of class r if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R};X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R};X,\mu,\nu,r)$.
We denote by $PAP(\mathbb{R};X,\mu,\nu,r)$, the space of all such functions.

Definition 3.9. $\mu,\nu \in \mathcal{M}$. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R}\times X_1 \to X_2$ is called uniformly (μ,ν)-pseudo almost periodic of class r if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R}\times X_1;X_2)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu,r)$.

We denote by $PAP(\mathbb{R}\times X_1;X_2,\mu,\nu,r)$, the space of all such functions.

4 Properties of (μ,ν)-pseudo almost periodic functions of class r

Lemma 4.1. *Assume that ($\boldsymbol{H_2}$) holds. The space $\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$ endowed with the uniform topology norm is a Banach space.*

Proof. We can see that $\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$ is a vector subspace of $BC(\mathbb{R};X)$. To complete the proof, it is enough to prove that $\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$ is closed in $BC(\mathbb{R};X)$. Let $(z_n)_n$ be a sequence in $\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$ such that $\lim_{n\to+\infty} z_n = z$ uniformly in $\mathbb{R}$. From $\nu(\mathbb{R}) = +\infty$, it follows $\nu([-\tau,\tau]) > 0$ for τ sufficiently large. Let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|z_n - z\|_\infty < \varepsilon$. Let $n \geq n_0$, then we have

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z(\theta)|\Big)d\mu(t) &\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_n(\theta)-z(\theta)|\Big)d\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_n(\theta)|\Big)d\mu(t)\\
&\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{t\in\mathbb{R}}|z_n(t)-z(t)|\Big)d\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_n(\theta)|\Big)d\mu(t)\\
&\leq \|z_n - z\|_\infty \times \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} + \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_n(\theta)|\Big)d\mu(t).
\end{aligned}$$

We deduce that

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z(\theta)|\Big)d\mu(t) \leq \alpha\varepsilon \text{ for any } \varepsilon > 0.\blacksquare$$

From the definition of $PAP(\mathbb{R};X,\mu,\nu,r)$, we deduce the following result.

Proposition 4.2. $\mu \in \mathcal{M}$. *The space $PAP(\mathbb{R};X,\mu,\nu,r)$ endowed with the uniform topology norm is a Banach space.*

Next result is a characterization of μ-ergodic functions of class r.

Theorem 4.3. *Assume that $(\boldsymbol{H_2})$ holds and let $\mu,\nu \in \mathcal{M}$ and I be a bounded interval (eventually $I = \emptyset$). Assume that $f \in BC(\mathbb{R},X)$. Then the following assertions are equivalent:*
i) $f \in \mathcal{E}(\mathbb{R},X,\mu,\nu,r)$.
ii) $\displaystyle\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f(\theta)|\Big)d\mu(t) = 0.$
iii) For any $\varepsilon > 0$, $\displaystyle\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t\in[-\tau,\tau]\setminus I : \sup_{\theta\in[t-r,t]}|f(\theta)| > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau]\setminus I)} = 0.$

Proof. *i)* $\Leftrightarrow$ *ii)* Denote by $A = \nu(I)$, $B = \displaystyle\int_I\Big(\sup_{\theta\in[t-r,t]}|f(\theta)|\Big)d\mu(t)$. We have A and $B \in \mathbb{R}$, since the interval I is bounded and the function f is bounded and continuous. For $\tau > 0$ such that $I \subset [-\tau,\tau]$ and $\nu([-\tau,\tau]\setminus I) > 0$, we have

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f(\theta)|\Big)d\mu(t) &= \frac{1}{\nu([-\tau,\tau])-A}\Big[\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}|f(\theta)|\Big)d\mu(t) - B\Big]\\
= \frac{\nu([-\tau,\tau])}{\nu([-\tau,\tau])-A}\Big[\frac{1}{\nu([-r,r])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}|f(\theta)|\Big)d\mu(t) - \frac{B}{\nu([-\tau,\tau])}\Big].
\end{aligned}$$

From above equalities and the fact that $\nu(\mathbb{R}) = +\infty$, we deduce that *ii*) is equivalent to

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |f(\theta)| \Big) d\mu(t) = 0,$$

that is *i*).
iii) $\Rightarrow$ *ii*) Denote by A_τ^ε and B_τ^ε the following sets

$$A_\tau^\varepsilon = \Big\{ t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} |f(\theta)| > \varepsilon \Big\} \text{ and } B_\tau^\varepsilon = \Big\{ t \in [-\tau,\tau] \setminus I) : \sup_{\theta\in[t-r,t]} |f(\theta)| \le \varepsilon \Big\}.$$

Assume that *iii*) holds, that is

$$\lim_{\tau\to+\infty} \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)} = 0. \tag{4.1}$$

From the equality

$$\int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f(\theta)| \Big) d\mu(t) = \int_{A_\tau^\varepsilon} \Big(\sup_{\theta\in[t-r,t]} |f(\theta)| \Big) d\mu(t) + \int_{B_\tau^\varepsilon} \Big(\sup_{\theta\in[t-r,t]} |f(\theta)| \Big) d\mu(t),$$

we deduce that for τ sufficiently large

$$\frac{1}{\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f(\theta)| \Big) d\mu(t) \le \|f\|_\infty \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)} + \varepsilon \frac{\mu(B_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}.$$

By using $(\mathbf{H_2})$, it follows that

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |f(\theta)| \Big) d\mu(t) \le \alpha\varepsilon, \text{ for any } \varepsilon > 0,$$

consequently (ii) holds.
ii) $\Rightarrow$ *iii*) Assume that *ii*) holds. From the following inequality

$$\begin{aligned}
\int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f(\theta)| \Big) d\mu(t) &\ge \int_{A_\tau^\varepsilon} \Big(\sup_{\theta\in[t-r,t]} |f(\theta)| \Big) d\mu(t) \\
\frac{1}{\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f(\theta)| \Big) d\mu(t) &\ge \varepsilon \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)} \\
\frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f(\theta)| \Big) d\mu(t) &\ge \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)},
\end{aligned}$$

for τ sufficiently large, we obtain equation (4.1), that is *iii*).■
From $\mu \in \mathcal{M}$, we formulate the following hypotheses.
$(\mathbf{H_3})$ For all a, b and $c \in \mathbb{R}$, such that $0 \le a < b \le c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \ge \delta_0 \Rightarrow \mu(a+\delta, b+\delta) \le \alpha_0 \mu(\delta, c+\delta).$$

$(\mathbf{H_4})$ For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a+\tau : a \in A\} \le \beta\mu(A) \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

We have the following results due to [6]

Lemma 4.4. *[6] Hypothesis ($\boldsymbol{H_4}$) implies ($\boldsymbol{H_3}$).*

Proposition 4.5. *[5, 8] $\mu,\nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_3}$) and $f \in PAP(\mathbb{R};X,\mu,\nu)$ be such that*

$$f = g + h$$

where $g \in AP(\mathbb{R},X)$ and $h \in \mathcal{E}(\mathbb{R},X,\mu,\nu)$. Then

$$\{g(t),\ t \in \mathbb{R}\} \subset \overline{\{f(t),\ t \in \mathbb{R}\}} \text{ (the closure of the range of f).}$$

Corollary 4.6. *[8] Assume that ($\boldsymbol{H_3}$) holds. Then the decomposition of a (μ,ν)-pseudo almost periodic function in the form $f = g + \phi$ where $g \in AP(\mathbb{R};X)$ and $\phi \in \mathcal{E}(\mathbb{R};X,\mu,\nu)$, is unique.*

The following corollary is a consequence of Theorem 4.3.

Proposition 4.7. *Let $\mu,\nu \in \mathcal{M}$. Assume ($\boldsymbol{H_3}$) holds. Then the decomposition of a (μ,ν)-pseudo-almost periodic function $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R};X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R};X,\mu,\nu,r)$, is unique.*

Proof. In fact, since as a consequence of Corollary 4.6, the decomposition of a (μ,ν)-pseudo-almost periodic function $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R};X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R};X,\mu,\nu)$, is unique. Since $PAP(\mathbb{R};X,\mu,\nu,r) \subset PAP(\mathbb{R};X,\mu,\nu)$, we get the desired result.■

Definition 4.8. Let $\mu_1,\mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2, denoting this as $\mu_1 \sim \mu_2$ if there exist constants α and $\beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that

$$\alpha\mu_1(A) \leq \mu_2(A) \leq \beta\mu_1(A), \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

From [6] $\sim$ is a binary equivalence relation on $\mathcal{M}$. the equivalence class of a given measure $\mu \in \mathcal{M}$ will then be denoted by

$$cl(\mu) = \{\varpi \in \mathcal{M} :\ \mu \sim \varpi\}.$$

Theorem 4.9. *Let $\mu_1,\mu_2,\nu_1,\nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $PAP(\mathbb{R};X,\mu_1,\nu_1,r) = PAP(\mathbb{R};X,\mu_2,\nu_2,r)$.*

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$ there exist some constants $\alpha_1,\alpha_2,\beta_1,\beta_2 > 0$ and a bounded interval I (eventually $I = \emptyset$) such that $\alpha_1\mu_1(A) \leq \mu_2(A) \leq \beta_1\mu_1(A)$ and $\alpha_2\nu_1(A) \leq \nu_2(A) \leq \beta_2\nu_1(A)$ for each $A \in \mathcal{B}$ satisfies $A \cap I = \emptyset$ i.e

$$\frac{1}{\beta_2\nu_1(A)} \leq \frac{1}{\nu_2(A)} \leq \frac{1}{\alpha_2\nu_1(A)}.$$

Since $\mu_1 \sim \mu_2$ and $\mathcal{B}$ is the Lebesgue σ-field, we obtain for τ sufficiently large, it follows that

$$\begin{aligned}\frac{\alpha_1\mu_1\Big(\Big\{t\in[-\tau,\tau]\setminus I:\ \sup_{\theta\in[t-r,t]}|f(\theta)|>\varepsilon\Big\}\Big)}{\beta_2\nu_1([-\tau,\tau]\setminus I)} &\leq \frac{\mu_2\Big(\Big\{t\in[-\tau,\tau]\setminus I:\ \sup_{\theta\in[t-r,t]}|f(\theta)|>\varepsilon\Big\}\Big)}{\nu_2([-\tau,\tau]\setminus I)}\\ &\leq \frac{\beta_1\mu_1\Big(\Big\{t\in[-\tau,\tau]\setminus I:\ \sup_{\theta\in[t-r,t]}|f(\theta)|>\varepsilon\Big\}\Big)}{\alpha_2\nu_1([-\tau,\tau]\setminus I)}\end{aligned}$$

By using Theorem 4.3 we deduce that $\mathcal{E}(\mathbb{R},X,\mu_1,\nu_1,r)=\mathcal{E}(\mathbb{R},X,\mu_2,\nu_2,r)$. From the definition of a (μ,ν)-pseudo almost periodic function, we deduce that $PAP(\mathbb{R};X,\mu_1,\nu_1,r)=PAP(\mathbb{R};X,\mu_2,\nu_2,r)$.■

Let $\mu,\nu\in\mathcal{M}$ we denote by

$$cl(\mu,\nu)=\{\varpi_1,\varpi_2\in\mathcal{M}:\ \mu\sim\varpi_2 \text{ and } \nu\sim\varpi_2\}.$$

Proposition 4.10. *[8] Let $\mu,\nu\in\mathcal{M}$ satisfy ($\boldsymbol{H_4}$). Then $PAP(\mathbb{R},X,\mu,\nu)$ is invariant by translation, that is $f\in PAP(\mathbb{R},X,\mu,\nu)$ implies $f_\alpha\in PAP(\mathbb{R},X,\mu,\nu)$ for all $\alpha\in\mathbb{R}$.*

In what follows, we prove some preliminary results concerning the composition of (μ,ν)-pseudo almost periodic functions of class r.

Theorem 4.11. *Let $\mu,\nu\in\mathcal{M}$, $\phi\in PAP(\mathbb{R}\times X_1;X_2,\mu,\nu,r)$ and $h\in PAP(\mathbb{R};X_1,\mu,\nu,r)$. Assume that there exists a function $L_\phi:\mathbb{R}\to[0,+\infty[$ sastisfies*

$$|\phi(t,x_1)-\phi(t,x_2)|\leq l_\phi(t)|x_1-x_2| \ \ \textit{for}\ \ t\in\mathbb{R}\ \ \textit{and for}\ \ x_1,x_2\in X_1. \tag{4.2}$$

If

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}L_\phi(\theta)\Big)d\mu(t)<\infty \ \textit{and}\ \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}L_\phi(\theta)\Big)\xi(t)d\mu(t)=0 \tag{4.3}$$

for each $\xi\in\mathcal{E}(\mathbb{R},\mu,\nu)$ and for almost $\tau>0$, then the function $t\to\phi(t,h(t))$ belongs to $PAP(\mathbb{R};X_2,\mu,\nu,r)$.

Proof. Assume that $\phi=\phi_1+\phi_2$, $h=h_1+h_2$ where $\phi_1\in AP(\mathbb{R}\times X_1;X_2)$, $\phi_2\in\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu,r)$ and $h_1\in AP(\mathbb{R};X_1)$, $h_2\in\mathcal{E}(\mathbb{R};X_1,\mu,\nu,r)$. Consider the following decomposition

$$\phi(t,h(t))=\phi_1(t,h_1(t))+[\phi(t,h(t))-\phi(t,h_1(t))]+\phi_2(t,h_1(t)).$$

From [7, 18], $\phi_1(.,h_1(.))\in AP(\mathbb{R};X_2)$. It remains to prove that both $\phi(.,h(.))-\phi(.,h_1(.))$ and $\phi_2(.,h_1(.))$ belong to $\mathcal{E}(\mathbb{R};X_2,\mu,\nu,r)$.

Using equation (4.2), it follows that

$$\begin{aligned}&\frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\ \sup_{\theta\in[t-r,t]}|\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))|>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}\leq\frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\ \sup_{\theta\in[t-r,t]}(L_\phi(\theta)|h_2(\theta)|)>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}\\ &\leq\ \frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\Big(\sup_{\theta\in[t-r,t]}L_\phi(\theta)\Big)\Big(\sup_{\theta\in[t-r,t]}|h_2(\theta)|\Big)>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}.\end{aligned}$$

Since h_2 is (μ,ν)-ergodic of class r, Theorem 4.3 and equation (4.3) yield that for the above-mentioned ε, we have

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\Big(\sup_{\theta\in[t-r,t]}L_\phi(\theta)\Big)\Big(\sup_{\theta\in[t-r,t]}|h_2(\theta)|\Big)>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}=0,$$

and then we obtain

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\sup_{\theta\in[t-r,t]}|\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))|>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}=0, \tag{4.4}$$

By Theorem 4.3, equation (4.4) shows that $t\mapsto\phi(t,h(t))-\phi(t,h_1(t))$ is (μ,ν)-ergodic of class r.
Now to complete the proof, it is enough to prove that $t\mapsto\phi_2(t,h(t))$ is (μ,ν)-ergodic of class r. Since ϕ_2 is uniformly continuous on the compact set $K=\overline{\{h_1(t):\ t\in\mathbb{R}\}}$ with respect to the second variable x, we deduce that for given $\varepsilon>0$, there exists $\delta>0$ such that, for all $t\in\mathbb{R}$, ξ_1 and $\xi_2\in K$, one has

$$\|\xi_1-\xi_2\|\leq\delta\Rightarrow\|\phi_2(t,\xi_1(t))-\phi_2(t,\xi_2(t))\|\leq\varepsilon.$$

Therefore, there exist $n(\varepsilon)$ and $\{z_i\}_{i=1}^{n(\varepsilon)}\subset K$, such that

$$K\subset\bigcup_{i=1}^{n(\varepsilon)}B_\delta(z_i,\delta)$$

and then

$$\|\phi_2(t,h_1(t))\|\leq\varepsilon+\sum_{1}^{n(\varepsilon)}\|\phi_2(t,z_i)\|$$

Since

$$\forall i\in\{1,...,n(\varepsilon)\},\quad\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2(\theta,z_i)|\Big)d\mu(t)=0,$$

we deduce that

$$\forall\varepsilon>0,\quad\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2(\theta,h_1(t))|\Big)d\mu(t)\leq\varepsilon,$$

that implies

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2(\theta,h_1(t))|\Big)d\mu(t)=0.$$

Consequently $t\mapsto\phi_2(t,h(t))$ is (μ,ν)-ergodic of class r.■
For $\mu\in\mathcal{M}$ and $\alpha\in\mathbb{R}$, we denote μ_α the positive measure on $(\mathbb{R},\mathcal{B})$ defined by

$$\mu_\alpha(A)=\mu([a+\alpha:\ a\in A]) \tag{4.5}$$

Lemma 4.12. *[6] Let $\mu\in\mathcal{M}$ satisfy ($\boldsymbol{H_3}$). Then the measures μ and μ_α are equivalent for all $\alpha\in\mathbb{R}$.*

Lemma 4.13. *[6] ($\boldsymbol{H_3}$) implies*

$$\text{for all } \sigma > 0 \ \limsup_{\tau\to+\infty} \frac{\mu([-\tau-\sigma,\tau+\sigma])}{\mu([-\tau,\tau])} < +\infty.$$

We have the following result.

Theorem 4.14. *Assume that ($\boldsymbol{H_3}$) holds. Let $\mu,\nu \in \mathcal{M}$ and $\phi \in PAP(\mathbb{R};X,\mu,\nu,r)$, then the function $t \to \phi_t$ belongs to $PAP(C([-r,0];X),\mu,\nu,r)$.*

Proof. Assume that $\phi = g + h$ where $g \in AP(\mathbb{R};X)$ and $h \in \mathcal{E}(\mathbb{R};X,\mu,\nu,r)$. Then we can see that, $\phi_t = g_t + h_t$ and g_t is almost periodic. Let us denote by

$$M_\alpha(\tau) = \frac{1}{\nu_\alpha([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\|h(\theta)\|\Big)d\mu_\alpha(t),$$

where μ_α and ν_α are the positive measures defined by equation (4.5). By using Lemma 4.12, it follows that μ_α and μ are equivalent and ν_α and ν are also equivalent. then by using Theorem 4.9 we have $\mathcal{E}(\mathbb{R};X,\mu_\alpha,\nu_\alpha,r) = \mathcal{E}(\mathbb{R};X,\mu,\nu,r)$, therefore $h \in \mathcal{E}(\mathbb{R};X,\mu_\alpha,\nu_\alpha,r)$, that is

$$\lim_{\tau\to+\infty} M_\alpha(\tau) = 0, \ \text{ for all } \alpha \in \mathbb{R}.$$

On the other hand, for $r > 0$ we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\Big[\sup_{\xi\in[-r,0]}\|h(\theta+\xi)\|\Big]\Big)d\mu(t) \leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-2r,t]}\|h(\theta)\|\Big)d\mu(t)\\
\leq\ & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-2r,t-r]}\|h(\theta)\| + \sup_{\theta\in[t-r,t]}\|h(\theta)\|\Big)d\mu(t)\\
\leq\ & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau-r}^{+\tau-r}\Big(\sup_{\theta\in[t-r,t]}\|h(\theta)\|\Big)d\mu(t+r) + \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\|h(\theta)\|\Big)d\mu(t)
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau-r}^{+\tau+r}\Big(\sup_{\theta\in[t-r,t]}\|h(\theta)\|\Big)d\mu(t+r) + \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\|h(\theta)\|\Big)d\mu(t)\\
\leq\ & \Big[\frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\Big]\times\frac{1}{\nu([-\tau-r,\tau+r]}\int_{-\tau-r}^{+\tau+r}\Big(\sup_{\theta\in[t-r,t]}\|h(\theta)\|\Big)d\mu(t+r)\\
& + \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\|h(\theta)\|\Big)d\mu(t).
\end{aligned}$$

Consequently

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\Big[\sup_{\xi\in[-r,0]}\|h(\theta+\xi)\|\Big]\Big)d\mu(t) \ \leq\ & \Big[\frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\Big]\times M_r(\tau+r)\\
& + \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\|h(\theta)\|\Big)d\mu(t),
\end{aligned}$$

which shows using Lemma 4.12 and Lemma 4.13 that ϕ_t belongs to $PAP(C([-r,0];X),\mu,\nu,r)$. Thus, we obtain the desired result.■

5 (μ,ν)-Pseudo almost periodic solutions of class r

In what follows, we will be looking at the existence of bounded integral solutions of class r of equation (1.1).

Proposition 5.1. *[15] Assume that ($\boldsymbol{H_0}$) and ($\boldsymbol{H_1}$) hold and the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic. If $f\in BC(\mathbb{R};X)$, then there exists a unique bounded solution u of equation* (1.1) *on $\mathbb{R}$, given by*

$$u_t=\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \ \textit{for}\ \ t\in\mathbb{R},$$

where Π^s and Π^u are the projections of C_0 onto the stable and unstable subspaces, respectively.

Proposition 5.2. *[11] Let $h\in AP(\mathbb{R};X)$ and Γ be the mapping defined for $t\in\mathbb{R}$ by*

$$\Gamma h(t)=\Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s))ds\Big](0).$$

Then $\Gamma h\in AP(\mathbb{R},X)$.

Theorem 5.3. *Let $\mu,\nu\in\mathcal{M}$ satisfy ($\boldsymbol{H_3}$) and $g\in\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$. Then $\Gamma g\in\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$.*

Proof. In fact, for $\tau>0$ we get

$$\begin{aligned}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\Gamma h(\theta)|ds\Big)d\mu(t) &\leq \overline{M}\widetilde{M}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta}e^{-\omega(\theta-s)}|\Pi^s|\,|g(s)|ds\Big)d\mu(t)\\ &\quad+\overline{M}\widetilde{M}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{\theta}^{+\infty}e^{\omega(\theta-s)}|\Pi^u|\,|g(s)|ds\Big)d\mu(t)\\ &\leq \overline{M}\widetilde{M}|\Pi^s|\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}e^{\omega r}\int_{-\infty}^{\theta}e^{-\omega(t-s)}|g(s)|ds\Big)d\mu(t)\\ &\quad+\overline{M}\widetilde{M}|\Pi^u|\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{\theta}^{+\infty}e^{\omega(t-s)}|g(s)|ds\Big)d\mu(t).\end{aligned}$$

On the one hand using Fubini's theorem, we have

$$\begin{aligned}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}e^{\omega r}\int_{-\infty}^{\theta}e^{-\omega(t-s)}|g(s)|ds\Big)d\mu(t) &\leq \int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}e^{\omega r}\int_{-\infty}^{t}e^{-\omega(t-s)}|g(s)|ds\Big)d\mu(t)\\ &\leq e^{\omega r}\int_{-\tau}^{\tau}\int_{-\infty}^{t}e^{-\omega(t-s)}|g(s)|dsd\mu(t)\\ &\leq e^{\omega r}\int_{-\tau}^{\tau}\int_{0}^{+\infty}e^{-\omega s}|g(t-s)|dsd\mu(t)\\ &\leq e^{\omega r}\int_{0}^{+\infty}e^{-\omega s}\int_{-\tau}^{\tau}|g(t-s)|d\mu(t)ds.\end{aligned}$$

By using Proposition 4.10, we deduce that

$$\lim_{\tau\to+\infty}\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g(t-s)|d\mu(t)\to 0 \text{ for all } s\in\mathbb{R}^{+} \text{ and } \frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g(t-s)|d\mu(t)\to 0\leq e^{-\omega s}|g|_{\infty}.$$

Since g is a bounded function, then the function $s\mapsto e^{-\omega s}|g|_{\infty}$ belongs to $L^{1}([0,+\infty[)$, in view of the Lebesgue dominated convergence theorem, it follows that

$$e^{\omega r}\lim_{\tau\to+\infty}\int_{0}^{+\infty}e^{-\omega s}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g(t-s)|d\mu(t)ds=0.$$

On the other hand by Fubini's theorem, we also have

$$\begin{aligned}
\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{\theta}^{+\infty}e^{\omega(t-s)}|g(s)|ds\Big)d\mu(t) &\leq \int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{t-r}^{+\infty}e^{\omega(t-s)}|g(s)|ds\Big)d\mu(t)\\
&\leq \int_{-\tau}^{\tau}\int_{t-r}^{+\infty}e^{\omega(t-s)}|g(s)|dsd\mu(t)\\
&\leq \int_{-\tau}^{\tau}\int_{-\infty}^{r}e^{\omega s}|g(t-s)|dsd\mu(t)\\
&\leq \int_{-\infty}^{r}e^{\omega s}\int_{-\tau}^{\tau}|g(t-s)|d\mu(t)ds.
\end{aligned}$$

Since the function $s\mapsto e^{\omega s}|g|_{\infty}$ belongs to $L^{1}(]-\infty,r])$, resoning like above, it follows that

$$\lim_{\tau\to+\infty}\int_{-\infty}^{r}e^{\omega s}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g(t-s)|d\mu(t)ds=0.$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|(\Gamma g)(\theta)|\Big)d\mu(t)=0.$$

Thus, we obtain the desired result.■

For the existence of (μ,ν)-pseudo almost periodic solution of class r, we make the following assumption.

(H_5) $f:\mathbb{R}\to X$ is in $cl(\mu,\nu)$-pseudo almost periodic of class r.

Proposition 5.4. *Assume (**H_0**), (**H_1**), (**H_3**) and (**H_5**) hold. Then equation* (1.1) *has a unique $cl(\mu,\nu)$-pseudo almost periodic solution of class r.*

Proof. Since f is a (μ,ν)-pseudo almost periodic function, f has a decomposition $f=f_1+f_2$ where $f_1\in AP(\mathbb{R};X)$ and $f_2\in\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$. Using Proposition 5.1, Proposition 5.2 and Theorem 5.3, we get the desired result. ■

Our next objective is to show the existence of (μ,ν)-pseudo almost periodic solutions of class r for the following problem

$$u'(t)=Au(t)+L(u_t)+f(t,u_t) \text{ for } t\in\mathbb{R} \tag{5.1}$$

where $f:\mathbb{R}\times C\to X$ is continuous.

For the sequel, we make the following assumption.

($\mathbf{H_6}$) Let $\mu,\nu\in\mathcal{M}$ and $f:\mathbb{R}\times C([-r,0];X))\to X$ $cl(\mu,\nu)$-pseudo almost periodic of class r such that there exists a continuous function $L_f:\mathbb{R}\to[0,+\infty[$ such that

$$|f(t,\varphi_1)-f(t,\varphi_2)|\le L_f(t)|\varphi_1-\varphi_2| \quad \text{for all } \ t\in\mathbb{R} \ \text{ and } \ \varphi_1,\varphi_2\in C([-r,0];X)$$

and L_f satisfies (4.3).

Theorem 5.5. *Assume (**H_0**), (**H_1**), (**H_2**), (**H_4**) and (**H_6**) hold. If*

$$\overline{M}\widetilde{M}\sup_{t\in\mathbb{R}}\Big(|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)ds+|\Pi^u|\int_{t}^{+\infty}e^{\omega(t-s)}L_f(s)ds\Big)<1.$$

Then equation (5.1) *has a unique $cl(\mu,\nu)$-pseudo almost periodic solution of class r.*

Proof. Let x be a function in $PAP(\mathbb{R};X,\mu,\nu,r)$, from Theorem 4.14 the function $t\to x_t$ belongs to $PAP(C([-r,0];X),\mu,r)$. Hence Theorem 4.11 implies that the function $g(.):=f(.,x_.)$ is in $PAP(\mathbb{R};X,\mu,r)$. Consider the mapping

$$\mathcal{H}:PAP(\mathbb{R};X,\mu,\nu,r)\to PAP(\mathbb{R};X,\mu,\nu,r)$$

defined for $t\in\mathbb{R}$ by

$$(\mathcal{H}x)(t)=\Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0f(s,x_s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0f(s,x_s))ds\Big](0).$$

From Proposition 5.1, Proposition 5.2 and taking into account Theorem 5.3, it suffices now to show that the operator $\mathcal{H}$ has a unique fixed point in $PAP(\mathbb{R};X,\mu,r)$. Let $x_1,x_2\in PAP(\mathbb{R};X,\mu,\nu,r)$. Then we have

$$\begin{aligned}
|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)| &\le \Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f((s,x_{1s}))-f((s,x_{1s}))]ds\Big|\\
&\quad+\Big|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^s(t-s)\Pi^u(\widetilde{B}_\lambda X_0[f((s,x_{2s}))-f((s,x_{2s}))]ds\Big|\\
&\le \overline{M}\widetilde{M}\Big(|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)|x_{1s}-x_{2s}|ds+|\Pi^u|\int_{t}^{+\infty}e^{\omega(t-s)}L_f(s)|x_{1s}-x_{2s}|ds\Big)\\
&\le \overline{M}\widetilde{M}\sup_{t\in\mathbb{R}}\Big(|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)ds+|\Pi^u|\int_{t}^{+\infty}e^{\omega(t-s)}L_f(s)ds\Big)|x_1-x_2|.
\end{aligned}$$

This means that $\mathcal{H}$ is a strict contraction. Thus by Banach's fixed point theorem, $\mathcal{H}$ has a unique fixed point u in $PAP(\mathbb{R};X,\mu,\nu,r)$. We conclude that equation (5.1), has one and only one $cl(\mu,\nu)$-pseudo almost periodic solution of class r.■

Proposition 5.6. *Assume (**H_0**), (**H_1**), (**H_2**) and, (**H_4**) and f is lipschitz continuous with respect the second argument. If*

$$Lip(f)<\frac{\omega}{\overline{M}\widetilde{M}(|\Pi^s|+|\Pi^u|)}$$

then equation (5.1) *has a unique $cl(\mu,\nu)$-pseudo almost periodic solution of class r, where $Lip(f)$ is the lipschitz constant of f.*

Proof. Let us pose $k = Lip(f)$, we have

$$
\begin{aligned}
|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)| &\leq \overline{M}\widetilde{M}\Big(|\Pi^s|\int_{-\infty}^{t} e^{-\omega(t-s)}k|x_{1s}-x_{2s}|ds + |\Pi^u|\int_{t}^{+\infty} e^{\omega(t-s)}k|x_{1s}-x_{2s}|ds\Big) \\
&\leq \frac{k\overline{M}\widetilde{M}(|\Pi^s|+|\Pi^u|)}{\omega}|x_1-x_2|.
\end{aligned}
$$

Consequently $\mathcal{H}$ is a strict contraction if $k < \dfrac{\omega}{\overline{M}\widetilde{M}(|\Pi^s|+|\Pi^u|)}$.■

6 Application

For illustration, we propose to study the existence of solutions for the following model

$$
\begin{cases}
\dfrac{\partial}{\partial t}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_{-r}^{0} G(\theta)z(t+\theta,x))d\theta + (\sin t + \sin(\sqrt{2}t)) + \arctan(t) \\
\qquad\qquad + \displaystyle\int_{-r}^{0} h(\theta, z(t+\theta,x))d\theta \text{ for } t\in\mathbb{R} \text{ and } x\in[0,\pi] \\
z(t,0) = z(t,\pi) = 0 \text{ for } t\in\mathbb{R},
\end{cases}
\tag{6.1}
$$

where $G : [-r,0] \to \mathbb{R}$ is a continuous function and $h : [-r,0]\times\mathbb{R} \to \mathbb{R}$ is continuous and lipschitzian with respect to the second argument. To rewrite equation (6.1) in the abstract form, we introduce the space $X = C_0([0,\pi];\mathbb{R})$ of continuous function from $[0,\pi]$ to $\mathbb{R}^+$ equipped with the uniform norm topology. Let $A : D(A) \to X$ be defined by

$$
\begin{cases}
D(A) = \{y \in X \cap C^2([0,\pi],\mathbb{R}) : \ y'' \in X\} \\
Ay = y''.
\end{cases}
$$

Then A satisfied the Hille-Yosida condition in X. Moreover the part A_0 of A in $\overline{D(A)}$ is the generator of strongly continuous compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$. It follows that ($\mathbf{H_0}$) and ($\mathbf{H_1}$) are satisfied.
We define $f : \mathbb{R}\times C \to X$ and $L : C \to X$ as follows

$$
\begin{aligned}
f(t,\varphi)(x) &= \sin t + \sin(\sqrt{2}t) + \arctan(t) + \int_{-r}^{0} h(\theta,\varphi(\theta)(x))d\theta \ \text{ for } x\in[0,\pi] \ \text{ and } \ t\in\mathbb{R}, \\
L(\varphi)(x) &= \int_{-r}^{0} G(\theta)\varphi(\theta)(x))d\theta \ \text{ for } \ -r\leq\theta\leq 0 \ \text{ and } \ x\in[0,\pi].
\end{aligned}
$$

Let us pose $v(t) = z(t,x)$. Then equation (6.1) takes the following abstract form

$$
v'(t) = Av(t) + L(v_t) + f(t,v_t) \text{ for } t\in\mathbb{R}. \tag{6.2}
$$

Consider the measures μ and ν where its Radon-Nikodym derivative are respectively ρ_1, ρ_2 : $\mathbb{R} \to \mathbb{R}$ defined by

$$
\rho_1(t) = \begin{cases} 1 \ \text{ for } \ t > 0 \\ e^t \ \text{ for } \ t \leq 0. \end{cases}
$$

and

$$\rho_2(t) = |t| \text{ for } t \in \mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\nu(t) = \rho_2(t)dt$ where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \int_A \rho_2(t)dt \text{ for } A \in \mathcal{B}.$$

From [6] $\mu, \nu \in \mathcal{M}$, μ, ν satisfy Hypothesis $(\mathbf{H_4})$ and $\sin t + \sin(\sqrt{2}t) + \dfrac{\pi}{2}$ is almost periodic. We have

$$\limsup_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty} \frac{\int_{-\tau}^{0} e^t dt + \int_0^\tau dt}{2\int_0^\tau t dt} = \limsup_{\tau\to+\infty} \frac{1-e^{-\tau}+\tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H_2})$ is satisfied.
For all $t \in \mathbb{R}$, $\dfrac{-\pi}{2} \le \arctan t \le \dfrac{\pi}{2}$ therefore, for all $\theta \in [t-r, t]$, $\arctan(t-r) \le \arctan\theta$. It follows $|\arctan\theta - \dfrac{\pi}{2}| = \dfrac{\pi}{2} - \arctan\theta \le |\arctan(t-r) - \dfrac{\pi}{2}| = \dfrac{\pi}{2} - \arctan(t-r)$, hence $\sup\limits_{\theta\in[t-r,t]} |\arctan\theta - \dfrac{\pi}{2}| \le |\arctan(t-r) - \dfrac{\pi}{2}|$. On the one hand, we have the following:

$$\frac{1}{\nu([-\tau,\tau])}\int_0^{+\tau} |\arctan(t-r) - \frac{\pi}{2}|dt = \frac{\pi}{2\nu([-\tau,\tau])}\tau + \frac{1}{\nu([-\tau,\tau])}\int_{-r}^{+\tau-r} \arctan t dt$$
$$= \frac{\frac{\pi}{2}\tau + \ln(\sqrt{1+(\tau-r)^2}) - \ln(\sqrt{1+r^2}) - (\tau-r)\arctan(\tau-r) + r\arctan r}{\tau^2} \to 0 \text{ as } \tau \to +\infty.$$

On the other hand we have

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{0} |\arctan(t-r) - \frac{\pi}{2}|e^t dt \le \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{0} \pi e^t dt \to 0 \text{ as } \tau \to +\infty.$$

Consequently

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} \sup_{\theta\in[t-r,t]} |\arctan\theta - \frac{\pi}{2}|d\mu(t) = 0.$$

It follows that $t \mapsto \arctan t - \dfrac{\pi}{2}$ is (μ,ν)-ergodic of class r, consequently, f is uniformly (μ,ν)-pseudo almost periodic of class r. Moreover, L is a bounded linear operator from C to X.
Let k be the lipschiz constant of h, then for every $\varphi_1, \varphi_2 \in C$ and $t \ge 0$, we have

$$\begin{aligned} |f(t,\varphi_1) - f(t,\varphi_2)| &= \sup_{0\le x\le\pi} |f(t,\varphi_1)(x) - f(t,\varphi_2)(x)| \\ &\le kr \sup_{\substack{-r<\theta\le 0 \\ 0\le x\le\pi}} |\varphi_1(\theta)(x) - \varphi_2(\theta)(x)|. \end{aligned}$$

Consequently, we conclude that f is Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost periodic of class r.
For the hyberbolicity, we suppose that
$(\mathbf{H_7})$ $\displaystyle\int_{-r}^{0} |G(\theta)|d\theta < 1.$

Lemma 6.1. *[15] Assume that* $(\boldsymbol{H_6})$ *holds. Then the semigroup* $(\mathcal{U}(t))_{t\geq 0}$ *is hyperbolic.*

Then by Corollary 5.6 we deduce the following result.

Theorem 6.2. *Under the above assumptions, if* $Lip(h)$ *is small enough, then equation* (6.2) *has a unique* $cl(\mu,\nu)$*-pseudo almost periodic solution* v *of class* r*.*

7 (μ,ν)-pseudo almost automorphic functions

In this section, we recall some properties about pseudo almost automorphic functions. Let $BC(\mathbb{R},X)$ be the space of all bounded and continuous function from $\mathbb{R}$ to X equipped with the uniform topology norm.

Definition 7.1. A bounded continuous function $\phi:\mathbb{R}\to X$ is called almost automorphic if for each real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t)=\lim_{n\to+\infty}\phi(t+s_n)$$

is well defined for each $t\in\mathbb{R}$ and

$$\lim_{n\to+\infty}g(t-s_n)=\phi(t)$$

for each $t\in\mathbb{R}$.

We denote by $AA(\mathbb{R},X)$, the space of all such functions.

Proposition 7.2. *[16]* $AA(\mathbb{R},X)$ *equipped with the sup norm is a Banach space.*

Definition 7.3. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi:\mathbb{R}\times X_1\to X_2$ is called almost automorphic in $t\in\mathbb{R}$ uniformly for each x in X_1 if for every real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t,x)=\lim_{n\to+\infty}\phi(t+s_n,x)\ \text{ in }\ X_2$$

is well defined for each $t\in\mathbb{R}$ and each $x\in X_1$ and

$$\lim_{n\to+\infty}g(t-s_n,x)=\phi(t,x)\ \text{ in }\ X_2$$

for each $t\in\mathbb{R}$ and for every $x\in X_1$.

Denote by $AA(\mathbb{R}\times X_1;X_2)$ the space of all such functions.

Definition 7.4. A bounded continuous function $\phi:\mathbb{R}\to X$ is called compact almost automorphic if for each real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t)=\lim_{n\to+\infty}\phi(t+s_n)\ \text{ and }\ \lim_{n\to+\infty}g(t-s_n)=\phi(t)$$

uniformly on compact subsets of $\mathbb{R}$.

We denote by $AA_c(\mathbb{R};X)$, the space of all such functions.
It is well known that $AA_c(\mathbb{R};X)$ is closed subsets of $(BC(\mathbb{R},X),|\,.|_\infty)$.
In view of the above, the proof of the next lemma is straightforward.

Lemma 7.5. *$AA_c(\mathbb{R};X)$ equipped with the sup norm is a Banach space.*

Definition 7.6. Let X_1 and X_2 be two Banach spaces. A continuous function $\phi:\mathbb{R}\times X_1\to X_2$ is called compact almost automorphic in $t\in\mathbb{R}$ if every real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t,x)=\lim_{n\to+\infty}\phi(t+s_n,x) \text{ and } \lim_{n\to+\infty}g(t-s_n,x)=\phi(t,x) \text{ in } X_2$$

where the limits are uniform on compact subsets of $\mathbb{R}$ for each $x\in X_1$.

Denote by $AA_c(\mathbb{R}\times X_1;X_2)$ the space of all such functions.

Definition 7.7. A bounded continuous function $\phi:\mathbb{R}\to X$ is called (μ,ν)-pseudo almost automorphic if $\phi=\phi_1+\phi_2$ where $\phi_1\in AA(\mathbb{R};X)$ and $\phi_2\in\mathcal{E}(\mathbb{R};X,\mu,\nu)$.

We denote by $PAA(\mathbb{R};X)$, the space of all such functions.

Definition 7.8. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi:\mathbb{R}\times X_1\to X_2$ is called uniformly (μ,ν)-pseudo almost automorphic if $\phi=\phi_1+\phi_2$, where $\phi_1\in AA(\mathbb{R}\times X_1;X_2)$ and $\phi_2\in\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu)$.

We denote by $PAA(\mathbb{R}\times X_1;X_2)$, the space of all such functions.
We now introduce some new spaces used in the sequel.

Definition 7.9. A bounded continuous function $\phi:\mathbb{R}\to X$ is called (μ,ν)-pseudo compact almost automorphic if $\phi=\phi_1+\phi_2$ where $\phi_1\in AA_c(\mathbb{R};X)$ and $\phi_2\in\mathcal{E}(\mathbb{R};X,\mu,\nu)$.

We denote by $PAA_c(\mathbb{R};X)$, the space of all such functions.

Definition 7.10. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi:\mathbb{R}\times X_1\to X_2$ is called uniformly (μ,ν)-pseudo compact almost automorphic if $\phi=\phi_1+\phi_2$, where $\phi_1\in AA_c(\mathbb{R}\times X_1;X_2)$ and $\phi_2\in\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu)$.

We denote by $PAA_c(\mathbb{R}\times X_1;X_2)$, the space of all such functions.

Definition 7.11. A bounded continuous function $\phi:\mathbb{R}\to X$ is called (μ,ν)-pseudo almost automorphic of class r (respectively (μ,ν)-pseudo compact almost automorphic of class r) if $\phi=\phi_1+\phi_2$ where $\phi_1\in AA(\mathbb{R};X)$ and $\phi_2\in\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$ (respectively if $\phi=\phi_1+\phi_2$ where $\phi_1\in AA_c(\mathbb{R};X)$ and $\phi_2\in\mathcal{E}(X,\mu,\nu,r)$).

We denote by $PAA(\mathbb{R};X,\mu,\nu,r)$ (respectively $PAA_c(\mathbb{R};X,\mu,\nu,r)$) the space of all such functions.

Definition 7.12. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi:\mathbb{R}\times X_1\to X_2$ is called uniformly (μ,ν)-pseudo almost automorphic of class r (respectively uniformly pseudo compact almost automorphic of class r) if $\phi=\phi_1+\phi_2$, where $\phi_1\in AA(\mathbb{R}\times X_1;X_2)$ and $\phi_2\in\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu,r)$ (respectively if $\phi=\phi_1+\phi_2$, where $\phi_1\in AA_c(\mathbb{R}\times X_1;X_2)$ and $\phi_2\in\mathcal{E}(\mathbb{R}\times X_1;X_2,,\mu,\nu,r)$).

We denote by $PAA(\mathbb{R}\times X_1;X_2,\mu,\nu,r)$ (respectively $PAA_c(\mathbb{R}\times X_1;X_2,\mu,\nu,r)$) the space of all such functions.
From the definition of $PAA(\mathbb{R};X,\mu,\nu,r)$, we easily deduce the following result.

Proposition 7.13. $\mu\in\mathcal{M}$. *The space* $PAA(\mathbb{R};X,\mu,\nu,r)$ *endowed with the uniform topology morm is a Banach space.*

Proposition 7.14. *[8]* $\mu,\nu\in\mathcal{M}$ *and* $f\in PAA(\mathbb{R};X,\mu,\nu)$ *be such that*

$$f=g+h$$

where $g\in AA(\mathbb{R},X)$ *and* $h\in\mathcal{E}(\mathbb{R},X,\mu,\nu)$. *If* $PAA(\mathbb{R};X,\mu,\nu)$ *is translation invariant, then*

$$\{g(t),\ t\in\mathbb{R}\}\subset\overline{\{f(t),\ t\in\mathbb{R}\}}\ \textit{(the closure of the range of f).}$$

Corollary 7.15. *[8] Let* $\mu,\nu\in\mathcal{M}$. *Assume that* $PAA(\mathbb{R},X,\mu,\nu)$ *is translation invariant. Then the decomposition of a* (μ,ν)*-pseudo almost automorphic function in the form* $f=g+\phi$ *where* $g\in AA(\mathbb{R};X)$ *and* $\phi\in\mathcal{E}(\mathbb{R};X,\mu,\nu)$*, is unique.*

The following corollary is a consequence of Corollary 7.15.

Proposition 7.16. *Let* $\mu,\nu\in\mathcal{M}$. *Assume that* $PAA(\mathbb{R},X,\mu,\nu)$ *is translation invariant. Then the decomposition of a* (μ,ν)*-pseudo-almost automorphic function* $\phi=\phi_1+\phi_2$*, where* $\phi_1\in AA(\mathbb{R};X)$ *and* $\phi_2\in\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$*, is unique.*

Proof. In fact, since as a consequence of Corollary 7.15, the decomposition of a (μ,ν)-pseudo-almost automorphic function $\phi=\phi_1+\phi_2$, where $\phi_1\in AA(\mathbb{R};X)$ and $\phi_2\in\mathcal{E}(\mathbb{R};X,\mu,\nu)$, is unique. Since $PAA(\mathbb{R};X,\mu,\nu,r)\subset PAA(\mathbb{R};X,\mu,\nu)$, we get the desired result.■

Theorem 7.17. *Let* $\mu_1,\mu_2,\nu_1,\nu_2\in\mathcal{M}$. *If* $\mu_1\sim\mu_2$ *and* $\nu_1\sim\nu_2$*, then* $PAA(\mathbb{R};X,\mu_1,\nu_1,r)=PAA(\mathbb{R};X,\mu_2,\nu_2,r)$.

Proof. The proof is the same like Theorem 4.9.■

Proposition 7.18. *[8] Let* $\mu\in\mathcal{M}$ *satisfy* (***H₄***)*. Then* $PAA(\mathbb{R},X,\mu,\nu)$ *is invariant by translation, that is* $f\in PAP(\mathbb{R},X,\mu,\nu)$ *implies* $f_\alpha\in PAA(\mathbb{R},X,\mu,\nu)$ *for all* $\alpha\in\mathbb{R}$.

Theorem 7.19. *Assume that* ($\boldsymbol{H_3}$) *holds. Let* $\mu\in\mathcal{M}$ *and* $u\in PAA_c(\mathbb{R};X,\mu,\nu,r)$*, then the function* $t\to u_t$ *belongs to* $PAA_c(C([-r,0];X),\mu,\nu,r)$.

Proof. Assume that $u=g+\varphi$ where $g\in AA(\mathbb{R};X)$ and $\varphi\in\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$. We can see that $u_t=g_t+\varphi_t$. We want to show that $g_t\in AA_c(C([-r,0];X))$ and $\varphi_t\in\mathcal{E}(C([-r,0];X),\mu,\nu,r)$. From [10] (Theorem 4.2), the function $s\to g_s$ belongs to $AA_c(C)$. Using Theorem 4.14, we conclude that the function $t\to u_t$ belongs to $PAA_c(C([-r,0];X),\mu,\nu,r)$.■

Proposition 7.20. *[10] Let $h \in AA_c(\mathbb{R}, X)$ and Γ be the mapping defined for $t \in \mathbb{R}$ by*

$$\Gamma h(t) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s))ds\Big](0).$$

Then $\Gamma h \in AA_c(\mathbb{R}, X)$.

($\mathbf{H_8}$) $f : \mathbb{R} \to X$ is $cl(\mu,\nu)$-pseudo almost automorphic of class r.

Theorem 7.21. *Assume ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$), ($\boldsymbol{H_4}$) and ($\boldsymbol{H_8}$) hold. Then equation* (1.1) *has a unique $cl(\mu,\nu)$-pseudo almost automorphic solution of class r.*

Proof. Since f is a (μ,ν)-pseudo compact almost automorphic function, f has a decomposition $f = f_1 + f_2$ where $f_1 \in AA_c(\mathbb{R}; X)$ and $f_2 \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$. Using Proposition 5.1, Proposition 5.2, Theorem 5.3 and Theorem 7.16, we get the desired result.■
Our next objective is to show the existence of (μ,ν)-pseudo almost automorphic solutions of class r for the equation (5.1).
To prove our result, we need some preliminary results concerning the composition of (μ,ν)-pseudo almost automorphic functions of class r.

Theorem 7.22. *Let $\mu,\nu \in \mathcal{M}$, $\phi = \phi_1 + \phi_2 \in PAA(\mathbb{R}\times X; X, \mu, \nu, r)$ with $\phi_1 \in AA(\mathbb{R}\times X; X)$, $\phi_2 \in \mathcal{E}(\mathbb{R}\times X; X, \mu, \nu, r)$ and $h \in PAA(\mathbb{R}; X, \mu, \nu, r)$. Assume:*
i) $\phi_1(t, x)$ is uniformly continuous on any bounded subset uniformly for $t \in \mathbb{R}$.
ii) there exist a nonnegative function $L_\phi \in L^p(\mathbb{R})$, $(1 \le p \le \infty)$ such that

$$|\phi(t, x_1) - \phi(t, x_2)| \le L_\phi(t)|x_1 - x_2|, \quad \textit{for all } t \in \mathbb{R} \quad \textit{and for all } x_1, x_2 \in X. \tag{7.1}$$

If

$$\beta = \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]} L_\phi(\theta)\Big)d\mu(t) < \infty \tag{7.2}$$

then the function $t \to \phi(t, h(t))$ belongs to $PAA(\mathbb{R}; X, \mu, \nu, r)$.

Proof. Assume that $\phi = \phi_1 + \phi_2$, $h = h_1 + h_2$ where $\phi_1 \in AA(\mathbb{R}\times X; X)$, $\phi_2 \in \mathcal{E}(\mathbb{R}\times X; X, \mu, \nu, r)$ and $h_1 \in AA(\mathbb{R}; X)$, $h_2 \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$. Consider the following decomposition

$$\phi(t, h(t)) = \phi_1(t, h_1(t)) + [\phi(t, h(t)) - \phi(t, h_1(t))] + \phi_2(t, h_1(t)).$$

From [5], $\phi_1(., h_1(.)) \in AA(\mathbb{R}; X_2)$. It remains to prove that both $\phi(., h(.)) - \phi(., h_1(.))$ and $\phi_2(., h_1(.))$ belong to $\mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$. Clearly, $\phi(t, h(t)) - \phi(t, h_1(t))$ is bounded and continuous. We can assume $|\phi(t, h(t)) - \phi(t, h_1(t))| \le N$, $\forall t \in \mathbb{R}$. Since $h(t)$, $h_1(t)$ are bounded, we can choose a bounded subset $B \subset \mathbb{R}$ such that $h(\mathbb{R}), h_1(\mathbb{R}) \subset B$. Under assumption ($ii$), for a given $\varepsilon > 0$, $|x_1 - x_2| \le \varepsilon$, implies that $|\phi(t, x_1) - \phi(t, x_2)| \le \varepsilon L_\phi(t)$, for all $t \in \mathbb{R}$. Since for $\alpha \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$, Lemma 3.5 yields that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]}\mu(M_{\tau,\varepsilon}(\alpha)) = 0.$$

So

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))|\Big)d\mu(t) = \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\alpha)}\Big(\sup_{\theta\in[t-r,t]}|\phi(\theta,h(\theta))\\
&-\phi(\theta,h_1(\theta))|\Big)d\mu(t)+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(\alpha)}\Big(\sup_{\theta\in[t-r,t]}|\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))|\Big)d\mu(t)\\
\leq\ &\frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\alpha)}d\mu(t)+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(\alpha)}\Big(\sup_{\theta\in[t-r,t]}|L_\phi(\theta)|\Big)d\mu(t)\\
&\leq\ \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\alpha)}d\mu(t)+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}|L_\phi(\theta)|\Big)d\mu(t)\\
&\leq\ \frac{N}{\nu([-\tau,\tau])}M_{\tau,\varepsilon}(\alpha)+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}|L_\phi(\theta)|\Big)d\mu(t).
\end{aligned}$$

Which implies that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))|\Big)d\mu(t)\leq\varepsilon\beta \text{ for any } \varepsilon>0,$$

which shows that $t\mapsto\phi(t,h(t))-\phi(t,h_1(t))$ is (μ,ν)-ergodic of class r.
To prove that $t\mapsto\phi_2(t,h(t))$ is (μ,ν)-ergodic of class r, we process like in the proof of Theorem 4.11.■

In what follows, we suppose that:
(**H₉**) The instable space $U\equiv\{0\}$.
(**H₁₀**) $f:\mathbb{R}\times C\to X$ is uniformly $cl(\mu,\nu)$-pseudo compact almost automorphic such that there exists a function $L_f\in L^p(\mathbb{R},\mathbb{R}^+)$, with $1\leq p<+\infty$, such that

$$|f(t,\varphi_1)-f(t,\varphi_2)|\leq L_f(t)|\varphi_1-\varphi_2|,\quad \text{for all } t\in\mathbb{R},\ \varphi_1,\varphi_2\in C([-r,0];X_0))$$

where L_f satisfies **ii)** of Theorem 7.22.

Theorem 7.23. *Assume (**H_0**), (**H_1**), (**H_4**), (**H_8**) and (**H_9**) hold. Then equation* (5.1) *has a unique $cl(\mu,\nu)$-pseudo compact almost automorphic mild solution of class r.*

Proof. Let x be a function in $PAA_c(\mathbb{R};X,\mu,\nu,r)$, from Theorem 7.19 the function $t\to x_t$ belongs to $PAA_c(C([-r,0];X),\mu,r)$. Hence Theorem 7.22 implies that the function $g(.):=f(.,x_.)$ is in $PAA_c(\mathbb{R};X,\mu,\nu,r)$. Since the instable space $U\equiv\{0\}$, then $\Pi^u\equiv 0$. Consider now the mapping

$$\mathcal{H}:PAA_c(\mathbb{R};X,\mu,\nu,r)\to PAA_c(\mathbb{R};X,\mu,\nu,r)$$

defined for $t\in\mathbb{R}$ by

$$(\mathcal{H}x)(t)=\Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s,x_s))ds\Big](0).$$

From, Proposition 5.1, Theorem 5.3, Theorem 7.20 and Theorem 7.22, we can infer that $\mathcal{H}$ maps $PAA_c(\mathbb{R};X,\mu,\nu,r)$ into $PAA_c(\mathbb{R};X,\mu,\nu,r)$. It suffices now to show that the operator $\mathcal{H}$

has a unique fixed point in $PAA_c(\mathbb{R};X,\mu,\nu,r)$.
Case 1: $L_f \in L^1(\mathbb{R})$, $(p=1)$.
Let $x_1, x_2 \in PAA_c(\mathbb{R};X,\mu,\nu,r)$. Then we have

$$\begin{aligned}
|\,\mathcal{H}x_1(t)-\mathcal{H}x_2(t)| &\leq \Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s,x_{1s})-f(s,x_{2s})))ds\Big| \\
&\leq \overline{M}\widetilde{M}|\Pi^s|\,|x_1-x_2|\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)ds \\
&\leq \overline{M}\widetilde{M}|\Pi^s|\,|x_1-x_2|\int_{-\infty}^{t}L_f(s)ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
|\,\mathcal{H}^2x_1(t)-\mathcal{H}^2x_2(t)| &\leq \Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s,\mathcal{H}x_{1t})-f(s,\mathcal{H}x_{2t}))ds\Big| \\
&\leq (\overline{M}\widetilde{M}|\Pi^s|)^2|x_1-x_2|\int_{-\infty}^{t}L_f(s)\int_{-\infty}^{s}L_f(\delta)d\delta ds \\
&\leq \frac{(\overline{M}\widetilde{M}|\Pi^s|)^2}{2}\Big(\int_{-\infty}^{t}L_f(s)ds\Big)^2|x_1-x_2|.
\end{aligned}$$

Induction on n in the same way, gives

$$|\,\mathcal{H}^nx_1-\mathcal{H}^nx_2|\leq\frac{(\overline{M}\widetilde{M}|\Pi^s|)^n}{n!}\Big(\int_{-\infty}^{t}L_f(s)ds\Big)^n|x_1-x_2|.$$

Therefore

$$|\,\mathcal{H}^nx_1-\mathcal{H}^nx_2|\leq\frac{(\overline{M}\widetilde{M}|\Pi^s|\,|L_f|_{L^1(\mathbb{R})})^n}{n!}|x_1-x_2|.$$

Let n_0 be such that $\dfrac{(\overline{M}\widetilde{M}|\Pi^s|\,|L_f|_{L^1(\mathbb{R})})^{n_0}}{n_0!}<1$. By Banach's fixed point Theorem, $\mathcal{H}$ has a unique fixed point and this fixed point satisfies the integral equation

$$u_t=\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0f(s,u_s))ds.$$

Case 2: $L_f \in L^p(\mathbb{R})$; $(1<p<\infty)$.
First, put

$$\mu(t)=\int_{-\infty}^{t}(L_f(s))^pds.$$

Then we define an equivalent norm over $PAA(\mathbb{R},X)$ as follows,

$$|f|_c=\sup_{t\in\mathbb{R}}e^{-c\mu(t)}|f(t)|,$$

where c is a fixed positive number to be precised later. Using the Hölder inequality we have

$$\begin{aligned}
|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)| &\leq \Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s,x_{1s})-f(s,x_{2s})))ds\Big| \\
&\leq \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)|x_{1s}-x_{2s}|ds \\
&\leq \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}e^{-c\mu(s)}e^{c\mu(s)}L_f(s)|x_{1s}-x_{2s}|ds \\
&\leq \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}\Big(e^{-\omega(t-s)}e^{c\mu(s)}L_f(s)\Big)\sup_{s\in\mathbb{R}}\Big(e^{-c\mu(s)}|x_1(s)-x_2(s)|\Big)ds \\
&\leq \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}\Big(e^{-\omega(t-s)}e^{c\mu(s)}L_f(s)ds\Big)|x_1-x_2|_c \\
&\leq \overline{M}\widetilde{M}|\Pi^s|\Big(\int_{-\infty}^{t}e^{pc\mu(s)}(L_f(s))^p ds\Big)^{\frac{1}{p}}\Big(\int_{-\infty}^{t}e^{-\omega q(t-s)}ds\Big)^{\frac{1}{q}}|x_1-x_2|_c
\end{aligned}$$

$$\begin{aligned}
&\leq \overline{M}\widetilde{M}|\Pi^s|\Big(\int_{-\infty}^{t}e^{pc\mu(s)}\mu'(s)ds\Big)^{\frac{1}{p}}\Big(\int_{-\infty}^{t}e^{-\omega q(t-s)}ds\Big)^{\frac{1}{q}}|x_1-x_2|_c \\
&\leq \overline{M}\widetilde{M}|\Pi^s|\Big(\frac{1}{(pc)^{\frac{1}{p}}}\times\frac{1}{(\omega q)^{\frac{1}{q}}}\Big)e^{c\mu(t)}|x_1-x_2|_c.
\end{aligned}$$

Consequently

$$|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)|_c \leq \frac{\overline{M}\widetilde{M}|\Pi^s|}{(pc)^{\frac{1}{p}}\times(\omega q)^{\frac{1}{q}}}|x_1-x_2|_c.$$

Fix $c>0$ so large, then the function $c\mapsto\dfrac{1}{(pc)^{\frac{1}{p}}}$ converges to 0 when c converges to $+\infty$. It follows that for $c>0$ so large enough we have $\dfrac{\overline{M}\widetilde{M}|\Pi^s|}{(pc)^{\frac{1}{p}}\times(\omega q)^{\frac{1}{q}}}<1$. Thus $\mathcal{H}$ is a contractive mapping. Using the same argument as in Theorem 3.3 of [17], we conclude that there is a $cl(\mu,\nu)$-unique pseudo almost automorphic integral solution to equation (5.1) which ends the proof.■

Proposition 7.24. *Assume (**H₀**), (**H₁**), (**H₂**), (**H₄**) hold and f is Lipschitz continuous with respect the second argument. If*

$$Lip(f)<\frac{\omega}{\overline{M}\widetilde{M}|\Pi^s|},$$

then equation (5.1) *has a unique $cl(\mu,\nu)$-pseudo almost automorphic solution of class r, where $Lip(f)$ is the Lipschitz constant of f.*

Proof. Let us pose $k = Lip(f)$, we have

$$\begin{aligned}
|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)| &\leq \Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s,x_{1s}) - f(s,x_{2s})))ds\Big| \\
&\leq |\Pi^s|\overline{M}\widetilde{M}|x_1 - x_2|k\Big(\int_{-\infty}^{t} e^{-\omega(t-s)}\Big) \\
&\leq \frac{|\Pi^s|\overline{M}\widetilde{M}|x_1 - x_2|k}{\omega}.
\end{aligned}$$

Consequently $\mathcal{H}$ is a strict contraction if $k < \dfrac{\omega}{\overline{M}\widetilde{M}|\Pi^s|}$.■

8 Application

For illustration, we propose to study the existence of solutions for the following model

$$\begin{cases}
\dfrac{\partial}{\partial t}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_{-r}^{0} G(\theta)z(t+\theta,x))d\theta + \sin\Big(\frac{1}{2+\cos t + \cos\sqrt{2}t}\Big) + \arctan(t) \\
\qquad\qquad + \displaystyle\int_{-r}^{0} h(\theta, z(t+\theta,x))d\theta \text{ for } t \in \mathbb{R} \text{ and } x \in [0,\pi] \\
z(t,0) = z(t,\pi) = 0 \text{ for } t \in \mathbb{R},
\end{cases} \tag{8.1}$$

where $G : [-r,0] \to \mathbb{R}$ is a continuous function and $h : [-r,0]\times\mathbb{R} \to \mathbb{R}$ is continuous and lipschitzian with respect to the second argument. To rewrite equation (8.1) in the abstract form, we introduce the space $X = C_0([0,\pi];\mathbb{R})$ of continuous function from $[0,\pi]$ to $\mathbb{R}^+$ equipped with the uniform norm topology. Let $A : D(A) \to X$ be defined by

$$\begin{cases}
D(A) = \{y \in X \cap C^2([0,\pi],\mathbb{R}) : \ y'' \in X\} \\
Ay = y''.
\end{cases}$$

Then A satisfied the Hille-Yosida condition in X. Moreover the part A_0 of A in $\overline{D(A)}$ is the generator of strongly continuous compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$. It follows that $(\mathbf{H_0})$ and $(\mathbf{H_1})$ are satisfied.
We define $f : \mathbb{R}\times C \to X$ and $L : C \to X$ as follows

$$\begin{aligned}
f(t,\varphi)(x) &= \sin\Big(\frac{1}{2+\cos t + \cos\sqrt{2}t}\Big) + \arctan(t) + \int_{-r}^{0} h(\theta,\varphi(\theta)(x))d\theta \ \text{ for } x \in [0,\pi] \ \text{ and } \ t \in \mathbb{R}, \\
L(\varphi)(x) &= \int_{-r}^{0} G(\theta)\varphi(\theta)(x))d\theta \ \text{ for } \ -r \leq \theta \leq 0 \ \text{ and } \ x \in [0,\pi].
\end{aligned}$$

Let us pose $v(t) = z(t,x)$. Then equation (8.1) takes the following abstract form

$$v'(t) = Av(t) + L(v_t) + f(t,v_t) \text{ for } t \in \mathbb{R}. \tag{8.2}$$

Consider the measure μ,ν be defined like in Section 6.
From [12], $\sin\Big(\dfrac{1}{2+\cos t + \cos\sqrt{2}t}\Big) - \dfrac{\pi}{2}$ is compact almost automorphic. In Section 6, we

show that $t \mapsto \arctan t - \dfrac{\pi}{2}$ is (μ,ν)-ergodic of class r, consequently, f is uniformly (μ,ν)-pseudo almost automorphic of class r. Moreover, L is a bounded linear operator from C to X.
Let k be the lipschiz constant of h, then for every $\varphi_1, \varphi_2 \in C$ and $t \geq 0$, we have

$$\begin{aligned} |f(t,\varphi_1) - f(t,\varphi_2)| &= \sup_{0\leq x\leq\pi} |f(t,\varphi_1)(x) - f(t,\varphi_2)(x)| \\ &\leq kr \sup_{\substack{-r<\theta\leq 0\\ 0\leq x\leq\pi}} |\varphi_1(\theta)(x) - \varphi_2(\theta)(x)|. \end{aligned}$$

Consequently, we conclude that f is Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost periodic of class r.
(**H$_7$**) implies (**H$_9$**), then by Proposition 7.24 we deduce the following result.

Theorem 8.1. *Under the above assumptions, if $Lip(h)$ is small enough, then equation* (8.2) *has a unique $cl(\mu,\nu)$-pseudo almost automorphic solution v of class r.*

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