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Topology of Manifolds with Asymptotically Nonnegative Ricci Curvature

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Abstract

In this paper, we study the topology of complete noncompact Riemannian manifolds with asymptotically nonnegative Ricci curvature. We show that a complete noncompact manifold M with asymptotically nonnegative Ricci curvature and sectional curvature $K(x) \geq -\frac{C}{d_p(x)^{\alpha}}$ is diffeomorphic to the Euclidean space $\mathbb{R}^n$ under some conditions on the density of rays starting from the base point p or on the volume growth of geodesic balls in M.

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1 Introduction

One of most important problems in Riemannian geometry is to find conditions under which manifold is of finite topological type: A noncompact manifold M is said to have finite topological type if there exists a compact domain $\Omega \subset M$ with boundary such that $M \setminus \Omega$ is homeomorphic to $\partial\Omega \times [0,\infty[$. The fundamental notion involved in such a finite topological type result is that of the critical point of a distance function. Let p be a fix point and set $d_p(x) = d(p,x)$. A point $x \neq p$ is called critical point of d_p if, for any v in the tangent space T_xM, there is a minimal geodesic γ from x to p forming an angle less or equal to $\pi/2$ with $\gamma'(0)$. We denote by $Crit_p$ the criticality radius of M at p, i.e the smallest critical value of the function d_p.

The isotopy lemma says that the absence of critical point ensures that the manifold is diffeomorphic to the euclidean space $\mathbb{R}^n$. In several papers it has been proved results for manifolds with nonnegative curvature.

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X. Menguy in [6] and J. Sha and D.Yang in [8] constructed manifolds with nonnegative Ricci curvature and infinite topological type. Hence a natural question is under what additional conditions are manifolds with nonnegative Ricci curvature of finite topological type? Under volume growth, diameter or density of rays conditions, some results on the geometry and topology of complete noncompact manifolds with nonnegative Ricci curvature were obtained .

Let K denotes the sectional curvature of M and fix a point $p \in M$. For $r > 0$ let

$$k_p(r) = \inf_{M \setminus B(p,r)} K$$

where $B(p,r)$ is the open geodesic ball around p with radius r and the infimum is taken over all sections at points on $M \setminus B(p,r)$.

If (M,g) is a complete noncompact Riemannian manifold, we say that M is of sectional curvature decay at most quadratic if $k_p(r) \geq -\frac{C}{r^\alpha}$ for some $C > 0$, $\alpha \in [0,2]$ and all $r > 0$.

In this paper we see the case of manifolds with asymptotically nonnegative Ricci curvature and with sectional curvature decay almost quadratically.

A complete noncompact Riemannian manifold M is said to have an asymptotically nonnegative sectional curvature (resp. Ricci curvature) if there exists a point p, called base point, and a monotone decreasing positive function λ such that $\int_0^{+\infty} s\lambda(s)ds = b_0 < +\infty$ and for any point x in M we have

$$K(x) \geq -\lambda(d_p(x)) \; (resp.\; Ric_M(x) \geq -(n-1)\lambda(d_p(x)))$$

where d_p is the distance to p and $Ric_M(x)$ is the Ricci curvature of M at the point x.

Let $B(x,r)$ denote the metric ball of radius r and centre x in M and $B(\overline{x},r)$ denote the similar metric ball in the simply connected noncompact complete manifold with sectional curvature $-\lambda(d_{\overline{p}}(\overline{x}))$ at the point $\overline{x}$.

The volume comparison theorem proved in [4] says that the function $r \mapsto \frac{volB(x,r)}{volB(\overline{x},r)}$ is monotone decreasing. Set

$$\alpha_x = \lim_{r \to +\infty} \frac{volB(x,r)}{volB(\overline{x},r)} \text{ and } \alpha_M = \inf_{x \in M} \alpha_x.$$

We say that M is of large volume growth if $\alpha_M > 0$.

In [1] U. Abresch proved that manifolds with asymptotically nonnegative sectional curvature are of finite topolological type.

Let R_p denote the set of all rays issuing from p and $S(p,r)$ the sphere of radius r and centre p. Set $H(p,r) = \max_{x \in S(p,r)} d(x, R_p)$. By definition, we have $H(p,r) \leq r$.

In [10] Q. Wang and C. Xia proved that if a manifold M is with quadratic sectional curvature decay, then there exists a constant δ such that if $H(p,r) < \delta r$ then M is diffeomorphic to $\mathbb{R}^n$.

In this paper we prove the following theorem:

Theorem 1.1. *Given $c > 0$ and $\alpha \in [0,2]$; suppose that M is an n-dimensional complete noncompact Riemannian manifold with $Ric_M(x) \geq -(n-1)\lambda(d_p(x))$ and $K(x) \geq -\frac{C}{d_p(x)^\alpha}$, $Crit_p \geq r_0$ then there exists a positive constant $\delta_0 > 0$ such that if $H(p,r) < \delta_0 r^{\beta/2}$ then M is diffeomorphic to $\mathbb{R}^n$ where $\beta = \frac{2}{n} + \alpha(1 - \frac{1}{n})$.*

Remark 1.2. (i) Theorem 1.1 is an improvement of theorem 1.1 [10] where nonnegative Ricci curvature was assumed and sectional curvature $K_p(r) \geq -\frac{C}{(1+r)^\alpha}$.

(ii) For $\alpha = 0$ theorem1.1 is a generalisation of lemma 3.1 [12].

In [10] Q. Wang and C. Xia proved the following theorem (Theorem 1.3)

Theorem 1.3. *Given $\alpha \in [0,2]$, positive numbers r_0 and C, and an integer $n \geq 2$, there exists an $\epsilon = \epsilon(n, r_0, C, \alpha) > 0$ such that any complete Riemannian n-manifold M with Ricci curvature $Ric_M \geq 0$, $\alpha_M > 0$, $Crit_p \geq r_0$ and*

$$K(x) \geq -\frac{C}{(1+d_p(x))^\alpha}, \; \frac{volB(p,r)}{\omega_n r^n} \leq \left(1 + \frac{\epsilon}{r^{(n-2+\frac{1}{n})(1-\frac{\alpha}{2})}}\right)\alpha_M$$

for some $p \in M$ and all $r \geq r_0$ is diffeomorphic to $\mathbb{R}^n$.

In this paper we prove a more general result:

Theorem 1.4. *Given $c > 0$ and $\alpha \in [0,2]$; suppose that M is an n-dimensional complete noncompact Riemannian manifold with $Ric_M(x) \geq -(n-1)\lambda(d_p(x))$ and $K(x) \geq -\frac{C}{d_p(x)^\alpha}$, $Crit_p \geq r_0$ then there exists a positive constant $\epsilon = \epsilon(C, \alpha, r_0)$ such that if*

$$\frac{volB(p,r)}{volB(\overline{p},r)} \leq \left(1 + \frac{\epsilon}{r^{(n-2+\frac{1}{n})(1-\frac{\alpha}{2})}}\right)\alpha_p \tag{1.1}$$

then M is diffeomorphic to $\mathbb{R}^n$.

2 Preliminaries

Let p and q be two points of a complete Riemannian manifold M. The excess function e_{pq} is defined by: $e_{pq}(x) = d_p(x) + d_q(x) - d(p,q)$. In [2] U. Abresch and D. Gromoll gave an explicit upper bound of the excess function in manifolds with curvature bounded below. They proved the following lemma:

Lemma 2.1. *(Proposition 3.1 [2]) Let M be an n−dimensional complete Riemannian manifold $n \geq 3$, let γ be a minimal geodesic joining the base point p and another point $q \in M$, let x be a third point in M. Suppose $d(p,q) \geq 2d_p(x)$ and, moreover, that there exists a nonincreasing function $\lambda : [0, +\infty[\to [0, +\infty[$ such that $C(\lambda) = \int_0^\infty r\lambda(r)dr$ converges and $Ric_M(x) \geq -(n-1)\lambda(d_p(x))$ at all points $x \in M$. Then the height of the triangles can be bounded from below in terms of $d_p(x)$ and excess $e_{pq}(x)$. More precisely,*

$$s = d(x, R_p) \geq min\left\{\frac{1}{6}d_p(x), \frac{d_p(x)}{(1+8b_0)^{1/2}}, C_0 d_p(x)^{1/n}(2e_{pq}(x))^{1-\frac{1}{n}})\right\} \tag{2.1}$$

where $C_0 = \frac{4}{17}\frac{n-2}{n-1}(\frac{5}{1+8b_0})^{1/n}$.

Lemma 2.2 (lemma 3.4 [4])**.** *Let (M,g) be a complete noncompact Riemannian manifold with asymptotically nonnegative Ricci curvature with base point p.*

Then for all $x \in M$ and all numbers R', R with $0 < R' < R$ we have

$$\frac{volB(x,R)}{volB(x,R')} \leq \frac{volB(\overline{x},R)}{volB(\overline{x},R')} \leq \begin{cases} e^{(n-1)b_0}\left(\frac{R}{R'}\right)^n & if \quad 0 < R < r = d(p,x) \\ e^{(n-1)b_0}\left(\frac{R+r}{R'}\right)^n & if \quad R \geq r \end{cases} \tag{2.2}$$

where $B(\overline{x}, s)$ is the ball in $\overline{M}$ with center $\overline{x}$ and radius s.

Let $R_{p,t}$ be the set of the unit initial tangent vectors to the geodesics starting from p which are minimized at least to t and $R^c_{p,t}$ its complementary set. We have

$$\lim_{t\to+\infty} R_{p,t} = R_p.$$

Note

$$B_{R_{p,r}}(p,r) = \left\{x \in B(p,r)/\exists\gamma : [0,s] \to M, \gamma(0) = p, \gamma(s) = x \text{ and } \gamma'(0) \in U_p\right\}.$$

The following two lemmas are generalizations of the lemma 2.2 above.

Lemma 2.3 (Lemma 3.9 [5])**.** *Let (M,g) be a Riemannian complete noncompact manifold such that $Ric_M \geq -(n-1)\lambda(d_p(x))$ and Σ_p be a closed subset of U_p. Then the function $r \mapsto \frac{volB_{\Sigma_p}(p,r)}{volB(\overline{p},r)}$ is non increasing.*

In [5] we proved the following lemma:

Lemma 2.4. *Let (M,g) be a Riemannian complete noncompact manifold such that $Ric_M \geq -(n-1)\lambda(d_p(x))$. Then, for any $r > 0$ $\frac{volB_{R_{p,r}}(p,r)}{volB(\overline{p},r)} \geq \alpha_p$.*

3 Proofs

Proof of theorem 1.1

To prove the theorem1.1, it suffices to show that d_p has no critical point other than p. Let x be a point of M. Set $r = d(p,x)$; $s = d(x,R_p)$. Since R_p is closed there exists a ray γ issuing from p such that $s = d(x,\gamma)$. Set $q = \gamma(t_0)$ for $t_0 \geq 2r$. Let σ_1 and σ_2 be geodesics joining x to p and q respectively.

Set $\tilde{p} = \sigma_1(\delta r^{\alpha/2})$; $\tilde{q} = \sigma_2(\delta r^{\alpha/2})$ with

$$\delta < min\left\{C_0^n, \frac{r_0^{1-\beta/2}}{20}, \frac{r_0^{1-\beta/2}}{\sqrt{1+8b_0}}, C_0^n r_0^{1-\beta/2}\right\}. \tag{3.1}$$

Consider the triangle $(x,\tilde{p},\tilde{q})$; if y is a point on this triangle, then

$$d(p,y) \geq d(p,x) - d(x,y) \geq d(p,x) - d(\tilde{p},x) - d(\tilde{p},y) \geq d(p,x) - 2\delta r^{\alpha/2}.$$

Since $\beta \geq \alpha$ (we can suppose $r_0 > 1$ by dilating M if necessary), we have

$$d(p,y) \geq d(p,x) - 2\delta r^{\beta/2} \geq r(1 - 2\delta r^{\frac{\beta}{2}-1}) \geq r(1 - 2\delta r_0^{\frac{\beta}{2}-1}) \geq r/4. \tag{3.2}$$

Hence $y \in M \setminus B(p,r/4)$ and $K_M(y) \geq -\frac{4^\alpha C}{r^\alpha}$.

Thus the triangle $(x,\tilde{p},\tilde{q}) \subset M \setminus B(p,\frac{r}{4})$. Set $\theta = \measuredangle\sigma_1'(0),\sigma_2'(0)$.

Applying the Toponogov's theorem to the triangle $(x,\tilde{p},\tilde{q})$ we have:

$$\cosh\left(\frac{2^\alpha C^{1/2}}{r^{\alpha/2}}d(\tilde{p},\tilde{q})\right) \leq \cosh^2\left(\frac{2^\alpha C^{1/2}}{r^{\alpha/2}}d(\tilde{p},x)\right) - \sinh^2\left(\frac{2^\alpha C^{1/2}}{r^{\alpha/2}}d(\tilde{p},x)\right)\cos\theta \tag{3.3}$$

Since $s < \delta r^{\beta/2}$, we deduce from inequalities (2.1) and (3.1)

$$C_0 r^{1/n}(2e_{pq}(x))^{1-\frac{1}{n}} < \delta r^{\beta/2},$$

hence

$$e_{pq}(x) \leq \frac{\delta^{n/n-1}}{2C_0^{n/n-1}} r^{\alpha/2} \leq \frac{\delta}{2}.r^{\alpha/2}. \tag{3.4}$$

By triangle inequality, we have

$$\begin{array}{ll} d(\tilde{p},\tilde{q}) & \geq d(p,q) - d(p,\tilde{p}) - d(q,\tilde{q}) \\ & \geq d(p,q) - d(p,x) + d(\tilde{p},x) - d(x,q) + d(\tilde{q},x) \quad . \\ & \geq 2\delta r^{\alpha/2} - e_{pq}(x). \end{array} \tag{3.5}$$

Hence

$$d(\tilde{p},\tilde{q}) \geq 2\delta r^{\alpha/2} - \frac{\delta}{2} r^{\alpha/2} \geq \frac{3}{2}\delta r^{\alpha/2}. \tag{3.6}$$

From inequalities (3.3) and (3.6) we deduce

$$cosh\left(\frac{3}{2}C^{1/2}2^{\alpha}\delta\right) \leq cosh^2\left(C^{1/2}2^{\alpha}\delta\right) - sinh^2\left(C^{1/2}2^{\alpha}\delta\right)cos\theta.$$

Therefore

$$sinh^2\left(C^{1/2}2^{\alpha}\delta\right)cos\theta \leq cosh^2\left(C^{1/2}2^{\alpha}\delta\right) - cosh\left(\frac{3}{2}C^{1/2}2^{\alpha}\delta\right)$$

Let X_0 be the solution of the equation $cosh^2 2X - cosh3X = 0$. If $\delta_0 < \frac{X_0}{2^{\alpha-1}}$ then $\theta > \frac{\pi}{2}$ which means that x is not a critical point of d_p and the conclusion follows.

Proof of theorem 1.4

If $y(t)$ denotes the function given by the Jacobi equation

$$y''(t) = \lambda(t)y(t)$$

in the simply connected manifold with sectional curvature $-\lambda(d(\overline{p},\overline{x}))$ at the point $\overline{x}$ then (see [4])

$$t \leq y(t) \leq e^{b_0}t \tag{3.7}$$

and it follows that

$$\omega_n r^n \leq volB(\overline{p},r) \leq \omega_n e^{(n-1)b_0} r^n. \tag{3.8}$$

In one hand we have:

Let $x \in M$, $x \neq p$; set $s = d(x,R_p)$ and $R_p^c = U_p \setminus \Sigma_p(\infty)$. Thus

$$B(x,\frac{s}{2}) \subset B_{R_p^c}(p,r+\frac{s}{2}) \setminus B(p,r-\frac{s}{2}).$$

Hence

$$volB(x,\frac{s}{2}) \leq volB_{R_p^c}(p,r+\frac{s}{2}) - volB(p,r-\frac{s}{2}) \tag{3.9}$$

$$\leq volB_{R_p^c}(p,r+\frac{s}{2})-volB_{R_p^c}(p,r-\frac{s}{2})$$

$$\leq volB_{R_p^c}(p,r-\frac{r}{2})\left(\frac{volB_{R_p^c}(p,r+\frac{s}{2})}{volB_{R_p^c}(p,r-\frac{r}{2})}-1\right).$$

We deduce from lemma 2.3

$$volB(x,\frac{s}{2})\leq volB_{R_p^c}(p,r-\frac{s}{2})\left(\frac{volB(\overline{p},r+\frac{s}{2})}{volB(\overline{p},r-\frac{s}{2})}-1\right) \tag{3.10}$$

Using the inequality (3.8) we have:

$$volB(x,\frac{s}{2})\leq\frac{volB_{R_p^c}(p,r-\frac{s}{2})}{(r-\frac{s}{2})^n}e^{(n-1)b_0}\left((r+s/2)^n-(r-s/2)^n\right) \tag{3.11}$$

$$\leq volB_{R_p^c}(p,r-\frac{s}{2})e^{(n-1)b_0}\left(\left(\frac{r+s/2}{r-s/2}\right)^n-1\right)\leq volB_{R_p^c}(p,r-\frac{s}{2})e^{(n-1)b_0}\left((1+\frac{2s}{r})^n-1\right)$$

$$\leq volB_{R_p^c}(p,r-\frac{s}{2})e^{(n-1)b_0}.\frac{s}{r}(3^n-1).$$

In other hand we have

$$volB_{R_p^c}(p,r-\frac{s}{2})=volB(p,r-s/2)-volB_{R_p}(p,r-s/2)$$

By (2.4) we have

$$volB_{R_p}(p,r-s/2)\geq\alpha_p volB(\overline{p},r-s/2). \tag{3.12}$$

From (3) and (3.12) we deduce

$$volB(x,s/2)\leq\left[volB(p,r-s/2)-\alpha_p volB(\overline{p},r-s/2)\right].e^{(n-1)b_0}.\frac{s}{r}(3^n-1).$$

By (1.1) we have

$$volB(x,s/2)\leq\frac{\epsilon\alpha_p}{(r-s/2)^{(n-2+\frac{1}{n})(1-\frac{\alpha}{2})}}e^{(n-1)b_0}\frac{s}{r}3^n volB(\overline{p},r-\frac{s}{2}). \tag{3.13}$$

From (3.8) and (3.13) we have

$$volB(x,s/2)\leq\epsilon\alpha_p e^{2(n-1)b_0}s3^n\omega_n r^{(n-1)(\frac{1}{n}+\frac{\alpha}{2}(1-\frac{1}{n}))}. \tag{3.14}$$

We claim that

$$volB(x,s/2)\geq\frac{\omega_n\alpha_p}{6^n e^{(n-1)b_0}}s^n. \tag{3.15}$$

Indeed we have $B(p,r)\subset B(x,2r)$, and by (2.2) we deduce

$$\frac{volB(p,r)}{volB(x,s/2)}\leq\frac{volB(x,2r)}{volB(x,s/2)}\leq\frac{B(\overline{x},2r)}{volB(\overline{x},s/2)} \tag{3.16}$$

$$\leq e^{(n-1)b_0}\left(\frac{2r+s}{s/2}\right)^n\leq e^{(n-1)b_0}6^n\left(\frac{r}{s}\right)^n. \tag{3.17}$$

Thus

$$volB(x, s/2) \geq \frac{s^n volB(p, r)}{6^n e^{(n-1)b_0} r^n}. \tag{3.18}$$

Hence from (3.8), lemma2.4 and (3.18) the conclusion follows.

Thus from (3.14) and the inequality (3.15) we have

$$s^{n-1} \leq \epsilon 18^n e^{3(n-1)b_0} r^{(n-1)(\frac{1}{n}+\frac{\alpha}{2}(1-\frac{1}{n}))}$$

which means that

$$s \leq \epsilon^{1/(n-1)} 18^{n/(n-1)} e^{3b_0} r^{\frac{1}{n}+\frac{\alpha}{2}(1-\frac{1}{n})}.$$

Then it suffices to take $\epsilon < \frac{\delta^{n-1}}{18^n e^{3(n-1)b_0}}$.

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