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On Irreducibility of an Induced Representation of a Simply Connected Nilpotent Lie Group

Adjiey Jean-Luc Koffi*
FHB University of Cocody, Department of Mathematics and computer,
Abidjan, Côte d'Ivoire (Ivory Coast)

Kinvi Kangni†
FHB University of Cocody, Department of Mathematics and computer,
Abidjan, Côte d'Ivoire (Ivory Coast)

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Abstract. Let G be a simply connected nilpotent Lie group, $\mathcal{G}$ the finite-dimensional Lie algebra of G, $\mathcal{V}$ a finite-dimensional vector space over $\mathbb{C}$ or $\mathbb{R}$, and H a connected Lie subgroup of G such that the Lie algebra of H is a subordinate subalgebra to an element π of $Hom(\mathcal{G}, gl(\mathcal{V}))$. In this work, we construct an irreducible representation χ_π of H such that the induced of χ_π on G is irreducible.

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1 Introduction

Let $\mathcal{G}$ be a finite-dimensional Lie algebra, $\mathcal{V}$ a finite-dimensional $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $Hom(\mathcal{G}, gl(\mathcal{V}))$ the space of linear operators of $\mathcal{G}$ into $gl(\mathcal{V})$, the Lie algebra of endomorphisms of $\mathcal{V}$.

Let $B : \mathcal{G}\times\mathcal{G} \longrightarrow gl(\mathcal{V})$ be an alternating bilinear map of $\mathcal{G}\times\mathcal{G}$ into $gl(\mathcal{V})$.
For each Lie subalgebra $\mathfrak{h}$ of $\mathcal{G}$, *the orthogonal* of $\mathfrak{h}$ with respect to B, denoted by $\mathfrak{h}^B$ is defined by: $\mathfrak{h}^B = \{X \in \mathcal{G} \,/\, B(X, \mathfrak{h}) = 0\}$ and we have: $\mathcal{G}^B \subset \mathfrak{h}^B$.
The Lie subalgebra $\mathfrak{h}$ of $\mathcal{G}$ is said to be totally isotropic with respect to B if $\mathfrak{h} \subset \mathfrak{h}^B$, and maximal totally isotropic with respect to B if $\mathfrak{h} = \mathfrak{h}^B$.

Let G be the simply connected Lie group with Lie algebra $\mathcal{G}$, π an element of $Hom(\mathcal{G}, gl(\mathcal{V}))$ and χ_π a generalization of a character of a Lie subgroup of G with Lie algebra $\mathfrak{h}$. The aim of our work

*e-mail address: adjieyjlk@yahoo.fr
†e-mail address: kinvi.kangni@univ-fhb.edu.ci

is to define the notions of subordinate subalgebra and polarization on the space $Hom(\mathcal{G}, gl(\mathcal{V}))$, and study the irreducibility of the representation of G induced by χ_π, denoted by $\rho(\pi, \mathfrak{h}, \mathcal{G})$, in the case where G is a nilpotent Lie group.

2 Polarizations at a linear operator

Let π be an element of $Hom(\mathcal{G}, gl(\mathcal{V}))$. We consider the alternating bilinear map associated to π denoted by B_π defined of $\mathcal{G}\times\mathcal{G}$ into $gl(\mathcal{V})$ by:

$$B_\pi(X,Y) = \pi([X,Y]), \forall X, Y \in \mathcal{G}. \tag{2.1}$$

For each Lie subalgebra $\mathfrak{h}$ of $\mathcal{G}$, the orthogonal of $\mathfrak{h}$ with respect to B_π is $\mathfrak{h}^{B_\pi}$ denoted by $\mathfrak{h}^\pi$.
In particular, the orthogonal of $\mathcal{G}$ with respect to B_π is the kernel of B_π denoted by $\mathcal{G}(\pi)$ i.e. $\mathcal{G}(\pi) = \mathcal{G}^\pi$.

Definition 2.1. A Lie subalgebra $\mathfrak{h}$ of $\mathcal{G}$ is *subordinate to* the operator π if $\pi([\mathfrak{h},\mathfrak{h}]) = 0$.
The set of all Lie subalgebras of $\mathcal{G}$ *subordinate to* π will be denoted by $Sub(\pi)$.
The Lie subalgebra $\mathfrak{h}$ of $\mathcal{G}$ is a *polarization* at π if $\mathfrak{h}$ is maximal totally isotropic with respect to B_π.
The set of all polarizations at π will be denoted by $Pol(\pi)$.

We will establish a relation between polarization at $\pi \in Hom(\mathcal{G}, gl(\mathcal{V}))$ and polarization at a linear form $f \in \mathcal{G}^*$ such that $\pi = f \otimes u$ where $u \in gl(\mathcal{V})$.

Theorem 2.2. *Let π be an element of $Hom(\mathcal{G}, gl(\mathcal{V}))$ and $\mathfrak{h}$ a Lie subalgebra of $\mathcal{G}$ subordinate to π.*
Let $(f,u) \in \mathcal{G}^* \times gl(\mathcal{V})$ such that $\pi = f \otimes u$.
We have: $\mathfrak{h}^\pi = \mathfrak{h}^f$, $Sub(\pi) = Sub(f)$ and $Pol(\pi) = Pol(f)$.

Proof. For all Lie subalgebra $\mathfrak{h}$ of $\mathcal{G}$ and for all $X \in \mathcal{G}$, we have:

$$\begin{aligned} X \in \mathfrak{h}^\pi &\iff \pi([X,\mathfrak{h}]) = 0 \iff f\otimes u([X,\mathfrak{h}]) = 0 \iff f([X,\mathfrak{h}])u = 0 \\ &\iff f([X,\mathfrak{h}]) = 0 \iff X \in \mathfrak{h}^f, \end{aligned}$$

$$\begin{aligned} \mathfrak{h} \in Sub(\pi) &\iff \pi([\mathfrak{h},\mathfrak{h}]) = 0 \iff f\otimes u([\mathfrak{h},\mathfrak{h}]) = 0 \iff f([\mathfrak{h},\mathfrak{h}])u = 0 \\ &\iff f([\mathfrak{h},\mathfrak{h}]) = 0 \iff \mathfrak{h} \in Sub(f), \end{aligned}$$

$$\mathfrak{h} \in Pol(\pi) \iff \mathfrak{h}^\pi \subset \mathfrak{h} \iff \mathfrak{h}^f \subset \mathfrak{h} \iff \mathfrak{h} \in Pol(f), \text{ since } \mathfrak{h}^\pi = \mathfrak{h}^f.$$

□

Remark 2.3. *In the case where G* is a simply connected nilpotent Lie group *with finite-dimentinal Lie algebra $\mathcal{G}$, for all $\pi \in Hom(\mathcal{G}, gl(\mathcal{V}))$* , the set $Pol(\pi)$ is not empty.

3 Irreducibility of an induced representation

Let G be a simply connected Lie group with Lie algebra $\mathcal{G}$, $\mathcal{V}$ a finite-dimensional $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $\pi : \mathcal{G} \longrightarrow gl(\mathcal{V})$ a linear operator of $\mathcal{G}$ into $gl(\mathcal{V})$, $\mathfrak{h}$ a Lie subalgebra of $\mathcal{G}$ subordinate to π and H the connected Lie subgroup of G with Lie algebra $\mathfrak{h}$.We denote by χ_π the representation of H in $\mathcal{V}$ which is definied by:

$$\chi_\pi(\exp X) = e^{i\pi(X)}, \forall X \in \mathfrak{h}. \tag{3.1}$$

We denote by $\rho(\pi, \mathfrak{h}, \mathcal{G})$ the unitary representation of G induced by the representation χ_π of H i.e. $\rho(\pi, \mathfrak{h}, \mathcal{G}) = Ind_{H\uparrow G}\chi_\pi$.

In the following, we assume that G is a *simply connected nilpotent* Lie group.

Lemma 3.1. *Let $\pi \in Hom(\mathcal{G}, gl(\mathcal{V}))$ and $\mathfrak{h} \in Sub(\pi)$. If the representation $\rho(\pi, \mathfrak{h}, \mathcal{G})$ of G is irreducible then $\mathfrak{h}$ contains the center of the Lie algebra $\mathcal{G}$.*

Proof. Let's suppose that the representation $\rho(\pi, \mathfrak{h}, \mathcal{G})$ of G is irreducible.
Let $\mathcal{Z}$ be the center of $\mathcal{G}$ and $\mathcal{Z}'$ supplementary of $\mathfrak{h} \cap \mathcal{Z}$ in $\mathcal{Z}$. We have $\mathcal{Z} = \mathcal{Z}' \oplus (\mathfrak{h} \cap \mathcal{Z})$ and we denote by $\mathfrak{h}' = \mathfrak{h} + \mathcal{Z} = \mathfrak{h} \oplus \mathcal{Z}'$. Let $H' = \exp \mathfrak{h}'$, $Z' = \exp \mathcal{Z}'$, and $H = \exp \mathfrak{h}$ be the analytic subgroups of G with Lie algebras $\mathfrak{h}'$, $\mathcal{Z}'$ and $\mathfrak{h}$ respectively. Let $q' : \mathfrak{h} \longrightarrow \mathfrak{h}'/\mathcal{Z}'$ be the restriction to $\mathfrak{h}$ of the canonical homomorphism $p' : \mathfrak{h}' \longrightarrow \mathfrak{h}'/\mathcal{Z}'$, and e the neutral element of G. Since $\mathfrak{h}' = \mathfrak{h} \oplus \mathcal{Z}'$, q' is a Lie algebra isomorphism. q' is the differential at e of the homomorphism $q : H \longrightarrow H'/Z'$, the restriction to H of the canonical homomorphism $p : H' \longrightarrow H'/Z'$. Then, q is a covering map of H'/Z', and since H'/Z' is a simply connected Lie group, q is a Lie group isomorphism. Let $s : H'/Z' \longrightarrow H$ be the inverse isomorphism of q. The map s is an analytic section of p, i.e. $p \circ s = Id_{H'/Z'}$.
Let $\varphi : H' \longrightarrow Z' \times H$ be the analytic map which is defined by

$$\varphi(x) = \left(xs\left(p\left(x^{-1}\right)\right), s(p(x))\right) \ , \ \forall x \in H'. \tag{3.2}$$

The analytic map $\theta : Z' \times H \longrightarrow H'$ which is defined by

$$\theta(z, h) = zh \ , \ \forall (z, h) \in Z' \times H, \tag{3.3}$$

is an isomorphism of the direct product $Z' \times H$ into H', and φ is the inverse isomorphism of θ. Hence, $H' \cong Z' \times H$.
Let's suppose that $\mathfrak{h}$ does not contain the center $\mathcal{Z}$ of $\mathcal{G}$. Then the dimension of $\mathcal{Z}'$ is strictly positive and the representation $\rho(\pi, \mathfrak{h}, \mathfrak{h}')$ of H' is not irreducible. Indeed, if we consider the representation $\rho(\pi, \mathfrak{h}, \mathfrak{h}')$ on the space $L^2(Z')$, its restriction to Z' is the regular representation of Z', and its restriction to H is scalar. Hence, any closed subspace of $L^2(Z')$ which is invariant by Z' is also invariant by H'. Therefore, the representation $\rho(\pi, \mathfrak{h}, \mathfrak{h}')$ is not irreducible. Moreover, since $Ind_{H'\uparrow G}\rho(\pi, \mathfrak{h}, \mathfrak{h}') = Ind_{H'\uparrow G}\left(Ind_{H\uparrow H'}\chi_\pi\right) = Ind_{H\uparrow G}\chi_\pi = \rho(\pi, \mathfrak{h}, \mathcal{G})$, then $\rho(\pi, \mathfrak{h}, \mathcal{G})$ is not irreducible. It follows that $\mathcal{Z} \subset \mathfrak{h}$. □

Remark 3.2. When $\dim \mathcal{V} = 1$, we have $Hom(\mathcal{G}, gl(\mathcal{V})) \cong \mathcal{G}^*$ and it has been proved by A. A. Kirillov, that for all $\pi \in \mathcal{G}^*$, there exists a polarization $\mathfrak{h}$ at π such that the representation $\rho(\pi, \mathfrak{h}, \mathcal{G})$ is irreducible, and $\rho(\pi, \mathfrak{h}, \mathcal{G})$ is irreducible if and only if $\mathfrak{h}$ is a polarization at π (Cf.[7], [14], [15]).

Theorem 3.3. *Let G be a non-abelian simply connected nilpotent Lie group with finite-dimentinal Lie algebra $\mathcal{G}$, $\mathcal{Z}$ the center of $\mathcal{G}$, $\mathcal{V}$ a finite-dimensional $\mathbb{K}$-vector space of dimension ≥ 2, and an operator $\pi \in Hom(\mathcal{G}, gl(\mathcal{V}))$ such that $\mathcal{Z} \cap \ker(\pi) \neq \{0\}$.*
1) *There exists a polarization $\mathfrak{h}$ at π such that the representation $\rho(\pi, \mathfrak{h}, \mathcal{G})$ is irreducible.*
2) *If a non-abelian Lie subalgebra $\mathfrak{h} \in Sub(\pi)$, the representation $\rho(\pi, \mathfrak{h}, \mathcal{G})$ of G is irreducible if and only if $\mathfrak{h}$ is a polarization at π.*

Proof. 1) If $\mathcal{G}$ is abelian, $\mathcal{G}$ is the only polarization at π, and the representation $\rho(\pi,\mathcal{G},\mathcal{G})$ is not irreducible by the lemmas of Schur (Cf. [3], [7], [8]) since the dimension of $\rho(\pi,\mathcal{G},\mathcal{G})$ is $\dim\mathcal{V}\geq 2$. Consequently, there is no polarization $\mathfrak{h}$ at π such that the representation $\rho(\pi,\mathfrak{h},\mathcal{G})$ is irreducible.

We assume that $\mathcal{G}$ is non-abelian and $\mathcal{Z}\cap\ker(\pi)\neq\{0\}$.

If $\dim\mathcal{G}=3$, since $\mathcal{G}$ is nilpotent non-abelian, we have $\dim\mathcal{Z}=1$ and π is trivial on $\mathcal{Z}$.
Let (X_0,Y_0,Z_0) be a basis of $\mathcal{G}$ such that Z_0 generates $\mathcal{Z}$ and $[X_0,Y_0]=Z_0$. We have $\pi(Z_0)=0$, and
$\forall\ X=a_1X_0+a_2Y_0+a_3Z_0\in\mathcal{G}$ and
$\forall\ Y=b_1X_0+b_2Y_0+b_3Z_0\in\mathcal{G}$ with $a_i,b_i\in\mathbb{K}$, $\forall i\in\{1,2,3\}$, we have:

$$\begin{aligned}\pi([X,Y]) &= \pi([a_1X_0+a_2Y_0+a_3Z_0,b_1X_0+b_2Y_0+b_3Z_0])\\ &= \pi([a_1X_0,b_2Y_0]+[a_2Y_0,b_1X_0])\\ &= (a_1b_2-a_2b_1)\pi([X_0,Y_0])\\ &= (a_1b_2-a_2b_1)\pi(Z_0)\\ &= 0.\end{aligned}$$

Hence, $\mathcal{G}^{\pi}=\mathcal{G}$ and so $\mathcal{G}$ is a polarization at π.

Since $\rho(\pi,\mathcal{G},\mathcal{G})=\chi_\pi$ and G is non-abelian, the only operators which commute with $\chi_\pi(\exp X)$ for all $X\in\mathcal{G}$ are the scalar multiples of the identity of $\mathcal{V}$. Consequently, the representation $\rho(\pi,\mathcal{G},\mathcal{G})$ is irreducible by the lemmas of Schur.

Let's suppose that for all non-abelian nilpotent Lie algebra $\mathcal{G}_0$ with center $\mathcal{Z}_0$ such that $\dim\mathcal{G}_0<\dim\mathcal{G}$, and $\pi_0\in Hom(\mathcal{G}_0,gl(\mathcal{V}))$ such that $\mathcal{Z}_0\cap\ker(\pi_0)\neq\{0\}$, there exists a polarization $\mathfrak{h}_0$ at π_0 such that the representation $\rho(\pi_0,\mathfrak{h}_0,\mathcal{G}_0)$ is irreducible.
Then, there exists a polarization $\mathfrak{h}$ at π such that the representation $\rho(\pi,\mathfrak{h},\mathcal{G})$ is irreducible. Indeed:
If $\mathcal{G}(\pi)=\mathcal{G}$, then $\mathcal{G}$ is the only polarization at π and the representation $\rho(\pi,\mathcal{G},\mathcal{G})$ is irreducible.
If $\mathcal{G}(\pi)\neq\mathcal{G}$, and if $\mathcal{Z}'=\mathcal{Z}\cap\ker(\pi)$, the operator $\pi\in Hom(\mathcal{G},gl(\mathcal{V}))$ induces an operator π' of the nilpotent Lie algebra $\mathcal{G}'=\mathcal{G}/\mathcal{Z}'$ into the space $gl(\mathcal{V})$ such that $\pi=\pi'\circ p$ where $p:\mathcal{G}\longrightarrow\mathcal{G}'$ is the canonical surjection of $\mathcal{G}$ onto $\mathcal{G}'$.
Let $\mathcal{Z}(\mathcal{G}')$ be the center of $\mathcal{G}'$ and $\mathcal{Z}^2\mathcal{G}=\{X\in\mathcal{G}\ /\ [X,\mathcal{G}]\subset\mathcal{Z}\}$. We have:

$$\begin{aligned}\mathcal{Z}(\mathcal{G}') &= \left\{\overline{X}\in\mathcal{G}'\ /\ \forall\overline{Y}\in\mathcal{G}',\left[\,\overline{X},\overline{Y}\,\right]=\overline{0}\right\}\ (\text{where } \overline{X}=p(X),\forall X\in\mathcal{G})\\ &= \left\{\overline{X}\in\mathcal{G}'\ /\ \forall Y\in\mathcal{G},[X,Y]\in\mathcal{Z}\cap\ker(\pi)\right\}\\ &= \left\{\overline{X}\in\mathcal{G}'\ /\ [X,\mathcal{G}]\subset\mathcal{Z}\ and\ \pi([X,\mathcal{G}])=0\right\}\\ &= \left\{\overline{X}\in\mathcal{G}'\ /\ X\in\mathcal{Z}^2\mathcal{G}\cap\mathcal{G}(\pi)\right\}.\end{aligned}$$

Since $\mathcal{G}(\pi)\neq\mathcal{G}$, we have $\mathcal{Z}(\mathcal{G}')\neq\mathcal{G}'$ and so $\mathcal{G}'$ is non-abelian.

$$\begin{aligned}\ker(\pi') &= \left\{\overline{X}\in\mathcal{G}'\ /\ \pi'\left(\overline{X}\right)=0\right\}\\ &= \left\{\overline{X}\in\mathcal{G}'\ /\ \pi(X)=0\right\}\\ &= \left\{\overline{X}\in\mathcal{G}'\ /\ X\in\ker(\pi)\right\}.\end{aligned}$$

Then, we have $\mathcal{Z}(\mathcal{G}')\cap\ker(\pi')=\left\{\overline{X}\in\mathcal{G}'\ /\ X\in\mathcal{Z}^2\mathcal{G}\cap\mathcal{G}(\pi)\cap\ker(\pi)\right\}$.

Since $\mathcal{Z}'\subsetneq\mathcal{Z}^2\mathcal{G}\cap\mathcal{G}(\pi)\cap\ker(\pi)$, then, we have $\mathcal{Z}(\mathcal{G}')\cap\ker(\pi')\neq\left\{\overline{0}\right\}$.
Since $\dim\mathcal{G}'<\dim\mathcal{G}$, by the inductive hypothesis, there exists a polarization $\mathfrak{h}'$ at π' such that

$\rho(\pi', \mathfrak{h}', \mathcal{G}')$ is a representation irreducible of the simply connected nilpotent Lie group G' with Lie algebra $\mathcal{G}'$.
Let $\mathfrak{h} = p^{-1}(\mathfrak{h}')$ a Lie subalgebra of $\mathcal{G}$.
For all X , $Y \in \mathfrak{h}$, since $\mathfrak{h}' \in Sub(\pi')$ and $p(X), p(Y) \in \mathfrak{h}'$ we have:

$$\pi([X,Y]) = \pi' \circ p([X,Y]) = \pi'([p(X), p(Y)]) = 0.$$

Therefore $\mathfrak{h} \in Sub(\pi)$.

For any $X \in \mathfrak{h}^{\pi} = \{X \in \mathcal{G} \,/\, \pi([X,\mathfrak{h}]) = 0\}$, we have:

$$\begin{aligned} X \in \mathfrak{h}^{\pi} &\Longrightarrow \pi([X,\mathfrak{h}]) = 0 \\ &\Longrightarrow \pi'([p(X), p(\mathfrak{h})]) = 0 \\ &\Longrightarrow \pi'([p(X), \mathfrak{h}']) = 0 \\ &\Longrightarrow p(X) \in \mathfrak{h}'^{\pi'} = \mathfrak{h}' (\text{since } \mathfrak{h}' \in Pol(\pi'),\, \mathfrak{h}'^{\pi'} = \mathfrak{h}') \\ &\Longrightarrow X \in p^{-1}(\mathfrak{h}') = \mathfrak{h}. \end{aligned}$$

Hence $\mathfrak{h}$ is a polarization at π.

The representation $\rho(\pi, \mathfrak{h}, \mathcal{G})$ of G is irreducible. Indeed, let $Z' = \exp \mathcal{Z}'$, $H = \exp \mathfrak{h}$ be the connected Lie subgroups of G with Lie algebras $\mathcal{Z}'$, $\mathfrak{h}$ respectively, and $G' = G/Z'$ the simply connected nilpotent Lie group with Lie algebra $\mathcal{G}' = \mathcal{G}/\mathcal{Z}'$. Let $q : G \longrightarrow G'$ be the canonical morphism of G onto G'. We have $\rho(\pi, \mathfrak{h}, \mathcal{G}) = \rho(\pi', \mathfrak{h}', \mathcal{G}') \circ q$. Moreover, since $\rho(\pi', \mathfrak{h}', \mathcal{G}')$ is irreducible, the representation $\rho(\pi, \mathfrak{h}, \mathcal{G})$ is irreducible.

2) For $\dim \mathcal{G} = 3$, since $\mathcal{G}$ is non-abelian nilpotent Lie algebra, we have $\dim \mathcal{Z} = 1$ and π is trivial on $\mathcal{Z}$. Let (X_0, Y_0, Z_0) be a basis of $\mathcal{G}$ such that Z_0 generates $\mathcal{Z}$ and $[X_0, Y_0] = Z_0$. The Lie subalgebras of $\mathcal{G}$ which contain the center $\mathcal{Z}$ are: $\mathcal{Z}$, $\mathcal{G}$, and those of the form $\mathfrak{h}_{(\alpha,\beta,\gamma)} = \mathbb{K}(\alpha X_0 + \beta Y_0 + \gamma Z_0) \oplus \mathbb{K} Z_0$ where $\alpha, \beta, \gamma \in \mathbb{K}$ such that $(\alpha X_0 + \beta Y_0 + \gamma Z_0, Z_0)$ is linearly independent.
We have $\mathfrak{h}^{\pi}_{(\alpha,\beta,\gamma)} = \mathcal{G}$, $\mathcal{Z}^{\pi} = \mathcal{G}$ and $\mathcal{G}^{\pi} = \mathcal{G}$. Hence, $\mathcal{G}$ is the only polarization at π and $\rho(\pi, \mathcal{G}, \mathcal{G})$ is irreducible. Also, since $\mathfrak{h}_{(\alpha,\beta,\gamma)}$ and $\mathcal{Z}$ are abelian, the representations χ_{π} defined respectively on $H_{(\alpha,\beta,\gamma)} = \exp \mathfrak{h}_{(\alpha,\beta,\gamma)}$ and $Z = \exp \mathcal{Z}$, are not irreducible. Therefore, the representations $\rho\left(\pi, \mathfrak{h}_{(\alpha,\beta,\gamma)}, \mathcal{G}\right)$ and $\rho(\pi, \mathcal{Z}, \mathcal{G})$ of G are not irreducible. Consequently, the representation $\rho(\pi, \mathfrak{h}, \mathcal{G})$ of G is irreducible if and only if $\mathfrak{h}$ is a polarization at π, for all $\mathfrak{h} \in Sub(\pi)$.

Let's suppose that for all non-abelian nilpotent Lie algebra $\mathcal{G}_0$ of center $\mathcal{Z}_0$ such that $\dim \mathcal{G}_0 < \dim \mathcal{G}$ and $\pi_0 \in Hom(\mathcal{G}_0, gl(\mathcal{V}))$ such that $\mathcal{Z}_0 \cap \ker(\pi_0) \neq \{0\}$, we have: the representation $\rho(\pi_0, \mathfrak{h}_0, \mathcal{G}_0)$ is irreducible if and only if $\mathfrak{h}_0$ is a polarization at π_0, for all non-abelian Lie subalgebra $\mathfrak{h}_0 \in Sub(\pi_0)$.
Then, the representation $\rho(\pi, \mathfrak{h}, \mathcal{G})$ of G is irreducible if and only if $\mathfrak{h}$ is a polarization at π, for all $\mathfrak{h} \in Sub(\pi)$. Indeed:
If $\mathcal{G}(\pi) = \mathcal{G}$, then $\mathcal{G}$ is the only polarization at π and the representation $\rho(\pi, \mathcal{G}, \mathcal{G})$ is irreducible.
If $\mathcal{G}(\pi) \neq \mathcal{G}$, and if $\mathcal{Z}' = \mathcal{Z} \cap \ker(\pi)$, the operator $\pi \in Hom(\mathcal{G}, gl(\mathcal{V}))$ induces an operator π' of the nilpotent Lie algebra $\mathcal{G}' = \mathcal{G}/\mathcal{Z}'$ into the space $gl(\mathcal{V})$ such that $\pi = \pi' \circ p$ where $p : \mathcal{G} \longrightarrow \mathcal{G}'$ is the canonical surjection of $\mathcal{G}$ onto $\mathcal{G}'$. We have $\mathcal{Z}(\mathcal{G}') \cap \ker(\pi') \neq \{0\}$ where $\mathcal{Z}(\mathcal{G}')$ is the center of $\mathcal{G}'$.
$\mathfrak{h}' = p(\mathfrak{h}) \in Sub(\pi')$ since $\pi'([\mathfrak{h}', \mathfrak{h}']) = \pi'([p(\mathfrak{h}), p(\mathfrak{h})]) = \pi' \circ p([\mathfrak{h}, \mathfrak{h}]) = \pi([\mathfrak{h}, \mathfrak{h}]) = 0$.
Moreover, if $\mathfrak{h} \in Pol(\pi)$ then $\mathfrak{h}' \in Pol(\pi')$. Indeed, let's suppose that $\mathfrak{h}$ is a polarization at π, for all

$X \in \mathfrak{h}$, we have:

$$\begin{aligned} p(X) \in \mathfrak{h}'^{\pi'} &\Longrightarrow \pi'([p(X), \mathfrak{h}']) = 0 \\ &\Longrightarrow \pi' \circ p([X, \mathfrak{h}]) = 0 \\ &\Longrightarrow \pi([X, \mathfrak{h}]) = 0 \\ &\Longrightarrow X \in \mathfrak{h}^{\pi} = \mathfrak{h} (\text{since } \mathfrak{h} \in Pol(\pi),\ \mathfrak{h}^{\pi} = \mathfrak{h}) \\ &\Longrightarrow p(X) \in p(\mathfrak{h}) = \mathfrak{h}'. \end{aligned}$$

Consequently, $\mathfrak{h}'^{\pi'} \subset \mathfrak{h}'$ and hence $\mathfrak{h}' \in Pol(\pi')$.

Conversely, if $\mathfrak{h}' \in Pol(\pi')$ then $p^{-1}(\mathfrak{h}') = \mathfrak{h} + \mathcal{Z}' \in Pol(\pi)$. Indeed, we suppose that $\mathfrak{h}' \in Pol(\pi')$. Then for all $X \in \mathcal{G}$, we have:

$$\begin{aligned} X \in \left(p^{-1}(\mathfrak{h}')\right)^{\pi} &\Longleftrightarrow X \in \left(p^{-1}(\mathfrak{h}')\right)^{\pi' \circ p} \\ &\Longleftrightarrow \pi'([p(X), p(\mathfrak{h} + \mathcal{Z}')]) = 0 \\ &\Longleftrightarrow \pi'([p(X), \mathfrak{h}']) = 0 \\ &\Longleftrightarrow p(X) \in \mathfrak{h}'^{\pi'} = \mathfrak{h}' (\text{since } \mathfrak{h}' \in Pol(\pi'),\ \mathfrak{h}'^{\pi'} = \mathfrak{h}') \\ &\Longleftrightarrow X \in p^{-1}(\mathfrak{h}') = \mathfrak{h} + \mathcal{Z}'. \end{aligned}$$

Consequently $\left(p^{-1}(\mathfrak{h}')\right)^{\pi} = p^{-1}(\mathfrak{h}')$. Therefore $p^{-1}(\mathfrak{h}') = \mathfrak{h} + \mathcal{Z}' \in Pol(\pi)$. Hence,

$$if\ \mathcal{Z}' \subset \mathfrak{h}\ then : \mathfrak{h} \in Pol(\pi) \Longleftrightarrow \mathfrak{h}' \in Pol(\pi'). \qquad \textbf{(I)}$$

Let $Z' = \exp \mathcal{Z}'$, $H = \exp \mathfrak{h}$ be the connected Lie subgroups of G with Lie algebras $\mathcal{Z}'$, $\mathfrak{h}$ respectively, and $G' = G/Z'$ the simply connected nilpotent Lie group with Lie algebra $\mathcal{G}' = \mathcal{G}/\mathcal{Z}'$.
Let $q : G \longrightarrow G'$ be the canonical morphism of G onto G'. If $\mathcal{Z}' \subset \mathfrak{h}$, then $\rho(\pi', \mathfrak{h}', \mathcal{G}')$ is the representation of G' such that

$$\rho(\pi, \mathfrak{h}, \mathcal{G}) = \rho(\pi', \mathfrak{h}', \mathcal{G}') \circ q. \qquad \textbf{(a)}$$

Hence,

$$if\ \mathcal{Z}' \subset \mathfrak{h}\ then : \rho(\pi, \mathfrak{h}, \mathcal{G})\ is\ irreducible \Longleftrightarrow \rho(\pi', \mathfrak{h}', \mathcal{G}')\ is\ irreducible. \qquad \textbf{(II)}$$

Indeed:
Let's suppose that the representation $\rho(\pi, \mathfrak{h}, \mathcal{G})$ is irreducible.
Then, $\rho(\pi', \mathfrak{h}', \mathcal{G}')$ is irreducible. If not, the representation $\rho(\pi', \mathfrak{h}', \mathcal{G}')$ would admit a nontrivial closed invariant subspace W, i.e. $\rho(\pi', \mathfrak{h}', \mathcal{G}')_{\bar{x}}(W) \subset W$, $\forall x \in G$ (where $\overline{x} = q(x) \in G'$). Then, we would have:

$$\begin{aligned} \rho(\pi', \mathfrak{h}', \mathcal{G}')_{\bar{x}}(W) \subset W &\Longrightarrow \rho(\pi', \mathfrak{h}', \mathcal{G}')_{q(x)}(W) \subset W \\ &\Longrightarrow \rho(\pi', \mathfrak{h}', \mathcal{G}') \circ q_x(W) \subset W \\ &\Longrightarrow \rho(\pi, \mathfrak{h}, \mathcal{G})_x(W) \subset W \text{ (by } (\mathbf{a})). \end{aligned}$$

Hence, W would be a nontrivial closed invariant subspace with respect to $\rho(\pi, \mathfrak{h}, \mathcal{G})$.

Conversely, let's suppose that the representation $\rho(\pi', \mathfrak{h}', \mathcal{G}')$ is irreducible.
Then, $\rho(\pi, \mathfrak{h}, \mathcal{G})$ is irreducible. If not, the representation $\rho(\pi, \mathfrak{h}, \mathcal{G})$ would admit a nontrivial closed invariant subspace F, and for all $x \in G$, we would have:

$$\begin{aligned} \rho(\pi, \mathfrak{h}, \mathcal{G})_x(F) \subset F &\Longrightarrow \rho(\pi', \mathfrak{h}', \mathcal{G}')_{q(x)}(F) \subset F \text{ (by } (\mathbf{a})) \\ &\Longrightarrow \rho(\pi', \mathfrak{h}', \mathcal{G}')_{\bar{x}}(F) \subset F \text{ (where } \overline{x} = q(x) \in G'). \end{aligned}$$

Hence, F would be a nontrivial closed invariant subspace with respect to $\rho(\pi',\mathfrak{h}',\mathcal{G}')$.

If $\rho(\pi,\mathfrak{h},\mathcal{G})$ is irreducible then $\mathcal{Z}\subset\mathfrak{h}$ by the *lemma 3.1*, and hence we have $\mathcal{Z}'\subset\mathfrak{h}$. Therefore, the representation $\rho(\pi',\mathfrak{h}',\mathcal{G}')$ of G' is irreducible by (**II**). Moreover, since $\dim\mathcal{G}'<\dim\mathcal{G}$, by the inductive hypothesis, we have $\mathfrak{h}'\in Pol(\pi')$. Hence $\mathfrak{h}\in Pol(\pi)$ by (**I**).
If $\mathfrak{h}\in Pol(\pi)$ then $\mathcal{Z}\subset\mathfrak{h}$, and therefore $\mathcal{Z}'\subset\mathfrak{h}$. Hence $\mathfrak{h}'\in Pol(\pi')$ by (**I**). Since $\dim\mathcal{G}'<\dim\mathcal{G}$, by the inductive hypothesis, the representation $\rho(\pi',\mathfrak{h}',\mathcal{G}')$ of G' is irreducible. Consequently, the representation $\rho(\pi,\mathfrak{h},\mathcal{G})$ of G is irreducible by (**II**). □

Corollary 3.4. *Let G be a non-abelian simply connected nilpotent Lie group with finite-dimentinal Lie algebra $\mathcal{G}$, $\mathcal{Z}$ the center of $\mathcal{G}$, $\mathcal{V}$ a finite-dimensional $\mathbb{K}$-vector space of dimension ≥ 2, $\pi\in Hom(\mathcal{G},gl(\mathcal{V}))$ such that $\mathcal{Z}\cap\ker(\pi)\neq\{0\}$, $\mathfrak{h}$ a non-abelian Lie subalgebra of $\mathcal{G}$ subordinate to π and $(f,u)\in\mathcal{G}^*\times gl(\mathcal{V})$ such that $\pi=f\otimes u$. The representation $\rho(\pi,\mathfrak{h},\mathcal{G})$ of G is irreducible if and only if the representation $\rho(f,\mathfrak{h},\mathcal{G})$ of G is irreducible.*

Proof.

$$\begin{aligned}\rho(\pi,\mathfrak{h},\mathcal{G}) \text{ is irreducible} &\iff \mathfrak{h}\in Pol(\pi) \text{ (by the } \textit{theorem 3.3})\\ &\iff \mathfrak{h}\in Pol(f) \text{ (by the } \textit{theorem 2.2})\\ &\iff \rho(f,\mathfrak{h},\mathcal{G}) \text{ is irreducible (Cf.[7])}\end{aligned}$$

□

4 Examples

We suppose that $\dim\mathcal{V}=2$. Let (v_1,v_2) be a basis of $\mathcal{V}$ and (u_1,u_2,u_3,u_4) a basis of $gl(\mathcal{V})$ where u_1, u_2, u_3 and u_4 are the endomorphisms of $\mathcal{V}$ such that their matrices with respect to (v_1,v_2) are respectively: $\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and $\begin{pmatrix}0&0\\0&1\end{pmatrix}$.

Let U_4 be the unipotent standard Lie group of order 4 with Lie algebra $\mathcal{U}_4$ i.e.:

$$U_4=\left\{\begin{pmatrix}1&a_1&a_2&a_3\\0&1&a_4&a_5\\0&0&1&a_6\\0&0&0&1\end{pmatrix}\in GL(4,\mathbb{R}),\ a_i\in\mathbb{R},\ \forall i\in\{1,...,6\}\right\} \text{ and}$$

$$\mathcal{U}_4=\left\{\begin{pmatrix}0&b_1&b_2&b_3\\0&0&b_4&b_5\\0&0&0&b_6\\0&0&0&0\end{pmatrix}\in\mathcal{M}(4,\mathbb{R}),\ b_i\in\mathbb{R},\ \forall i\in\{1,...,6\}\right\}.$$

Let $(X_1,X_2,X_3,X_4,X_5,X_6)$ be a basis of $\mathcal{U}_4$ where the elements X_1, X_2, X_3, X_4, X_5 and X_6 are defined as follows:

$$X_1=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ X_2=\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ X_3=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$X_4=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ X_5=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\ X_6=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

The non trivial Lie brackets among basis elements are described as follows:

$$[X_1, X_4] = X_2, \quad [X_1, X_5] = [X_2, X_6] = X_3, \quad \text{and} \quad [X_4, X_6] = X_5.$$

We consider the Lie algebra $\mathcal{G} = \mathbb{K}X_1 \oplus \mathbb{K}X_2 \oplus \mathbb{K}X_3 \oplus \mathbb{K}X_4 \oplus \mathbb{K}X_5$, a Lie subalgebra of $\mathcal{U}_4$, and $G = \exp\mathcal{G}$.

The center of $\mathcal{G}$ is $\mathcal{Z} = \mathbb{K}X_2 \oplus \mathbb{K}X_3$. We denote by $\mathfrak{h}_1$ and $\mathfrak{h}_2$ the Lie subalgebras of $\mathcal{G}$ such that :

$\mathfrak{h}_1 = \mathbb{K}X_1 \oplus \mathbb{K}X_2 \oplus \mathbb{K}X_3 \oplus \mathbb{K}X_4$ and $\mathfrak{h}_2 = \mathbb{K}X_1 \oplus \mathbb{K}X_2 \oplus \mathbb{K}X_3 \oplus \mathbb{K}(X_4 + X_5)$.

Let π_1, π_2, π_3, and π_4 be the linear operators defined on $\mathcal{G}$ into $gl(\mathcal{V})$ by:

$$\begin{cases} \pi_1(X_1) = u_1 \\ \pi_1(X_2) = 0 \\ \pi_1(X_3) = u_3 \\ \pi_1(X_4) = 0 \\ \pi_1(X_5) = 0 \end{cases}, \begin{cases} \pi_2(X_1) = 0 \\ \pi_2(X_2) = u_2 \\ \pi_2(X_3) = -u_2 \\ \pi_2(X_4) = 0 \\ \pi_2(X_5) = 0 \end{cases}, \begin{cases} \pi_3(X_1) = u_1 \\ \pi_3(X_2) = 0 \\ \pi_3(X_3) = 0 \\ \pi_3(X_4) = u_2 \\ \pi_3(X_5) = u_3 \end{cases} \text{and} \begin{cases} \pi_4(X_1) = 0 \\ \pi_4(X_2) = u_2 \\ \pi_4(X_3) = u_3 \\ \pi_4(X_4) = 0 \\ \pi_4(X_5) = 0 \end{cases}$$

We have:

$$\mathcal{Z} \cap \ker(\pi_1) \neq \{0\}, \mathcal{Z} \cap \ker(\pi_2) \neq \{0\},$$
$$\mathcal{Z} \cap \ker(\pi_3) \neq \{0\}, \mathcal{Z} \cap \ker(\pi_4) = \{0\}.$$

$\mathfrak{h}_1$ and $\mathfrak{h}_2$ are polarizations respectively at π_1 and π_2. Indeed:
For all $X = \sum_{i=1}^{5} t_i X_i \in \mathcal{G}$, with $t_i \in \mathbb{K}$, $\forall i \in \{1, ..., 5\}$, we have:

$$\begin{aligned} [X, X_2] &= [X, X_3] = 0, \\ [X, X_1] &= t_4 [X_4, X_1] + t_5 [X_5, X_1] = -t_4 X_2 - t_5 X_3, \\ [X, X_4 + X_5] &= t_1 [X_1, X_4] + t_1 [X_1, X_5] = t_1 X_2 + t_1 X_3. \end{aligned}$$

Then, we have:

$$\begin{aligned} \pi_2([X, X_2]) &= \pi_2([X, X_3]) = 0, \\ \pi_2([X, X_1]) &= \pi_2(-t_4 X_2 - t_5 X_3) = -t_4 u_2 + t_5 u_2 = (t_5 - t_4) u_2, \\ \pi_2([X, X_4 + X_5]) &= \pi_2(t_1 X_2 + t_1 X_3) = t_1 u_2 - t_1 u_2 = 0. \end{aligned}$$

Since $(X_1, X_2, X_3, X_4 + X_5)$ is a basis of $\mathfrak{h}_2$, we have:

$$\begin{aligned} X \in \mathfrak{h}_2^{\pi_2} &\Longleftrightarrow \pi_2([X, \mathfrak{h}_2]) = 0 \\ &\Longleftrightarrow \begin{cases} \pi_2([X, X_1]) = 0 \\ \pi_2([X, X_2]) = 0 \\ \pi_2([X, X_3]) = 0 \\ \pi_2([X, X_4 + X_5]) = 0 \end{cases} \\ &\Longleftrightarrow (t_5 - t_4) u_2 = 0 \\ &\Longleftrightarrow t_4 = t_5 \\ &\Longleftrightarrow X = t_1 X_1 + t_2 X_2 + t_3 X_3 + t_4 (X_4 + X_5) \\ &\Longleftrightarrow X \in \mathfrak{h}_2. \end{aligned}$$

Therefore, $\mathfrak{h}_2^{\pi_2} = \mathfrak{h}_2$. Moreover, in the same way, we prove that $\mathfrak{h}_1^{\pi_1} = \mathfrak{h}_1$.

Hence the representations $\rho(\pi_1, \mathfrak{h}_1, \mathcal{G})$ and $\rho(\pi_2, \mathfrak{h}_2, \mathcal{G})$ of G are irreducible.

$\mathfrak{h}_1$ and $\mathfrak{h}_2$ are subordinate subalgebras to π_3 but they are not polarizations at π_3. Indeed, we have $\mathfrak{h}_1^{\pi_3} = \mathcal{G}$ and $\mathfrak{h}_2^{\pi_3} = \mathcal{G}$. Hence the representations $\rho(\pi_3, \mathfrak{h}_1, \mathcal{G})$ and $\rho(\pi_3, \mathfrak{h}_2, \mathcal{G})$ of G are not irreducible by the *theorem 3.3*.

$\mathcal{G}$ is a polarization at π_3 since $\mathcal{G}^{\pi_3} = \mathcal{G}$.

The Lie subalgebras of $\mathcal{G}$ of the form $\mathfrak{h}_3 = \mathbb{K}\left(\sum_{i=1}^{5} \alpha_i X_i\right) \oplus \mathbb{K}X_2 \oplus \mathbb{K}X_3$ with $\alpha_i \in \mathbb{K}$, $\forall i \in \{1, ..., 5\}$ and $\alpha_1 \neq 0$, such that the family $\left(X_2, X_3, \sum_{i=1}^{5} \alpha_i X_i\right)$ is linearly independent, are polarizations at π_4. Indeed: For all $X = \sum_{i=1}^{5} t_i X_i \in \mathcal{G}$, with $t_i \in \mathbb{K}$, $\forall i \in \{1, ..., 5\}$, we have

$$X \in \mathfrak{h}_3^{\pi_4} \iff \begin{cases} \alpha_1 t_4 = t_1 \alpha_4 \\ \alpha_1 t_5 = t_1 \alpha_5 \end{cases}$$

$$\iff X = t_1\left(X_1 + \frac{\alpha_4}{\alpha_1} X_4 + \frac{\alpha_5}{\alpha_1} X_5\right) + t_2 X_2 + t_3 X_3.$$

Therefore $\mathfrak{h}_3^{\pi_4} = \mathbb{K}\left(X_1 + \frac{\alpha_4}{\alpha_1} X_4 + \frac{\alpha_5}{\alpha_1} X_5\right) \oplus \mathbb{K}X_2 \oplus \mathbb{K}X_3$.
Since $X_1 + \frac{\alpha_4}{\alpha_1} X_4 + \frac{\alpha_5}{\alpha_1} X_5 = \frac{1}{\alpha_1}\left(\sum_{i=1}^{5} \alpha_i X_i\right) - \frac{\alpha_2}{\alpha_1} X_2 - \frac{\alpha_3}{\alpha_1} X_3 \in \mathfrak{h}_3$, we have $\mathfrak{h}_3^{\pi_4} = \mathfrak{h}_3$.
Hence, $\mathfrak{h}_3$ is a polarization at π_4.

However, the representation $\rho(\pi_4, \mathfrak{h}_3, \mathcal{G})$ of G is not irreducible. Indeed:
the *theorem 3.3* can not be applied in this case, since $\mathcal{Z} \cap \ker(\pi_4) = \{0\}$.
Since $\mathfrak{h}_3$ is abelian, the representation χ_{π_4} of $H_3 = \exp \mathfrak{h}_3$ in $\mathcal{V}$ is not irreducible by the lemmas of Schur.
Therefore, $\rho(\pi_4, \mathfrak{h}_3, \mathcal{G})$ is not irreducible.

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