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On the Completeness of the Root Vectors of Dissipative Dirac Operators with Transmission Conditions

Hüseyin Tuna*
Mehmet Akif Ersoy University, Department of Mathematics,
15100, Burdur, Turkey

Aytekin Eryilmaz†
Nevşehir HacıBektaş Veli University, Department of Mathematics,
50300 Nevşehir, Turkey

Abstract. In this article, we consider dissipative Dirac system in the limit-circle case. Then using the Livsic's theorem , we prove the completeness of the system of root vectors for dissipative Dirac system with transmission conditions

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1 INTRODUCTION

We will consider the Dirac system

$$l_1(y) := J\frac{dy(x)}{dx} + B(x)y(x) = \lambda A(x)y(x), \quad x \in I, \tag{1.1}$$

where λ is a complex spectral parameter and $I = I_1 \cup I_2$, $I_1 := [0,c)$, $I_2 := (c,1]$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $A(x) = \begin{pmatrix} a(x) & c(x) \\ c(x) & b(x) \end{pmatrix}$, $B(x) = \begin{pmatrix} 0 & p(x) \\ q(x) & 0 \end{pmatrix}$, $A(x) > 0$ (for almost all $x \in I$); elements of the matrices $A(x)$ and $B(x)$ are real valued, continuous functions on I and $q(1) \neq 1$. Equation (1.1) is the radial wave equation for a relativistic particle in a central field and is of interest in physics [9]. Spectral properties of (1.1) have been investigated in [22]-[28].

Boundary-value problems with transmission conditions arise in different branches of mathematics, radio, electronics, geophysics, mechanics, and other fields of natural science and technology [1]. Discontinuous Sturm-Liouville problems were investigated in [13], [29], [31], [33].

The first general results on completeness property of non-homogeneous string with dissipative boundary condition was obtained by Krein and Nudelman [14]. The recent publications [19], [16],

*e-mail address: hustuna@gmail.com
†e-mail address: eryilmazaytekin@gmail.com

[17], [18] devoted to the questions of completeness and spectral synthesis for general $n\times n$ first order systems of ODE (see also references therein). In [19], [16], [17] it was shown that the completeness property for some classes of boundary conditions essentially depends on boundary values of the potential matrix and explicit conditions of the completeness were found. In particular, in [19], an example of incomplete dissipative 2×2 Dirac operator was constructed. It was shown in [17], [18] that the resolvent of any complete dissipative Dirac type operator admits the spectral synthesis. Moreover, explicit conditions of the dissipativity and completeness of such operators were found. It is also worth to mention recent papers [4], [5], [6], [7], [8] devoted to the Riesz basis property for 2×2 Dirac operator (see also references therein). In this paper, using Livsic's theorem we prove the system of all eigenfunctions and associated functions of the Dirac operator. A similar way was employed earlier in the Sturm- Liouville operator case in [2], [3], [11], [12], [30], [32].

To pass from the differential expression $l(y) := A^{-1}(x)l_1(y)$ $(x\in I)$ to operators we introduce the Hilbert space $H_1 := L^2_A(I;E)$ $\left(E := \mathbb{C}^2\right)$ of vector valued functions with values in $\mathbb{C}^2$and with the inner product

$$\langle y,z\rangle = \int_0^1 \langle A(x)y(x), z(x)\rangle_E dx.$$

Denote by D the linear set of all vectors $y\in H_1$ such that y_1 and y_2 are locally absolutely continuous functions on I and $l(y)\in H_1$. We define the operator L on D by the equality $Ly = ly$.

For two arbitrary vectors $y,z\in D$, we have Green's formula

$$\langle Ly,z\rangle - \langle y,Lz\rangle = [y,z]_{c-} - [y,z]_0 + [y,z]_1 - [y,z]_{c+} \tag{1.2}$$

where $[y,z]_x := W_x[y,\overline{z}] = y_1(x)\overline{z_2(x)} - y_2(x)\overline{z_1(x)}$.

In H, we consider the dense linear set D'_0 consisting of smooth, compactly supported vector-valued functions on I. Denote by L'_0 the restriction of the operator L to D'_0. It follows from (1.2) that L'_0 is symmetric. Consequently, it is closable. Its closure is denoted by L_0. The operators L_0 and L are called the minimal and maximal operators, respectively [20].

We assume that L_0 satisfies the Weyl's limit circle case.

Denote by

$$\begin{aligned}
u(x,\lambda) &= \begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix},\ v(x,\lambda) = \begin{pmatrix} v_1(x,\lambda) \\ v_2(x,\lambda) \end{pmatrix},\\
u_1(x,\lambda) &= \begin{cases} u_{11}(x,\lambda), & x\in I_1 \\ u_{12}(x,\lambda), & x\in I_2 \end{cases},\ u_2(x,\lambda) = \begin{cases} u_{21}(x,\lambda), & x\in I_1 \\ u_{22}(x,\lambda), & x\in I_2 \end{cases}\\
v_1(x,\lambda) &= \begin{cases} v_{11}(x,\lambda), & x\in I_1 \\ v_{12}(x,\lambda), & x\in I_2 \end{cases},\ v_2(x,\lambda) = \begin{cases} v_{21}(x,\lambda), & x\in I_1 \\ v_{22}(x,\lambda), & x\in I_2 \end{cases},\ I_1 = [0,c),\ I_2 = (c,1],
\end{aligned}$$

the solutions of the equation

$$l(y) = \lambda y,\ x\in I \tag{1.3}$$

satisfying the initial conditions

$$\begin{aligned}
u_{12}(0,\lambda) &= \cos\alpha,\ u_{22}(0,\lambda) = \sin\alpha,\\
v_{12}(0,\lambda) &= -\sin\alpha,\ v_{22}(0,\lambda) = \cos\alpha,
\end{aligned}$$

and

$$\begin{aligned}
u_{11}(c-,\lambda) &= \delta_1 u_{12}(c+,\lambda),\ u_{21}(c-,\lambda) = \delta_2 u_{22}(c+,\lambda),\\
v_{11}(c-,\lambda) &= \delta_1 v_{12}(c+,\lambda),\ v_{21}(c-,\lambda) = \delta_2 v_{22}(c+,\lambda),
\end{aligned}$$

where $\delta_1,\ \delta_2$ and α are some real numbers with $\delta_1\delta_2 \neq 0$

The Wronskian of the two solutions (1.3) doesn't depend on x, and the two solutions of this equation are linearly independent if and only if their wronskian is nonzero. It is clear that

$$W_x[u,v] = W_0[u,v] = 1,\ x \in I.$$

Lemma 1.1. *Let $[u,v]_x = 1\ \ (a \le x \le b)$ for some real solutions $u(x)$ and $v(x)$ of equation $l(y) = 0$. Then, one has the equality*

$$[y,z]_x = [y,u]_x[\bar{z},v]_x - [y,v]_x[\bar{z},u]_x, \quad y,z \in D. \tag{1.4}$$

Proof. Since the functions $u(x)$ and $v(x)$ are real valued and $[u,v]_x = 1\ \ (a \le x \le b)$, we obtain

$$\begin{aligned}
[y,u]_x[z,v]_x - [y,v]_x[z,u]_x &= (y_1(x)u_2(x) - y_2(x)u_1(x))(\bar{z}_1(x)v_2(x) - \bar{z}_2(x)v_1(x)) \\
&\quad - (y_1(x)v_2(x) - y_2(x)v_1(x))(\bar{z}_1(x)u_2(x) - \bar{z}_2(x)u_1(x)) \\
&= y_1(x)u_2(x)\bar{z}_1(x)v_2(x) - y_1(x)u_2(x)\bar{z}_2(x)v_1(x) \\
&\quad - y_2(x)u_1(x)\bar{z}_1(x)v_2(x) + y_2(x)u_1(x)\bar{z}_2(x)v_1(x) \\
&\quad - y_1(x)v_2(x)\bar{z}_1(x)u_2(x) + y_1(x)v_2(x)\bar{z}_2(x)u_1(x) \\
&\quad + y_2(x)v_1(x)\bar{z}_1(x)u_2(x) - y_2(x)v_1(x)\bar{z}_2(x)u_1(x) \\
&= -y_1(x)u_2(x)\bar{z}_2(x)v_1(x) - y_2(x)u_1(x)\bar{z}_1(x)v_2(x) \\
&\quad - y_1(x)v_2(x)\bar{z}_1(x)u_2(x) + y_1(x)v_2(x)\bar{z}_2(x)u_1(x) \\
&\quad + y_2(x)v_1(x)\bar{z}_1(x)u_2(x) \\
&= (-y_1(x)\bar{z}_2(x) + y_2(x)\bar{z}_1(x))(u_2(x)v_1(x) - u_1(x)v_2(x)) \\
&= [y,z]_x.
\end{aligned}$$

□

The identity (1.2) is well known for Sturm-Liouville operators.

Since L_0 satisfies the Weyl's limit circle case, $u,v \in H_1$, and moreover $u,v \in D$. The solutions $u(x,\lambda)$ and $v(x,\lambda)$ form a fundamental system of (1.3) and they are entire functions of λ (see [20]). Let $u(x) = u(x,0)$ and $v(x) = v(x,0)$ the solutions of the equation $l(y) = 0$ satisfying the initial conditions

$$\begin{aligned}
u_{12}(0) &= \cos\alpha,\ u_{22}(0) = \sin\alpha, \\
v_{12}(0) &= -\sin\alpha,\ v_{22}(0) = \cos\alpha.
\end{aligned}$$

Let us consider the functions $y \in D$ satisfying the conditions

$$y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \tag{1.5}$$

$$[y,u]_1 - h[y,v]_1 = 0, \tag{1.6}$$

$$y_1(c-) = \delta_1 y_1(c+) \tag{1.7}$$

$$y_2(c-) = \delta_2 y_2(c+) \tag{1.8}$$

where $\Im m\, h > 0,\ \alpha \in \mathbb{R}$.

2 Preliminaries

Let A denote the linear non-selfadjoint operator in the Hilbert space with domain $D(A)$. A complex number λ_0 is called an eigenvalue of the operator A if there exists a non-zero element $y_0 \in D(A)$ such that $Ay_0 = \lambda_0 y_0$; in this case, y_0 is called the eigenvector of A for λ_0. The eigenvectors for λ_0 span a subspace of $D(A)$, called the eigenspace for λ_0.

The element $y \in D(A),\ y \neq 0$ is called a root vector of A corresponding to the eigenvalue λ_0 if $(T-\lambda_0 I)^n y = 0$ for some $n \in \mathbb{N}$. The root vectors for λ_0 span a linear subspace of $D(A)$, is called the root lineal for λ_0. The algebraic multiplicity of λ_0 is the dimension of its root lineal. A root vector is called an associated vector if it is not an eigenvector. The completeness of the system of all eigenvectors and associated vectors of A is equivalent to the completeness of the system of all root vectors of this operator.

An operator A is called dissipative if $\Im m\langle Ax, x\rangle \geq 0,\ (\forall x \in D(A))$. A bounded operator is dissipative if and only if

$$\Im m A = \frac{1}{2i}(A - A^*) \geq 0.$$

Let A be an arbitrary compact operator acting in the Hilbert space H. Let $\left\{\mu_j(A)\right\}$ be a sequence of all nonzero eigenvalues of A arranged by considering algebraic multiplicity and with decreasing modulus, where $\nu(A)$ $(\leq \infty)$ is a sum of algebraic multiplicities of all nonzero eigenvalues of A. If A is a nuclear operator, then $\sum_{j=1}^{\nu(A)} \left|\mu_j(A)\right| < +\infty$ and if A is a Hilbert - Schmidt operator, then $\sum_{j=1}^{\nu(A)} \left|\mu_j(A)\right|^2 < +\infty$. We will denote the class of all nuclear and Hilbert - Schmidt operator in H by σ_1 and σ_2, respectively. If $A \in \sigma_1$, then $\sum_{j=1}^{\nu(A)} \mu_j(A)$ is called the trace of A and is denoted by spA.

The determinant

$$det(I - \mu A) = \prod_{j=1}^{\nu(A)} \left[I - \mu\mu_j(A)\right],\ A \in \sigma_1$$

is called the characteristic determinant of A and is denoted by $D_A(\mu)$. $D_A(\mu)$ is an entire function of μ.

For any $A \in \sigma_2$, the product

$$\widetilde{D}_A(\mu) = \prod_{j=1}^{\nu(A)} \left[I - \mu\mu_j(A)\right] e^{\mu\mu_j(A)} \tag{2.1}$$

is also an entire function of μ, called the regularized characteristic determinant of A.

If the operator $I - \mu A$ has a bounded inverse defined on the whole space H, then the complex number μ is called an F -regular point (regular in the sense of Fredholm) for A.

Let A and B be linear bounded operators in H and $A - B \in \sigma_1$. If the point μ is an F -regular point of B, then

$$(I - \mu A)(I - \mu B)^{-1} = I - \mu(A - B)(I - \mu B)^{-1}$$

where $\mu(A - B)(I - \mu B)^{-1} \in \sigma_1$. Consequently, the determinant

$$D_{A/B}(\mu) = \det\left[(I - \mu A)(I - \mu B)^{-1}\right]$$

makes sense and is called the determinant of perturbation of the operator B by the operator $K = A - B$.

Theorem 2.1 ([10, p.172])**.** *If $A,\ B \in \sigma_2,\ A - B \in \sigma_1$ and μ is an F-regular point of B, then*

$$D_{A/B}(\mu) = \frac{\widetilde{D}_A(\mu)}{\widetilde{D}_B(\mu)} e^{\mu sp(B-A)}.$$

Theorem 2.2 ([10, p.177])**.** *If A and B are bounded dissipative operator and $A - B \in \sigma_1$, then for any $\beta_0 \in \left(0, \frac{\pi}{2}\right)$, the limit*

$$\lim_{\rho\to\infty} \frac{\ln\left|D_{A/B}\left(\rho e^{i\beta}\right)\right|}{\rho} = 0$$

converges uniformly in β on the interval $\left(\frac{\pi}{2} - \beta_0, \frac{\pi}{2} + \beta_0\right)$.

Definition 2.3. Let f be an entire function. If for each $\varepsilon > 0$ there exists a finite constant $C_\varepsilon > 0$, such that

$$|f(\lambda)| \le C_\varepsilon e^{\varepsilon|\lambda|},\ \ \lambda \in \mathbb{C} \tag{2.2}$$

then f is called an entire function of order ≤ 1 of growth and minimal type.

From (2.2), it is clear that

$$\lim_{|\lambda|\to\infty} \sup \frac{1}{|\lambda|} \ln|f(\lambda)| \le 0. \tag{2.3}$$

It is known that each function f, having properties (2.2) and $f(0) = -1$, has the representation

$$f(\lambda) = -\lim_{r\to\infty} \prod_{|\lambda_j|\le r} \left(1 - \frac{\lambda}{\lambda_j}\right), \tag{2.4}$$

and also the limit $\lim_{r\to\infty} \prod\limits_{|\lambda_j|\le r} \frac{1}{\lambda_j}$ exists and is finite [12], [21], [34].

Theorem 2.4 (Livšic [10, p.226])**.** *Let A be compact dissipative operator on H and let $A_{\Im m} \in \sigma_1$ where $A_{\Im m} = \frac{1}{2}(A - A^*)$. The system of all root vectors of A be complete in H, if and only if*

$$\sum_{j=1}^{\nu(A)} \Im m \mu_j(A) = spA_{\Im m}.$$

3 Main Results

In this section, let us define a Hilbert space and an operator whose root vectors coincide with those of problem (1.3)-(1.8).

Let H be the Hilbert space $H := \left\{ y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} :\ y_1(x),\ y_2(x) \in H_1 \right\}$. The inner product of H is defined by

$$\langle y(.), z(.)\rangle_H := \int_0^c y^T(x)\bar{z}(x)\,dx + \delta_1\delta_2 \int_c^1 y^T(x)\bar{z}(x)\,dx$$

where ^{T} denotes the matrix transpose, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},\ z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \in H,\ y_i(.), z_i(.) \in H_1,\ i = 1, 2.$

Let us adopt the notations:

$$\begin{aligned} S_-(y) &:= [y,u]_1 - h[y,v]_1, \\ S_+(y) &:= y_1(0)\cos\alpha + y_2(0)\sin\alpha, \\ S_1(y) &:= y_1(c-) - \delta_1 y_1(c+), \\ S_2(y) &:= y_2(c-) - \delta_2 y_2(c+). \end{aligned}$$

We construct the operator $A : H \to H$ with domain

$$D(A) := \left\{ \begin{array}{l} f \in H : f \in AC_{loc}(I),\ f(c\pm) \text{ one sided limit exists and are finite,} \\ l(y) \in H,\ S_-(f) = 0,\ S_+(f) = 0,\ S_1(f) = 0, S_2(f) = 0,\ Ay = l(y) \end{array} \right\}.$$

Thus, we can pose the boundary-value problems (1.3)-(1.8) in H as $Ay = \lambda y,\ y \in D(A)$. It is clear that the eigenvalues and root lineals A and L coincide.

Let $\psi(x,\lambda) = \begin{pmatrix} \psi_1(x,\lambda) \\ \psi_2(x,\lambda) \end{pmatrix}$,

$$\psi_1(x,\lambda) = \begin{cases} \psi_{11}(x,\lambda), & x \in I_1 \\ \psi_{12}(x,\lambda), & x \in I_2 \end{cases},\ \psi_2(x,\lambda) = \begin{cases} \psi_{21}(x,\lambda), & x \in I_1 \\ \psi_{22}(x,\lambda), & x \in I_2 \end{cases}$$

be solutions of (1.1) given in the introduction. Let us define $\omega_1(\lambda) := W[\psi_1, v_1]_x\ (x \in I_1)$ and $\omega_2(\lambda) := W[\psi_2, v_2]_x\ (x \in I_2)$. If we set $\omega := \omega_1 = \delta_1\delta_2\omega_2$, then ω becomes an entire function that its zeros coincide with the eigenvalues of the operator A. So A has discrete spectrum and possible limit points can only at infinity.

We set $z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$ where

$$z_1(x) = \begin{cases} z_{11}(x), & x \in I_1 \\ z_{12}(x), & x \in I_2 \end{cases},\ z_2(x) = \begin{cases} z_{21}(x), & x \in I_1 \\ z_{22}(x), & x \in I_2 \end{cases},$$

$$\begin{aligned} z_{11}(x) &= u_{11}(x) - hv_{11}(x),\ (x \in I_1), \\ z_{12}(x) &= u_{12}(x) - hv_{12}(x),\ (x \in I_2), \\ z_{21}(x) &= u_{21}(x) - hv_{21}(x),\ (x \in I_1), \\ z_{22}(x) &= u_{22}(x) - hv_{22}(x),\ (x \in I_2). \end{aligned}$$

It is clear that the solution $z(x)$ satisfies both transmission conditions (1.7), (1.8) and the boundary condition (1.6). Similarly, the solution $v(x)$ satisfies the boundary condition (1.5) and both transmission conditions (1.7), (1.8).

It is clear that

$$\begin{aligned} [y,z]_x &= [y,u]_x[\bar{z},v]_x - [y,v]_x[\bar{z},u]_x\ (x \in I_1), \\ [y,z]_x &= \delta_1\delta_2([y,u]_x[\bar{z},v]_x - [y,v]_x[\bar{z},u]_x)\ (x \in I_2). \end{aligned} \tag{3.1}$$

Theorem 3.1. *The operator A is dissipative in H.*

Proof. Let $\eta \in D$, then by Lagrange identity we get

$$\langle A\eta,\eta\rangle - \langle \eta, A\eta\rangle = [\eta,\eta]_{c-} - [\eta,\eta]_0 + \delta_1\delta_2[\eta,\eta]_1 - \delta_1\delta_2[\eta,\eta]_{c+}. \tag{3.2}$$

Since $\eta_0 \in D$, we have

$$[\eta,\eta]_0 = 0,\ [\eta,\eta]_{c-} = \delta_1\delta_2[\eta,\eta]_{c+} \tag{3.3}$$

From Lemma 1,

$$\begin{aligned}[\eta,\eta]_1 &= \delta_1\delta_2\left([\eta,u]_1[\overline{\eta},v]_1 - [\eta,v]_1[\overline{\eta},u]_1\right)\\ &= \delta_1\delta_2 2i\Im m\, h\left([\eta,v]_1\right)^2.\end{aligned} \tag{3.4}$$

From (3.3) and (3.4)

$$\Im m\langle A\eta,\eta\rangle = (\delta_1\delta_2)^2\,\Im m\, h\left([\eta,v]_1\right)^2, \tag{3.5}$$

and so A is dissipative in H. □

It follows from Theorem 4, all the eigenvalues of A lie in the closed upper half plane $\Im m\lambda \geq 0$.

Theorem 3.2. *The operator A has not any real eigenvalue.*

Proof. Suppose that the operator A has a real eigenvalue λ_0. Let $\eta_0(x) = \eta(x,\lambda_0)$ be the corresponding eigenfunction. Since $\Im m\,(A\eta_0,\eta_0)) = \Im m\left(\lambda_0\|\eta_0\|^2\right)$, we get from (3.5) that $[\eta_0,v]_1 = 0$. By the boundary condition (1.6), we have $[\eta_0,u]_1 = 0$. Thus

$$[\eta_0(x,\lambda_0),u]_1 = [\xi_0(x,\lambda_0),v]_1 = 0. \tag{3.6}$$

From Lemma 1 with $\xi_0(x) = \xi(x,\lambda_0)$,

$$1 = \delta_1\delta_2[\eta_0,\xi_0]_1 = [\eta_0,u]_1[\xi_0,v]_1 - [\eta_0,v]_1[\xi_0,u]_1.$$

By the equality (3.6), the right -hand side is equal to 0. This contradiction proves the theorem. □

From Theorem 5, there exist the inverse operator A^{-1}. We shall find the operator A^{-1}. For $y \in D(A)$, the equation $Ay = f(x)$ is equivalent to he non homogeneous differential equation

$$l(y) = f(x),\ x \in I$$

subject to the boundary conditions

$$\begin{aligned} y_1(0)\cos\alpha + y_2(0)\sin\alpha &= 0,\\ [y,u]_1 - h[y,v]_1 &= 0,\end{aligned}$$

$$\begin{aligned} y_1(c-) &= \delta_1 y_1(c+)\\ y_2(c-) &= \delta_2 y_2(c+)\end{aligned}$$

where $f(x) = \begin{cases} f_1(x), & x \in I_1\\ f_2(x), & x \in I_2\end{cases}$, $f(x) \in H,\ \delta_1\delta_2 > 0$.

Let

$$G(x,t) = \begin{cases} v(x)z^T(t), & 0 \leq t \leq x \leq 1,\ x \neq c,\ t \neq 0\\ v(t)z^T(x), & 0 \leq x \leq t \leq 1,\ x \neq c,\ t \neq 0\end{cases} \tag{3.7}$$

where ^{T} denotes the matrix transpose. Then we have

$$y(x) = \langle G(x,t),\overline{f}\rangle_H,$$

where $\overline{f(x)} = \begin{cases} \overline{f_1(x)}, & x \in I_1\\ \overline{f_2(x)}, & x \in I_2\end{cases}$

The integral operator K defined by the formula

$$Kf = \langle G(x,t), \overline{f(t)} \rangle_H \quad (f \in H) \tag{3.8}$$

is a compact linear operator in the space H. K is a Hilbert Schmidth operator. It is evident that $K = A^{-1}$. Consequently the root lineals of the operator A and K coincide and, therefore, the completeness in H of the system of all eigenvectors and associated vectors of A is equivalent to the completeness of those for K. Since the algebraic multiplicity of nonzero eigenvalues of a compact operator is finite, each eigenvector of A may have only a finite number of linear independent associated vectors.

Let

$$\begin{aligned} \tau_1(\lambda) &: = [\varphi_1(x,\lambda), u_1(x)]_1, \\ \tau_2(\lambda) &: = [\varphi_1(x,\lambda), v_1(x)]_1, \\ \tau(\lambda) &: = \tau_1(\lambda) - h\tau_2(\lambda). \end{aligned} \tag{3.9}$$

It is clear that

$$\sigma_p(A) = \{\lambda : \lambda \in \mathbb{C},\ \tau(\lambda) = 0\}$$

where $\sigma_p(A)$ denotes the set of all eigenvalues of A. Since for arbitrary b $(c \le b < 1)$, the function $\varphi_1(b,\lambda)$ is entire function of λ of order ≤ 1 (see [15]), consequently, $\tau(\lambda)$ have the same property. Then $\tau(\lambda)$ is entire functions of the order ≤ 1 of growth, and of minimal type.

Since $z(x) = u(x) - hv(x)$, setting $h = h_1 + ih_2$ $(h_1, h_2 \in \mathbb{R})$, we get from (3.8) in view of (3.7) that $K = K_1 + iK_2$, where

$$K_1 f = \langle G_1(x,t), \overline{f(t)} \rangle,\ K_2 f = \langle G_2(x,t), \overline{f(t)} \rangle$$

and

$$G_1(x,t) = \begin{cases} v(x)[u(t) - h_1 v(t)], & 0 \le t \le x \le 1,\ x \ne c,\ t \ne 0 \\ v(t)[u(x) - h_1 v(x)], & 0 \le t \le x \le 1,\ x \ne c,\ t \ne 0 \end{cases}$$

$$G_2(x,t) = -h_2 v(x) v(t),\ h_2 = \Im m h > 0.$$

The operator K_1 is the self-adjoint Hilbert–Schmidt operator in H, and K_2 is the self-adjoint one dimensional operator in H.

Let A_1 denote the operator in H generated by the differential expression l and the boundary conditions

$$\begin{aligned} y_1(0)\cos\alpha + y_2(0)\sin\alpha &= 0, \\ [y,u]_1 - h_1[y,v]_1 &= 0, \end{aligned}$$

$$\begin{aligned} y_1(c-) &= \delta_1 y_1(c+) \\ y_2(c-) &= \delta_2 y_2(c+) \end{aligned}$$

where $\delta_1\delta_2 > 0$.

It is easy to verify that K_1 is the inverse A_1.

Let $T = -K$ and $T = T_1 + iT_2$, where $T_1 = -K_1,\ T_2 = -K_2$. We will denote by λ_j and γ_k the eigenvalues of the operators A and A_1, respectively. Then the eigenvalues of T are $\frac{-1}{\lambda_j}$ and the eigenvalues of T_1 are $\frac{-1}{\gamma_k}$. $\Im m \gamma_k = 0$ for all k, since L_1 is a self-adjoint operator.

Theorem 3.3. $\sum_j \Im m \left(\frac{-1}{\lambda_j} \right) = spT_2$.

Proof. Let $A = T_1$ and $B = T$. Substituting this in the Theorem 1, we obtain

$$D_{T_1/T}(\mu) = \frac{\widetilde{D}_{T_1}(\mu)}{\widetilde{D}_T(\mu)} e^{\mu sp T_2}, \tag{3.10}$$

and by (2.1) we get

$$\widetilde{D}_T(\mu) = \prod_j \left(1 + \frac{\mu}{\lambda_j}\right) e^{-\frac{\mu}{\lambda_j}}, \; \widetilde{D}_{T_1}(\mu) = \prod_k \left(1 + \frac{\mu}{\gamma_k}\right) e^{-\frac{\mu}{\gamma_k}}.$$

We set

$$\tau(\mu) := \tau_1(\mu) - h\tau_2(\mu), \; \Gamma(\mu) := \tau_1(\mu) - h_1\tau_2(\mu),$$

where the functions $\tau_1(\mu)$ and $\tau_2(\mu)$ are defined by (3.9). The eigenvalues of K and K_1 coincide with the root of functions $\tau(\mu)$ and $\Gamma(\mu)$, respectively. The functions $\tau(\mu)$ and $\Gamma(\mu)$ are entire functions of order ≤ 1 of growth and minimal type and $\tau(0) = \Gamma(0) = -1$. Then

$$\tau(\mu) = -\prod_j \left(1 + \frac{\mu}{\lambda_j}\right), \; \Gamma(\mu) = -\prod_k \left(1 + \frac{\mu}{\gamma_k}\right),$$

by (2.3). So

$$\widetilde{D}_T(\mu) = -\tau(-\mu) e^{-\mu \sum_j \left(\frac{1}{\lambda_j}\right)}, \; \widetilde{D}_{T_1}(\mu) = -\Gamma(-\mu) e^{-\mu \sum_k \left(\frac{1}{\gamma_k}\right)},$$

and from 3.10) we find

$$D_{T_1/T}(\mu) = \frac{\Gamma(-\mu)}{\tau(-\mu)} . \frac{e^{-\mu \sum_k \left(\frac{1}{\gamma_k}\right)}}{e^{-\mu \sum_j \left(\frac{1}{\lambda_j}\right)}} e^{i\mu sp T_2}.$$

Putting here $\mu = it \; (t > 0)$, then we get

$$\frac{1}{t} \ln \left| D_{T_1/T}(it) \right| = \frac{1}{t} \ln |\Gamma(-it)| - \frac{1}{t} \ln |\tau(-it)| - \sum_j \Im m \left(\frac{1}{\lambda_j}\right) - spT_2. \tag{3.11}$$

From (2.3) and Theorem 2 we obtain that

$$\lim_{t\to\infty} \frac{1}{t} \ln \left| D_{T_1/T}(it) \right| = 0, \tag{3.12}$$

$$\lim_{t\to\infty} \sup \frac{1}{t} \ln |\Gamma(-it)| \leq 0, \tag{3.13}$$

$$\lim_{t\to\infty} \sup \frac{1}{t} \ln |\tau(-it)| \leq 0. \tag{3.14}$$

On the other hand, taking into consideration that for $t > 0$,

$$\left|1 + \frac{it}{\lambda_j}\right|^2 = 1 + 2t \frac{\Im m \lambda_j}{\left|\lambda_j\right|^2} + \frac{t^2}{\left|\lambda_j\right|^2} \geq 1, \; \left|1 + \frac{it}{\gamma_k}\right|^2 = 1 + \frac{t^2}{|\gamma_k|^2} \geq 1,$$

we have $|\Gamma(-it)| \geq 1, \; |\tau(-it)| \geq 1$ for all $t > 0$. Consequently,

$$\frac{1}{t} \ln |\Gamma(-it)| \geq 0, \; \frac{1}{t} \ln |\tau(-it)| \geq 0,$$

and from (3.13)-(3.14) it follows that

$$\lim_{t\to\infty}\sup\frac{1}{t}\ln|\Gamma(-it)| = \lim_{t\to\infty}\sup\frac{1}{t}\ln|\tau(-it)| = 0. \tag{3.15}$$

Hence we get, by (3.11), (3.12), (3.15) that

$$\sum_j \Im m\left(\frac{-1}{\lambda_j}\right) = spT_2.$$

□

Theorem 3.4. *The system of all root vectors of the dissipative operator T (also of K) is complete in H.*

Proof. From Theorem 6, the operator T carries out all the conditions of Livšic's theorem on completeness. □

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