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Topological Structure of the Solutions Set of Impulsive Semilinear Differential Inclusions with Nonconvex Right-Hand Side

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Abstract

In this paper, we study the topological structure of solution sets for the following first-order impulsive evolution inclusion with initial conditions:

$$\begin{cases} y'(t)-Ay(t) & \in & F(t,y(t)), \quad \text{a.e. } t\in J\backslash\{t_1,\ldots,t_m\}, \\ y(t_k^+)-y(t_k^-) & = & I_k(y(t_k^-)), \quad k=1,\ldots,m \\ y(0) & = & a\in E, \end{cases}$$

where $J := [0,b]$ and $0 = t_0 < t_1 < \ldots < t_m < b$, A is the infinitesimal generator of a C_0-semigroup of linear operator $T(t)$ on a separable Banach space E and F is a set-valued map. The functions I_k characterize the jump of the solutions at impulse points t_k $(k = 1,\ldots,m)$. The continuous selection of the solution set is also investigated.

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1 Introduction

Differential equations with impulses were considered for the first time by Milman and Myshkis [41] and then followed by a period of active research which culminated with the

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monograph by Halanay and Wexler [34]. Many phenomena and evolution processes in the field of physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. (see for instance [1, 40] and the references therein). These short perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, chemical technology, mechanics (jump discontinuities in velocity), electrical engineering, medicine, and biology. These perturbations may be seen as impulses. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations and inclusions. Various mathematical results (existence, asymptotic behavior,...) have been obtained so far (see [6, 11, 40, 44] and the references therein).

Given a real separable Banach space E with norm $\|\cdot\|$, consider the following problem

$$\begin{cases} y'(t)-Ay(t) & \in & F(t,y(t)), \quad \text{a.e. } t\in J\backslash\{t_1,\dots,t_m\}, \\ \Delta y_{t=t_k} & = & I_k(y(t_k^-)), \quad k=1,\dots,m \\ y(0) & = & a, \end{cases} \tag{1.1}$$

where $0=t_0<t_1<\ldots<t_m<t_{m+1}=b$, $J=[0,b]$, $F:\ J\times E\to\mathcal{P}(E)$ is a multifunction, and the operator A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, $I_k\in C(E,E)\,(k=1,\dots,m)$, and $\Delta y|_{t=t_k}=y(t_k^+)-y(t_k^-)$. The notations $y(t_k^+)=\lim\limits_{h\to 0^+}y(t_k+h)$ and $y(t_k^-)=\lim\limits_{h\to 0^+}y(t_k-h)$ stand for the right and the left limits of the function y at $t=t_k$, respectively.

In 1890, Peano [43] proved that the Cauchy problem for ordinary differential equations has local solutions although the uniqueness property does not hold in general. In case where the uniqueness does not hold, Keneser [39] proved in 1923 that the solution set is a continuum, i.e. closed and connected. In 1942, Aronszajn [5] improved this result for differential inclusions in the sense that he showed that the solution set is compact and acyclic, and he specified this continuum to be an $R_\delta-$set. An analogous result was obtained for differential inclusions by serval authors; see for instance [3, 10, 16, 17, 24, 31, 49].

Very recently, the topological and geometric structures of the solution sets for impulsive differential equations and inclusions on compact and non-compact intervals were investigated in [2, 19, 20, 21, 22, 23, 29, 32, 33, 35] where the solution set is shown to be an absolute retract, contractible, acyclic, and $R_\delta-$ set.

Our goal is to investigate the topological structures of the solution set of Problem (1.1), where the right-hand side is not necessarily convex valued.

2 Preliminaries

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper. Let $(E,\|\cdot\|)$ be a separable Banach space, $J:=[0,b]$ an interval in $\mathbb{R}$ and $C(J,E)$ the Banach space of all continuous functions from J into E with the suppremum norm

$$\|y\|_\infty=\sup\{\|y(t)\|:\ 0\leq t\leq b\}.$$

Set

$$\begin{aligned} AC(J,E) &= \{y : [0,b] \to E \text{ absolutely continuous,} \\ & \quad y(t) = y(0) + \int_0^t y'(s)ds, \text{ and } y' \in L^1([0,b],E)\}. \end{aligned}$$

$B(E)$ refers to the Banach space of linear bounded operators from E into E with the usual norm

$$\|N\|_{B(E)} = \sup\{\|N(y)\| : \|y\| = 1\}.$$

A function $y\colon J \to E$ is called measurable provided for every open subset $U \subset E$, the set $y^{-1}(U) = \{t \in J\colon y(t) \in U\}$ is Lebesgue measurable. A measurable function $y\colon J \to E$ is Bochner integrable if $\|y\|$ is Lebesgue integrable. For properties of the Bochner integral, see e.g. Yosida [48]. In what follows, $L^1(J,E)$ denotes the Banach space of functions $y\colon J \longrightarrow E$, which are Bochner integrable with norm

$$\|y\|_{L^1} = \int_0^b \|y(t)\|dt.$$

Denote by $\mathcal{P}(E) = \{Y \subset E\colon Y \neq \emptyset\}$, $\mathcal{P}_{cl}(E) = \{Y \in \mathcal{P}(E)\colon Y \text{ closed}\}$, $\mathcal{P}_b(E) = \{Y \in \mathcal{P}(E)\colon Y \text{ bounded}\}$, $\mathcal{P}_{cv}(E) = \{Y \in \mathcal{P}(E)\colon Y \text{ convex}\}$, $\mathcal{P}_{cp}(E) = \{Y \in \mathcal{P}(E)\colon Y \text{ compact}\}$, and $\mathcal{P}_{wkcp}(E) = \{Y \in \mathcal{P}(E)\colon Y \text{ weakly compact}\}$.

Definition 2.1. A multi-valued map $F : J \to \mathcal{P}_{cl}(Y)$ is said measurable provided for every open $U \subset Y$, the set

$$F^{+1}(U) = \{x \in J : F(x) \subset U\}$$

is Lebesgue measurable.

An characterization of the measurability of the multifunction is given by the following lemma

Lemma 2.2. ([15, 30]) *The mapping F is measurable if and only if for each $x \in Y$, the function $\zeta : J \to [0,+\infty)$ defined by*

$$\zeta(t) = \text{dist}(x,F(t)) = \inf\{\|x-y\| : y \in F(t)\}, \quad t \in J,$$

is Lebesgue measurable.

The following two lemmas are needed in this paper. The first one is the celebrated Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 2.3. *([30], Theorem 19.7) Let Y be a separable metric space and $F : J \to \mathcal{P}_{cl}(Y)$ a measurable multivalued map. Then F has a measurable selection.*

First, consider the Hausdorff pseudo-metric

$$H_d\colon \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$$

defined by

$$H_d(A,B) = \max\left\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(A,b)\right\}$$

where $d(A,b) = \inf_{a \in A} d(a,b)$ and $d(a,B) = \inf_{b \in B} d(a,b)$. $(\mathcal{P}_{b,cl}(E), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

Definition 2.4. A multi-valued operator $N\colon\ E \to \mathcal{P}_{cl}(E)$ is called

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y), \quad \text{for each} \;\; x,\ y \in E,$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Notice that if N is $\gamma-$Lipschitz, then for every $\gamma' > \gamma$,

$$N(x) \subset N(y) + \gamma' d(x,y) B(0,1),\ \forall\, x,y \in E,$$

where $B(0,1)$ denotes the unit ball in the space E.

Definition 2.5. A is called $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $I \times D$ where I is Lebesgue measurable in J and D is Borel measurable in E.

Definition 2.6. A subset $A \subset L^1(J,E)$ is decomposable if for all $u,v \in A$ and for every Lebesgue measurable set $I \subset J$, we have:

$$u\chi_I + v\chi_{J\setminus I} \in A,$$

where χ_A stands for the characteristic function of the set A. We denote Dco the family of decomposable sets.

More details on the previous sets can be found in the monograph by Fryszkowski [28].

Let $F:\ J \times E \to \mathcal{P}(E)$ be a multi-valued map with nonempty closed values. Assign to F the multi-valued operator $\mathcal{F}:\ C(J,E) \to \mathcal{P}(L^1(J,E))$ defined by

$$\mathcal{F}(y) = \{v \in L^1(J,E) : v(t) \in F(t, y(t)),\ \text{a.e. } t \in J\}.$$

The operator $\mathcal{F}$ is called the Nemyts'kiĭ operator associated to F.

Definition 2.7. Let $F : J \times E \to \mathcal{P}(E)$ be a multi-valued map with nonempty compact values. We say that F is of lower semi-continuous type (*l.s.c.* type) if its associated Nemyts'kiĭ operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Next, we state a classical selection theorem due to Bressan and Colombo.

Lemma 2.8. *(see [13, 18, 36]) Let X be a separable metric space and let E be a Banach space. Then every l.s.c. multi-valued operator $N : X \to \mathcal{P}_{cl}(L^1(J,E))$ with closed decomposable values has a continuous selection, i.e. there exists a continuous single-valued function $f : X \to L^1(J,E)$ such that $f(x) \in N(x)$ for every $x \in X$.*

Let us introduce the following hypothesis.

($\mathcal{H}_1$) $F : J \times E \longrightarrow \mathcal{P}(E)$ is a nonempty compact valued multi-valued map such that
(a) the mapping $(t,y) \mapsto F(t,y)$ is $\mathcal{L}\otimes\mathcal{B}$ measurable;
(b) the mapping $y \mapsto F(t,y)$ is lower semi-continuous for a.e. $t \in J$.

The following lemma is crucial in the proof of our existence theorem.

Lemma 2.9. *(see e.g. [18, 27]) Let $F : J \times E \to \mathcal{P}_{cp}(E)$ be an integrably bounded multivalued map satisfying $(\mathcal{H}_1)$. Then F is of lower semi-continuous type.*

Definition 2.10. A space X is called an absolute retract (written as $X \in AR$) provided that for every space Y, a closed subset $B \subset Y$, and a continuous map $f : B \to X$, there exists a continuous extension $\widetilde{f} : Y \to X$ of f over Y, i.e., e $\widetilde{f}(x) = f(x)$ for every $x \in B$.

We need the following result owed to Bressan-Cellina-Fryszkowski.

Lemma 2.11. *([12], Thm. 2) Let E be a Banach space, $X = L^1(T, E)$, for some measurable space T, and $N : E \to \mathcal{P}(X)$ a contraction map with decomposable values. Then $Fix(N)$ is an absolute retract.*

For further readings and details on multivalued analysis, we refer to the books by Aubin and Cellina [7], Aubin and Frankowska [8], Deimling [18], Górniewicz [30], Hu and Papageorgiou [36, 37], Kamenskii [38], Smirnov [45], and Tolstonogov [46].

3 Continuous selection and absolute retract of solution set

In this section we present some proprieties of the solution set of the Problem (1.1) where the right-hand side is not necessarily convex. First, we give some auxiliary lemmas.

Let $f \in L^1(J, E)$, and assume that the operator A generates a semigroup $T(t)$. We consider the following impulsive problem

$$\begin{cases} y'(t) - Ay(t) &= f(t), \quad \text{a.e. } t \in J, \\ \Delta y_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, \ldots, \\ y(0) &= a, \end{cases} \tag{3.1}$$

where A is the infinitesimal generator of a C_0-semigroup of linear operator $T(t)$ on a separable Banach space E.

Note the solution of the problem (3.1) by $y(a, f)$.

$$PC := \{y : J \to E : y \in AC((t_k, t_{k+1}), E),\ k = 0, \ldots m-1\}.$$

Then $(PC, \|\cdot\|_{PC})$ is a Banach space with norm

$$\|y\|_{PC} = \|y\|_\infty + \|y'\|_{L^1}.$$

Definition 3.1. A function $y(a, f) \in PC$ is called mild solution of Problem (3.1) if

$$y(a, f) = T(t)a + \int_0^t T(t-s)f(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(a, f)(t_k)), \quad t \in J.$$

Definition 3.2. A function $y \in PC$ is called mild solution of Problem (1.1) if there exists $f \in L^1(J, E)$ such that

$$y(t) = T(t)a + \int_0^t T(t-s)f(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)), \quad t \in J,$$

and

$$f(t) \in F(t, y(t)),\ a.e.\ t \in J.$$

Note that $\Omega = \{y(a,f) : (a,f) \in E \times L^1(J,E)\}$ is the appropriate space for mild solutions of the problem (3.1). We define the following operator $\overline{L} : E \times L^1(J,E) \to \Omega$ by

$$\overline{L}(v)(t) = \begin{cases} L_0(a,v)(t), & \text{if } t \in [0,t_1], \\ L_1(a,v)(t), & \text{if } t \in (t_1,t_2], \\ \dots & \dots \\ L_{m-1}(a,v)(t), & \text{if } t \in (t_{m-1},t_m], \\ L_m(a,v)(t), & \text{if } t \in (t_m,b], \end{cases} \tag{3.2}$$

where

$$\begin{aligned} L_0(a,v)(t) &= T(t)a + \int_0^t T(t-s)v(s)ds,\ t \in [0,t_1] \\ L_1(a,v)(t) &= T(t-t_1)[L_0(v)(t_1) + I_1(L_0(v)(t_1))] + \int_{t_1}^t T(t-s)v(s)ds,\ t \in (t_1,t_2] \\ L_2(a,v)(t) &= T(t-t_2)[L_1(v)(t_2) + I_2(L_1(v)(t_2))] + \int_{t_2}^t T(t-s)v(s)ds,\ t \in (t_2,t_3] \\ &\dots \\ L_{m-1}(a,v)(t) &= T(t-t_m)[L_{m-2}(v)(t_{m-1}) + I_{m-1}(L_{m-2}(v)(t_{m-1}))] \\ &\quad + \int_{t_m}^t (T(t-s)v(s)ds,\ t \in (t_m,b]. \end{aligned}$$

From (3.2), we can easily check that

$$\overline{L}(a,v)(t) = T(t)a + \sum_{0<t_k<t} T(t-s)I_k(L_{k-1}(v)(t_k)) + \int_0^t T(t-s)v(s)ds, \text{ a.e. } t \in J.$$

Lemma 3.3. *The map $\overline{L} : E \times L^1(J,E) \to \Omega$ defined by $\overline{L}(a,v) = y(a,v)$ is one-to-one, where $y(.,.)$ is solution of the problem (3.1).*

Proof. For the construction of the operator $\overline{L}$ we have $\overline{L}(a,v) = y(a,v)$. Now, we prove that $\overline{L}$ is one-to-one.

1. Let $v_1, v_2 \in L^1(J,E)$ be such that $\overline{L}(v_1) = \overline{L}(v_2)$. For $t \in [0,t_1]$, we have

   $$\int_0^t T(t-s)v_1(s)ds = \int_0^t T(t-s)v_2(s)ds.$$

   For $t \in (t_1,t_2]$ we have

   $$\begin{aligned} &T(t-t_1)[L_0(v_1)(t_1) + I_1(L_0(v_1)(t_1))] + \int_{t_1}^t T(t-s)v_1(s)ds \\ = \ &T(t-t_2)[L_0(v_2)(t_1) + I_1(L_0(v_2)(t_1))] + \int_{t_1}^t T(t-s)v_2(s)ds. \end{aligned}$$

   For $t \in (t_2,t_3]$ we have

   $$\begin{aligned} &T(t-t_2)[L_1(v_1)(t_2) + I_2(L_1(v_1)(t_2))] + \int_{t_2}^t T(t-s)v_1(s)ds \\ = \ &T(t-t_2)[L_1(v_2)(t_2) + I_2(L_1(v_2)(t_2))] + \int_{t_2}^t T(t-s)v_2(s)ds. \end{aligned}$$

We continuous this process, and get for $t \in (t_m, b]$

$$\begin{aligned} & T(t-t_m)[L_{m-1}(v_1)(t_{m-1}) + I_{m-1}(L_{m-1}(v_1)(t_{m-1}))] + \int_{t_{m-1}}^{t} T(t-s)v_1(s)ds \\ = \;& T(t-t_m)[L_{m-1}(v_2)(t_{m-1}) + I_{m-1}(L_{m-1}(v_2)(t_{m-1}))] + \int_{t_{m-1}}^{t} T(t-s)v_2(s)ds. \end{aligned}$$

Let $t \in J_1 = [0, t_1]$, then

$$\int_0^t T(t-s)[v_1(s) - v_2(s)]ds = 0. \tag{3.3}$$

Thus, for each $h > 0$, we have

$$T(h)\int_0^t T(t-s)[v_1(s) - v_2(s)]ds = 0 \Rightarrow \int_0^t T(t+h-s)[v_1(s) - v_2(s)]ds = 0. \tag{3.4}$$

From (3.3) and (3.4) we obtain

$$\begin{aligned} & \int_t^{t+h} T(t+h-s)[v_1(s) - v_2(s)]ds \\ = \;& \int_0^t T(t+h-s)[v_1(s) - v_2(s)]ds \\ & + \int_t^{t+h} T(t+h-s)[v_1(s) - v_2(s)]ds \\ = \;& \int_0^{t+h} T(t+h-s)[v_1(s) - v_2(s)]ds = 0. \end{aligned}$$

Hence

$$\frac{1}{h}\int_t^{t+h} T(t+h-s)[v_1(s) - v_2(s)]ds = 0. \tag{3.5}$$

Let J_1^* be the set of all $t \in (0, t_1)$ such that

$$\lim_{h\to 0} \frac{1}{2h}\int_{t-h}^{t+h} \|v_1(t) - v_2(t) - v_1(s) + v_2(s)\|ds = 0, \tag{3.6}$$

and we observe that $J_1 \backslash J_1^*$ has Lebesgue measure $m(J_1 \backslash J_1^*) = 0$ (see [25] p. 217). Let $t \in J_1^*$ be arbitrary, we have

$$\begin{aligned} & \frac{1}{h}\int_t^{t+h} T(t+h-s)[v_1(s) - v_2(s)]ds \\ = \;& \frac{1}{h}\int_t^{t+h} T(t+h-s)[v_1(s) - v_2(s) - v_1(t) + v_2(t)]ds \\ & + \frac{1}{h}\int_t^{t+h} T(t+h-s)[v_1(t) - v_2(t)]ds. \end{aligned}$$

By (3.5), we have

$$\begin{aligned}&\frac{1}{h}\int_t^{t+h} T(t+h-s)[v_1(t)-v_2(t)-v_1(s)+v_2(s)]ds\\ =\ &\frac{1}{h}\int_t^{t+h} T(t+h-s)[v_1(t)-v_2(t)]ds.\end{aligned}$$

Set

$$\mu(h)=\frac{1}{h}\int_0^h T(s)[v_1(t)-v_2(t)]ds.$$

$$\begin{aligned}\mu(h)\ &\le\ \frac{1}{h}\int_t^{t+h} \|T(t+h-s)\|_{B(E)}\|v_1(t)-v_2(t)-v_1(s)+v_2(s)\|ds\\ &\le\ \frac{1}{h}\int_t^{t+h} \|T(t+h-s)\|_{B(E)}\|v_1(t)-v_2(t)-v_1(s)+v_2(s)\|ds\\ &\le\ \frac{Me^{2\omega h}}{h}\int_{t-h}^{t+h} \|v_1(t)-v_2(t)-v_1(s)+v_2(s)\|ds.\end{aligned}$$

Hence

$$\mu(h)\le\frac{2Me^{2\omega h}}{2h}\int_{t-h}^{t+h} \|v_1(t)-v_2(t)-v_1(s)+v_2(s)\|ds.$$

In view of (3.6) we concluded that

$$\lim_{h\to 0}\mu(h)=0\Rightarrow v_1(t)-v_2(t)=0.$$

Since t is arbitrary and $m(J_1\backslash J_1^*)=0$, we have

$$v_1-v_2=0\Rightarrow v_1=v_2 \text{ on } J_1.$$

For $t\in(t_1,t_2]$ we have

$$\int_{t_1}^t T(t-s)[v_1(s)-v_2(s)]ds,\qquad t\in(t_1,t_2]=J_2. \tag{3.7}$$

Thus, for each $h>0$, we have

$$\int_{t_1}^t T(t+h-s+t_1)[v_1(s)-v_2(s)]ds=0 \tag{3.8}$$

From (3.7) and (3.8) we obtain

$$\begin{aligned}&\int_t^{t+h} T(t+h-s+t_1)[v_1(s)-v_2(s)]ds\\ =\ &\int_{t_1}^t T(t+h-s+t_1)[v_1(s)-v_2(s)]ds\\ &+\int_t^{t+h} T(t+h-s+t_1)[v_1(s)-v_2(s)]ds\\ =\ &\int_{t_1}^{t+h} T(t+h-s+t_1)[v_1(s)-v_2(s)]ds=0.\end{aligned}$$

Hence

$$\frac{1}{h}\int_t^{t+h} T(t+h-s+t_1)[v_1(s)-v_2(s)]ds = 0. \tag{3.9}$$

Let J_2^* be the set of all $t \in (t_1,t_2)$ such that

$$\lim_{h\to 0}\frac{1}{2h}\int_{t-h}^{t+h} \|v_1(t)-v_2(t)-v_1(s)+v_2(s)\|ds = 0, \tag{3.10}$$

and we observe that $J_2\backslash J_2^*$ has Lebesgue measure $m(J_2\backslash J_2^*) = 0$ (see [25] p. 217). Let $t \in J_2^*$ be arbitrary, we have

$$\begin{aligned}
& \frac{1}{h}\int_t^{t+h} T(t+h-s+t_1)[v_1(s)-v_2(s)]ds \\
= & \frac{1}{h}\int_t^{t+h} T(t+h-s)[v_1(s)-v_2(s)-v_1(t)+v_2(t)]ds \\
& +\frac{1}{h}\int_t^{t+h} T(t+h-s+t_1)[v_1(t)-v_2(t)]ds.
\end{aligned}$$

By (3.9), we have

$$\begin{aligned}
& \frac{1}{h}\int_t^{t+h} T(t+h-s+t_1)[v_1(t)-v_2(t)-v_1(s)+v_2(s)]ds \\
= & \frac{1}{h}\int_t^{t+h} T(t+h-s+t_1)[v_1(t)-v_2(t)]ds.
\end{aligned}$$

Set

$$\mu_1(h) = \frac{1}{h}\int_{t_1}^{t_1+h} T(s)[v_1(t)-v_2(t)]ds.$$

$$\begin{aligned}
\mu_1(h) & \leq \frac{1}{h}\int_t^{t+h} \|T(t+h-s+t_1)\|_{B(E)}\|v_1(t)-v_2(t)-v_1(s)+v_2(s)\|ds \\
& \leq \frac{1}{h}\int_t^{t+h} \|T(t+h-s+t_1)\|_{B(E)}\|v_1(t)-v_2(t)-v_1(s)+v_2(s)\|ds \\
& \leq \frac{Me^{2\omega h}}{h}\int_{t-h}^{t+h} \|v_1(t)-v_2(t)-v_1(s)+v_2(s)\|ds.
\end{aligned}$$

Hence

$$\mu_1(h) \leq \frac{2Me^{2\omega(h+t_1)}}{2h}\int_{t-h}^{t+h} \|v_1(t)-v_2(t)-v_1(s)+v_2(s)\|ds.$$

In view of (3.10) we conclude that

$$\lim_{h\to 0}\mu(h) = 0 \Rightarrow v_1(t)-v_2(t) = 0.$$

Since t is arbitrary and $m(J_2\backslash J_2^*) = 0$, we have

$$v_1 - v_2 = 0 \Rightarrow v_1 = v_2 \text{ on } J_2.$$

We continue this process until we arrive that

$$v_1 = v_2 \text{ on } J_m.$$

This implies that

$$v_1 = v_2 \text{ on } J.$$

For each $(a, f) \in E \times L^1(J, E)$ we denote $y(a, f)$ the solution of the problem (3.1). Set

$$\Omega = \{y(a, f) : \ (a, f) \in E \times L^1(J, E)\},$$

is a metric space with distance

$$d(y(a_1, f_1), y(a_2, f_2)) = \|y(a_1, f_1) - y(a_2, f_2)\|_{PC} + \|f_1 - f_2\|_{L^1}. \tag{3.11}$$

Lemma 3.4. Ω *is a complete metric space.*

Proof. Since $\overline{L}$ is one-to-one, the distance (3.11) makes sense. It remains to show that Ω with the distance d, is complete. Let $\{y_n(a_n, f_n)\}_{n\geq 1}$ be Cauchy sequence in Ω. For $n, m \in \mathbb{N}$ we have

$$d(y_n(a_n, f_n), y_m(a_m, f_m)) = \|y_n(a_n, f_n) - y_m(a_m, f_m)\|_{PC} + \|f_n - f_m\|_{L^1}.$$

Since L^1 is a Banach space, there exist $f \in L^1$ such that

$$\|f_n - f\|_{L^1} \to 0 \text{ as } n \to \infty.$$

From the definition of y_n we have

$$\|a_n - a_m\| = \|T(0)a_n - T(0)a_m\| \leq \|y_n(a_n, f_n) - y_m(a_m, f_m)\|.$$

Thus $\{a_n\}_{n\geq 1}$ is a Cauchy sequence in E then there exists $a \in E$ such that

$$\|a_n - a\| \to 0 \text{ as } n \to \infty.$$

We consider the following impulsive problem

$$\left\{\begin{array}{rcll} y'(t) - Ay(t) &=& f(t), & \text{a.e. } t \in J, \\ \Delta y_{t=t_k} &=& I_k(y(t_k^-)), & k = 1, \ldots, \\ y(0) &=& a. & \end{array}\right. \tag{3.12}$$

The solution of Problem (3.12) is defined by

$$y(a, f) = T(t)a + \int_0^t T(t-s)f(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(a, f)(t_k)), \ \ t \in J.$$

Now we prove that the sequence $\{y_n\}_{n\geq 1}$ converges to $y(a, f)$. Let $t \in [0, t_1]$. We have

$$\|y_n(a_n, f_n)(t) - y(a, f)(t)\| \ \ \leq \ \ Me^{\omega b}\|a_n - a\| + Me^{\omega b} \int_0^b \|f_2(s) - f_1(s)\|ds.$$

Then

$$\begin{aligned}\sup_{t\in[0,t_1]}\|y_n(a_n,f_n)(t)-y(a,f)(t)\| &\le Me^{\omega b}\|a_n-a\| \\ &\quad +Me^{\omega b}\int_0^b\|f_n(s)-f(s)\|ds\to 0 \text{ as } n\to\infty.\end{aligned}$$

This implies that

$$\|y_n(a_n,f_n)(t_1)-y(a,f)(t_1)\|\to 0 \text{ as } n\to\infty$$

and by the continuity of I_1 we have

$$\begin{aligned}\|y_n(a_n,f_n)(t_1^+)-y(a,f)(t_1^+)\| &\le \|y_n(a_n,f_n)(t_1)-y(a,f)(t_1)\| \\ &\quad +\|I_1(y_n(a_n,f_n)(t_1))-I_1(y(a,f)(t_1))\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

For $t\in(t_1,t_2]$ we have

$$\begin{aligned}\sup_{t\in[t_1,t_2]}\|y_n(a_n,f_n)(t)-y(a,f)(t)\| &\le Me^{\omega b}\|y_n(a_n,f_n)(t_1)-y(a,f)(t_1)\| \\ &\quad +Me^{\omega b}\|I_1(y_n(a_n,f_n)(t_1))-I_1(y(a,f)(t_1))\| \\ &\quad +Me^{\omega b}\int_0^b\|f_n(s)-f(s)\|ds\to 0 \text{ as } n\to\infty.\end{aligned}$$

We continue this process until we arrive

$$\begin{aligned}\sup_{t\in[t_m,b]}\|y_n(a_n,f_n)(t)-y(a,f)(t)\| &\le Me^{\omega b}\|y_n(a_n,f_n)(t_m)-y(a,f)(t_m)\| \\ &\quad +Me^{\omega b}\|I_m(y_n(a_n,f_n)(t_m))-I_m(y(a,f)(t_m))\| \\ &\quad +Me^{\omega b}\int_0^b\|f_n(s)-f(s)\|ds\to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence $\{y_n(a_n,f_n)\}_{n\ge 1}$ converges to $y(a,f)$ in Ω. Let

$$G(a,f)=\{v\in L^1(J,E): v(t)\in F(t,y(a,f)(t)) \text{ a.e. } t\in J\},$$

and

$$G_*(a)=\{f\in L^1(J,E): f\in G(a,f)\}.$$

Lemma 3.5. *Assume that the multi-valued map $F: J\times E\to\mathcal{P}_{cp}(E)$ is such that $t\to F(t,.)$ is a measurable and*

($\mathcal{H}_2$) *There exist $p\in L^1(J,\mathbb{R}^+)$ such that*

$$H_d(F(t,x),F(t,y))\le p(t)\|x-y\|, \text{ for } x,y\in E \text{ and a.e. } t\in J_k$$

and

$$F(t,0)\subset p(t)\overline{B}(0,1), \text{ for a.e. } t\in J,$$

hold. Then F is of lower semicontinuous type and $G: E\times L^1(J,E)\to\mathcal{P}(Dco)$.

Remark 3.6.

(a) $(\mathcal{H}_2)$ implies that the multifunction F has at most the linear growth

$$\|F(t,x)\|_{\mathcal{P}} \leq p(t)(1+\|x\|),\ p \in L^1(J,\mathbb{R}^+),\ \text{a.e. } t \in J,\ x \in E.$$

Proof of Lemma 3.5. The multivalued map $t \to F(t,.)$ is measurable and $x \to F(.,x)$ is H_d-continuous. In addition $F(.,.)$ has compact values; hence $G(.,.)$ is measurable, continuous. Since the measurable multifunction F is integrably bounded, Lemma 2.9 implies that the Nemyts'kiĭ operator $\mathcal{F}$ has decomposable values. Let $(a,f) \in E \times L^1(J,E)$, then there exists a unique $y(a,f) \in PC$ such that $y(a,f)$ is a solution of the problem (3.1), by the definition of $\mathcal{F}$ and G we have $\mathcal{F}(y(a,f)) = G(a,f)$, then for every $(a,f) \in E \times L^1(J,E)$ we obtain that $G(a,f) \in \mathcal{P}(Dco)$.

Theorem 3.7. *$F : J \times E \to \mathcal{P}_{cp}(E)$; $t \to F(t,x)$ is measurable for each $x \in E$. Assume that $(\mathcal{H}_2)$ and the following condition*

$(\mathcal{A}_1)$ *there exist $c_k > 0$, $k = 1,\dots,m$ such that*

$$\|I_k(x) - I_k(y)\| \leq c_k\|x-y\| \text{ for every } x,y \in E$$

hold, then there exists a continuous function $L_* :\ \overline{\Omega} \to PC$ satisfying for each $a \in E$ the following properties

$$\begin{aligned} L_*(a,y) \in S(a) & \quad \text{for each } y \in \Omega_* \\ L_*(a,f) &= y,\ \text{for each } y \in S(a), \end{aligned}$$

where

$$\overline{\Omega} = \{(a,y) \in E \times \Omega :\ a \in E,\ y \in \Omega_*\}$$

and

$$\Omega_* = \{y \in \Omega : y(0) = a\}.$$

Remark 3.8. By conditions of Theorem 3.7 the solution set of Problem (1.1) is not empty. For the proof, we can see [11, 20, 35].

Proof. From Lemma 3.5, we have that, for every $(a,f) \in E \times L^1(J,E)$, $G(a,f) \in \mathcal{P}(Dco)$ with decomposable values. Now we prove that G is Hausdorff continuous. Let (a_n, f_n) be a sequence in $E \times L^1(J,E)$ converging to (a,f). Let $\epsilon > 0$ and $g_n \in G(a_n, f_n)$. From $(\mathcal{H}_2)$ tells us that

$$H_d(F(t,y(a,f)(t)), F(t,y_n(a_n,f_n)(t))) \leq p(t)\|y_n(a_n,f_n)(t) - y(a,f)(t)\| + \frac{\epsilon}{2},\ \text{a.e. } t \in J.$$

Hence there is $w \in F(t,y(a,f)(t))$ such that

$$\|g_n(t) - w\| \leq p(t)\|y(a,f)(t) - y(a_n,f_n)(t)\| + \frac{\epsilon}{2}, \quad t \in J.$$

Consider the mapping $U_n : J \to \mathcal{P}(E)$ defined by

$$U_n(t) = \{w \in E : \|f(t) - w\| \leq p(t)\|y(a_n,f_n)(t) - y(a,f)(t)\| + \frac{\epsilon}{2}\},\ t \in J,$$

that is $U_n(t) = \overline{\mathcal{B}}(g_n(t), p(t)\|y(a_n, f_n)(t) - y(a, f)(t)\| + \frac{\epsilon}{2})$. Since $g, l, y, \bar{y}$ are measurable, Theorem III.4.1 in [15] tells us that the closed ball U is measurable.

Finally the set $V(t) = U(t) \cap F(t, \overline{y}(t))$ is nonempty since it contains w. Therefore the intersection multi-valued operator V is measurable with nonempty, closed values (see [8, 15, 30, 47, 49]). By Lemma 2.3, there exists a function $\overline{g}_n(t)$, which is a measurable selection for V. Thus $\overline{g}_n(t) \in F(t, y(a, f)(t))$ and

$$\|g_n(t) - \overline{g}_n(t)\| \leq p(t)\|y(a_n, f_n)(t) - y(a, f)(t)\| + \frac{\epsilon}{2}, \text{ for a.e. } t \in J.$$

Thus for a.e. $t \in J$

$$\begin{aligned} \|g_n(t) - \overline{g}_n(t)\| &\leq p(t)\|y(a_n, f_n)(t) - y(a, f)(t)\|ds + \frac{\epsilon}{2} \\ &\leq Me^{\omega b}p(t)\|a_n - a\| + Me^{\omega b}p(t)\|f_n - f\|_{L^1} \\ &\quad Me^{\omega b}p(t)\sum_{k=1}^{m}\|I_k(y(a_n, f_n))(t_k) - I_k(y(a, f))(t_k)\| + \frac{\epsilon}{2}. \end{aligned}$$

Hence

$$\begin{aligned} \|g_n - \overline{g}_n\|_{L^1} &\leq Me^{\omega b}\|p\|_{L^1}\|a_n - a\| + Me^{\omega b}\|p\|_{L^1}\|f_n - f\|_{L^1} \\ &\quad Me^{\omega b}\|p\|_{L^1}\sum_{k=1}^{m}\|I_k(y(a_n, f_n))(t_k) - I_k(y(a, f))(t_k)\| + b\frac{\epsilon}{2}. \end{aligned}$$

By an analogous relation, obtained by interchanging the roles of $y(a_n, f_n)$ and $y(a, f)$, we finally arrive at

$$\begin{aligned} H_d(G(a_n, f_n), G(a, f)) &\leq Me^{\omega b}\|p\|_{L^1}\|a_n - a\| + Me^{\omega b}\|p\|_{L^1}\|f_n - f\|_{L^1} \\ &\quad Me^{\omega b}\|p\|_{L^1}\sum_{k=1}^{m}\|I_k(y(a_n, f_n))(t_k) - I_k(y(a, f))(t_k)\| + b\frac{\epsilon}{2}. \end{aligned}$$

Using the fact that $I_k,\ k = 1, \ldots, m$ are continuous functions, we have

$$H_d(G(a_n, f_n), G(a, f)) \to 0 \text{ as } n \to \infty.$$

Now, show that $a \to G(a, .)$ is H_d- Lipschitz. Let $f_1, f_2 \in L^1(J, E)$. Denote $y(a, f_1)$ and $y(a, f_2)$ the mild solutions of the problem (3.1) respectively, thus for $t \in [0, t_1]$ we have

$$\|y(a, f_1)(t) - y(a, f_2)(t)\| \leq Me^{\omega t_1}\int_0^t \|f_1(s) - f_2(s)\|ds. \tag{3.13}$$

From $(\mathcal{A}_1)$ we obtain

$$\|I_1(y(a, f_1)(t_1)) - I_1(y(a, f_2)(t_1))\| \leq c_1 Me^{\omega t_1}\int_0^{t_1} \|f_1(s) - f_2(s)\|ds. \tag{3.14}$$

For $t \in (t_1, t_2]$ we have

$$\begin{aligned} \|y(a_1, f_1)(t) - y(a_2, f_2)(t)\| &\leq \|y(a, f_1)(t_1) - y(a, f_2)(t_1)\| \\ &\quad + \|I_1(y(a, f_1)(t_1)) - I_1(y(a, f_2)(t_1))\| \\ &\quad + \int_{t_1}^{t} \|T(t-s)\|_{B(E)} \|f_1(s) - f_2(s)\| ds \\ &\leq Me^{\omega t} \int_0^t \|f_1(s) - f_2(s)\| ds \\ &\quad + c_1 M e^{\omega t_1} \int_0^{t_1} \|f_1(s) - f_2(s)\| ds. \end{aligned}$$

Hence for $t \in (t_{p-1}, t_p],\ p = 1, \ldots, m$, we have

$$\begin{aligned} \|y(a_1, f_1)(t) - y(a_2, f_2)(t)\| &\leq Me^{\omega t} \int_0^t \|f_1(s) - f_2(s)\| ds \\ &\quad + \sum_{k=1}^{p-1} c_k M e^{\omega t_k} \int_{t_{k-1}}^{t_k} \|f_1(s) - f_2(s)\| ds. \end{aligned} \tag{3.15}$$

Let $v_1 \in G(a, f_1)$ and $\epsilon > 0$. The function

$$F(t, y(a, f_2)) \cap B(v_1(t), p(t)\|y(a, f_1) - y(a, f_2)\| + \frac{\epsilon}{b})$$

is measurable, then there exists $v_2 \in F(t, y(a, f_2))$ such that

$$\|v_1(t) - v_2(t)\| \leq p(t)\|y(a, f_1) - y(a, f_2)\| + \frac{\epsilon}{b}.$$

From (3.15), we have

$$\begin{aligned} \|v_1(t) - v_2(t)\| &\leq p(t) Me^{\omega t} \int_0^t \|f_1(s) - f_2(s)\| ds \\ &\quad + p(t) \sum_{k=1}^{p-1} c_k M e^{\omega t_k} \int_{t_{k-1}}^{t_k} \|f_1(s) - f_2(s)\| ds + \frac{\epsilon}{b}. \end{aligned}$$

Since $t_{p-1} < t$, we have

$$\|v_1(t) - v_2(t)\| \leq p(t) \left(Me^{\omega b} + \sum_{k=1}^{m} c_k e^{\omega t_k} \right) \int_0^t \|f_1(s) - f_2(s)\| ds + \frac{\epsilon}{b}. \tag{3.16}$$

Let $\tau > Me^{\omega b} + \sum_{k=1}^{m} c_k e^{\omega t_k} = D$, $m(t) = \int_0^t p(s) ds$, and let L^1 be the weighted space of Lebesgue measurable functions such that $\int_0^b \|v(t)\| e^{-\tau m(t)} dt < \infty$. Endowed with norm

$$\|v\|_1 = \int_0^b \|v(t)\| e^{-\tau m(t)} dt, \text{ for } v \in L^1,$$

it becomes a Banach space. From (3.16), we have

$$\begin{aligned}
\|v_1 - v_2\|_1 &= \int_0^b e^{-\tau m(t)}\|v_1(t) - v_2(t)\|dt \\
&\le \int_0^b p(t)e^{-\tau m(t)} D dt \int_0^t \|f_1(s) - f_2(s)\|ds \\
&= \frac{-D}{\tau}\int_0^b [e^{-\tau m(t)}]' dt \int_0^t \|f_1(s) - f_2(s)\|ds \\
&= \frac{-D}{\tau}\left\{ e^{-\tau m(b)} \int_0^b \|f_1(t) - f_2(t)\|dt - \int_0^b \|f_1(t) - f_2(t)\|dt \right\} + \epsilon. \\
&\le \frac{D}{\tau}\int_0^b \|f_1(t) - f_2(t)\|dt + \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we get

$$\|v_1 - v_2\|_1 \le \frac{D}{\tau}\|f_1 - f_2\|_1.$$

By an analogous relation obtained by interchanging the roles of v_1 and v_2, we finally obtain that for every $f_1, f_2 \in L^1(J, E)$

$$H_d(G(a, f_1), G(a, f_2)) \le \frac{D}{\tau}\|f_1 - f_2\|_1.$$

Hence there exist continuous single-valued map $\phi : E \times L^1(J, E) \to L^1(J, E)$ such that

$$\phi(a, f) \in G(a, f) \text{ for each } f \in L^1$$

and

$$\phi(a, f) = f \text{ for every } f \in G_*(a, f).$$

Let $(a, y) \in \overline{\Omega}$ be arbitrary. There exists a unique $y \in \Omega_*$ for some $f \in L^1(J, E)$ where $y(a, f)$ denotes the mild solution of (3.1). Hence $(a, y) = (a, y(a, f))$. We consider the following function

$$L_*(a, y(a, f))(t) = T(t)a + \int_0^t T(t - s)\phi(a, f)(s)ds + \sum_{0<t_k<t} T(t - t_k)I_k(y(a, f)(t_k)),\ t \in J.$$

Now we prove that the function L_* is continuous. Let $(a, y(a, f)), (a_0, y(a_0, f_0)) \in E \times \overline{\Omega}$, then

$$\begin{aligned}
&\|L_*(a, y(a, f))(t) - L_*(a_0, y(a_0, f_0))(t)\| \\
\le\ & Me^{\omega b}\|a - a_0\| \\
&+ Me^{\omega b}\int_0^t \|\phi(a, f)(s) - \phi(a_0, f_0)(s)ds \\
&+ Me^{\omega b}\sum_{0<t_k<t} \|I_k(y(a, f)(t_k)) - I_k(y(a_0, f_0)(t_k)\|
\end{aligned}$$

Thus

$$\begin{aligned}
\|L_*(a,y(a,f))-L_*(a_0,y(a_0,f_0))\|_{PC} \quad &\leq \quad Me^{\omega b}\|a-a_0\| \\
&\quad +Me^{\omega b}\int_0^b \|\phi(a,f)(s)-\phi(a_0,f_0)(s)ds \\
&\quad +Me^{\omega b}\sum_{k=1}^{m} c_k\|y(a,f)(t_k))-y(a_0,f_0)(t_k)\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
& d(L_*(a,y(a,f)),L_*(a_0,y(a_0,f_0)) \\
\leq \quad & Me^{\omega b}\|a-a_0\|+Me^{\omega b}\int_0^b \|\phi(a,f)(s)-\phi(a_0,f_0)(s)ds \\
& +Me^{\omega b}\sum_{k=1}^{m} c_k\|y(a,f)(t_k))-y(a_0,f_0)(t_k)\| \\
& +\|f-f_0\|_{L^1} \to 0 \text{ as } (a,y(a,f)) \to (a_0,y(a_0,f_0)).
\end{aligned}$$

Hence L_* is continuous. Now, we prove that the solutions set is an absolute retract.

Theorem 3.9. *Let $F : J\times E \to \mathcal{P}_{cp}(E)$ be multivalued map. Assume that $t \to F(t,.)$ is a measurable, $(\mathcal{H}_2)$ and*

$(\mathcal{A}_2)$ *there exist $c_k > 0,\ k = 1,\dots,m$ such that*

$$I_k(x) = c_k x, \textit{ for each } x \in E,$$

then the solution set of Problem (1.1) is an absolute retract.

Proof. Using the fact that $I_k,\ k = 1,\dots,m$ are linear functions, then $\overline{L}$ is linear. Also it clear that $\overline{L}(E\times L^1(J,E)) = \Omega$. Moreover since $\overline{L}$ is one-to-one then from Lemma 3.4 Ω is a Banach space with norm

$$\|y(a,f)\|_\Omega = \|y(a,f)\|_{PC}+\|f\|_{L^1}, \text{ for each } (a,f) \in E\times L^1(J,E).$$

By using Theorem 3.7, we can prove that there exist $L_* : \Omega \to PC$ such that

$$\begin{aligned}
L_*(a,y) \in S(a) \qquad & \text{for each } y \in \Omega_* \\
L_*(a,f) \quad = \quad & y, \ \text{ for each } y \in S(a),
\end{aligned}$$

where

$$\overline{\Omega} = \{(a,y) \in E\times\Omega :\ a \in E,\ y \in \Omega_*\},$$

and

$$\Omega_* = \{y \in \Omega : y(0) = a\}.$$

It is clear that $S(a) \subset \Omega_*$. Since $\overline{L}$ is linear continuous and bijective, then $\overline{L}$ is an homomorphism

$$S(a) = (L\circ G_*)(a).$$

By the same method as used in Theorem 3.7 we can easily prove that G_* is a contraction. Hence from (2.11) $S(a)$ is an absolute retract of Ω_* Since Ω_* is closed convex of a Banach space Ω. Then Ω_* is retract of Ω (see, [14]), this implies that $S(a)$ is retract.

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