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EXISTENCE OF STABLE PERIODIC ORBITS FOR A PREDATOR-PREY MODEL WITH BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE AND DELAY

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Abstract

In this paper a Beddington-DeAngelis predator-prey model with time lag for predator is proposed and analyzed. Mathematical analysis regard to boundedness of solutions, nature of equilibria, uniform persistence, and stability are analyzed. We show that if the positive equilibrium is unstable, an orbitally asymptotically stable periodic solution exists.

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1 Introduction

Generally a predator-prey model has the structure

$$\begin{aligned} N'(t) &= f(N)N - g(N,P)P \\ P'(t) &= -\nu P + \varepsilon g(N,P)P, \end{aligned} \tag{1.1}$$

where $N(t)$, $P(t)$ represent the population density of prey and predator at time t, respectively. $f(N)$ is the growth rate of prey and it is assumed that $f(0) > 0$ and has exactly one positive zero, say K, which is called the carrying capacity of prey. In most cases, the prey is always assumed to grow logistically, i.e., $f(N) = r(1 - N/K)$. The constant r is called intrinsic growth rate of prey. The function $g(N,P)$ is the functional response of the model, i.e., the rate at which an individual predator consumes prey. The parameter ε describes the efficiency of the predator in converting consumed prey into predator offspring, while ν is the predator mortality rate. In this paper we will assume that the functional response is the so-called Beddington-DeAngelis one; i.e.

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$$g(N,P) = \frac{\mu N}{\alpha + \beta N + \gamma P}. \tag{1.2}$$

This type of functional response was introduced by Beddington [1] and DeAngelis et al. [7], independently. It is similar to the well known Holling type II functional response but has an extra term γP in the denominator which models mutual interference between predators. It can be derived mechanistically via considerations of time utilization [1, 15] or spatial limits on predation [5]. The system (1.1) with the functional response (1.2) can take the form (after rescaling)

$$\begin{aligned} N' &= N(1-N) - \frac{aNP}{1+bN+cP}, \\ P' &= -dP + \frac{mNP}{1+bN+cP}. \end{aligned} \tag{1.3}$$

System (1.3) and the analogous systems with diffusion have received much attention in the literature in recent years [3, 4, 8, 9, 10, 11, 12, 20]. The studies [3, 9, 10] present a complete classification of the global dynamics of (1.3), if there is no positive steady state, the boundary steady state $(1,0)$ is globally attracting; if (1.3) admits a positive steady state but it is unstable, then there is a unique limit cycle; otherwise, the positive steady state is the global attractor.

In this paper, we will consider the model (1.3) with delay, which is described by the integro-differential system

$$\begin{aligned} N' &= N(1-N) - \frac{aNP}{1+bN+cP}, \\ P' &= -dP + m\int_{-\infty}^{t} \delta \frac{N(\tau)P(\tau)}{1+bN(\tau)+cP(\tau)} e^{-\delta(t-\tau)} d\tau \end{aligned} \tag{1.4}$$

where the exponential weight function satisfies

$$\int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} d\tau = \int_{0}^{\infty} \delta e^{-\delta s} ds = 1.$$

We are assuming in a more realistic fashion that the present level of the predator affects instantaneously the growth of the prey, but that the growth of the predator is influenced by the amount of prey in the past. More precisely, the predator grow up depending on the weight average time of the functional response over the past by means of the function $Q(t)$ given by the integral

$$Q(t) = \int_{-\infty}^{t} \delta \frac{N(\tau)P(\tau)}{1+bN(\tau)+cP(\tau)} e^{-\delta(t-\tau)} d\tau. \tag{1.5}$$

Clearly, this assumption implies that the influence of the past fades away exponentially and the number $1/\delta$ might be interpreted as the measure of the influence of the past. So, the smaller the $\delta > 0$, the longer the interval in the past in which the values of N are taken into account; see [2, 6, 14].

The integro-differential system (1.4) can be transformed (see [6, 14]) into the system of differential equations on the interval $[0,\infty)$,

$$\begin{aligned} N' &= N(1-N) - \frac{aNP}{1+bN+cP}, \\ P' &= -dP + mQ, \\ Q' &= -\delta Q + \frac{\delta NP}{1+bN+cP}. \end{aligned} \tag{1.6}$$

We understand the relationship between the two systems as follows: If $(N,P):[0,\infty)\longrightarrow \mathbb{R}^3$ is the solution of (1.4) corresponding to a continuous and bounded initial functions $(\widetilde{N},\widetilde{P}):(-\infty,0]\longrightarrow\mathbb{R}^2$, then $(N,P,Q):[0,\infty)\longrightarrow\mathbb{R}^3$ is a solution of (1.6) with $N(0)=\widetilde{N}(0)$, $P(0)=\widetilde{P}(0)$, and

$$Q(0) = \int_{-\infty}^{0} \delta \frac{\widetilde{N}(\tau)\widetilde{P}(\tau)}{1+b\widetilde{N}(\tau)+c\widetilde{P}(\tau)} e^{-\delta\tau} d\tau.$$

Conversely, if (N,P,Q) is any solution of (1.6) defined on the entire real line and bounded on $(-\infty,0]$, then Q is given by (1.5) so (N,P) satisfies (1.4).
The main concern of this paper is to study the dynamics of the system (1.6). More concretely, we will show that if the positive equilibrium of the system (1.6) is unstable, an orbitally asymptotically stable periodic solution exists.

2 Preliminaries

In this section we will summarize the main facts related to our research. Let us consider the system of differential equations

$$x' = F(x), \quad x \in D \tag{2.1}$$

where D is an open subset of $\mathbb{R}^3$ and F is twice continuously differentiable in D. The noncontinuable solution of (2.1) satisfying $x(0)=x_0$ is denoted by $x(\cdot,x_0)$, the positive (negative) semi-orbit through x_0 is denoted by $\varphi^+(x_0)$ ($\varphi^-(x_0)$), and the orbit through x_0 is denoted by $\varphi(x_0)=\varphi^-(x_0)\cup\varphi^+(x_0)$. We use the notation $\omega(x_0)$ ($\alpha(x_0)$) for the positive (negative) limit set of $\varphi^+(x_0)$ ($\varphi^-(x_0)$) provided the later semi-orbit has compact closure in D.
System (2.1) is said to be competitive in D if the Jacobian matrix of F at x, $F'(x)$, has non-positive off-diagonal elements

$$\frac{\partial F_i}{\partial x_j} \le 0, \quad i \ne j,$$

at each point of D. System (2.1) is said to be competitive and irreducible in D provided that de Jacobian matrix is an irreducible matrix at each point $x\in D$ and (2.1) is competitive in D. Recall that an $n\times n$ matrix A is irreducible if for each nonempty proper subset I of $N=\{1,2,\ldots,n\}$ there exist $i\in I$ and $j\in N-I$ such that $A_{ij}\ne 0$.
For vectors x and y in $\mathbb{R}^3$ the inequality $x\ll y$ ($x\le y$) means that $x_i<y_i$ ($x_i\le y_i$) holds for all i and $x<y$ means that $x\le y$ but $x\ne y$. Two vectors x and y are related if either $x\le y$ or

$y \leq x$ and are unrelated otherwise. The open set D is said to be p convex provided that for every x and y belonging to D for which $x \leq y$ the line segment joining x and y belongs to D. The following theorem is proved in [16].

Theorem 2.1. *Let* (2.1) *be a competitive system in $D \subset \mathbb{R}^n$ y suppose that D contains a unique equilibrium point p which is hyperbolic and assume that $F'(p)$ is irreducible. Suppose further that $W^s(p)$, the stable manifold of p, is one-dimensional. If $q \in D \backslash W^s(p)$ and $\varphi^+(q)$ has compact closure in D, then $\omega(q)$ is a nontrivial periodic orbit.*

The existence of an orbitally stable periodic solution can also be proved. We introduce the following hypotheses.

(H1) System (2.1) is dissipative: For each $x \in D$, $\varphi^+(x)$ has compact closure in D. Moreover, there exists a compact subset B of D with the property that for each $x \in D$ there exists $T(x)$ such that $x(t,x) \in B$ for $t \geq T(x)$.

(H2) System (2.1) is competitive and irreducible en D.

(H3) D is an open, p-convex subset of $\mathbb{R}^3$.

(H4) D contains a unique equilibrium point x^* and $det(F'(x^*)) < 0$.

The following result holds (see [17]).

Theorem 2.2. *Let* (H1) *through* (H4) *hold. Then either*

(a) x^* is stable or,

(b) there exists a nontrivial orbitally stable periodic orbit in D. In addition, let us assume that F is analytic in D. If x^* is unstable then there is at least one but no more than finitely many periodic orbits for (2.1) *and at least one of these is orbitally asymptotically stable.*

Our system (1.6) can be transformed into a competitive system. Let us denote by $u = (N,P,Q)^T$, $v = (x,y,z)^T$, and $H = diag[1,1,-1]$. The transformation $v = Hu$ in the system (1.6) results in

$$\begin{aligned} x' &= x(1-x) - \frac{axy}{1+bx+cy}, \\ y' &= -dy - mz, \\ z' &= -\delta z - \frac{\delta xy}{1+bx+cy}. \end{aligned} \tag{2.2}$$

Let F denote the right-hand side of (2.2). Then the Jacobian of F is given by

$$F'(v) = \begin{bmatrix} 1-2x-\dfrac{ay(1+cy)}{(1+bx+cy)^2} & -\dfrac{ax(a+bx)}{(1+bx+cy)^2} & 0 \\ 0 & -d & -m \\ -\dfrac{\delta y(1+cy)}{(1+bx+by)^2} & -\dfrac{\delta x(1+bx)}{(1+bx+cy)^2} & -\delta \end{bmatrix}.$$

Obviously (2.2) is competitive and irreducible in the open region $D = \{(x,y,z) \in \mathbb{R}^3 : x > 0, y > 0, z < 0\}$. Our main results will follow from this observation and the above theorems. The equilibria of the system (1.6) consist of two trivial critical points $E_0 = (0,0,0)$ and $E_1 = (1,0,0)$ in the boundary of $\Omega = \{(N,P,Q) \in \mathbb{R}^3 : N \geq 0, P \geq 0, Q \geq 0\}$, and a unique nontrivial equilibrium point $E^* = (N^*, P^*, Q^*)$ if and only if the following condition is true

$$m > d(b+1). \tag{2.3}$$

In this case, we have

$$\begin{aligned} N^* &= \frac{cm - a(m-bd) + \sqrt{(a(m-bd)-cm)^2 + 4adcm}}{2cm}, \\ P^* &= \frac{1}{cd}((m-bd)N^* - d), \quad Q^* = \frac{d}{m}P^*. \end{aligned}$$

The stability properties of E_0 and E_1 can be determined by their linearizations. Let $J(E_i)$ denote the Jacobian matrices evaluated at E_i. Then

$$J(E_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d & m \\ 0 & 0 & -\delta \end{pmatrix},$$

and

$$J(E_1) = \begin{pmatrix} -1 & -\dfrac{a}{1+b} & 0 \\ \\ 0 & -d & m \\ \\ 0 & \dfrac{\delta}{1+b} & -\delta \end{pmatrix}.$$

Thus, E_0 is a saddlepoint, having two negative eigenvalues and one positive eigenvalue. If $m > d(1+b)$ then E_1 also a saddlepoint with two negative eigenvalues and one positive eigenvalue.
Is easy to show that

Lemma 2.3. 1. *The N-axis and the (P,Q) plane are invariant under flow induced by the system* (1.6).

2. *The intersection of the stable manifold of E_0 with Ω consists of all points $(0,P,Q)$ such that $P \geq 0$ and $Q \geq 0$.*

3 Dissipativeness and Uniform Persistence

The main goal of this section is to give conditions implying that the predator and prey persist indefinitely, i.e., that neither becomes extinct.
The following theorem shows that all solutions of system (1.6) are bounded and therefore defined for all $t \geq 0$.

Theorem 3.1. Ω *is positively invariant under the flow induced by* (1.6)*. Moreover,* (1.6) *is pointwise dissipative and the absorbing set (into which every solution eventually enters and remains) is given by* $B = [0,1]\times[0,\frac{m}{dc}]\times[0,\frac{1}{c}]$.

Proof. Standard and simple arguments show that solutions of the system (1.6) always exist and stay positive.
Now, let us prove that the solutions of the system (1.6) are bounded for $t \geq 0$.
Taking into account the first equation of the system (1.6) we obtain that

$$N' \leq N(1-N),$$

implying that

$$\limsup_{t\longrightarrow\infty} N(t) \leq 1. \tag{3.1}$$

This means that for any $\eta > 0$, exist $T_\eta > 0$ such that $0 < N(t) < 1+\eta$ for $t \geq T_\eta$.
From the third equation of the system (1.6),

$$Q' + \delta Q = \frac{\delta NP}{1+bN+cP} \leq \frac{\delta}{c}N \leq \frac{\delta}{c}(1+\eta), \quad \text{if } t \geq T_\eta,$$

which implies that

$$Q(t) \leq Q(T_\eta)e^{-\delta(t-T_\eta)} + \frac{1}{c}(1+\eta)[1-e^{-\delta(t-T_\eta)}], \quad \text{for all } t \geq T_\eta,$$

therefore

$$\limsup_{t\longrightarrow\infty} Q(t) \leq \frac{1}{c}(1+\eta). \tag{3.2}$$

Since η is arbitrary, we have

$$\limsup_{t\longrightarrow\infty} Q(t) \leq \frac{1}{c}. \tag{3.3}$$

Now, given $\varepsilon > 0$, exits $T_\varepsilon > 0$ such that $0 < Q(t) < 1/c+\varepsilon$ if $t \geq T_\varepsilon$. By using the second equation of the system (1.6)

$$P' + dP = mQ \leq m\left(\frac{1}{c}+\varepsilon\right), \quad \text{if } t \geq T_\varepsilon.$$

Hence, for $t \geq T_\varepsilon$ we have

$$P(t) \leq P(T_\varepsilon)e^{-d(t-T_\varepsilon)} + \frac{m}{d}\left(\frac{1}{c}+\varepsilon\right)[1-e^{-d(t-T_\varepsilon)}] \quad \text{for all } t \geq T_\varepsilon,$$

therefore

$$\limsup_{t\longrightarrow\infty} P(T) \leq \frac{m}{dc}$$

This completes the proof of our claim. □

We now show that the system (1.6) persist indefinitely if $m > d(b+1)$. Mathematically, we use the theory of uniform persistence (see [19]).

Theorem 3.2. *Assume that $m > d(b+1)$. Then exists $\eta > 0$ such that*

$$\liminf_{t\longrightarrow\infty} N(t) > \eta, \quad \liminf_{t\longrightarrow\infty} P(t) > \eta, \quad \liminf_{t\longrightarrow\infty} Q(t) > \eta,$$

for all solutions of the systems (1.6) *starting in* Ω *with positive initial data.*

Proof. We use Theorem 4.6 of [19], employing the notation of that result and the notation $u = (N, P, Q)$. Let X_2 the union of the nonnegative N-axis and the (P, Q) plane, with $P \geq 0$ and $Q \geq 0$. Let $X_1 = \Omega \cap X_2^c$. We need to prove that solutions starting in X_1 are eventually bounded away from X_2, uniformly with respect to the initial data. The compactness assumption $(C_{4,2})$ of Theorem 4.6 holds with B as in Theorem 3.1 (for small positive δ as defined in [19]). Define $\emptyset_2 = \cup_{u\in X_2}\omega(u)$. According to Lemma 2.3, $\emptyset_2$ consist of the equilibria E_0 and E_1 and hence it has an acyclity isolated covering $M = M_0 \cup M_1$, where $M_i = E_i$ for $i = 0, 1$. Here, acyclity of M means that there do not exist points $u_i \in X_2$ with $\alpha(u_1) = E_0$, $\omega(u_1) = E_0$, $\alpha(u_2) = E_2$ and $\omega(u_2) = E_2$. In fact, it is the latter u_2 which cannot exist by Lemma 2.3. Isolatedness of M_i means that these sets are isolated in Ω, that is, there exists open sets U_i of M_i in Ω such that M_i is the maximal invariant set in U_i. This Holds since each E_i is hyperbolic. We must also show that each M_i is a weak repeller for X_1: for all $u(0) \in X_1$, $\limsup_{t\to\infty} |u(t) - E_i| > 0$. Suppose, for contradiction, that a solution $u(t)$ with $u(0) \in X_1$ satisfies $\lim_{t\to\infty} u(t) = E_0$. Then $u(0)$ belongs to the stable manifold of E_1. But the intersection of the latter with Ω consists of the (P, Q)-plane by Lemma 2.3 so we have a contradiction to $u(0) \in X_1$. Suppose now that $\lim_{t\to\infty} u(t) = E_1$, let us linearize the system (1.6) about E_1, obtaining the following system

$$\begin{aligned} P' &= -dP + mQ, \\ Q' &= \frac{\delta}{1+b}P - \delta Q. \end{aligned} \tag{3.4}$$

Taking into account that $m > d(b+1)$ and by using comparison techniques we obtain that $P(t)$ and $Q(t)$ cannot tend to zero as t tends to infinity, which is a contradiction. Hence, by Theorem 4.6 of [19], X_2 is a strong repeller for X_1. □

4 Global stability of E_1

Theorem 4.1. *If $m < d(b+1)$ then E_1 is globally asymptotically stable.*

Proof. From (3.1), for any $\eta > 0$ sufficiently small, exist $T_\eta > 0$ such that $N(t) < 1 + \eta$ if $t \geq T_\eta$.
From the last two equations of the system (1.6), we get

$$\begin{aligned} P' &= -dp + mQ \\ Q' &= -\delta Q + \frac{\delta N p}{1 + bN + cP} < -\delta Q + \frac{\delta(1+\eta)}{1+b}P \end{aligned} \tag{4.1}$$

We consider the comparison equations

$$
\begin{aligned}
u'(t) &= -du + mv, \\
v'(t) &= -dv + \frac{\delta(1+\eta)}{1+b}u.
\end{aligned} \tag{4.2}
$$

It easy to show that if $m < d(b+1)$ for any solution of (4.2) with nonnegative initial values we have $u(t) \longrightarrow 0$ and $v(t) \longrightarrow 0$, as $t \longrightarrow \infty$. Let $0 < P(0) \le u(0)$, $0 < Q(0) \le v(0)$. If $(u(t), v(t))$ is a solution of the system (4.2) with initial value $(u(0), v(0))$, then by the comparison theorem we have $P(t) \le u(t)$, $Q(t) \le v(t)$ for all $t \ge 0$. Hence $P(t) \longrightarrow 0$ and $Q(t) \longrightarrow 0$, as $t \longrightarrow \infty$.
Having in mind that $P(t) \longrightarrow 0$ as $t \longrightarrow \infty$, we get that for any $\varepsilon > 0$, exist $T_\varepsilon > 0$ such that

$$
\frac{P}{1+cP} < \frac{\varepsilon}{1+c\varepsilon}, \quad \forall t \ge T_\varepsilon.
$$

By using the first equation of the system (1.6) we obtain that

$$
\begin{aligned}
N' &= N(1-N) - \frac{aNP}{1+bN+cP} \ge N(1-N) - \frac{aNP}{1+cP} \\
&\ge N(1-N) - \frac{aN\varepsilon}{1+c\varepsilon} = N\left(1 - N - \frac{a\varepsilon}{1+c\varepsilon}\right),
\end{aligned}
$$

which implies that

$$
\liminf_{t \longrightarrow \infty} N(t) \ge 1 - \frac{a\varepsilon}{1+c\varepsilon}.
$$

Since ε is arbitrary, we get

$$
\liminf_{t \longrightarrow \infty} N(t) \ge 1.
$$

This completes the proof of our claim. □

5 Stability of the nontrivial equilibrium

We want to determine the stability of the unique nontrivial equilibrium E^*. It is convenient to examine the equivalent problem of the stability of the unique nontrivial equilibrium $v^* = HE^*$ of (2.2). The Jacobian matrix of F evaluated in v^* is given by

$$
F'(v^*) = \begin{bmatrix} a_{11} & -aa_{12} & 0 \\ 0 & -d & -m \\ -\delta a_{31} & -\delta a_{12} & -\delta \end{bmatrix},
$$

where

$$
a_{11} = \frac{bd}{m}(1-x^*) - x^*, \quad a_{12} = \frac{x^*(1+bx^*)}{(1+bx^*+cy^*)^2}, \quad a_{31} = \frac{y^*(1+cy^*)}{(1+bx^*+cy^*)^2},
$$

$x^* = N^*$, $y^* = P^*$ and the characteristic equation is given by

$$P_\delta(\lambda) = \lambda^3 + q_1(\delta)\lambda^2 + q_2(\delta)\lambda + q_3(\delta),$$

where

$$\begin{aligned} q_1(\delta) &= \delta + d - a_{11} \\ q_2(\delta) &= \delta\Big(\frac{c}{a}(1-x^*) - a_{11}\Big) - da_{11}, \\ q_3(\delta) &= \delta\Big(-\frac{c}{a}(1-x^*)a_{11} + maa_{12}a_{31}\Big). \end{aligned}$$

It is worth noting that the coordinates of the critical point v^* are independent of δ.

Lemma 5.1. *Suppose that*

$$\frac{bd}{bd+m} < x^*,$$

then exactly one of the following hold:

1. *All the roots of $P_\delta(\lambda)$ have negative real part.*
2. *There is one negative eigenvalue and a pair of nonzero purely imaginary eigenvalue (if and only if $q_1 > 0$, $q_2 > 0$ and $q_1q_2 = q_3$).*
3. *There is one negative eigenvalue and a pair of eigenvalues with positive real part.*

Proof. Since $(bd)/(bd+m) < x^*$ then $\det F'(v^*) = -q_3(\delta) < 0$. As the product of the eigenvalues is negative, we conclude that an even number (0 or 2) of eigenvalues have positive real part and zero cannot be an eigenvalue. In the nonhyperbolic case (ii), one sees that $\eta^2 = q_2 = q_3/q_1$ by substituting $\lambda = i\eta$ into $P_\delta(\lambda) = 0$. □

The Routh-Hurwitz criteria give necessary and sufficient conditions for (i). We will be particulary interested in finding conditions for v^* to be hyperbolic and unstable because Theorem 2.2 implies the existence of periodic orbits. Of course, the Hopf Bifurcations Theorem my apply but it leads to the existence of small-amplitude periodic orbits.
Let us define the function

$$\phi(\delta) = q_1(\delta)q_2(\delta) - q_3(\delta) = A\delta^2 + B\delta + C,$$

where

$$\begin{aligned} A &= \frac{c}{a}(1-x^*) - a_{11} \\ B &= \frac{dc}{a}(1-x^*) - 2da_{11} + a_{11}^2 - maa_{12}a_{31} \\ C &= -da_{11}(d - a_{11}). \end{aligned}$$

Applying the Routh-Hurwitz criteria, we get the following theorem.

Theorem 5.2. *Let us assume that*

$$\frac{bd}{bd+m} < x^*.$$

Then $A > 0$, $C > 0$, and $q_i(\delta) > 0$ for $i = 1,2,3$. Exactly one of the following holds:

1. *All the roots of $P_\delta(\lambda)$ have a negative real part for all $\delta > 0$.*
2. *There exist $0 < \delta_1 \leq \delta_2$ such that for $\delta \in (0,\delta_1) \cap (\delta_2,\infty)$ the roots of $P_\delta(\lambda)$ have real negative real part, and $P_\delta(\lambda)$ has a negative root and two complex roots with positive part for all $\delta \in (\delta_1,\delta_2)$. This case holds if and only if $B < 0$ and $B^2 - 4AC \geq 0$.*

6 Existence of a stable periodic orbit

Our main result below gives sufficient conditions that almost every solution is asymptotically periodic.

Theorem 6.1. *Suppose that*

$$\frac{bd}{bd+m} < x^*.$$

Assume that the unique nontrivial equilibrium E^* is hyperbolic and unstable. Then it has a one-dimensional stable manifold $W^s(E^*)$. Furthermore, there exist an asymptotically orbitally stable periodic orbit, and the omega limit set of every solution $(N(t),P(t),Q(t))$ with $N(0) > 0$, $P(0) > 0$, and $(N(0),P(0),Q(0)) \notin W^s(E^*)$ is a nonconstant periodic orbit.

Proof. We apply Theorems 2.1 and 2.2 to the transformed system (2.2). From Lemma 5.1 we see that the stable manifold of E^* is one-dimensional. The existence of an orbitally asymptotically stable periodic orbit follows from Theorem 2.1 and the analyticity of the vector field. Note que (H1) holds by Theorem 2.1 and Theorem 3.2 (the latter must be translated appropriately to system (2.2)). In particular, we take the domain D as in Section 2. Using Theorem 3.2, Theorem 2.1 implies the final assertion. □

7 Discussion

The Beddington-DeAngelis form of functional response has some of the same qualitative features as the ratio-dependent. A ratio-dependent version of (1.3) would be the form

$$\begin{aligned} N' &= N(1-N) - \frac{aNP}{bN+cP}, \\ P' &= -dP + \frac{mNP}{bN+cP}. \end{aligned}$$

The ratio-dependent form also incorporates mutual interference by predators, but it has somewhat singular behavior at low densities and has been criticized on other groups, see [13] for a mathematical analysis and the references in [5] for some aspect of the debate among biologists about ratio dependence. The Beddington-DeAngelis form of functional response has some of the same qualitative features as the ratio-dependent form but avoids some of the behaviors of ratio-dependent models at low densities which have been the source of controversy (see [5, 13, 18]).

Figure 1. $\delta = 4$. Initial value I1= $(0.2, 0.2, 0.1)$. Initial value I2= $(0.4, 0.4, 0.02)$

Figure 2. $\delta = 15$. Initial value I1= $(0.1, 0.1, 0.01)$.

In this paper, a Beddington-DeAngelis predador-prey model with time lag for predator is proposed and investigated. Using results about competitive system, we prove that there exist and orbitally asymptotically stable periodic orbit when (2.2) is permanent and the positive equilibrium is unstable. Comparing our results with the results of Cantrell and Cosner [3] and Hwang [9, 10] we know that this is a new phenomenon, and this shows that the time lag may be the cause of periodic oscillations in the populations. Finally the following numeric examples shows that the feasibility of our results.
Let us pick $a = 3, b = 1, c = 0.01, d = 0.1, m = 2$. Under this selection of parameters the critical point is given by

$$(N^*, P^*, Q^*) = (0.05280707500, 0.3334425000, 0.01667212500).$$

In this parameter configuration the nontrivial equilibrium point is hyperbolic with a one-dimensional stable manifold just for $\delta \in (0.000687597, 9.708583324)$, therefore for this values of δ, E^* is unstable and an orbitally asymptotically stable periodic solution exist (see Fig. 1). If $\delta \in (0, 0.000687597) \cup (9.708583324, \infty)$ then all roots of the characteristic equation have negative real parts and therefore E^* is asymptotically stable (see Fig. 2).

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