African Diaspora Journal of Mathematics
Volume 16, Number 2, pp. 23–30 (2014)

ISSN 1539-854X
www.math-res-pub.org/adjm

On Simultaneous Characterization of the Set of Elements of Good Approximation in Metric Spaces

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Abstract

If W is a subset of a metric space (X,d) then for a given $\varepsilon > 0$, an element $y_0 \in W$ is called a good approximation or ε−approximation for $x \in X$ if $d(x,y_0) \leq d(x,W)+\varepsilon$. We denote by $P_{W,\varepsilon}(x)$ the set of all such $y_0 \in W$ i.e. $P_{W,\,\varepsilon}(x) = \{y \in W : d(x,y) \leq d(x,W)+\varepsilon\}$. In particular, for $\varepsilon = 0$ we get the set of all best approximations to x in W. Given a subset M of W, what are the necessary and sufficient conditions in order that every element $y_0 \in M$ is an element of good approximation to x by the elements of W? The paper mainly deals with this problem of simultaneous characterization of elements of good approximation in metric spaces. The proved results extend and generalize several known results on the subject.

AMS Subject Classification: [2010]41A50, 41A65.

Keywords: Good approximation; convex metric space; convex set; approximatively compact set; quasi Chebyshev set.

1 Introduction

For a subset G of a metric space (X,d) and $x \in X$, one of the main problems of approximation theory is to find elements $g_0 \in G$ such that $d(x,\ g_0) = inf\ \{d(x,g) : g \in G\} \equiv d(x,G)$. Each such element $g_0 \in G$, called a best approximation to x in G, is an exact solution (or optimal approximation) to the problem. A somewhat varied problem is to seek an element $g_0 \in G$ to be an approximate solution to the above problem in the following sense: Given $\varepsilon > 0$, find an element $g_0 \in G$ such that $d(x,g_0) \leq d(x,G)+\varepsilon$. Such a g_0, called a good approximation (or ε-approximation) to x in G, is an approximate solution (or almost optimal approximation) to the problem. Since for $\varepsilon > 0$, good approximation always exist, it is sufficient to find good approximation if one cannot find best approximation because in numerical analysis such elements sometimes serve the purpose.

The duality theory in normed linear spaces has helped a lot in developing a fairly large theory of approximation in normed linear spaces (see e.g. Singer [13]). Hahn-Banach

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Theorem has been an important tool in proving many results on approximation in normed linear spaces. Because of lack of duality theory, the theory of approximation has been comparatively very less developed in linear metric spaces and metric spaces. C.Mustăta [5], [6], Narang [7],[8], Narang and Chandok [9], Narang and Khanna [10] , G. Pantelidis [11] and few others have discussed some results on best approximation in linear metric spaces and metric spaces by considering a space of functions similar to the conjugate space X^* of a normed linear space X. Motivated by the results on the characterization of elements of best approximation proved by Mustăta [5],[6], Pantelidis [11], Singer [13],and the notion of good approximation introduced by R.C. Buck [1], some results on the characterization of elements of best approximation and good approximation (ε-approximation) have been proved in linear metric spaces and metric spaces in [7], [8], [10] and [11]. Continuing the study, this paper mainly deals with simultaneous characterization of a set of elements of good approximation in metric spaces i.e. the following problem: Given a subset W of a metric space (X,d), $x \in X$ and $M \subset W$, what are the necessary and sufficient conditions in order that every element $g \in M$ is an element of good approximation of x by means of elements of W? The results proved in this paper generalize some of the results proved in [6], [10], [12], [13] and of others.

2 PRELIMINARIES

In this section we discuss some elementary properties and related concepts concerning elements of ε-approximation.

Let W be a nonempty subset of a metric space (X,d) and ε be a positive real number. A point $y_0 \in W$ is said to be an **ε-approximation** or **good approximation**(respectively, **best approximation**) for $x \in X$ if

$$d(x,y_0) \leq d(x,W)+\varepsilon \text{ (respectively, } d(x,y_0) = d(x,W)).$$

For $x \in X$, let

$$P_{W,\varepsilon}(x) = \{y \in W : d(x,y) \leq d(x,W)+\varepsilon\}$$

and

$$P_W(x) = \{y \in W : d(x,y) = d(x,W)\}.$$

It is clear that $P_W(x) = \cap_{\varepsilon>0} P_{W,\varepsilon}(x)$ for all $\varepsilon > 0$. Concerning the set $P_{W,\varepsilon}(x)$, we have:

(1) $P_{W,\varepsilon}(x)$ is a non-empty and bounded subset of X.

Proof. By the definition of $d(x,W)$, there exists $y_0 \in W$ such that $d(x,y_0) \leq d(x,W) + \varepsilon$ for any given $\varepsilon > 0$ and so $P_{W,\varepsilon}(x)$ is a non-empty. Now we show that $P_{W,\varepsilon}(x)$ is bounded.

Let $y_1, y_2 \in P_{W,\varepsilon}(x)$. Consider

$$\begin{aligned} d(y_1,y_2) &\leq d(y_1,x)+d(x,y_2) \\ &\leq d(x,W)+\varepsilon+d(x,W)+\varepsilon = 2d(x,W)+2\varepsilon. \end{aligned}$$

This gives $d(y_1,y_2) \leq 2[d(x,W)+\varepsilon] < \infty$ for all $y_1, y_2 \in P_{W,\varepsilon}(x)$ and hence $P_{W,\varepsilon}(x)$ is bounded.

(2) If W is a closed subset of X then so is $P_{W,\varepsilon}(x)$.

Proof. Let y be a limit point of $P_{W,\varepsilon}(x)$. Then there exist a sequence $< y_n >$ in $P_{W,\varepsilon}(x)$ such that $< y_n > \rightarrow y$. Consider

$$\begin{aligned} d(x,y) &= d(x, \lim y_n) \\ &= \lim d(x,y_n) \\ &\leq \lim[d(x,W)+\varepsilon] \\ &= d(x,W)+\varepsilon. \end{aligned}$$

This gives $y \in P_{W,\varepsilon}(x)$.

A subset W of a metric space (X,d) is called $\varepsilon-$**quasi Chebyshev** in X if $P_{W,\varepsilon}(x)$ is compact in X for each $x \in X$.

(3) Every $\varepsilon-$quasi Chebyshev subset is closed.

Proof: Let W be an $\varepsilon-$quasi Chebyshev subset of a metric space (X,d) and $y \in \overline{W}$. Then $y \in X$ and there exists a sequence $< y_n >$ in W such that $< y_n > \rightarrow y$. Therefore there exists a positive integer m_0 such that

$$d(y_n, y) < \varepsilon \text{ for all } n \geq m_0.$$

i.e. $d(y_n,y) < d(y,W)+\varepsilon$ for all $n \geq m_0$ as $y \in \overline{W}$ implies $d(y,W) = 0$. This implies that $< y_n >_{n=m_0}^{+\infty}$ is a sequence in the compact set $P_{W,\varepsilon}(y)$ and so there exists a subsequence $< y_{n_i} >$ such that $< y_{n_i} > \rightarrow y_0 \in P_{W,\varepsilon}(y) \subseteq W$. Consequently, $y = y_0 \in W$ and hence W is closed.

A sequence $< y_n >$ in W is called $\varepsilon-$ **minimizing** (respectively, **minimizing**) if $\lim d(x,y_n) \leq d(x,W)+\varepsilon$ (respectively, $\lim d(x,y_n) = d(x,W)$). The set W is called $\varepsilon-$**approximatively compact**[9], (respectively, **approximatively compact**[2]) if for each $x \in X$, each $\varepsilon-$minimizing (respectively, minimizing) sequence has a subsequence converging to an element of W.W is called **quasi-Chebyshev** if $P_W(x)$ is compact.

(4) [9] If W is an $\varepsilon-$approximatively compact set in a metric space (X,d), then $P_{W,\varepsilon}(x)$ is a nonempty compact set i.e. W is $\varepsilon-$quasi Chebyshev subset of X.

(5) [13] If W is an approximatively compact subset of a metric space (X,d), then $P_W(x)$ is compact and so W is quasi-Chebyshev.

(6) If W is a proximinal and $\varepsilon-$ quasi Chebyshev set for every $\varepsilon > 0$ in a metric space (X,d) then W is quasi-Chebyshev.

Proof. Let $< y_n >$ be a sequence in $P_W(x)$. Then $< y_n > \subseteq P_{W,\varepsilon}(x)$ for all $\varepsilon > 0$. Since W is $\varepsilon-$quasi Chebyshev, $< y_n >$ has a subsequence $< y_{n_i} > \rightarrow y_0 \in P_{W,\varepsilon}(x)$ for all $\varepsilon > 0$. This implies

$$d(x,y_{n_i}) \leq d(x,W)+\varepsilon for\ all\ \varepsilon > 0.$$

Letting $\varepsilon \rightarrow 0$, we get $d(x,y_0) \leq d(x,W)$. This gives $y_0 \in P_W(x)$. Hence $P_W(x)$ is quasi-Chebyshev.

Note. Converse is not true (see Example 2.3 [12]).

For a metric space (X,d), a continuous mapping $W : X \times X \times [0,1] \rightarrow X$ is said to be a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0, 1]$, we have

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)$$

for all $u \in X$. The metric space (X, d) together with a convex structure W is called a **convex metric space** [14].

A subset K of a convex metric space (X, d) is said to be a **convex set** [14] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$.

(7) If M is a convex subset of a convex metric space (X, d) then $P_{M, \varepsilon}(x)$ is also convex.

Proof. Let $y_1, y_2 \in P_{M, \varepsilon}(x)$ and $\lambda \in [0, 1]$. Consider

$$\begin{aligned} d(x, W(y_1, y_2, \lambda)) &\leq \lambda d(x, y_1) + (1 - \lambda) d(x, y_2) \\ &\leq \lambda[d(x, M) + \varepsilon] + (1 - \lambda)[d(x, M) + \varepsilon] \\ &= d(x, M) + \varepsilon. \end{aligned}$$

This gives $W(y_1, y_2, \lambda) \in P_{M, \varepsilon}(x)$ and hence $P_{M, \varepsilon}(x)$ is convex as $W(y_1, y_2, \lambda) \in M$.

3 Simultaneous Characterization of a set of elements of ε−approximation

This section deals with the problem of simultaneous characterization of a set of elements of ε−approximation in metric spaces.

Let (X, d) be a metric space and x_0 be a fixed point of X. The set

$$X_0^\# = \{f : X \to \mathbb{R}, \sup_{x,y \in X,\, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < \infty, f(x_0) = 0\}$$

with the usual operations of addition and multiplication by real scalars, normed by

$$||f||_X = \sup_{x,y \in X,\, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}, f \in X_0^\#$$

is a Banach space (even a conjugate Banach space [4]).

We shall be using the following theorem on the characterization of elements of ε−approximation:

Theorem 3.1. [7] *Let Y be a subset of a metric space (X, d) such that $x_0 \in Y$ and let $x \in X \backslash Y, y_0 \in Y$ and $\varepsilon > 0$. Then $y_0 \in P_{Y,\epsilon}(x)$ if and only if there exists an $f \in X_0^\#$ such that*

(i) $||f||_X = 1$

(ii) $f \mid Y = 0$

(iii) $|f(x) - f(y_0)| \geq d(x, y_0) - \varepsilon.$

Applying Theorem 3.1, we prove the following:

Theorem 3.2. *Let Y be a subset of a metric space (X,d) such that $x_0 \in Y$ and let $x \in X \setminus Y$, $M \subset Y$ and $\epsilon > 0$. Then $M \subset P_{Y,\epsilon}(x)$ if and only if there exists $f \in X_0^{\#}$ satisfying*

(i) $\|f\|_X = 1$

(ii) $f \mid Y = 0$

(iii) $|f(x) - f(y)| \geq d(x,y) - \varepsilon$. *for all* $y \in M$.

Proof. Suppose $M \subset P_{Y,\epsilon}(x)$ and $y_0 \in M$. Then $y_0 \subset P_{Y,\epsilon}(x)$ and so by Theorem 3.1, there exists $f \in X_0^{\#}$ defined by $f(z) = d(z,Y), z \in X$ such that $\|f\|_X = 1$, $f \mid Y = 0$ and $|f(x) - f(y_0)| \geq d(x,y_0) - \varepsilon$, $y_0 \in M$.

Let $y \in M$ be arbitrary. Then $y \in P_{Y,\epsilon}(x)$. Consider

$$|f(x) - f(y)| = |f(x)| = d(x,Y) \geq d(x,y) - \varepsilon, \ by\ definition.$$

Conversely, suppose there exists $f \in X_0^{\#}$ satisfying (i), (ii) and (iii) and let $y_0 \in M$. Consider

$$\begin{aligned} d(x,y_0) &\leq |f(x) - f(y_0)| + \varepsilon \\ &= |f(x) - f(y)| + \varepsilon \text{ for all } y \in Y \\ &\leq \|f\|_X\, d(x,y) + \varepsilon \text{ for all } y \in Y \\ &= d(x,y) + \varepsilon \text{ for all } y \in Y \end{aligned}$$

This implies $d(x,y_0) \leq d(x,y) + \varepsilon$ i.e $y_0 \in P_{Y,\epsilon}(x)$. Hence $M \subset P_{Y,\epsilon}(x)$.

Remarks 1 Let Y be a subspace of a linear metric space (X,d) with a translation invariant metric d and $x_0 = 0$, the additive identity of X, then one can choose the function f in the preceding discussion such that $f \in X^v$, where

$$X^v = \{f : X \to \mathbb{R}, \sup_{x \in X \setminus \{0\}} \frac{|f(x)|}{d(x,0)} < \infty, f(0) = 0, f\ subadditive\},$$

is the cone of subadditive functions in X (see [11]).

Moreover, f satisfies

(a)$|f(x) - f(y)| = |d(x,Y) - d(y,Y)|$

$\leq d(x,y)$ for all $x,y \in X$.

(b) $f(x-g) = d(x-g,Y) = d(x,g+Y) = d(x,Y)$

$\geq d(x,g) - \varepsilon$ for all $g \in Y$ and $M \subset P_{Y,\epsilon}(x)$.

2. Theorem 3.2 gives Theorems 2.1, 2.2 and 3.1 of [10].

3. If Y is a subspace of a normed linear space X then one can choose $f \in X^*$, the conjugate space of X (see Singer [13])and above theorem gives Lemmas 1.2 and 2.2 of [12] and Theorem 6.12 of [13] on $\varepsilon-$approximation.

Theorem 3.3. *Let Y be a subset of a metric space $(X,d), x \in X, y_0 \in Y$ and $\varepsilon > 0$ be given. Then $y_0 \in P_{Y,\epsilon}(x)$ if and only $d(x,y_0) \leq d_{Y^\perp}(x,y_0) + \varepsilon$, where*

$$d_{Y^\perp}(x,y_0) = \sup\{\frac{|f(x)-f(y_0)|}{||f||_X}, f \in Y^\perp \backslash \{0\}\}, Y^\perp = \{f : f \in X_0^\#, f\backslash Y = 0\}$$

*Proof.*Let $y_0 \in P_{Y,\epsilon}(x)$ then there exists $f \in Y^\perp$ such that

$$||f||_X = 1, |f(x) - f(y_0)| \geq d(x,y_0) - \varepsilon.$$

Consider

$$d_{Y^\perp}(x,y_0) = \sup_{g \in Y^\perp \backslash \{0\}} \frac{|g(x)-g(y_0)|}{||g||_X} \geq \frac{|f(x)-f(y_0)|}{||f||_X} \geq d(x,y_0) - \varepsilon$$

i.e. $d(x,y_0) \leq d_{Y^\perp}(x,y_0) + \varepsilon$.

Conversely, suppose $d(x,y_0) \leq d_{Y^\perp}(x,y_0) + \varepsilon$
i.e. $d(x,y_0) - \varepsilon \leq d_{Y^\perp}(x,y_0)$

$$= \sup_{f \in Y^\perp \backslash \{0\}} \frac{|f(x)-f(y_0)|}{||f||_X}$$

$$= \sup_{f \in Y^\perp \backslash \{0\}} \frac{|f(x)-f(y)|}{||f||_X} \text{ for all } y \in Y$$

$$\leq d_{Y^\perp}(x,y) \text{ for all } y \in Y$$

$$\leq d(x,y) \text{ for all } y \in Y \text{ (see [6]).}$$

This gives $d(x,y_0) \leq d(x,y) + \varepsilon$ i.e. $y_0 \in P_{Y,\epsilon}(x)$.

In the particular case, for $\varepsilon = 0$ Theorem 3.3 gives:

Corollary 3.4. [12] *Let Y be a subspace of a normed linear space X, $x \in X, g_0 \in Y$ and $\varepsilon > 0$ be given. Then $g_0 \in P_{W,\epsilon}(x)$ if and only if $||x - g_0|| \leq ||x - g_0||_{Y^\perp} + \varepsilon$,where*

$$||x - g_0||_{Y^\perp} = \sup\{|f(x - y_0)| : ||f|| \leq 1, f \in Y^\perp\}.$$

Corollary 3.5. [6] *Let Y be a subset of a metric space (X,d) and $y_0 \in Y, x \in X\backslash Y$. Then $y_0 \in Y$ is an element of best approximation for x by elements of Y if and if only $d_{Y^\perp}(x,y_0) = d(x,y_0)$.*

Theorem 3.6. *Let Y be a subset of a metric space (X,d) such that $y_0 \in Y$ and let $x \in X, M \subset Y$ and $\varepsilon > 0$ be given. Then $y_0 \in P_{Y,\epsilon}(x)$ if and only if*

$$d(x,y) \leq d_{Y^\perp}(x,y) + \varepsilon \textit{ for all } y \in M.$$

*Proof.*Let $M \subset P_{Y,\epsilon}(x)$. Then there exists $f \in Y^\perp$ such that

$$||f||_X = 1, |f(x) - f(y)| \geq d(x,y) - \varepsilon \text{ for all } y \in M.$$

We have

$$d_{Y^\perp}(x,y) = \sup_{g \in Y^\perp \backslash \{0\}} \frac{|g(x)-g(y)|}{||g||_X} \geq \frac{|f(x)-f(y)|}{||f||_X} \geq d(x,y) - \varepsilon$$

i.e. $d(x,y) \leq d_{Y^\perp}(x,y) + \varepsilon$ for all $y \in M$.

Conversely, suppose $d(x,y) \leq d_{Y^\perp}(x,y) + \varepsilon$ for all $y \in M$. Then for any $y_0 \in M$ and $z \in Y$, we have

$$d(x,y_0) \leq d_{Y^\perp}(x,y_0) + \varepsilon = \sup_{f \in Y^\perp \backslash \{0\}} \frac{|f(x)-f(y_0)|}{||f||_X} + \varepsilon$$

$$= \sup_{f\in Y^{\perp}\setminus\{0\}} \frac{|f(x)-f(z)|}{\|f\|_X} + \varepsilon \text{ for all } z \in Y$$
$$= d_{Y^{\perp}}(x,z) + \varepsilon \text{ for all } z \in Y$$
$$\leq d(x,z) + \varepsilon \text{ for all } z \in Y$$

This gives $d(x,y_0) \leq d(x,Y) + \varepsilon$ i.e. $y_0 \in P_{Y,\epsilon}(x)$.

In the particular case, for $\varepsilon = 0$ Theorem 3.6. gives:

Corollary 3.7. *Let Y be a subset of a metric space (X,d) such that $y_0 \in Y$ and let $x \in X\setminus Y$. Then $y_0 \in Y$ is an element of best approximation for x by elements of Y if and only if $d_{Y^{\perp}}(x,y_0) = d(x,y_0)$.*

4 Acknowledgements

The author is thankful to the learned referees for very valuable suggestions leading to an improvement of the paper. The author is also thankful to University Grants Commission for sanctioning Major Research Project (F.No. 39-48/2010(SR)) under which this research work has been carried out.

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