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Cancellation of the Singularities of the Heat Equation Restricted to a Finite Bandwich

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Abstract. The cancellation of the singularities of the heat equation in a polygonal domain with cracks is analyzed. Using a density result, a bi-orthogonality property of a family of finite eigenfunctions of the Laplacian and Holmgren's theorem, we obtain a regular solution of the heat equation by an internal control.

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1 Introduction

We consider a bounded polygonal domain Ω of $\mathbb{R}^2$ with cracks whose boundary Γ is a union of the edges Γ_j for $0 \leq j \leq n$. We denote by S_j the vertex between Γ_{j-1} and Γ_j for $1 \leq j \leq n$ and S_0 the

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vertex between Γ_n and Γ_0.
For $T > 0$, we set $Q = \Omega\times(0,T)$ and $\Sigma = \Gamma\times(0,T)$. Let us consider the following linear heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f & \text{in } Q, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma. \end{cases} \tag{1.1}$$

The existence of nonconvex angles of the boundary Γ of Ω produce singularities even if the right-hand term and the data of the equation are smooth.
Consider the set $W = D(-\Delta) = \{u \in H_0^1(\Omega); -\Delta u \in L^2(\Omega)\}$ endowed with the graph norm. We have the following result:
for any $f \in L^2(Q)$ and $u_0 \in H_0^1(\Omega)$, the problem (1.1) has a unique solution in $L^2(0,T;W)\cap H^1(0,T;L^2(\Omega))$.
For more details, see [5].
If Γ is C^2 then $W = H^2(\Omega)\cap H_0^1(\Omega)$. In our case, due to the presence of nonconvex angles, W is not embedded in $H^2(\Omega)$. More precisely, if $N \in \mathbb{N}^*$ is the number of nonconvex angles of the boundary of Ω, from [5], there exist N functions $u_1,...,u_N$ in $H_0^1(\Omega)\backslash H^2(\Omega)$ called singular solutions such that $W = H^2(\Omega)\cap H_0^1(\Omega)\oplus Span\{u_1,...,u_N\}$.
The solution u of the problem (1.1) may be broken into the sum

$$u(x,t) = u^r(x,t) + \sum_{i=1}^{N} c_i(t)u_i(x), \tag{1.2}$$

where $u^r(x,t) \in L^2(0,T;H^2(\Omega))$ and $c_i \in L^2(0,T), i = 1,...,N$; are the singularity coefficients.
So far, there is no way of killing singularities by acting on an arbitrary small part of the domain for the heat equation. Here we propose a method to regularize the solution of problem (1.1).
The aim of this paper is to cancel or control the singularity coefficients in a space generated by a family of finite eigenfunctions of $L^2(\Omega)$. More precisely, we prove that there exist m regular functions $(g_i)_{1\le i\le m}$ with compact support in ϖ and m functions $(\theta_i)_{1\le i\le m}$ such that for any $f \in L^2(Q)$ and $v_0 \in H_0^1(\Omega)\cap H^2(\Omega)$ the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = f - \displaystyle\sum_{i=1}^{m} \theta_i(t)g_i(x) & \text{in } Q, \\ v(0,x) = v_0(x) & \text{in } \Omega, \\ v = 0, & \text{on } \Sigma. \end{cases} \tag{1.3}$$

has a unique solution in $L^2(0,T;H^2(\Omega))$.
Such a problem has already been studied in the literature but only in the stationary case, see [1, 9].
Similar problem was studied by the authors for the wave equation [2].
Our problem is quite different from the one studied in [4] where topological optimization method is used to study a numerical aspect of the Dirichlet problem in a polygonal domain. In fact, in [4], the model considered is stationary. Moreover the objective of their paper is not to cancel the singularities but to make the solution of their model as close as possible to a desired state in the space $L^2(\Omega)$.
But it is important to underline that a change of the geometry and the topology of the domain could imply variations of the coefficients of the singularities (see [4]). Therefore, one interesting question is: how is it possible to both control the state of the system and minimise or cancel the singularities?
The method developed in [4] could give some answers.
This paper is organized as follows. Section 2 is devoted to the density theorem. In section 3 we study the bi-orthogonality properties of the eigenfunctions . In section 4, we establish the cancellation result.

2 Density theorem

Let H be a Hilbert space equipped with an inner product $(.,.)_H$.

Theorem 2.1. *(Density property). Let H be a Hilbert space, D a dense subspace of H and $E = \{e_0, e_1, \cdots, e_m\}$ a linearly independent subset of H. Then, there exists $\{d_0, d_1, \cdots, d_m\}$ in D such that for any $i, j \in \{0, 1, \cdots, m\}$, $(e_i, d_j)_H = \delta_{ij}$.*

The proof of the Theorem 2.1 requires the following lemma :

Lemma 2.2. *[3]*
Let X be a vector space and $\varphi_0, \varphi_1, ..., \varphi_m$ be linear forms on X not all null such that:

$$\bigcap_{i=1}^{m} \ker \varphi_i \subset \ker \varphi_0; \tag{2.1}$$

then, there exist real numbers $\lambda_1, \lambda_2, ..., \lambda_m$ not all null such that $\varphi_0 = \sum_{i=1}^{m} \lambda_i \varphi_i$.

Proof of Theorem 2.1: The proof will be done in two steps.
Step 1. We will show that for any $i \in \{0, 1, ..., m\}$, there exits $d_i^* \in D$ such that $\left\langle e_i, d_i^* \right\rangle_H = 1$. We proceed by contradiction. Suppose that there is $i_0 \in \{0, 1, ..., m\}$, such that

$$\langle e_{i_0}, d \rangle_H = 0, \ \forall \ d \in D.$$

Since D is dense in H, there exists a sequence $(d_i^n)_{n\in\mathbb{N}}$ in D, such that $\lim\limits_{n\to+\infty} d_i^n = e_{i_0}$.
So $\left\langle e_{i_0}, d_i^n \right\rangle_H = 0, \forall n \in \mathbb{N}$ and when $n \longrightarrow +\infty$, we have $\left\| e_{i_0} \right\| = 0$, which contradicts the fact that the family $\{e_0, e_1, \cdots, e_m\}$ is linearly independent. Therefore there exists $d_i^{n_0} \in D$ such that $\left\langle e_{i_0}, d_i^{n_0} \right\rangle_H = \alpha \neq 0$. Setting $d_i^* = \dfrac{d_i^{n_0}}{\alpha}$ one has $\left\langle e_{i_0}, d_i^* \right\rangle_H = 1$.

Step 2. We proceed by induction on m. For $m = 0$, there is $d_0 \in D$ such that $\langle e_0, d_0 \rangle = 1$, (see step 1). Let us check that, there is a subset $\{d_0, d_1\}$ of D such that

$$\forall i, j \in \{0, 1\}, \ \left\langle e_i, d_j \right\rangle_H = \delta_{ij}.$$

Suppose that

$$\forall \ d \in D, \ \langle e_1, d \rangle = 0 \Longrightarrow \langle e_0, d \rangle = 0.$$

Let x in H such that $\langle e_1, x \rangle = 0$. Since D is dense in H, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in D such that $x = \lim\limits_{n\to+\infty} x_n$. Choosing $d_1^* \in D$ such that $\left\langle e_1, d_1^* \right\rangle = 1$ and setting

$$d^n = x_n - \langle x_n, e_1 \rangle_H \, d_1^*,$$

one has $d^n \in D$ and

$$\begin{aligned} \langle e_1, d^n \rangle_H &= \langle e_1, x_n \rangle_H - \langle x_n, e_1 \rangle_H \left\langle e_1, d_1^* \right\rangle_H \\ &= \langle e_1, x_n \rangle_H - \langle x_n, e_1 \rangle_H = 0. \end{aligned}$$

So that, for all $n \in \mathbb{N}$, $\langle e_0, d^n \rangle = 0$.
Since $\lim\limits_{n\to+\infty} d^n = \lim\limits_{n\to+\infty} (x_n - \langle x_n, e_1 \rangle_H \, d_1^*) = x$, we obtain

$$\langle e_0, x \rangle_H = \lim_{n\to+\infty} \langle e_0, d^n \rangle_H = 0.$$

From Lemma2.2, there is $\lambda \in \mathbb{R}$ such that $e_0 = \lambda e_1$, which is impossible. Consequently, there is $d_0 \in D$, such that $\langle e_1, d_0 \rangle_H = 0$ and $\langle e_0, d_0 \rangle_H = 1$.
Suppose now that for $d \in D$, $\langle e_0, d \rangle_H = 0 \Longrightarrow \langle e_1, d \rangle_H = 0$. Arguing in the same way, one gets that, this is impossible. Then there exists $d_1 \in D$ such that $\langle e_0, d_1 \rangle_H = 0$ and $\langle e_1, d_1 \rangle_H = 1$. So, we construct inductively the set $F = \{d_0, d_1, ..., d_m\}$ in H such that E and F are bi-orthogonal.

3 Bi-orthogonality property of eigenfunctions

Consider the eigenvalue problem:

$$\begin{cases} -\Delta v = \lambda v, \\ v \in H_0^1(\Omega). \end{cases} \tag{3.1}$$

It is well known that the eigenvalues problem (3.1) admits two kinds of eigenfunctions: the regular eigenfunctions which are in $H^2(\Omega) \cap H_0^1(\Omega)$ and the singular eigenfunctions which are in $H_0^1(\Omega)$ but not in $H^2(\Omega)$.
Moreover, one has $L^2(\Omega) = V_1 \oplus V_2$ where V_1 is the space generated by the singular eigenfunctions and V_2 is the space generated by the other eigenfunctions. For more details see [8, 10].
Let $(w_i)_{i\geq 1}$ be a complete orthonormal family of singular eigenfunctions of $-\Delta$ and $(\alpha_i)_{i\geq 1}$ the corresponding eigenvalues in increasing order. Let now $(\lambda_i)_{i\geq 1}$ be the sequence of the unrepeated eigenvalues in increasing order. We denote by $(w_{ik})_{1\leq k\leq r(i)}$ the family of the eigenfunctions corresponding to λ_i, $i \geq 1$.
For $m \in \mathbb{N}^*$, we set

$$F_m = \text{Span}\{w_{ik}, 1 \leq k \leq r(i), 1 \leq i \leq m\}$$

Remark 3.1. The set $\{w_{ik}, 1 \leq k \leq r(i), 1 \leq i \leq m\}$ is linearly independent.

Proposition 3.2. *Let Ω be a nonempty domain of $\mathbb{R}^n$, ϖ a nonempty open subset of Ω. Assume that $\{w_1, ..., w_m\}$ is a set of some linearly independent singular eigenfunctions of $-\triangle$ such that to each eigenfunction w_i corresponds one eigenvalue λ_i such that:*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m.$$

Then, there exist C^∞ functions $(g_i)_{1\leq i\leq m}$ with compact support in ϖ such that:

$$\forall i, j \in \{1, ..., m\}, \int_\Omega w_i g_j dx = \delta_{ij}.$$

Proof. Let $H = L^2(\varpi)$. Let us prove that $w_1|_\varpi, ..., w_m|_\varpi$ are linearly independent. Assume that there exist real numbers $\alpha_1, ..., \alpha_m$ such that $\sum_{i=1}^m \alpha_i w_i = 0$ in ϖ.
Let $W = \sum_{i=1}^m \alpha_i w_i$. We have $-\Delta W = \sum_{i=1}^m \alpha_i \lambda_i w_i = 0$ on ϖ. Applying p times the Laplacian operator, it follows that $(-\Delta)^p W = \sum_{i=1}^m \alpha_i \lambda_i^p w_i = 0$ on $\varpi, \forall p \in \mathbb{N}^*$. Since w_m is not identically zero on ϖ , there exists $x_0 \in \varpi$ such that $w_m(x_0) \neq 0$. Hence from

$$\sum_{i=1}^m \alpha_i \lambda_i^p w_i = 0 \text{ on } \varpi, \forall p \in \mathbb{N}^*,$$

we have

$$\alpha_m w_m(x_0) + \sum_{i=1}^{m-1} \alpha_i (\frac{\lambda_i}{\lambda_m})^p w_i(x_0) = 0.$$

For $p \to +\infty$, one gets $\alpha_m = 0$. By iterating the same process, we obtain:

$$\alpha_i = 0, \ \forall i = 1, ..., m.$$

This proves that $w_1|_{\varpi}, ..., w_m|_{\varpi}$ are linearly independent.
Since $\mathcal{D}(\varpi)$ is dense in H, then by Theorem 2.1, there exist $g_1, ..., g_m \in \mathcal{D}(\Omega)$ with compact support in ϖ such that $\forall i, j \in \{1, ..., m\}$, $\int_\Omega w_i g_j dx = \delta_{ij}$. □

Proposition 3.3. *Let Ω be a nonempty domain of $\mathbb{R}^n$, ϖ a nonempty open subset of Ω. Assume that $\{(w_{ik})_{1\le k\le r(i)}\}$ is the sequence of singular eigenfunctions of $-\triangle$ corresponding to the eigenvalue λ_i. Then, the family $(w_{ik}|_{\varpi})_{1\le k\le r(i)}$ is linearly independent.*

Proof. Assume that there exist real numbers $\beta_1, \cdots, \beta_{r(i)}$ such that $\sum_{k=1}^{r(i)} \beta_k w_{ik}|_{\varpi} = 0$.

Setting $W = \sum_{k=1}^{r(i)} \beta_k w_{ik}|_{\varpi}$. Then W is a solution of

$$\begin{cases} -\Delta W = \lambda_i W & \text{in } \Omega, \\ W = 0 & \text{in } \Gamma. \end{cases}$$

As $W = 0$ on ϖ, the unicity theorem of Holmgren revised by Hormander [6] implies that $\sum_{k=1}^{r(i)} \beta_k w_{ik} = 0$ on Ω. Therefore $\beta_k = 0 \ \ \forall k = 1 \cdots r(i)$ and $(w_{ik}|_{\varpi})_{1\le k\le r(i)}$ is linearly independent. □

Now, we consider the general case

Theorem 3.4. *Let Ω be a nonempty domain of $\mathbb{R}^n$, ϖ a nonempty open subset of Ω. Assume that $\{w_1, ..., w_m\}$ is a set of linearly independent singular eigenfunctions of $-\triangle$.
Then, there exist C^∞ functions $(g_i)_{1\le i\le m}$ with compact support in ϖ such that:*

$$\forall i, j \in \{1, ..., m\}, \int_\Omega w_i g_j dx = \delta_{ij}.$$

Proof. Let $H = L^2(\varpi)$ and $\mu_1, ..., \mu_k$ with $k \le m$ be the unrepeated eigenvalues in increasing order of the the family $(\lambda_i)_{i\ge1}$. Let $(u_{il})_{1\le i\le m_l}$ be the family of the eigenfunctions of the set of $\{w_1, ..., w_m\}$ corresponding to μ_l.
Let us prove now that $(u_{il}|_{\varpi})_{\substack{1\le i\le m_l \\ 1\le l\le k}}$ is a family of linearly independent functions.

Assume that there exist real numbers $(\alpha_{il})_{\substack{1\le i\le m_l \\ 1\le l\le k}}$ such that: $\sum_{l=1}^{k}\sum_{i=1}^{m_l} \alpha_{il} u_{il} = 0$ in ϖ and let $W = \sum_{l=1}^{k}\sum_{i=1}^{m_l} \alpha_{il} u_{il}$. We have $-\Delta W = \sum_{l=1}^{k}\sum_{i=1}^{m_l} \mu_l \alpha_{il} u_{il} = 0$ on ϖ.

By reiterating the Laplacian p times, it follows that $\sum_{l=1}^{k}\sum_{i=1}^{m_l} \mu_l^p \alpha_{il} u_{il} = 0$ on $\varpi, \forall p \in \mathbb{N}^*$ Thanks to proposition 3.3, the family $(u_{ik}|_{\varpi})_{1\leq i\leq m_k}$ is linearly independent and there exist $x_1,\cdots,x_{m_k}\in\varpi$ such that the determinant:

$$\Delta_k=\begin{vmatrix} u_{1k}(x_1) & \cdots & u_{m_k k}(x_1)\\ \vdots & \ddots & \vdots \\ u_{1k}(x_{m_k}) & \cdots & u_{m_k k}(x_{m_k})\end{vmatrix}\neq 0.$$

We have for $j=1,...,m_k$,

$$\sum_{i=1}^{m_k}\alpha_{ik}u_{ik}\left(x_j\right)+\sum_{l=1}^{k-1}\sum_{i=1}^{m_l}\alpha_{il}\left(\frac{\mu_l}{\mu_k}\right)^p u_{il}\left(x_j\right)=0.$$

Letting $p\to+\infty$, it follows that $\sum_{i=1}^{m_k}\alpha_{ik}u_{ik}\left(x_j\right)=0\ \forall\ j.$

As $\Delta_k\neq 0$, one deduces that $\alpha_{ik}=0$ for $i=1,...m_k$.
Repeating this process, we get finally that

$$\alpha_{il}=0, i=1,...m_l; l=1,...,k.$$

This shows that $(u_{il})_{\substack{1\leq l\leq k\\ 1\leq i\leq m_l}}$ is lineary independent.

Since $\mathcal{D}(\varpi)$ is dense in H, then by Theorem 2.1, there exist $g_1,...,g_m\in\mathcal{D}(\Omega)$ with compact support in ϖ such that $\forall i,j\in\{1,...,m\}$, $\int_\Omega w_i g_j dx=\delta_{ij}$. □

Remark 3.5. The theorem 3.4 is valid even if Ω is a regular domain.

4 Cancellation of the singularities

Theorem 4.1. *Assume that ϖ is a nonempty open subset of Ω. Let $m\in\mathbb{N}^*$, $f\in L^2(Q)$, $v_0\in H_0^1(\Omega)$ and $0<t_0<T$. Then, there exist $(g_i)_{1\leq i\leq m}$, a family of C^∞ functions with compact support in ϖ, and m functions $(\theta_i)_{1\leq i\leq m}$, such that, the solution v of the problem*

$$\begin{cases}\dfrac{\partial v}{\partial t}-\Delta v=f+\displaystyle\sum_{i=1}^{m}\theta_i(t)g_i(x) & \text{in } Q,\\ v(0,x)=v_0(x) & \text{in } \Omega,\\ v=0, & \text{on } \Sigma.\end{cases}\tag{4.1}$$

belongs to $L^2(t_0,T;(H^2(\Omega)\cap H_0^1(\Omega))\cap F_m^\perp))$, where $F_m^\perp$ is the orthogonal of F_m.

Proof. For $m\in\mathbb{N}^*$ thanks to theorem 3.4, there exist m functions $(g_i)_{1\leq i\leq m}$, C^∞ with compact support in ϖ such that

$$\forall i,j\in\{1,...,m\},\int_\Omega w_i g_j dx=\delta_{ij}.$$

Using the Fourier decomposition (cf.[3]), we can write

$v_0(x)=\sum_{k=1}^{\infty}\beta_k w_k(x)+\sum_{i=1}^{\infty}\gamma_i\varphi_i(x),$

$f(t,x)=\sum_{k=1}^{\infty}f_k(t)w_k(x)+\sum_{i=1}^{\infty}\overline{f_i}(t)\varphi_i(x),$

$v(t,x) = \sum_{k=1}^{\infty} v_k(t)w_k(x) + \sum_{i=1}^{\infty} \overline{v_i}(t)\varphi_i(x).$

Then, the first equation of (4.1) becomes:

$$\begin{aligned}\sum_{k=1}^{+\infty}\Big(v_k'(t)+\lambda_k v_k(t)\Big)w_k(x)+\sum_{i=1}^{+\infty}\Big(\overline{v}_i'(t)+\lambda_k \overline{v}_i(t)\Big)\varphi_i(x) &= \sum_{k=1}^{+\infty} f_k(t)w_k(x)+\sum_{i=1}^{+\infty}\overline{f}_i(t)\varphi_i(x)\\ &+ \sum_{i=1}^{m}\theta_i(t)g_i(x)\end{aligned}$$

Multiplying (4.1) by $w_k(x)$ and integrating on Ω, we obtain that, for $k = 1,\cdots,m$, the function v_k is solution of the system

$$\begin{cases} v_k'(t)+\lambda_k v_k(t) = f_k(t)+\theta_k(t)\\ v_k(0) = \beta_k. \end{cases} \tag{4.2}$$

This gives that

$$\begin{aligned} v_k(t) &= \beta_k e^{-\lambda_k t} + \int_0^t e^{-\lambda_k(t-s)}(f_k+\theta_k)(s)ds\\ &= e^{-\lambda_k t}[\beta_k + \int_0^t e^{\lambda_k(s)}(f_k+\theta_k)(s)ds]. \end{aligned}$$

Taking

$$\theta_k(s) = \begin{cases} -f_k(s) - \dfrac{1}{t_0}\beta_k e^{-\lambda_k s} \text{ if } 0 \le s \le t_0\\ \\ -f_k(s) \text{ if } s > t_0 \end{cases},$$

one has for $t > t_0$, $v_k(t) = e^{-\lambda_k t}(\beta_k - \dfrac{\beta_k}{t_0}\displaystyle\int_0^{t_0} ds) = 0.$

Then $v_k(t) = 0\ \ \forall\, k \in \{1,\cdots,m\}, \forall\ \ t > t_0.$

Hence, $v(t,x) = \displaystyle\sum_{k=m+1}^{\infty} v_k(t)w_k(x) + \sum_{i=0}^{\infty}\overline{v_i}(t)\varphi_i(x) \in L^2(t_0,T;(H^2(\Omega)\cap H_0^1(\Omega))\cap F_m^{\perp}).$ □

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