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Asymptotic Behavior of Mild Solutions of Some Fractional Functional Integro-differential Equations

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Abstract. In this paper, we prove a new composition theorem for asymptotically antiperiodic and weighted pseudo antiperiodic functions. We also give some sufficient conditions to ensure invertibility of convolution operators in the space of antiperiodic functions. Then we prove the existence and uniqueness of asymptotically antiperiodic mild solutions to some fractional functional integro-differential equations in a Banach space using the Banach's fixed point theorem.

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1 Introduction

We are concerned in this paper with the existence of asymptotically antiperiodic mild solution of the following semilinear fractional integro-differential equation in a Banach space X which has been

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thoroughly studied in [13],

$$\begin{cases} u'(t) = \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_0^t (t-s)^{\alpha-2} Au(s)ds + F(t,u_t),\ t \geq 0, \\ \\ u_0 = \phi, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, $\phi \in \mathcal{B}$, an abstract space to be specified later, $A : D(A) \subseteq X \to X$ is a closed (not necessarily bounded) linear operator of sectorial type $\varpi < 0$, and $F : \mathbb{R}^+ \times \mathcal{B} \to X$ is a (jointly) continuous function. For any function $u : \mathbb{R} \to X$, the associated history function $t \to u_t$ for $t \geq 0$ is defined as $u_t : (-\infty, 0] \to X$ where $u_t(\theta) = u(t+\theta)$.

Eq.(1.1) has been thoroughly studied by E. Cuesta. He proved that this equation is well-posed when A is a sectorial operator in an appropriate sector of the complex plane (cf. [13]). Other studies are done in [19] for instance.

In addition Eq.(1.1) and the following one

$$D_t^\alpha v(t) = Av(t) + D_t^{\alpha-1} f(t, v_t),\ \ t \geq 0$$

where D_t^α $(1 < \alpha < 2)$ denotes the Riemann-Liouville derivative are *limiting equations* in the sense that their solutions are asymptotic as $t \to \infty$. Examples of such equations include fractional relaxation-oscillation equations (cf.[14]).

The study of the existence of antiperiodic solutions is one of the most attracting topics in the qualitative theory of differential equations due to its applications in biology, physics, engineering, and other sciences (see for instance [4, 5, 10, 11, 18, 33] and references therein).

Recently, Diagana et al [6] introduced the concept of weighted antiperiodic functions. Then they studied the existence and uniqueness of mild solutions to the nonautonomous differential equation

$$u'(t) = A(t)u(t) + g(t, u(t)),\ \ t \in \mathbb{R},$$

where $A(t)$ is a family of closed linear operators satisfying the so-called Acquistapace-Terreni conditions and such that $A(t)$ is periodic.

In [9], the authors studied the existence of asymptotically almost automorphic mild solutions to Eq.(1.1).

Following [31], N'Guérékata and Valmorin introduced the concept of asymptotically antiperiodic functions (these are functions which approach antiperiodic ones at infinity), and studied their properties in [32]. They also studied the existence of asymptotically antiperiodic mild solution of the following semilinear integro-differential equation in a Banach space X

$$u'(t) = Au(t) + \int_{-\infty}^t a(t-s)Au(s)ds + f(t, Cu(t)), \tag{1.2}$$

where $C : X \to X$ is a bounded linear operator, A is a closed linear (not necessarily bounded) operator defined in a Banach space X, and $a \in L^1_{loc}(\mathbb{R}^+)$ is a scalar-valued kernel.

Motivated by the above papers, we will establish a composition theorem for both asymptotically antiperiodic functions and weighted pseudo antiperiodic functions. Then we use the result obtained to prove the existence and uniqueness of asymptotically antiperiodic mild solutions to Eq.(1.1).

2 Preliminaries

In this paper, $(X, \|\cdot\|)$ will denote a Banach space, $\mathcal{B}(X)$, the space of all bounded linear operators $X \to X$, $BC(\mathbb{R}, X)$, the space of all bounded and continuous functions $\mathbb{R} \to X$, $C_0(\mathbb{R}, X)$, the space of all continuous functions $h : \mathbb{R} \to X$ such that $\lim_{|t|\to\infty} h(t) = 0$.

Definition 2.1. A closed linear operator A with domain $D(A)$ dense in a Banach space X is said to be sectorial of type ϖ and angle θ if there exist constants ϖ, and angle $\theta \in]0, \frac{\pi}{2}[$, $M > 0$ such that its resolvent exists outside the sector

$$\varpi + \Sigma_\theta := \{\lambda + \varpi : \lambda \in \mathbb{C}, |arg(-\lambda)| < \theta\}, \tag{2.1}$$

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \varpi|}, \quad \lambda \notin \varpi + \Sigma_\theta. \tag{2.2}$$

See Lunardi [20] for more details.

Definition 2.2. Let $\alpha > 0$ and A be a closed linear operator densely defined in X. Let $\rho(A)$ be the resolvent set of A. A will be called the generator of a solution operator if there exists $\varpi \in \mathbb{R}$ and a strongly continuous function $E_\alpha : \mathbb{R}^+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : Re\lambda > \varpi\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} E_\alpha(t) x dt, \; Re\lambda > \varpi, \; x \in X$$

In this case E_α is called solution operator generated by A.

Let's assume that A is sectorial with $0 \leq \theta \leq \pi(1 - \alpha/2)$, then A is the generator of a solution operator given by

$$E_\alpha(t) = \int_\Gamma e^{\lambda t} \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} d\lambda, \; t \geq 0$$

with Γ a suitable path lying outside the sector $\varpi + \Sigma_\theta$.

Lemma 2.3. *([13]) Let $1 < \alpha < 2$. Let $A : D(A) \subset X \to X$ be a sectorial operator in a complex Banach space X, satisfying hypothesis (2.1)-(2.2), for some $M > 0$, $\varpi < 0$ and $0 \leq \theta < \pi(1 - \alpha/2)$. Then there exists $C(\theta, \alpha) > 0$ depending solely θ and α, such that*

$$\|E_\alpha(t)\|_{\mathcal{L}(\mathbb{X})} \leq \frac{C(\theta, \alpha)M}{1 + |\varpi| t^\alpha}, \; t \geq 0. \tag{2.3}$$

Note that E_α is integrable. In the border cases $\alpha = 1$ and $\alpha = 2$ the family $E_\alpha(t)$ corresponds respectively to a C_0-semigroup and a cosine family.

Let's now describe the phase space. In what follows, $(\mathcal{B}, \|\cdot\|_\mathcal{B})$ will be a seminormed linear space of functions $]-\infty, 0] \to X$ satisfying the following fundamental axioms due to Kato and Hale:

- (P_0) If $x :]-\infty, T]$ is continuous on $I := [0, T]$ and $x_0 \in \mathcal{B}$, then for every $t \in I$, the following conditions hold:
  (i) $x_t \in \mathcal{B}$
  (ii) $\|x(t)\| \leq H \|x_t\|_\mathcal{B}$
  (iii) $\|x_t\|_\mathcal{B} \leq C_1(t) \sup_{0 \leq s \leq t} \|x(s)\| + C_2(t) \|x_0\|_\mathcal{B}$
  where $H \geq 0$ is a constant, $C_1 : [0, \infty[\to [0, \infty[$ is continuous, $C_2 : [0, \infty[\to [0, \infty[$ is locally bounded and H, C_1, C_2 are independent of $x(\cdot)$.

- (P_1) For the function $x(\cdot)$ in (P_0), x_t is a $\mathcal{B}$-valued continuous function on I.

- (P_1) The space $\mathcal{B}$ is complete.

Remark 2.4. Condition (ii) in (P_0) is equivalent to $\|\phi(0)\| \leq H \|\phi\|_\mathcal{B}$, for all $\phi \in \mathcal{B}$.

Definition 2.5. $\mathcal{B}$ will be called a fading memory if the following holds:

If $x : \mathbb{R} \to X$ is a continuous function on $[\sigma, +\infty)$ with $x_\sigma \in \mathcal{B}$ for some $\sigma \in \mathbb{R}$ such that $\|x(t)\| \to 0$ as $t \to +\infty$, then $\|x_t\| \to 0$ as $t \to +\infty$.

We recall some examples of phase spaces.

Example 2.6. E1. $BUC(]-\infty,0]), X)$ the Banach space of all bounded and uniformly continuous functions $\phi :]-\infty,0] \to X$ endowed with the supnorm.

E2. $C_0(]-\infty,0], X)$ the Banach space of all bounded and continuous functions $\phi :]-\infty,0] \to X$ such that $\lim_{\theta\to-\infty} \phi(\theta) = 0$ endowed with the norm

$$|\phi| := \sup_{\theta\leq 0} |\phi(\theta)|.$$

E3. $C_\gamma := \{\phi \in C(]-\infty,0], X) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } X\}$
endowed with the norm

$$|\phi| = \sup_{-\infty<\theta\leq 0} e^{\gamma\theta}|\phi(\theta)|.$$

E4. $C^\infty := \{\phi \in BC(]-\infty,0], X) : \lim_{\theta\to-\infty} \text{ exists in } X\}$
endowed with the norm $\|\phi\| = \sup_{\theta\leq 0} \|\phi(\theta)\|$

E5. $C^0 := \{\phi \in BC(]-\infty,0], X) : \lim_{\theta\to-\infty} = 0\}$ endowed with the norm $\|\phi\| = \sup_{\theta\leq 0} \|\phi(\theta)\|$

Note that among these examples, only C_γ is a fading memory.

3 Asymptotically antiperiodic and weighted pseudo antiperiodic functions

Definition 3.1. A function $f \in BC(\mathbb{R}, X)$ is said to be ω-antiperiodic (or simply antiperiodic) if there exists $\omega > 0$ such that $f(t+\omega) = -f(t)$ for all $t \in \mathbb{R}$. The least such ω will be called the antiperiod of f.

We will denote by $P_{\omega ap}(X)$, the space of all ω-antiperiodic functions $\mathbb{R} \to X$.

Theorem 3.2. *([32]) Let $f, f_1, f_2 \in P_{\omega ap}(X)$. Then the following are in $P_{\omega ap}(X)$, too.*

- *$f_1 + f_2, cf$, c is an arbitrary real number.*
- *$g(t) := (\frac{1}{f})(t)$, provided $f \neq 0$ on $\mathbb{R}$. Here $X = \mathbb{R}$*
- *$f_a(t) := f(t+a)$ a is an arbitrary real number*

Remark 3.3. It is clear that every ω-antiperiodic function is 2ω-periodic.

Remark 3.4. If $A \in \mathcal{B}(X)$, the space of all bounded linear operators $X \to X$ and f is an ω-antiperiodic X-valued function, then Af is also ω-antiperiodic.

A classical example of such function is

$$f(t) = \sum_{n=1}^{\infty} \frac{cos[(2n+1)t]}{n^2}, \quad t \in \mathbb{R}$$

which is π-anti periodic. See also [10, 32] for more examples.

Remark 3.5. Let $f : \mathbb{R} \to X$ and $h : \mathbb{R} \to \mathbb{R}$; the convolution function (if it does exist) of f with h is denoted by $f \star h$ and given by

$$(f \star h)(t) := \int_{\mathbb{R}} f(\sigma)h(t-\sigma)d\sigma = \int_{\mathbb{R}} f(t-\sigma)h(\sigma)d\sigma, \ \ \forall t \in \mathbb{R}.$$

Let $\varphi \in L^1$ and $\lambda \in \mathbb{C}$. Consider the operator $A_{\lambda,\varphi}$ defined by

$$A_{\lambda,\varphi}u := \lambda u + \varphi \star u$$

Then it is clear that $A_{\lambda,\varphi}(P_{\omega ap}(X)) \subset P_{\omega ap}(X)$. Moreover $A_{\lambda,\varphi}$ acts continuously in $P_{\omega ap}(X)$, that is there exists a constant $C > 0$ such that

$$||A_{\lambda,\varphi}u|| \le C||u||, \forall u \in P_{\omega ap}(X).$$

Let's now present a result on the invertibility of the convolution operators in $P_{\omega ap}(X)$. Consider

$$a(\xi) := \lambda + \hat{\varphi}(\xi)$$

where $\hat{\varphi}(\xi)$ is the Fourier transform of the function φ. $a(\xi)$ is the *symbol* of the operator $A_{\lambda,\varphi}$, where $\varphi \in L^1(\mathbb{R})$. And since $\lim_{\xi\to\infty} \varphi(\xi) = 0$, the symbol $a(\xi)$ is a well defined continuous function on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and $a(\infty) = \lambda$.

Now we state and prove

Theorem 3.6. *Suppose $\varphi \in L^1(\mathbb{R})$. Then the operator $A_{\lambda,\varphi}$ is invertible in $P_{\omega ap}(X)$ if $a(\xi) \neq 0$ for all $\xi \in \overline{\mathbb{R}}$.*

Proof. Suppose $a(\xi) \neq o$ for all $\xi \in \overline{\mathbb{R}}$. Then the function $\frac{1}{a(\xi)}$ is well-defined on $\overline{\mathbb{R}}$ and in view of the classical Wiener's theorem, we get

$$\frac{1}{a(\xi)} = \frac{1}{\lambda} + \hat{\psi}(\xi),$$

where $\psi \in L^1(\mathbb{R})$. It is easy to check the $A_{\psi,\frac{1}{\lambda}}$ is the inverse to the operator $A_{\lambda,\varphi}$ which acts in $P_{\omega ap}(X)$ in view of the above remark.

Theorem 3.7. *([32]) Let $f_n \in P_{\omega ap}(X)$, such that $f_n \to f$ uniformly on $\mathbb{R}$. Then $f \in P_{\omega ap}(X)$.*

Theorem 3.8. *([32]) $P_{\omega ap}(X)$ is a Banach space equipped with the supnorm.*

We will introduce the following definition which is slightly different from the one in [32].

Definition 3.9. A function $f \in BC(\mathbb{R}, X)$ is said to be asymptotically antiperiodic if there exist $g \in P_{\omega ap}(X)$ and $h \in C_0(\mathbb{R}, X)$, such that

$$f = g + h, \ \ \forall t \in \mathbb{R}.$$

g and h are called respectively the principal and corrective terms of f.

We will denote by $AP_{\omega ap}(X)$ the space of all asymptotically antiperiodic functions $\mathbb{R} \to X$. It is clear that it is a Banach space under the supnorm.

As in [6] we denote by $\mathcal{U}$ the collection of locally integrable functions (weights) $\mu : \mathbb{R} \to (0,\infty)$ such that $\mu > 0$ almost everywhere. For $\mu \in \mathcal{U}$, we let

$$\mu_T := \int_{Q_T} \mu(x)dx$$

where $Q_T := [-T,T]$, for $T > 0$.

Let

$$\mathcal{U}_\infty := \{\mu \in \mathcal{U} : \inf_{x\in\mathbb{R}} \mu(x) > 0, \ \lim_{T\to\infty} \mu_T = \infty\}.$$

Define $\mathcal{U}_\infty^{inv}$ as the set

$$\{\mu \in \mathcal{U}_\infty : \lim_{x\to\infty} \frac{\mu(x+\tau)}{\mu(x)} = \gamma(\tau), \ \lim_{T\to\infty} \frac{\mu_{T+|\tau|}}{\mu_T} = L(\tau), \ \forall\tau \in \mathbb{R}\}$$

where the functions γ and L satisfy the following hypothesis

$$\int_0^\infty \frac{\gamma(\xi)L(\xi)}{1+|\varpi|\xi^\alpha} d\xi < \infty, \tag{3.1}$$

and for $\mu \in \mathcal{U}_\infty$ let

$$PAP_0(X,\mu) := \{f \in BC(\mathbb{R},X) : \lim_{T\to\infty} \frac{1}{\mu_T} \int_{Q_T} \|f(s)\|\mu(s)ds = 0\}.$$

Definition 3.10. [6] Let $\mu \in \mathcal{U}_\infty^{inv}$. A function $f \in BC(\mathbb{R},X)$ is called weighted pseudo ω-antiperiodic if for some $\omega > 0$, f can be expressed as $f = g + h$ where $g \in P_{\omega ap}(X)$ and $h \in PAP_0(X,\mu)$.

We will denote $PP_{\omega ap}(X)$ the space of all such functions.

Remark 3.11. The decomposition of every function in $PP_{\omega ap}(X)$ is unique ([6]).

Now we present the main result of this section.

Theorem 3.12. *Let $F : \mathbb{R}\times\mathcal{B} \longrightarrow X$ be a continuous function such that:*

(i) $\forall(t,x) \in \mathbb{R}\times\mathcal{B},\ F(t+\omega,-x) = -F(t,x)$ *for some* $\omega > 0$*;*
(ii) $\exists K > 0, \forall(t,x,y) \in \mathbb{R}\times\mathcal{B}\times\mathcal{B},\ \|F(t,x) - F(t,y)\| \le K\|x-y\|$.

If $\mathcal{N}$ denotes the Nemytskii's superposition operator defined by

$$\mathcal{N}(\varphi)(\cdot) := F(\cdot,\varphi(\cdot)).$$

Then,

$$\mathcal{N}(AP_{\omega ap}(\mathcal{B})) \subset AP_{\omega ap}(X).$$

Proof. Let $\varphi \in AP_{\omega ap}(\mathcal{B})$. Then $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in P_{\omega ap}(\mathcal{B})$ and $\varphi_2 \in C_0(\mathbb{R},\mathcal{B})$. We have $\mathcal{N}(\varphi)(\cdot) = \mathcal{N}(\varphi_1)(\cdot) + f$ where $f = \mathcal{N}(\varphi)(\cdot) - \mathcal{N}(\varphi_1)(\cdot)$ is a continuous function which satisfies

$$\|f(t)\| \le K\|\varphi_2(t)\| \tag{3.2}$$

from **(ii)**.

Moreover $\mathcal{N}(\varphi_1)(\cdot)$ is in $P_{\omega ap}(X)$ from [32], Theorem 2.16.

Since $\lim_{|t|\to\infty} \varphi_2(t) = 0$, then from (3.2) the same holds for f showing that $f \in C_0(\mathbb{R},X)$, hence $\mathcal{N}(\varphi) \in AP_{\omega ap}(X)$.

Remark 3.13. If we define

$$F(t,x) := f(t)x$$

where $f(t) = \sum_{n=1}^{\infty} \frac{cos[(2n+1)t]}{n^2}$, $t \in \mathbb{R}$, $x \in \mathcal{B}$, then it is clear that condition (i) in the theorem is satisfied by F. Also condition (ii) is satisfied with

$$K = 2\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Theorem 3.14. *Let $F : \mathbb{R} \times X \longrightarrow X$ be a continuous function such that:*

(i) $\forall (t,x) \in \mathbb{R} \times X,\ F(t+\omega, -x) = -F(t,x)$;
(ii) $\exists K > 0, \forall (t,x,y) \in \mathbb{R} \times X \times X,\ \|F(t,x) - F(t,y)\| \le K\|x-y\|$.

If $\mathcal{N}$ denotes the Nemytskii's superposition operator defined by

$$\mathcal{N}(\varphi)(\cdot) := F(\cdot, \varphi(\cdot)).$$

Then,

$$\mathcal{N}(PP_{\omega ap}(X)) \subset PP_{\omega ap}(X).$$

Proof. Let $\varphi \in PP_{\omega ap}(X)$. Then $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in P_{\omega ap}(X)$ and $\varphi_2 \in PAP_0(\mathbb{R}, X)$.
As in the proof of Theorem 3.12, we have

$$\|f(t)\| \le K\|\varphi_2(t)\| \tag{3.3}$$

and $\mathcal{N}(\varphi_1)(\cdot)$ is in $P_{\omega ap}(X)$. It is also clear that $f \in BC(\mathbb{R}, X)$.

Now we have $\lim_{T\to\infty} \frac{1}{\mu(Q_T)} \int_{Q_T} \|\varphi_2(s)\|\mu(s)ds = 0$. It follows from (3.2) that

$$\frac{1}{\mu(Q_T)} \int_{Q_T} \|f(s)\|\mu(s)ds \le \frac{K}{\mu(Q_T)} \int_{Q_T} \|\varphi_2(s)\|\mu(s)ds,$$

which implies that

$$\lim_{T\to\infty} \frac{1}{\mu(Q_T)} \int_{Q_T} \|f(s)\|\mu(s)ds = 0.$$

Hence we have $f \in PAP_0(\mathbb{R}, X)$ and consequently $\mathcal{N}(\varphi) \in PP_{\omega ap}(X)$. The theorem is thus proved.

4 Semilinear fractional integro-differential equations

Now we consider the integro-differential equation with infinite delay

$$\begin{cases} u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} Au(s)ds + F(t, u_t),\ t \ge 0, \\ u_0 = \phi, \end{cases} \tag{4.1}$$

where $1 < \alpha < 2$, $\phi \in \mathcal{B}$, a phase space not necessarily a fading memory (cf. Section 2), $A : D(A) \subseteq X \to X$ is a sectorial operator and $F : \mathbb{R}^+ \times \mathcal{B} \to X$ is a (jointly) continuous function.

Definition 4.1. A bounded continuous function $u : \mathbb{R} \to X$ is said to be a mild solution to (4.1) if it satisfies the following.

$$u(t) = \begin{cases} \phi(t),\ \ t \in (-\infty, 0], \\ E_\alpha(t)\phi(0) + \int_0^t E_\alpha(t-s)F(s, u_s)ds,\ t \in \mathbb{R}^+. \end{cases} \tag{4.2}$$

Lemma 4.2. *Let $u \in P_{\omega ap}(X)$. Then the function*

$$v(t) := \int_{-\infty}^{t} E_\alpha(t-s)u(s)ds$$

is also in $P_{\omega ap}(X)$.

Proof. It is straightforward by an appropriate change of variable.

Lemma 4.3. *Let $u \in AP_{\omega ap}(X)$. Then the function*

$$v(t) := \int_{0}^{t} E_\alpha(t-s)u(s)ds$$

is also in $AP_{\omega ap}(X)$.

Proof. Let $u \in AP_{\omega ap}(X)$. Then $u = g+h$, where $g \in P_{\omega ap}(X)$, and $h \in C_0(X)$.
Now we can write $v(t) = G(t)+H(t)$ where

$$G(t) := \int_{-\infty}^{t} E_\alpha(t-s)g(s)ds$$

and

$$H(t) := \int_{0}^{t} E_\alpha(t-s)h(s)ds - \int_{-\infty}^{0} E_\alpha(t-s)g(s)ds.$$

$G(t) \in P_{\omega ap}(X)$ by Lemma 4.2. It remains to prove that $H(t) \in C_0(X)$.

Let $t > 0$ and $\epsilon > 0$ be given. Since $h(\cdot) \in C_0(X)$, there exists $T > 0$ such that $||h(s)|| < \epsilon$ for all $s \geq T$. So with $C := C(\theta,\alpha)$ as in (2.3) and $t > T$, we have.

$$\begin{aligned} ||H(t)|| &= ||\int_{0}^{T} E_\alpha(t-s)h(s)ds + \int_{T}^{t} E_\alpha(t-s)h(s)ds \\ &- \int_{-\infty}^{0} E_\alpha(t-s)g(s)ds|| \\ &\leq \int_{0}^{T} \frac{CM||h||_\infty}{1+|\varpi|(t-s)^\alpha}ds + \epsilon\int_{T}^{t} \frac{CM}{1+|\varpi|(t-s)^\alpha}ds \\ &+ \int_{-\infty}^{0} \frac{CM||g||_\infty}{1+|\varpi|(t-s)^\alpha}ds. \end{aligned}$$

On one hand we have

$$\begin{aligned} &\int_{0}^{T} \frac{CM||h||_\infty}{1+|\varpi|(t-s)^\alpha}ds + \int_{-\infty}^{0} \frac{CM||g||_\infty}{1+|\varpi|(t-s)^\alpha}ds \\ &\leq \frac{CM}{|\varpi|}(\int_{0}^{T} \frac{||h||_\infty}{(t-s)^\alpha}ds + \int_{-\infty}^{0} \frac{||g||_\infty}{(t-s)^\alpha}ds) \end{aligned}$$

Integrating we obtain

$$\frac{CM}{|\varpi|(\alpha-1)}[||h||_\infty((t-T)^{-\alpha+1} - t^{-\alpha+1}) + ||g||_\infty t^{-\alpha+1}].$$

Since $-\alpha+1 < 0$, then $\lim_{t\to\infty}(t-T)^{-\alpha+1} = \lim_{t\to\infty} t^{-\alpha+1} = 0$.

On the other hand, using the change of variable $u = |\varpi|(t-s)^\alpha$ we find

$$\begin{aligned}\int_T^t \frac{1}{1+|\varpi|(t-s)^\alpha}ds &= \frac{|\varpi|^{-1/\alpha}}{\alpha}\int_0^{|\varpi|(t-T)^\alpha}\frac{u^{(1/\alpha)-1}}{1+u}du\\ &\le \frac{|\varpi|^{-1/\alpha}}{\alpha}\int_0^{\infty}\frac{u^{(1/\alpha)-1}}{1+u}du\end{aligned}$$

Since $0<\frac{1}{\alpha}<1$, from $\int_0^\infty \frac{t^{\beta-1}}{t+1}dt = B(\beta,1-\beta) = \frac{\pi}{sin(\pi\beta)}$ $(0<\beta<1)$, where B denote the Euler function, we finally get

$$\int_T^t \frac{1}{1+|\varpi|(t-s)^\alpha}ds \le \frac{|\varpi|^{-1/\alpha}\pi}{\alpha sin(\frac{\pi}{\alpha})}.$$

The same argument can be made if $t<-T$. The proof is now complete.

Lemma 4.4. *Let $u \in AP_{\omega ap}(X)$. Then the function defined by*

$$w(t) := \int_{-\infty}^t E_\alpha(t-s)u(s)ds$$

is also in $AP_{\omega ap}(X)$.

Proof. Let $u \in AP_{\omega ap}(X)$. Then $u = g+h$, where $g \in P_{\omega ap}(X)$, and $h \in C_0(\mathbb{R},X)$. Thus $h \in BC(\mathbb{R},X)$.

Now we can write $w(t) = G(t)+H(t)$ where

$$G(t) := \int_{-\infty}^t E_\alpha(t-s)g(s)ds$$

and

$$H(t) := \int_{-\infty}^t E_\alpha(t-s)h(s)ds.$$

We have $G(t) \in P_{\omega ap}(X)$ by Lemma 4.2.

Using (2.3) with $C := C(\theta,\alpha)$ and the change of variable $\xi = t-s$ we get

$$||H(t)|| \le CM\int_{-\infty}^t \frac{||h(s)||}{1+|\varpi|(t-s)^\alpha}ds$$

and

$$||H(t)|| \le CM\int_0^\infty \frac{||h(t-\xi)||}{1+|\varpi|\xi^\alpha}d\xi.$$

Since for all ξ, $\lim_{|t|\to+\infty}||h(t-\xi)|| = 0$, h is bounded and $\int_0^\infty \frac{1}{1+|\varpi|\xi^\alpha}d\xi$ is finite, we have $\lim_{|t|\to+\infty}||H(t)|| = 0$ by Lebesgue's dominated convergence Theorem. Hence $H \in C_0(X)$.

This completes the proof.

Finally we study Eq.(4.1) where $\mathcal{B}$ is a phase space (not a fading memory).

Theorem 4.5. *Suppose conditions (i) and (ii) of Theorem 3.12 are satisfied. Suppose also that $C_1^* := \sup_{0\le t<\infty} C_1(t) < \infty$. Then Eq.(4.1) has a unique solution in $AP_{\omega ap}(X)$ provided $K < \frac{\alpha sin(\frac{\pi}{\alpha})}{CMC_1^*|\varpi|^{-1/\alpha}\pi}$.*

Proof. Let $u \in AP_{\omega ap}(X)$; then it is easy to see that $u_s \in AP_{\omega ap}(X)$, too. So by Theorem 3.12, $F(s,u_s) \in AP_{\omega ap}(X)$. And using Lemma 4.3, $\int_0^t E_\alpha(t-s)F(s,u_s)ds \in AP_{\omega ap}(X)$.

To prove the uniqueness, it suffices to consider the part of the solution on $t \geq 0$. To this end, let's define the operator $\Psi : AP_{\omega ap}(X) \to AP_{\omega ap}(X)$ by

$$(\Psi u)(t) := E_\alpha(t)\phi(0) + \int_0^t E_\alpha(t-s)F(s,u_s)ds,\ t \geq 0.$$

Since $E_\alpha(t)\varphi(0) \in C_0(\mathbb{R},X)$, we conclude that $(\Psi u)(t) \in AP_{\omega ap}(X)$. So the operator Ψ is well defined.

Now let $u, v \in AP_{\omega ap}(X)$ be solutions of Eq.(4.1); then $u_0 = v_0 = \varphi$ and

$$\begin{aligned}
||(\Psi u)(t) - (\Psi v)(t)|| &= ||\int_0^t E_\alpha(t-s)(F(s,u_s) - F(s,v_s))ds|| \\
&\leq K\int_0^t ||E_\alpha(t-s)||_{\mathcal{L}(X)}||u_s - v_s||_{\mathcal{B}}ds \\
&\leq K\int_0^t ||E_\alpha(t-s)||_{\mathcal{L}(X)}C_1(s) \\
x & \sup_{0\leq\sigma\leq s}||u(\sigma) - v(\sigma)||ds \\
&\leq K||u-v||_\infty \int_0^t C_1(s)||E_\alpha(t-s)||_{\mathcal{L}(X)}ds \\
&\leq KCM||u-v||_\infty \int_0^t \frac{C_1(s)}{1+|\varpi|(t-s)^\alpha}ds \\
&\leq KCMC_1^*||u-v||_\infty \int_0^\infty \frac{1}{1+|\varpi|(s)^\alpha}ds \\
&= KCMC_1^*\frac{|\varpi|^{-1/\alpha}\pi}{\alpha sin(\frac{\pi}{\alpha})}||u-v||_\infty.
\end{aligned}$$

Thus we obtain

$$||(\Psi u) - (\Psi v)||_\infty \leq KCMC_1^*\frac{|\varpi|^{-1/\alpha}\pi}{\alpha sin(\frac{\pi}{\alpha})}||u-v||_\infty.$$

We know conclude using the Banach contraction principle.

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