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Weighted Stepanov-Like Pseudo-Almost Periodic Functions in Lebesgue Space with Variable Exponents $L^{p(x)}$

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Abstract

In this paper we introduce and study a new class of functions called $S^{p,q(x)}$-pseudo-almost periodic (or weighted Stepanov-like pseudo-almost periodic functions with variable exponents), which generalizes the class of weighted Stepanov-like pseudo-almost periodic functions. Basic properties of these new spaces are established. The existence of weighted pseudo-almost periodic solutions to some first-order differential equations with $S^{p,q(x)}$-pseudo-almost periodic coefficients will also be studied.

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Keywords: weighted pseudo-almost periodicity; Lebesgue space with variable exponents; weighted Stepanov-like pseudo-almost periodicity with variable exponents.

1 Introduction

This paper is mainly motived by three sources. The first source is a paper by Diagana [6] in which Stepanov-like pseudo-almost periodic functions were introduced and studied. These functions were then utilized to study the existence of pseudo-almost periodic solutions to various classes of differential equations.

The second source, is a paper by Blot *et al.* [1] in which the concept of weighted pseudo-almost periodicity, using theoretical measure theory, was introduced and utilized to study the existence of weighted pseudo-almost periodic solutions to differential equations.

The third and last source is a recent paper by Diagana and Zitane [4] in which Stepanov-like pseudo-almost periodic functions were introduced in the Lebesgue space with variable exponents $L^{p(x)}$.

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The main objective of this paper consists of introducing and studying a new class of functions called weighted Stepanov-like pseudo-almost periodic functions with variable exponents, which generalizes the class of Stepanov-like pseudo-almost periodic functions introduced by Diagana and Zitane [4]. Basic properties of these new spaces are established. Next, we study the existence of weighted pseudo-almost periodic solutions of the following nonautonomous differential equations

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

$$u'(t) = A(t)u(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A(t) : D(A(t)) \subset \mathbb{X} \mapsto \mathbb{X}$ is a family of closed linear operators on a Banach space $\mathbb{X}$ satisfying the well-known Acquistapace-Terreni conditions, and $f : \mathbb{R} \mapsto \mathbb{X}$, $F : \mathbb{R}\times\mathbb{X} \mapsto \mathbb{X}$ are jointly continuous satisfying some additional assumptions.

2 μ-Pseudo-Almost Periodic Functions

Let $(\mathbb{X}, \|\cdot\|), (\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. Let $BC(\mathbb{R},\mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$). The space $BC(\mathbb{R},\mathbb{X})$ equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space. Furthermore, $C(\mathbb{R},\mathbb{Y})$ (respectively, $C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$). Let $B(\mathbb{X},\mathbb{Y})$ stand for the Banach space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with its natural operator topology $\|\cdot\|_{B(\mathbb{X},\mathbb{Y})}$; in particular, $B(\mathbb{X},\mathbb{X})$ is denoted by $B(\mathbb{X})$ (its corresponding norm will be denoted $\|\cdot\|_{B(\mathbb{X})}$).

In this section, we recall the concept of μ-pseudo-almost periodicity introduced by J. Blot *et al* [1].

Definition 2.1. (Bochner) A function $f \in C(\mathbb{R},\mathbb{X})$ is called almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t+\tau) - f(t)\| < \varepsilon$$

for each $t \in \mathbb{R}$.

The collection of all almost periodic functions from $\mathbb{R}$ to $\mathbb{X}$ will be denoted by $AP(\mathbb{X})$.

We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = \infty$ and $\mu([a,b]) < \infty$, for all $a,b \in \mathbb{R}\,(a \le b)$.

Definition 2.2. [1] Let $\mu \in \mathcal{M}$. A function $f \in BC(\mathbb{R},\mathbb{X})$ is said to be μ-ergodic if

$$\lim_{r\to\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \|f(t)\|\, d\mu(t) = 0$$

where $Q_r := [-r,r]$.

The collection of such functions will be denoted by $\mathcal{E}(\mathbb{X},\mu)$.

Proposition 2.3. *[1]* Let $\mu \in \mathcal{M}$. Then $(\mathcal{E}(\mathbb{X},\mu), \|\cdot\|_\infty)$ is a Banach space.

Theorem 2.4. *[1]* Let $\mu \in \mathcal{M}$ and I be a bounded interval (eventually $I = \emptyset$). Assume that $f \in BC(\mathbb{R}, \mathbb{X})$. Then the following assertions are equivalent:

(a) $f \in \mathcal{E}(\mathbb{X}, \mu)$;

(b) $\lim_{r\to\infty} \dfrac{1}{\mu([-r,r]\setminus I)} \displaystyle\int_{[-r,r]\setminus I} \|f(t)\| \, d\mu(t) = 0$;

(c) For any $\varepsilon > 0$, $\lim_{r\to\infty} \dfrac{\mu(\{t \in [-r,r]\setminus I : \|f(t)\| > \varepsilon\})}{\mu([-r,r]\setminus I)} = 0$.

Definition 2.5. [1] A function $f \in C(\mathbb{R}, \mathbb{X})$ is called μ-pseudo almost periodic if it can be expressed as $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in \mathcal{E}(\mathbb{X}, \mu)$. The collection of such functions will be denoted by $PAP(\mathbb{X}, \mu)$.

Let $\mathcal{N}_1$ denotes the set of all positive measure $\mu \in \mathcal{M}$ such that for all a, b and $c \in \mathbb{R}$ such that $0 \le a < b \le c$, there exist $\tau_0 \ge 0$ and $\alpha_0 > 0$ such that

$$|\tau| \ge \tau_0 \Rightarrow \mu((a+\tau, b+\tau)) \ge \alpha_0 \mu([\tau, c+\tau]).$$

And let $\mathcal{N}_2$ denotes the set of all positive measure $\mu \in \mathcal{M}$ such that for all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a+\tau : a \in A\}) \le \beta \mu(A) \text{ for all } A \in \mathcal{B} \text{ such that } A \cap I = \emptyset.$$

Theorem 2.6. *[1]* Let $\mu \in \mathcal{N}_1$. Then the decomposition of a μ-pseudo almost periodic function in the form $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in \mathcal{E}(\mathbb{X}, \mu)$ is unique.

Theorem 2.7. *[1]* Let $\mu \in \mathcal{N}_1$. Then $(PAP(\mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.

Theorem 2.8. *[1]* Let $\mu \in \mathcal{N}_2$. Then the space $\mathcal{E}(\mathbb{X}, \mu)$ is translation invariant, therefore $PAP(\mathbb{X}, \mu)$ is also translation invariant, that is, if $f \in PAP(\mathbb{X}, \mu)$ implies $f_\tau = f(\cdot + \tau) \in PAP(\mathbb{X}, \mu)$ for all $\tau \in \mathbb{R}$.

Definition 2.9. [2] A jointly continuous function $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{Y}$ if for each $\varepsilon > 0$ and any $K \subset \mathbb{Y}$ a bounded subset, there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|F(t+\tau, y) - F(t, y)\| < \varepsilon$$

for each $t \in \mathbb{R}$, $y \in K$.

The collection of such functions will be denoted by $AP(\mathbb{Y}, \mathbb{X})$.

Definition 2.10. [1] Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ is called μ-ergodic in t uniformly with respect to x in $\mathbb{X}$ if the following two conditions hold:

(a) for all x in $\mathbb{X}$, $f(\cdot, x) \in \mathcal{E}(\mathbb{Y}, \mu)$;

(b) f is uniformly continuous on each compact set $K \subset \mathbb{X}$ with respect to the second variable x.

We denote the space of all such functions by $\mathcal{E}(\mathbb{Y},\mathbb{X},\mu)$.

Definition 2.11. [1] Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ is called μ-pseudo almost periodic if it can be expressed as

$$f = g + \phi,$$

where $g \in AP(\mathbb{Y},\mathbb{X})$ and $\phi \in \mathcal{E}(\mathbb{Y},\mathbb{X},\mu)$. The collection of such functions will be denoted by $PAP(\mathbb{Y},\mathbb{X},\mu)$.

3 Weighted Stepanov-Like Pseudo-Almost Periodic Functions with Variable Exponents

In what follows, we recall the notion of Lebesgue spaces with variable exponents $L^{p(x)}(\mathbb{R},\mathbb{X})$ developed in [4, 5, 7, 9, 11].

Let $\Omega \subseteq \mathbb{R}$ be a subset and let $M(\Omega,\mathbb{X})$ denote the collection of all measurable functions $f : \Omega \mapsto \mathbb{X}$. Let us recall that two functions f and g of $M(\Omega,\mathbb{X})$ are equal whether they are equal almost everywhere. Set $m(\Omega) := M(\Omega,\mathbb{R})$ and fix $p \in m(\Omega)$.

Define

$$p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x), \quad p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x),$$

$$C_+(\Omega) := \Big\{p \in m(\Omega) : 1 < p^- \le p(x) \le p^+ < \infty, \text{ for each } x \in \Omega\Big\},$$

$$D_+(\Omega) := \left\{p \in m(\Omega) : 1 \le p^- \le p(x) \le p^+ < \infty, \text{ for each } x \in \Omega\right\},$$

$$\rho(u) = \rho_{p(x)}(u) = \int_\Omega \|u(x)\|^{p(x)} dx.$$

We then define the Lebesgue spaces with variable exponents $L^{p(x)}(\Omega,\mathbb{X})$ with $p \in C_+(\Omega)$, by

$$L^{p(x)}(\Omega,\mathbb{X}) := \Big\{u \in M(\Omega,\mathbb{X}) : \int_\Omega \|u(x)\|^{p(x)} dx < \infty\Big\}.$$

Define, for each $u \in L^{p(x)}(\Omega,\mathbb{X})$,

$$\|u\|_{p(x)} := \inf\left\{\lambda > 0 : \int_\Omega \left\|\frac{u(x)}{\lambda}\right\|^{p(x)} dx \le 1\right\}.$$

It can be shown that $\|\cdot\|_{p(x)}$ is a norm upon $L^{p(x)}(\Omega,\mathbb{X})$, which is referred to as the *Luxemburg norm*.

Remark 3.1. Let $p \in C_+(\Omega)$. If p is constant, then the space $L^{p(\cdot)}(\Omega,\mathbb{X})$, as defined above, coincides with the usual space $L^p(\Omega,\mathbb{X})$.

Proposition 3.2. *[7, 11]* Let $p \in C_+(\Omega)$. If $u, v \in L^{p(x)}(\Omega,\mathbb{X})$, then the following properties hold,

(a) $\|u\|_{p(x)} \ge 0$, with equality if and only if $u = 0$;

(b) $\rho_p(u) \le \rho_p(v)$ and $\|u\|_{p(x)} \le \|v\|_{p(x)}$ if $\|u\| \le \|v\|$;

(c) $\rho_p(u\|u\|_{p(x)}^{-1}) = 1$ if $u \neq 0$;

(d) $\rho_p(u) \leq 1$ if and only if $\|u\|_{p(x)} \leq 1$;

(e) If $\|u\|_{p(x)} \leq 1$, then

$$\Big[\rho_p(u)\Big]^{1/p^-} \leq \|u\|_{p(x)} \leq \Big[\rho_p(u)\Big]^{1/p^+}.$$

(f) If $\|u\|_{p(x)} \geq 1$, then

$$\Big[\rho_p(u)\Big]^{1/p^+} \leq \|u\|_{p(x)} \leq \Big[\rho_p(u)\Big]^{1/p^-}.$$

Theorem 3.3. *[7, 9]* Let $p \in C_+(\Omega)$. The space $(L^{p(x)}(\Omega,\mathbb{X}), \|\cdot\|_{p(x)})$ is a Banach space that is separable and uniform convex. Its topological dual is $L^{q(x)}(\Omega,\mathbb{X})$, where $p^{-1}(x)+q^{-1}(x) = 1$. Moreover, for any $u \in L^{p(x)}(\Omega,\mathbb{X})$ and $v \in L^{q(x)}(\Omega,\mathbb{R})$, we have

$$\Big\| \int_\Omega uv dx \Big\| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)\|u\|_{p(x)}.|v|_{q(x)}.$$

Corollary 3.4. *[11] Let $p, r \in D_+(\Omega)$. If the function q defined by the equation*

$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}$$

is in $D_+(\Omega)$, then there exists a constant $C = C(p,r) \in [1,5]$ such that

$$\|uv\|_{q(x)} \leq C\|u\|_{p(x)}.|v|_{r(x)},$$

for every $u \in L^{p(x)}(\Omega,\mathbb{X})$ and $v \in L^{r(x)}(\Omega,\mathbb{R})$.

Corollary 3.5. *[7] Let $mes(\Omega) < \infty$ where $mes(\cdot)$ stands for the Lebesgue measure and $p,q \in D_+(\Omega)$. If $q(\cdot) \leq p(\cdot)$ almost everywhere in Ω, then the embedding $L^{p(x)}(\Omega,\mathbb{X}) \hookrightarrow L^{q(x)}(\Omega,\mathbb{X})$ is continuous whose norm does not exceed $2(mes(\Omega)+1)$.*

Definition 3.6. [2] The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$ of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by $f^b(t,s) := f(t+s)$.

Remark 3.7. [2] (i) A function $\varphi(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, is the Bochner transform of a certain function f, $\varphi(t,s) = f^b(t,s)$, if and only if $\varphi(t+\tau, s-\tau) = \varphi(s,t)$ for all $t \in \mathbb{R}$, $s \in [0,1]$ and $\tau \in [s-1,s]$.

(ii) Note that if $f = h+\varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 3.8. [2] The Bochner transform $F^b(t,s,u)$, $t \in \mathbb{R}$, $s \in [0,1]$, $u \in \mathbb{X}$ of a function $F(t,u)$ on $\mathbb{R}\times\mathbb{X}$, with values in $\mathbb{X}$, is defined by $F^b(t,s,u) := F(t+s,u)$ for each $u \in \mathbb{X}$.

Definition 3.9. [2] Let $p \in [1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in $\mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p((0,1),\mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p d\tau\right)^{1/p}.$$

Note that for each $p \geq 1$, we have the following continuous inclusion:

$$(BC(\mathbb{X}), \|\cdot\|_\infty) \hookrightarrow (BS^p(\mathbb{X}), \|\cdot\|_{S^p}).$$

Definition 3.10. [4] Let $p \in C_+(\mathbb{R})$. The space $BS^{p(x)}(\mathbb{X})$ consists of all functions $f \in M(\mathbb{R}, \mathbb{X})$ such that $\|f\|_{S^{p(x)}} < \infty$, where

$$\begin{aligned} \|f\|_{S^{p(x)}} &= \sup_{t\in\mathbb{R}} \left[\inf\left\{ \lambda > 0 : \int_0^1 \left\| \frac{f(x+t)}{\lambda} \right\|^{p(x+t)} dx \leq 1 \right\} \right] \\ &= \sup_{t\in\mathbb{R}} \left[\inf\left\{ \lambda > 0 : \int_t^{t+1} \left\| \frac{f(x)}{\lambda} \right\|^{p(x)} dx \leq 1 \right\} \right]. \end{aligned}$$

Note that the space $\left(BS^{p(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x)}}\right)$ is a Banach space, which, depending on $p(\cdot)$, may or may not be translation-invariant.

Definition 3.11. [4] If $p, q \in C_+(\mathbb{R})$, we then define the space $BS^{p(x),q(x)}(\mathbb{X})$ as follows:

$$\begin{aligned} BS^{p(x),q(x)}(\mathbb{X}) &:= BS^{p(x)}(\mathbb{X}) + BS^{q(x)}(\mathbb{X}) \\ &= \left\{ f = h + \varphi \in M(\mathbb{R}, \mathbb{X}) : h \in BS^{p(x)}(\mathbb{X}) \text{ and } \varphi \in BS^{q(x)}(\mathbb{X}) \right\}. \end{aligned}$$

We equip $BS^{p(x),q(x)}(\mathbb{X})$ with the norm $\|\cdot\|_{S^{p(x),q(x)}}$ defined by

$$\|f\|_{S^{p(x),q(x)}} := \inf \left\{ \|h\|_{S^{p(x)}} + \|\varphi\|_{S^{q(x)}} : \quad f = h + \varphi \right\}.$$

Clearly, $\left(BS^{p(x),q(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x),q(x)}}\right)$ is a Banach space, which, depending on both $p(\cdot)$ and $q(\cdot)$, may or may not be translation-invariant.

Lemma 3.12. *[4] Let $p, q \in C_+(\mathbb{R})$. Then the following continuous inclusion holds,*

$$\left(BC(\mathbb{R}, \mathbb{X}), \|\cdot\|_\infty\right) \hookrightarrow \left(BS^{p(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x)}}\right) \hookrightarrow \left(BS^{p(x),q(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x),q(x)}}\right).$$

Definition 3.13. [2] Let $p \geq 1$ be a constant. A function $f \in BS^p(\mathbb{X})$ is said to be S^p-almost periodic (or Stepanov-like almost periodic) if $f^b \in AP(L^p((0,1), \mathbb{X}))$. That is, for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\sup_{t\in\mathbb{R}} \left(\int_0^1 \left\| f^b(t+\tau, s) - f^b(t, s) \right\|^p ds \right)^{1/p} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| f(s+\tau) - f(s) \right\|^p ds \right)^{1/p} < \varepsilon.$$

The collection of such functions will be denoted by $S^p_{ap}(\mathbb{X})$.

Remark 3.14. [4] There are some difficulties in defining $S^{p(x)}_{ap}(\mathbb{X})$ for a function $p \in C_+(\mathbb{R})$ that is not necessarily constant. This is mainly due to the fact that the space $BS^{p(x)}(\mathbb{X})$ is not always translation-invariant. In other words, the quantities $f^b(t+\tau, s)$ and $f^b(t, s)$ (for $t \in \mathbb{R},\ s \in [0,1]$) that are used in the definition of S^p-almost periodicity, do not belong to the same space, unless p is constant.

We now introduce the concept of weighted $S^{p,q(x)}$-pseudo-almost periodicity as follows:

Definition 3.15. Let $\mu \in \mathcal{M}$, $p \geq 1$ be a constant and let $q \in C_+(\mathbb{R})$. A function $f \in BS^{p,q(x)}(\mathbb{X})$ is said to be weighted $S^{p,q(x)}$-pseudo-almost periodic (or weighted Stepanov-like pseudo-almost periodic with variable exponents $p, q(x)$) if it can be decomposed as $f = h + \varphi$, where $h \in S^p_{ap}(\mathbb{X})$ and $\varphi^b \in \mathcal{E}(L^{q^b(x)}((0,1),\mathbb{X}),\mu)$, i.e.,

$$\lim_{r\to\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \inf\left\{\lambda > 0 : \int_0^1 \left\|\frac{\varphi(x+t)}{\lambda}\right\|^{p(x+t)} dx \leq 1\right\} d\mu(t) = 0.$$

The collection of such functions will be denoted by $S^{p,q(x)}_{pap}(\mathbb{X},\mu)$.

Proposition 3.16. *Let $r, s \geq 1, p, q \in D_+(\mathbb{R})$, $\mu \in \mathcal{M}$. If $s \leq r, q(\cdot) \leq p(\cdot)$ and $f \in BS^{r,p(x)}(\mathbb{X})$ is weighted $S^{r,p(x)}$-pseudo-almost periodic, then f is weighted $S^{s,q(x)}$-pseudo-almost periodic.*

Proof. Suppose f is weighted $S^{r,p(x)}$-pseudo-almost periodic. Thus f can be decomposed as $f = h + \varphi$, where $h^b \in AP(L^r((0,1),\mathbb{X}))$ and $\varphi^b \in \mathcal{E}(L^{p^b(x)}((0,1),\mathbb{X}),\mu)$.

Since $h^b \in AP(L^r((0,1),\mathbb{X}))$, for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|h^b(t+\tau) - h^b(t)\|_{S^r} \leq \varepsilon,$$

for each $t \in \mathbb{R}$.

In view of the continuous injection

$$L^r((0,1),\mathbb{X}) \hookrightarrow L^s((0,1),\mathbb{X}),$$

it follows that for each $t \in \mathbb{R}$

$$\|h^b(t+\tau) - h^b(t)\|_{S^s} \leq \|h^b(t+\tau) - h^b(t)\|_{S^r} \leq \varepsilon,$$

that is, $h \in AP(L^s((0,1),\mathbb{X}))$.

From $\mu(\mathbb{R}) = \infty$, we deduce the existence of $r_0 \geq 0$ such that $\mu(Q_r) > 0$ for all $r \geq r_0$. By using the fact that $\varphi^b \in \mathcal{E}(L^{p^b(x)}((0,1),\mathbb{X}),\mu)$ and Corollary 3.5, one has

$$\begin{aligned}&\frac{1}{\mu(Q_r)} \int_{Q_r} \inf\left\{\lambda > 0 : \int_0^1 \left\|\frac{\varphi(x+t)}{\lambda}\right\|^{q(x+t)} dx \leq 1\right\} d\mu(t)\\ &\leq \frac{4}{\mu(Q_r)} \int_{Q_r} \inf\left\{\lambda > 0 : \int_0^1 \left\|\frac{\varphi(x+t)}{\lambda}\right\|^{p(x+t)} dx \leq 1\right\} d\mu(t).\end{aligned}$$

that is $\varphi^b \in \mathcal{E}(L^{q^b(x)}((0,1),\mathbb{X}),\mu)$ and hence f is weighted $S^{s,q(x)}$-pseudo-almost periodic. □

Proposition 3.17. *Let $p \geq 1$ be a constant, $q \in C_+(\mathbb{R})$ and let $\mu \in \mathcal{N}_2$. Then $PAP(\mathbb{X},\mu) \subset S^{p,q(x)}_{pap}(\mathbb{X},\mu)$.*

Proof. Let $f \in PAP(\mathbb{X},\mu)$. Thus there exist two functions $h,\varphi : \mathbb{R} \to \mathbb{X}$ such that $f = h+\varphi$, where $h \in AP(\mathbb{X})$ and $\varphi \in \mathcal{E}(\mathbb{X},\mu)$. We first show that $h \in S^p_{ap}(\mathbb{X})$. Indeed, since $h \in AP(\mathbb{X})$, for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|h(t+\tau)-h(t)\| < \varepsilon$$

for each $t \in \mathbb{R}$.

Now

$$\int_t^{t+1} \Big\|h(s+\tau)-h(s)\Big\|^p ds \le \int_t^{t+1} \varepsilon^p dx = \varepsilon^p$$

for all $t \in \mathbb{R}$, which means that

$$\|h(\cdot+\tau)-h(\cdot)\|_{S^p} \le \varepsilon,$$

that is, $h^b \in AP(L^p((0,1),\mathbb{X}))$.

To complete the proof, we need to show that $\varphi^b \in \mathcal{E}(L^{q^b(x)}((0,1),\mathbb{X}),\mu)$. From $\mu(\mathbb{R}) = \infty$, we deduce the existence of $r_0 \ge 0$ such that $\mu(Q_r) > 0$ for all $r \ge r_0$.

Using (e)-(f) of Proposition 3.2, the usual Hölder inequality and Fubini's theorem it follows that

$$\begin{aligned}
&\int_{Q_r} \inf\left\{\lambda > 0 : \int_0^1 \Big\|\frac{\varphi(x+t)}{\lambda}\Big\|^{q(x+t)} dx \le 1\right\} d\mu(t)\\
&\le \int_{Q_r}\left(\int_0^1 \|\varphi(t+x)\|^{q(t+x)}\, dx\right)^{\gamma} d\mu(t)\\
&\le (\mu(Q_r))^{1-\gamma}\left[\int_{Q_r}\left(\int_0^1 \|\varphi(t+x)\|^{q(t+x)}\, dx\right) d\mu(t)\right]^{\gamma}\\
&\le (\mu(Q_r))^{1-\gamma}\left[\int_{Q_r}\left(\int_0^1 \|\varphi(t+x)\|.\,\|\varphi\|_\infty^{q(t+x)-1}\, dx\right) d\mu(t)\right]^{\gamma}\\
&\le (\mu(Q_r))^{1-\gamma}\Big(\|\varphi\|_\infty+1\Big)^{\frac{q^+-1}{\gamma}}\left[\int_{Q_r}\left(\int_0^1 \|\varphi(t+x)\|\, dx\right) d\mu(t)\right]^{\gamma}\\
&= (\mu(Q_r))^{1-\gamma}\Big(\|\varphi\|_\infty+1\Big)^{\frac{q^+-1}{\gamma}}\left[\int_0^1\left(\int_{Q_r} \|\varphi(t+x)\|\, d\mu(t)\right) dx\right]^{\gamma}\\
&= (\mu(Q_r))\Big(\|\varphi\|_\infty+1\Big)^{\frac{q^+-1}{\gamma}}\left[\int_0^1\left(\frac{1}{\mu(Q_r)}\int_{Q_r} \|\varphi(t+x)\|\, d\mu(t)\right) dx\right]^{\gamma},
\end{aligned}$$

where

$$\gamma = \begin{cases} \frac{1}{q^+} & \text{if } \|\varphi\| < 1,\\[2ex] \frac{1}{q^-} & \text{if } \|\varphi\| \ge 1. \end{cases}$$

Using the fact that $\mathcal{E}(\mathbb{X},\mu)$ is translation invariant and the (usual) Dominated Convergence Theorem, it follows that

$$\lim_{r\to+\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\inf\left\{\lambda>0:\int_0^1\left\|\frac{\varphi(x+t)}{\lambda}\right\|^{q(x+t)}dx\le 1\right\}d\mu(t)$$
$$\le\left(\|\varphi\|_\infty+1\right)^{\frac{q^+-1}{\gamma}}\left[\int_0^1\left(\lim_{r\to+\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\|\varphi(t+x)\|\,d\mu(t)\right)dx\right]^\gamma=0.$$

□

Theorem 3.18. *Let $p,q\ge 1$ be constants, $\mu\in\mathcal{M}$ and $f\in S^{p,q}_{pap}(\mathbb{X},\mu)$ be such that*

$$f=h+\varphi$$

where $h^b\in AP(L^p((0,1),\mathbb{X}))$ and $\varphi^b\in\mathcal{E}(L^q((0,1),\mathbb{X}),\mu)$. Then

$$\left\{h(t+.):t\in\mathbb{R}\right\}\subset\overline{\left\{f(t+.):t\in\mathbb{R}\right\}},\qquad in \quad BS^{p,q}(\mathbb{X}).$$

Proof. The proof follows along the same lignes as in [1, Theorem 2.24]. We prove it by contradiction. Indeed, if this is not true, then there exists $t_0\in\mathbb{R}$ and $\varepsilon>0$ such that

$$\|f(t+\cdot)-h(t_0+\cdot)\|_{S^{p,q}}>3\varepsilon,\quad\forall t\in\mathbb{R}.\tag{3.1}$$

Since $h^b\in AP(L^p((0,1),\mathbb{X}))$, there exists $l>0$ and for all $n\in\mathbb{Z}$, there exists $\tau_n\in[nl-t_0,nl-t_0+l]$ such that

$$\|h(t_0+\cdot+\tau_n)-h(t_0+\cdot)\|_{S^p}\le\varepsilon.\tag{3.2}$$

By using the uniform continuity on $\mathbb{R}$ of the almost periodic function h, there exists $K_0\in\mathbb{N}$ such that $K_0\ge 2$ and

$$\|h(t+\cdot)-h(t_0+\cdot+\tau_n)\|_{S^p}\le\varepsilon,\quad\forall t\in[t_0+\tau_n-\frac{l}{K_0},t_0+\tau_n+\frac{l}{K_0}].\tag{3.3}$$

From the following inequality

$$\begin{aligned}\|f(t+\cdot)-h(t_0+\cdot)\|_{S^{p,q}}&\le\|f(t+\cdot)-h(t+\cdot)\|_{S^{p,q}}+\|h(t+\cdot)-h(t_0+\cdot+\tau_n)\|_{S^{p,q}}\\&+\|h(t_0+\cdot+\tau_n)-h(t_0+\cdot)\|_{S^{p,q}}\\&=\|f(t+\cdot)-h(t+\cdot)\|_{S^{p,q}}+\|h(t+\cdot)-h(t_0+\cdot+\tau_n)\|_{S^p}\\&+\|h(t_0+\cdot+\tau_n)-h(t_0+\cdot)\|_{S^p}.\end{aligned}$$

and from (3.1)-(3.3), we deduce that

$$\|\varphi(t+\cdot)\|_{S^q}=\|\varphi(t+\cdot)\|_{S^{p,q}}=\|f(t+\cdot)-h(t+\cdot)\|_{S^{p,q}}>\varepsilon,\tag{3.4}$$

for all $t\in[t_0+\tau_n-\frac{l}{K_0},t_0+\tau_n+\frac{l}{K_0}]$.

Similarly, as in the proof of [1, Theorem 2.24], we obtain the existence of constants $\alpha_*>0$ and $n_*\in\mathbb{N},n_*\ge 1$, such that

$$|n|\ge n_*\Rightarrow\alpha_*\mu([nl,nl+l])\le\mu(\{t\in(nl,nl+l]:\|\varphi(t+\cdot)\|_{S^q}>\varepsilon\}).\tag{3.5}$$

Let $N \in \mathbb{N}$ be such that $N > n_*$. Denote by $\mathcal{S}$ the finite set of integers defined by

$$\mathcal{S} = \{-N, -N+1, ..., -n_* - 1\} \cup \{n_*, n_* + 1, ..., N-1\}.$$

By summing (3.5) on $\mathcal{S}$, we obtain

$$\alpha_* \sum_{n\in\mathcal{S}} \mu([nl, nl+l]) \leq \sum_{n\in\mathcal{S}} \mu(\{t \in (nl, nl+l] : \|\varphi(t+\cdot)\|_{S^q} > \varepsilon\}). \tag{3.6}$$

From the following inequalities:

$$\begin{aligned}\alpha_* \sum_{n\in\mathcal{S}} \mu([nl, nl+l]) &\geq \alpha_* \mu\Big(\bigcup_{n\in\mathcal{S}} [nl, nl+l]\Big)\\ &= \alpha_* \mu([-Nl, Nl] \setminus (-n_* l, n_* l)),\end{aligned}$$

$$\begin{aligned}\sum_{n\in\mathcal{S}} \mu(\{t \in (nl, nl+l] : \|\varphi(t+\cdot)\|_{S^q} > \varepsilon\}) &= \mu\Big(\bigcup_{n\in\mathcal{S}} \{t \in (nl, nl+l] : \|\varphi(t+\cdot)\|_{S^q} > \varepsilon\}\Big)\\ &= \mu(\{t \in (-Nl, Nl] \setminus (-n_* l, n_* l] : \|\varphi(t+\cdot)\|_{S^q} > \varepsilon\})\\ &\leq \mu(\{t \in [-Nl, Nl] \setminus (-n_* l, n_* l) : \|\varphi(t+\cdot)\|_{S^q} > \varepsilon\}),\end{aligned}$$

and from (3.6), we deduce that for all $N > n_*$

$$\alpha_* \mu([-Nl, Nl] \setminus (-n_* l, n_* l)) \leq \mu(\{t \in [-Nl, Nl] \setminus (-n_* l, n_* l) : \|\varphi(t+\cdot)\|_{S^q} > \varepsilon\}),$$

therefore we obtain

$$\lim_{N\to+\infty} \frac{\mu(\{t \in [-Nl, Nl] \setminus (-n_* l, n_* l) : \|\varphi(t+\cdot)\|_{S^q} > \varepsilon\})}{\mu([-Nl, Nl] \setminus (-n_* l, n_* l))} \geq \alpha_* > 0.$$

By using Theorem 2.4, it yields that $\varphi^b \notin \mathcal{E}(L^q((0,1), \mathbb{X}), \mu)$, which is a contradiction. □

Corollary 3.19. *Let $p, q \geq 1$ be constants and $\mu \in \mathcal{N}_1$. Then the decomposition of a $S^{p,q}$-μ-pseudo-almost periodic function in the form $f = h + \varphi$ where $h^b \in AP(L^p((0,1), \mathbb{X}))$ and $\varphi^b \in \mathcal{E}(L^q((0,1), \mathbb{X}), \mu)$, is unique.*

Proof. Suppose that $f = h_1 + \varphi_1 = h_2 + \varphi_2$ where $h_1^b, h_2^b \in AP(L^p((0,1), \mathbb{X}))$ and $\varphi_1^b, \varphi_1^b \in \mathcal{E}(L^q((0,1), \mathbb{X}), \mu)$. Then $0 = (h_1 - h_2) + (\varphi_1 - \varphi_2) \in S^{p,q}_{pap}(\mathbb{X}, \mu)$ where $h_1^b - h_2^b \in AP(L^p((0,1), \mathbb{X}))$ and $\varphi_1^b - \varphi_1^b \in \mathcal{E}(L^q((0,1), \mathbb{X}), \mu)$. From Theorem 3.18 we obtain $(h_1 - h_2)(\mathbb{R}) \subset \{0\}$, therefore one has $h_1 = h_2$ and $\varphi_1 = \varphi_2$.

□

Theorem 3.20. Let $p, q \geq 1$ be constants and $\mu \in \mathcal{N}_1$. The space $S^{p,q}_{pap}(\mathbb{X}, \mu)$ equipped with the norm $\|\cdot\|_{S^{p,q}}$ is a Banach space.

Proof. It suffices to prove that $S^{p,q}_{pap}(\mathbb{X}, \mu)$ is a closed subspace of $BS^{p,q}(\mathbb{X})$. Let $f_n = h_n + \varphi_n$ be a sequence in $S^{p,q}_{pap}(\mathbb{X}, \mu)$ with $(h_n^b)_{n\in\mathbb{N}} \subset AP(L^p((0,1), \mathbb{X}))$ and $(\varphi_n^b)_{n\in\mathbb{N}} \subset \mathcal{E}(L^q((0,1), \mathbb{X}), \mu)$ such that $\|f_n - f\|_{S^{p,q}} \to 0$ as $n \to \infty$. By Theorem 3.18, one has

$$\{h_n(t+.) : t \in \mathbb{R}\} \subset \overline{\{f_n(t+.) : t \in \mathbb{R}\}},$$

and hence

$$\|h_n\|_{S^p} = \|h_n\|_{S^{p,q}} \le \|f_n\|_{S^{p,q}} \quad \text{for all } n \in \mathbb{N}.$$

Consequently, there exists a function $h \in S^p_{ap}(\mathbb{X})$ such that $\|h_n - h\|_{S^p} \to 0$ as $n \to \infty$. Using the previous fact, it easily follows that the function $\varphi := f - h \in BS^q(\mathbb{X})$ and that $\|\varphi_n - \varphi\|_{S^q} = \|(f_n - h_n) - (f - h)\|_{S^q} \to 0$ as $n \to \infty$. From $\mu(\mathbb{R}) = \infty$, we deduce the existence of $r_0 \ge 0$ such that $\mu(Q_r) > 0$ for all $r \ge r_0$. Using the fact that $\varphi = (\varphi - \varphi_n) + \varphi_n$ and the triangle inequality, it follows that

$$\begin{aligned}
&\frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \left\|\varphi(\tau+t)\right\|^q d\tau\right)^{\frac{1}{q}} d\mu(t)\\
&\quad\le \frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \left\|\varphi(\tau+t)-\varphi_n(\tau+t)\right\|^q d\tau\right)^{\frac{1}{q}} d\mu(t)\\
&\quad\quad + \frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \left\|\varphi_n(\tau+t)\right\|^q d\tau\right)^{\frac{1}{q}} d\mu(t)\\
&\quad\le \|\varphi_n - \varphi\|_{S^q} + \frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \left\|\varphi_n(\tau+t)\right\|^q d\tau\right)^{\frac{1}{q}} d\mu(t).
\end{aligned}$$

Letting $r \to +\infty$ and then $n \to \infty$ in the previous inequality yields $\varphi^b \in \mathcal{E}(L^q((0,1),\mathbb{X}),\mu)$, that is, $f = h + \varphi \in S^{p,q}_{pap}(\mathbb{X},\mu)$. □

Definition 3.21. [1] Let $\mu_1, \mu_2 \in \mathcal{M}$. μ_1 is said to be equivalent to μ_2 ($\mu_1 \sim \mu_2$) if there exist constants $\alpha, \beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that

$$\alpha\mu_1(A) \le \mu_2(A) \le \beta\mu_1(A), \text{ for all } A \in \mathcal{B} \text{ such that } A \cap I = \emptyset.$$

Theorem 3.22. Let $\mu \in \mathcal{M}, p \ge 1$ be a constant, $q \in C_+(\mathbb{R})$ and $\mu_1, \mu_2 \in \mathcal{M}$. If μ_1 and μ_2 are equivalent then $S^{p,q(x)}_{pap}(\mathbb{X},\mu_1) = S^{p,q(x)}_{pap}(\mathbb{X},\mu_2)$.

Proof. The proof is similar to that of [1, Theorem 2.21]. Since $\mu_1 \sim \mu_2$, and $\mathcal{B}$ is the Lebesgue σ-field of $\mathbb{R}$, we obtain for r sufficiently large

$$\begin{aligned}
\frac{\alpha}{\beta}\frac{\mu_1(\{t \in Q_r \setminus I : \|f(t)\|_{S^{p,q(\cdot)}} > \varepsilon\})}{\mu(Q_r \setminus I)} &\le \frac{\mu_2(\{t \in Q_r \setminus I : \|f(t)\|_{S^{p,q(\cdot)}} > \varepsilon\})}{\mu(Q_r \setminus I)}\\
&\le \frac{\beta}{\alpha}\frac{\mu_1(\{t \in Q_r \setminus I : \|f(t)\|_{S^{p,q(\cdot)}} > \varepsilon\})}{\mu(Q_r \setminus I)}.
\end{aligned}$$

By using Theorem 2.4, we deduce that $\mathcal{E}(L^{q^b(x)}((0,1),\mathbb{X}),\mu_1) = \mathcal{E}(L^{q^b(x)}((0,1),\mathbb{X}),\mu_1)$. From the definition of a weighted $S^{p,q(x)}$-pseudo-almost periodic function it follows that

$$S^{p,q(x)}_{pap}(\mathbb{X},\mu_1) = S^{p,q(x)}_{pap}(\mathbb{X},\mu_2).$$

□

Definition 3.23. A function $F:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ with $F(.,u)\in BS^{p,q(x)}(\mathbb{X})$ for each $u\in\mathbb{Y}$, is said to be $S^{p,q(x)}$-μ-pseudo-almost periodic in $t\in\mathbb{R}$ uniformly in $u\in\mathbb{Y}$ if $t\mapsto F(t,u)$ is $S^{p,q(x)}$-μ-pseudo-almost periodic for each $u\in B$ where $B\subset\mathbb{Y}$ is an arbitrary bounded set.

This means, there exist two functions $G,H:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ such that $F=G+H$, where $G^b\in AP(\mathbb{Y},L^p((0,1),\mathbb{X}))$ and $H^b\in\mathcal{E}(\mathbb{Y},L^{q^b(x)}((0,1),\mathbb{X}),\mu)$, that is,

$$\lim_{r\to+\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\inf\left\{\lambda>0:\int_0^1\left\|\frac{H(x+t,u)}{\lambda}\right\|^{q(x+t)}dx\le 1\right\}d\mu(t)=0,$$

uniformly in $u\in B$ where $B\subset\mathbb{Y}$ is an arbitrary bounded set.

The collection of such functions will be denoted by $S^{p,q(x)}_{pap}(\mathbb{Y},\mathbb{X},\mu)$.

Let $Lip^r(\mathbb{Y},\mathbb{X})$ denote the collection of functions $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ satisfying: there exists a nonnegative function $L_f^b\in L^r(\mathbb{R})$ such that

$$\|f(t,u)-f(t,v)\|\le L_f(t)\|u-v\|_{\mathbb{Y}}\ \text{ for all }\ u,v\in\mathbb{Y},\ \ t\in\mathbb{R}. \tag{3.7}$$

Now, we recall the composition theorem for S^p_{ap} functions.

Theorem 3.24. *[8]* Let $p>1$ be a constant. We suppose that the following conditions hold:

(a) $f\in S^p_{ap}(\mathbb{R}\times\mathbb{X})\cap Lip^r(\mathbb{R},\mathbb{X})$ with $r\ge\max\{p,\frac{p}{p-1}\}$.

(b) $\phi\in S^p_{ap}(\mathbb{X})$ and there exists a set $E\subset\mathbb{R}$ with $mes\,(E)=0$ such that

$$K:=\overline{\{\phi(t):t\in\mathbb{R}\setminus E\}}$$

is compact in $\mathbb{X}$.

Then there exists $m\in[1,p)$ such that $f(\cdot,\phi(\cdot))\in S^m_{ap}(\mathbb{R}\times\mathbb{X})$.

To obtain the composition theorem for $S^{p(x)}_{pap}$ functions, we need the following lemma:

Lemma 3.25. Let $q>1$ be a constant, $\mu\in\mathcal{M}$ and $K\subseteq\mathbb{Y}$ be a compact subset. If $f\in Lip^q(\mathbb{Y},\mathbb{X})$ and $f^b\in\mathcal{E}(\mathbb{Y},L^q((0,1),\mathbb{X}),\mu)$, then $\widetilde{f}\in\mathcal{E}(\mathbb{X},\mu)$, where the function $\widetilde{f}$ is defined by

$$\widetilde{f}(t):=\left\|\sup_{u\in K}\|f(t+\cdot,u)\|\right\|_q \tag{3.8}$$

for all $t\in\mathbb{R}$.

Proof. We make extensive use of ideas of [8, Lemma 2.3]. Using the fact that $K\subset\mathbb{Y}$ is a compact subset, for any $\varepsilon>0$, there exists $x_1,x_2,....,x_k$ such that

$$K\subseteq\bigcup_{i=1}^{k}B(x_i,\varepsilon).$$

Using this argument along with the fact that $f\in Lip^q(\mathbb{Y},\mathbb{X})$, for all $u\in K$, there exists $x_{i(u)}\in\{x_1,x_2,....,x_k\}$ such that

$$\|f(t+s,u)\|\le\|f(t+s,u)-f(t+s,x_{i(u)})\|+\|f(t+s,x_{i(u)})\|\le L_f(t+s)\varepsilon+\|f(t+s,x_{i(u)})\|$$

for each $t \in \mathbb{R}$ and $s \in [0,1]$. Thus, we have

$$\sup_{u\in K} \|f(t+s,u)\| \le L_f(t+s)\varepsilon + \sum_{i=1}^{k} \|f(t+s, x_{i(u)})\|, \quad \forall t \in \mathbb{R}, \quad \forall s \in [0,1],$$

which yields

$$\widetilde{f}(t) = \left\| \sup_{u\in K} \|f(t+\cdot,u)\| \right\|_q \le \|L_f\|_{S^q}.\varepsilon + \sum_{i=1}^{k} \|f(t, x_{i(u)})\|_q, \quad \forall t \in \mathbb{R}. \tag{3.9}$$

Now using the fact that $f^b \in \mathcal{E}(\mathbb{Y}, L^q((0,1),\mathbb{X}),\mu)$, for the above $\varepsilon > 0$, there exists $r_0 > 0$ such that, for all $r > r_0$,

$$\frac{1}{\mu(Q_r)} \int_{Q_r} \left(\int_0^1 \left\| f(t+s, x_i) \right\|^q d\tau \right)^{\frac{1}{q}} d\mu(t) < \frac{\varepsilon}{k}, \quad i = 1,2,...,k.$$

This along with Eq. (3.9) yield

$$\frac{1}{\mu(Q_r)} \int_{Q_r} \widetilde{f}(t)\, d\mu(t) \le \left(\|L_f\|_{S^q} + 1 \right).\varepsilon,$$

and hence $\widetilde{f} \in \mathcal{E}(\mathbb{X},\mu)$. □

Theorem 3.26. *Let $p,q > 1$ be constants such that $p \le q$ and $\mu \in \mathcal{M}$. Suppose that the following conditions hold:*

(a) $f = g+h \in S^{p,q}_{pap}(\mathbb{Y},\mathbb{X},\mu)$ with $g^b \in AP(\mathbb{Y}, L^p((0,1),\mathbb{X}))$ and $h^b \in \mathcal{E}(\mathbb{Y}, L^q((0,1),\mathbb{X}),\mu)$. Further, $f,g \in Lip^r(\mathbb{Y},\mathbb{X})$ with $r \ge \max\{q, \frac{p}{p-1}\}$.

(b) $\phi = \alpha+\beta \in S^{p,q}_{pap}(\mathbb{Y})$ with $\alpha^b \in AP(L^p((0,1),\mathbb{Y}))$ and $\beta^b \in \mathcal{E}(L^q((0,1),\mathbb{Y}),\mu)$, and there exists a set $E \subset \mathbb{R}$ with $mes\,(E) = 0$ such that

$$K := \overline{\{\alpha(t) : t \in \mathbb{R} \setminus E\}}$$

is compact in $\mathbb{Y}$.

Then there exists $m \in [1,p)$ such that $f(\cdot,\phi(\cdot)) \in S^{m,m}_{pap}(\mathbb{Y},\mathbb{X},\mu)$.

Proof. We will make use of ideas of [8, Theorem 2.4]. Indeed, decompose f^b as follows:

$$f^b(\cdot,\phi^b(\cdot)) = g^b(\cdot,\alpha^b(\cdot)) + f^b(\cdot,\phi^b(\cdot)) - f^b(\cdot,\alpha^b(\cdot)) + h^b(\cdot,\alpha^b(\cdot)).$$

Using Theorem 3.24, it easily follows that there exists $m \in [1,p)$ with $\frac{1}{m} = \frac{1}{p} + \frac{1}{r}$ such that $g^b(\cdot,\alpha^b(\cdot)) \in AP(\mathbb{R} \times L^m((0,1),\mathbb{X}))$.

Set

$$\varphi^b(\cdot) = f^b(\cdot,\phi^b(\cdot)) - f^b(\cdot,\alpha^b(\cdot)).$$

Clearly, $\varphi^b \in \mathcal{E}(L^m((0,1),\mathbb{X}),\mu)$. Indeed, there exists $r_0 > 0$ such that, for all $r > r_0$,

$$\frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \|\varphi^b(t+s)\|^m ds\right)^{\frac{1}{m}} d\mu(t)$$

$$= \frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \|f^b(t+s,\phi^b(t+s)) - f^b(t+s,\alpha^b(t+s))\|^m ds\right)^{\frac{1}{m}} d\mu(t)$$

$$\leq \frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \left(L_f^b(t+s).\|\beta^b(t+s)\|\right)^m ds\right)^{\frac{1}{m}} d\mu(t)$$

$$\leq \|L_f^b\|_{S^r}.\left[\frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \|\beta^b(t+s)\|^p ds\right)^{\frac{1}{p}} d\mu(t)\right]$$

$$\leq \|L_f^b\|_{S^r}.\left[\frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \|\beta^b(t+s)\|^q ds\right)^{\frac{1}{q}} d\mu(t)\right].$$

Using the fact that $\beta^b \in \mathcal{E}(L^q((0,1),\mathbb{X}),\mu)$, it follows that $\varphi^b \in \mathcal{E}(L^m((0,1),\mathbb{X}),\mu)$.

Now using the fact that $h = f - g \in Lip^r(\mathbb{R},\mathbb{X}) \subset Lip^q(\mathbb{R},\mathbb{X})$, it follows by Lemma 3.25 that

$$\lim_{r\to+\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\left\|\sup_{u\in K}\|h(t+\cdot,u)\|\right\|_q d\mu(t) = 0,$$

which yields

$$\frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \|h^b(t+s,\alpha^b(t+s))\|^m ds\right)^{\frac{1}{m}} d\mu(t)$$

$$\leq \frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \|h^b(t+s,\alpha^b(t+s))\|^q ds\right)^{\frac{1}{q}} d\mu(t)$$

$$\leq \frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1 \left(\sup_{u\in K}\|h^b(t+s,u)\|\right)^q ds\right)^{\frac{1}{q}} d\mu(t) \to 0 \quad \text{as} \quad r\to\infty,$$

which means that $h^b(\cdot,\alpha^b(\cdot)) \in \mathcal{E}((L^m(0,1);\mathbb{X}),\mu)$. This completes the proof. □

Remark 3.27. A general composition theorem in $S^{p,q(x)}_{pap}(\mathbb{R}\times\mathbb{X})$ is unlikely as compositions of elements of $S^{p,q(x)}_{pap}(\mathbb{R}\times\mathbb{X},\mu)$ may not be well-defined unless $q(\cdot)$ is the constant function.

4 Exsitecne Results for Evolution Equations

Let $p, q > 1$ be constants such that $p \leq q$, $\vartheta \in C_+(\mathbb{R})$ and $\mu \in \mathcal{N}_1$. This section is devoted to the search of a μ-pseudo-almost periodic solutions to the abstract nonautonomous differential equations Eq. (1.1) and Eq. (1.2).

Throughout the rest of the paper we suppose that the following assumptiona hold:

(A.1) The family of closed linear operators $A(t)$, for $t \in \mathbb{R}$, on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the so-called Acquistapace-Terreni conditions;

namely, there exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $M_1, M_2 \geq 0$, and $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\|_{B(\mathbb{X})} \leq \frac{M_1}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\|_{B(\mathbb{X})} \leq M_2 |t - s|^\alpha |\lambda|^{-\beta}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} - \{0\} : |arg\lambda| \leq \theta\}$

(A.2) The evolution family $U(t,s)$ is exponentially stable. Namely, there exist some constants $M, \delta > 0$ such that

$$\|U(t,s)\|_{B(\mathbb{X})} \leq M e^{-\delta(t-s)}$$

for all $s, t \in \mathbb{R}$ with $t \geq s$. In addition, $R(\lambda_0, A(\cdot)) \in AP(\mathbb{R}, B(\mathbb{X}))$.

(A.3) $F = G + H \in S^{p,q}_{pap}(\mathbb{R} \times \mathbb{X}, \mu) \cap C(\mathbb{R} \times \mathbb{X})$ with $G^b \in AP(\mathbb{R} \times L((0,1), \mathbb{X}))$ and $H^b \in \mathcal{E}(\mathbb{R} \times L^q((0,1), \mathbb{X}), \mu)$. Moreover; $F, G \in Lip^r(\mathbb{R}, \mathbb{X})$ with

$$r \geq \max\left\{q, \frac{p}{p-1}\right\}.$$

Definition 4.1. Under (A.1)-(A.2), if $f : \mathbb{R} \to \mathbb{X}$ is a bounded continuous function, then a mild solution to Eq.(1.1) is a continuous function $u : \mathbb{R} \to \mathbb{X}$ satisfying

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\sigma) f(\sigma) d\sigma \tag{4.1}$$

for all $t, s \in \mathbb{R}$ and $t \geq s$.

Definition 4.2. Suppose (A.1)-(A.2) hold. If $F : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a bounded continuous function, then a mild solution to Eq.(1.2) is a continuous function $u : \mathbb{R} \to \mathbb{X}$ satisfying

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\sigma) F(\sigma, u(\sigma)) d\sigma \tag{4.2}$$

for all $t, s \in \mathbb{R}$ and $t \geq s$.

Lemma 4.3. *Under assumptions* (A.1)—(A.2)*, if* $h \in S^{p,\vartheta(x)}_{paa}(\mathbb{X}, \mu) \cap C(\mathbb{R}, \mathbb{X})$*, then the operator* Λ *defined by*

$$(\Lambda u)(t) := \int_{-\infty}^t U(t,\sigma) h(\sigma)\, d\sigma, \quad t \in \mathbb{R}$$

maps $PAP(\mathbb{X}, \mu)$ *into itself.*

Proof. Clearly, Λ is well defined. Moreover, let $u \in PAP(\mathbb{X}, \mu)$. Since $h \in S^{p,\vartheta(x)}_{paa}(\mathbb{X}, \mu) \cap C(\mathbb{R}, \mathbb{X})$, then $h = g + \varphi$, where $g^b \in AP(L^p((0,1), \mathbb{X}))$ and $\varphi^b \in \mathcal{E}(L^{\vartheta^b(x)}((0,1), \mathbb{X}), \mu)$. Then Λ can be decomposed as

$$(\Lambda u)(t) = X(t) + Y(t)$$

where

$$X(t) = \int_{-\infty}^{t} U(t,s)g(s)ds, \text{ and } Y(t) = \int_{-\infty}^{t} U(t,s)\varphi(s)ds.$$

Define for all $n = 1,2,....,$ the sequence of integral operators

$$X_n(t) := \int_{n-1}^{n} U(t,t-s)g(t-s)\,ds = \int_{t-n}^{t-n+1} U(t,s)g(s)\,ds,$$

and

$$Y_n(t) := \int_{n-1}^{n} U(t,t-s)\varphi(t-s)ds = \int_{t-n}^{t-n+1} U(t,s)\varphi(s)ds.$$

for each $t \in \mathbb{R}$.

Let us show that $X_n \in AP(\mathbb{X})$. Let $p^{'} > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Using the Hölder's inequality, it follows that

$$\begin{aligned}
\|X_n(t)\| &\le M \int_{t-n}^{t-n+1} e^{-\delta(t-\sigma)}\|g(\sigma)\|d\sigma \\
&\le M\left(\int_{t-n}^{t-n+1} e^{-p^{'}\delta(t-\sigma)}d\sigma\right)^{\frac{1}{p'}} \left(\int_{t-n}^{t-n+1} \|g(\sigma)\|^p d\sigma\right)^{\frac{1}{p}} \\
&\le \frac{M}{\sqrt[p']{p'\delta}}\left(e^{-p^{'}(n-1)\delta} - e^{-p^{'}n\delta}\right)^{\frac{1}{q}} \|g\|_{S^p} \\
&\le Me^{-n\delta}\sqrt[p']{\frac{1+e^{p'\delta}}{p'\delta}}\|g\|_{S^p} \\
&:= K_1 e^{-n\delta}\|g\|_{S^p}.
\end{aligned}$$

Since the series

$$K_1 \sum_{n=1}^{\infty} e^{-n\delta}$$

is convergent, we deduce from the well-known Weierstrass test that the sequence of functions $\sum_{n=1}^{\infty} X_n(t)$ is uniformly convergent on $\mathbb{R}$.

Using the fact that

$$X(t) = \sum_{n=1}^{\infty} X_n(t),$$

it follows that $X \in C(\mathbb{R},\mathbb{X})$. Moreover, for any $t \in \mathbb{R}$, we have

$$\|X(t)\| \le \sum_{n=1}^{\infty} \|X_n(t)\| \le C_{p'}(M,\delta)\|g\|_{S^p},$$

where $C_{p'}(M,\delta)$ depends only on the fixed constants $p^{'}, M$ and δ.

Since $g^b \in AP(L^p((0,1),\mathbb{X}))$, for each $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\sup_{t\in\mathbb{R}}\left(\int_{t}^{t+1} \left\|g(s+\tau) - g(s)\right\|^p ds\right)^{\frac{1}{p}} < \frac{\varepsilon}{C_{p'}(M,\delta)}.$$

Using triangle inequality, Hölder inequality and [10, Proposition 4.4], we obtain

$$\begin{aligned}
\|X(t+\tau)-X(t)\| &\le \left\| \int_{-\infty}^{t} U(t+\tau,s+\tau)g(s+\tau)\,ds - \int_{-\infty}^{t} U(t,s)g(s)\,ds \right\| \\
&\le \left\| \int_{-\infty}^{t} U(t+\tau,s+\tau)[g(s+\tau)-g(s)]\,ds \right\| \\
&+ \left\| \int_{-\infty}^{t} [U(t+\tau,s+\tau)-U(t,s)]g(s)\,ds \right\| \\
&\le M \sum_{n=1}^{\infty} \int_{n-1}^{n} e^{-\delta s} \left\| g(t-s+\tau)-g(t-s) \right\| ds \\
&+ \int_{-\infty}^{t} \left\| U(t+\tau,s+\tau)-U(t,s) \right\|_{B(\mathbb{X})} \|g(t-s)\|\,ds \\
&\le C_{p'}(M,\delta)\|g(t+\tau)-g(t)\|_{S^p} \\
&+ \int_{-\infty}^{t} \varepsilon e^{-\frac{\delta}{2}(t-s)} \|g(t-s)\|\,ds \\
&\le \varepsilon + \varepsilon. C_{p'}(\delta). \|g\|_{S^p} \\
&= (1+C_{p'}(\delta). \|g\|_{S^p})\varepsilon,
\end{aligned}$$

and therefore, $X \in AP(\mathbb{X})$.

Now, let us show that $Y_n \in \mathcal{E}(\mathbb{X},\mu)$. Indeed, let $d \in m(\mathbb{R})$ such that $d^{-1}(x)+\vartheta^{-1}(x) = 1$. From $\mu(\mathbb{R}) = \infty$, we deduce the existence of $r_0 \ge 0$ such that $\mu([-r,r]) > 0$ for all $r \ge r_0$. By using the Hölder inequality (Theorem 3.3), it follows that

$$\begin{aligned}
\|Y_n(t)\| &\le M \int_{t-n}^{t-n+1} e^{-\omega(t-s)} \|\varphi(s)\| ds \\
&\le M\Big(\frac{1}{d^-}+\frac{1}{\vartheta^-}\Big)\Big[\inf\Big\{\lambda>0 : \int_{t-n}^{t-n+1} \Big(\frac{e^{-\omega(t-s)}}{\lambda}\Big)^{d(s)} ds \le 1\Big\}\Big] \\
&\times \Big[\inf\Big\{\lambda>0 : \int_{t-n}^{t-n+1} \Big\|\frac{\varphi(s)}{\lambda}\Big\|^{\vartheta(s)} ds \le 1\Big\}\Big].
\end{aligned}$$

Now since

$$\begin{aligned}
\int_{t-n}^{t-n+1} \Big[\frac{e^{-\omega(t-s)}}{e^{-\omega(n-1)}}\Big]^{d(s)} ds &= \int_{t-n}^{t-n+1} \Big[e^{\omega(s-t+n-1)}\Big]^{d(s)} ds \\
&\le \int_{t-n}^{t-n+1} \Big[1\Big]^{d(s)} ds \\
&\le 1
\end{aligned}$$

it follows that $e^{-\omega(n-1)} \in \Big\{\lambda>0 : \int_{t-n}^{t-n+1} \big(\frac{e^{-\omega(t-s)}}{\lambda}\big)^{d(s)} ds \le 1\Big\}$, which shows that

$$\Big[\inf\Big\{\lambda>0 : \int_{t-n}^{t-n+1} \Big(\frac{e^{-\omega(t-s)}}{\lambda}\Big)^{d(s)} ds \le 1\Big\}\Big] \le e^{-\omega(n-1)}.$$

Consequently,

$$\|Y_n(t)\| \leq M\Big(\frac{1}{d^-} + \frac{1}{q^-}\Big)e^{-\omega(n-1)}\|\varphi\|_{S^{\vartheta(x)}}$$

Since the series

$$\sum_{n=1}^{\infty} e^{-\omega(n-1)}$$

is convergent, we deduce from the well-known Weierstrass test that the series

$$\sum_{k=1}^{\infty} Y_n(t)$$

is uniformly convergent on $\mathbb{R}$. Furthermore, from

$$Y(t) = \sum_{n=1}^{\infty} Y_n(t),$$

we deduce that $Y \in C(\mathbb{R},\mathbb{X})$, and

$$\|Y(t)\| \leq \sum_{n=1}^{\infty} \|Y_n(t)\| \leq K_1\|\varphi\|_{S^{\vartheta(x)}},$$

where $K_1 = M\Big(\frac{1}{d^-} + \frac{1}{\vartheta^-}\Big)\sum_{n=1}^{\infty} e^{-\omega(n-1)}$.

By using the following inequality

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{[-r,r]} \|Y(t)\|\, d\mu(t) &\leq \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|Y(t) - \sum_{n=1}^{\infty} Y_n(t)\|\, d\mu(t) \\ &+ \sum_{n=1}^{\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|Y_n(t)\|\, d\mu(t)\end{aligned}$$

we deduce that the uniform limit $Y(t) = \sum_{n=1}^{\infty} Y_n(t) \in \mathcal{E}(\mathbb{X},\mu)$. Therefore, $(\Lambda u) \in PAP(\mathbb{X},\mu)$. □

Using Lemma 4.3 one can prove the following theorems

Theorem 4.4. *Under assumptions* $(A.1)$—$(A.2)$*, if* $f \in S^{p,\vartheta(x)}_{paa}(\mathbb{X},\mu) \cap C(\mathbb{R},\mathbb{X})$*, then Eq.(1.1) has a unique* μ*-pseudo-almost periodic (mild) solution given by*

$$u(t) = \int_{-\infty}^{t} U(t,\sigma)f(\sigma)d\sigma, \quad t \in \mathbb{R}. \tag{4.3}$$

Proof. Define the function $u : \mathbb{R} \mapsto \mathbb{X}$ by

$$u(t) = \int_{-\infty}^{t} U(t,s)f(s)ds,\ t \in \mathbb{R}. \tag{4.4}$$

It is easy to check that u given in Eq. (4.4) satisfies Eq. (4.1) and hence it is a mild solution.

Since $f \in S^{p,q(x)}_{pap}(\mathbb{X},\mu) \cap C(\mathbb{R},\mathbb{X})$, from Lemma 4.3, we deduce that u given in Eq. (4.4) is in $PAP(\mathbb{X})$.

To complete the proof it remains to prove the uniqueness. By assumption there exist some constants $M, \delta > 0$ such that

$$\|U(t,s)\|_{B(\mathbb{X})} \leq Me^{-\delta(t-s)} \text{ for all } s,t \in \mathbb{R} \text{ with } t \geq s.$$

Assume that $u : \mathbb{R} \to \mathbb{X}$ is bounded and satisfies the homogeneous equation

$$u'(t) = A(t)u(t), \quad t \in \mathbb{R}, \tag{4.5}$$

Then $u(t) = U(t,s)u(s)$, for any $t \geq s$. Thus $\|u(t)\| \leq MKe^{-\delta(t-s)}$, where $\|u(s)\| \leq K$. Take a sequence of real numbers (s_n) such that $s_n \to -\infty$ as $n \to \infty$. For any $t \in \mathbb{R}$ fixed, one can find a subsequence $(s_{n_k}) \subset (s_n)$ such that $s_{n_k} < t$ for all $k = 1, 2, \ldots$. By letting $k \to \infty$, we get $u(t) = 0$. Now if u, v are bounded solutions to Eq.(1.1), then $w = u - v$ is a bounded solution to Eq.(4.5). In view of the above, $w = u - v = 0$ that is $u = v$. □

Theorem 4.5. Let $p, q > 1$ be constants such that $p \leq q$ and $\mu \in \mathcal{N}$. Then under assumptions (A.1)-(A.3), Eq.(1.2) has a unique μ-pseudo-almost periodic solutions whenever $\|L_F\|_{S^r}$ is small enough.

Proof. The proof is similar to that of [4, Theorem 6.4]. So, we omit it. □

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