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Abstract

This paper presents a couple of existence results, related to the classes of functional equations of the form $x+k*x=f$, or $\frac{d}{dt}[\dot{x}+k*x]=f$, with $f,x\in AP_r(R,C)=$ the space of almost periodic functions defined by

$$AP_r(R,C)=\left\{f:f\simeq\sum_{j=1}^{\infty}f_j e^{i\lambda_j t},\ f_j\in C,\ \lambda_j\in R,\ \sum_{j=1}^{\infty}|f_j|^r<\infty\right\},$$

the norm being given by $|f|_r=\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}$, for each $r\in[1,2]$. The convolution product $k*x$, $k\in L^1(R,C)$, $x\in AP_r(R,C)$ is defined by

$$(k*x)(t)=\sum_{j=1}^{\infty}x_j\left(\int_R k(s)\,e^{-\lambda_j s}\,ds\right)e^{i\lambda_j t},$$

where $x(t)\simeq\sum_{j=1}^{\infty}x_j e^{i\lambda_j t}$.

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1 Introduction

In several papers published recently, Corduneanu [4], Corduneanu and Li [8], Mahdavi [10], we have investigated several cases of functional differential equations, in regard to the existence of AP_r–almost periodic solutions. This concept of almost periodicity has been used in research by Shubin [11], in connection with partial differential operators. In

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Corduneanu [4], one finds their basic theory in the case of functions, $R \to C^n$, as well as applications to functional equations, of the same nature as those we shall consider in the present paper. If in the above quoted papers the type of equations investigated is often the differential functional type; in this paper we shall be also concerned with other types of functional equations, including integral ones, in nonlinear case.

2 A Second Kind Convolution Integral Equation

We shall consider the classical Fredholm type equation, on the real axis, of the form

$$x(t) + \int_R k(t-s)\,x(s)\,ds = f(t)\,,\ t \in R, \tag{2.1}$$

which in case of integrable kernel k, *i.e.*, $k \in L^1(R,C)$, can be easily treated by Fourier transform method. Indeed, there exists, under the integrability assumption, a conjugate kernel $\tilde{k} \in L^1(R,C)$, the only extra condition to be imposed being

$$1 + \tilde{k}(s) \neq 0\,,\ s \in R, \tag{2.2}$$

where

$$\tilde{k}(s) = \int_R k(t)\,e^{its}\,dt \tag{2.3}$$

is the Fourier transform of k.

For the proof of this assertion, based on a Wiener-Levy result, see motivation and references in Corduneanu [3].

Since the unique solution of the equation (2.1) is given by

$$x(t) = f(t) + \int_R \tilde{k}(t-s)\,f(s)\,ds, \tag{2.4}$$

when f belongs to a variety of function spaces on R, with values in C, the formula (2.4) can be applied for $f \in E(R,C)$, where E denotes any of the spaces $BC(R,C)$, $L^p(R,C)$, $1 \leq p \leq \infty$, $M(R,C)$ and many other function spaces, including the space $AP(R,C)$ of Bohr almost periodic functions. See details in Corduneanu [1].

Therefore, from (2.4), under condition (2.2), one derives the existence and uniqueness of solution to equation (2.1), for any $f \in AP(R,C)$.

In the case of the space $AP_r(R,C)$, as defined in Corduneanu [4] we cannot apply the classical and powerful method, with wide use in Fourier analysis, because we do not know how to integrate the items/series under integral. We have just a generalized Fourier series for x or f, namely

$$x(t) \simeq \sum_{j=1}^{\infty} x_j\,e^{i\lambda_j t}, \tag{2.5}$$

$$f(t) \simeq \sum_{j=1}^{\infty} f_j\,e^{i\lambda_j t},$$

the connection with its "sum" being still unknown. See lemma in Corduneanu [5, 6] for illustration of such a connection, under rather general assumptions. We shall follow the definition of the generalized convolution product appearing in the equation (2.1), when the factor $x(t)$ is replaced by the series in (2.5), for $x(t)$:

$$\begin{aligned}\int_R k(t-s)\,x(s)\,ds &= (k*x)(t)\\ &= \int_R k(t-s)\sum_{j=1}^{\infty} x_j e^{i\lambda_j s}\,ds\\ &= \sum_{j=1}^{\infty} x_j\left(\int_R k(t-s)\,e^{i\lambda_j s}\,ds\right)\\ &= \sum_{j=1}^{\infty} x_j\left(\int_R k(u)\,e^{-i\lambda_j u}\,du\right)e^{i\lambda_j t},\end{aligned}$$

assuming, of course, that all performed operations are justified. We shall prove now that, for any $x \in AP_r(R,C)$, $r \in [1,2]$, which means

$$\sum_{j=1}^{\infty} |x_j|^r \le +\infty, \tag{2.6}$$

the operations above are valid. Indeed, if one denotes

$$\tilde{x}_j = x_j \int_R k(u)\,e^{-i\lambda_j u}\,du\,,\ j \ge 1, \tag{2.7}$$

we see that

$$|\tilde{x}_j| = |x_j|\left|\int_R k(u)\,e^{-i\lambda_j u}\,du\right| \le |x_j| \int_R |k(u)|\,du, \tag{2.8}$$

and taking into consideration $k \in L^1(R,C)$, we find that $\tilde{x}_j$, $j \ge 1$ satisfy also (2.6). Hence, the above defined generalized convolution product makes sense, between an integrable function and an arbitrary function for $AP_r(R,C)$. Moreover, in the case $r = 1$, the series occurring in the definition of $k*x$ above, makes sense in the usual way (uniform convergence). See details in Corduneanu [4], for the assertion that the convolution with $k \in L^1(R,C)$ leaves invariant each $AP_r(R,C)$, $r \in [1,2]$.

Let us construct now the series of the form (2.6) for the solution of the equation (2.1). Substituting for $x(t)$ the series in (2.6), and keeping in mind the meaning of the (generalized) convolution product, one obtains the following identity:

$$\sum_{j=1}^{\infty} x_j e^{i\lambda_j t} + k*\left(\sum_{j=1}^{\infty} x_j e^{i\lambda_j t}\right) = \sum_{j=1}^{\infty} f_j e^{i\lambda_j t}, \tag{2.9}$$

which can take place iff

$$x_j + \left(\int_R k(u)\,e^{-i\lambda_j u}\,du\right)x_j = f_j\,,\ j \ge 1. \tag{2.10}$$

We have taken into account the fact that

$$\int_R k(t-s)\,e^{i\lambda_j s}\,ds = \left(\int_R k(u)\,e^{-i\lambda_j u}\,du\right) e^{i\lambda_j t}\ ,\ j \geq 1. \tag{2.11}$$

But the parenthesis in the right hand side of (2.11) equals $\tilde{k}(-\lambda_j)$, so (2.10) becomes

$$[1+\tilde{k}(-\lambda_j)]\,x_j = f_j\ ,\ j \geq 1. \tag{2.12}$$

On behalf of the assumption (2.2), taking into account that $-\lambda_j{}'s$ are reals, one sees that x_j can be uniquely determined from (2.12), $j \geq 1$. Hence, $x(t)$ can be uniquely determined in the form (2.5).

One can state and prove the following results regarding the equation (2.1), see also Corduneanu [3].

Theorem 2.1. *Let us consider the equation (2.1) in the space of almost periodic functions $AP_r(R,C)$, $r \in [1,2]$. Assume that $k \in L^1(R,C)$ is such that the frequency domain condition (2.2) is satisfied. Also, assume $f \in AP_r(R,C)$, for a fixed $r \in [1,2]$. Then, there exists a unique solution $x \in AP_r(R,C)$ to equation (2.1), whose Fourier series is*

$$\sum_{j=1}^{\infty}[1+\tilde{k}(-\lambda_j)]^{-1}\,f_j\,e^{i\lambda_j t}. \tag{2.13}$$

Proof. Follows directly from the above considerations, especially from formulas (2.9)–(2.13). It is necessary to provide the proof of the fact that the series (2.13) is defining an element of $AP_r(R,C)$. This will be the consequence of the fact that we can find $K > 0$, such that

$$|1+\tilde{k}(s)|^{-1} \leq K\ ,\ s \in R. \tag{2.14}$$

Obviously, (2.14) is apparently stronger than (2.2), which has been assumed in the derivation of the series (2.13), corresponding to the solution $x(t)$ of (2.1). On the other hand, (2.14) will follow from an inequality of the form

$$|1+\tilde{k}(s)| \geq m > 0\ ,\ s \in R. \tag{2.15}$$

Since $|\tilde{k}(s)| \to 0$ as $|s| \to \infty$, we can find a positive number M, with the property

$$|1+\tilde{k}(s)| > \frac{1}{2}\ ,\ |s| > M. \tag{2.16}$$

Now let us consider the *continuous* function $|1+\tilde{k}(s)|$ on the closed interval $|s| \leq M$, of the real axis, at a point $\bar{s} \in (-M, M)$ its minimum being attained, *i.e.*,

$$|1+\tilde{k}(\bar{s})| = \min|1+\tilde{k}(s)| = \alpha \geq 0. \tag{2.17}$$

But α must be positive, because, otherwise, we will have $1+\tilde{k}(\bar{s}) = 0$, in contradiction with the assumption (2.2). Hence, one can take K in (2.14) as $K = \max\{2, \alpha^{-1}\} > 0$. But this leads to the inequalities, in (2.13),

$$\left|[1+\tilde{k}(-\lambda_j)]^{-1}\,f_j\right|^r \leq K^r\,|f_j|^r\ ,\ j \geq 1. \tag{2.18}$$

The inequalities (2.18) prove that the solution $x(t)$ of (2.1), constructed above, is an element of $AP_r(R,C)$, as it has been assumed f.

This ends the proof of theorem 2.1.

Corollary 2.2. *The nonlinear counterpart of the equation (2.1), namely*

$$x(t)+(k*x)(t) = (fx)(t)\ ,\ t\in R, \tag{2.19}$$

where $f : AP_r(R,C) \to AP_r(R,C)$ *is an operator, generally nonlinear, satisfying a Lipschitz type condition*

$$|fx - fy|_r \le L|x-y|_r\ ,\ x,y \in AP_r, \tag{2.20}$$

with sufficiently small L, enjoys the properties of existence and uniqueness of solution in $AP_r(R,C)$*, under condition (2.3) for the kernel k.*

The proof of the corollary 2.2 can be carried out by contraction mapping principle in $AP_r(R,C)$, for the operator $x = Tu$, with x and u connected by the linear equation

$$x(t)+(k*x)(t) = (fu)(t)\ ,\ t\in R, \tag{2.21}$$

relying on the estimate like (2.18).

Several comments are in order with our assumptions and procedure of getting the solution for equation (2.1).

First, when several functions belonging to a space $AP_r(R,C)$, $r \in [1,2]$, it is not a restriction of generality if we assume that they all have the same exponentials (the same $\lambda_j' s$). This remark is valid even in case we deal with a family of such functions, if it is countable. The diagonal process shows us that we can assume, without loss of generality, that all these functions possess the same exponentials. Of course, it would be necessary to add some terms whose coefficients $(x_j\,' - s)$ are zero.

Second, the assumption (2.2) assures the validity of the theorem, regardless of the set of exponents $\{\lambda_j : \lambda_j \in R,\ j \ge 1\} \subset R$.

Third, the extreme cases for r, namely $r = 1$ and $r = 2$, are conducing to some classical spaces of almost periodic functions. In case $r = 1$, the Fourier series corresponds to the space (Poincaré) of almost periodic functions in the sense of Bohr, whose Fourier series is absolutely (hence, also uniformly on R) convergent. These series are the most often encountered in applications. $AP_r(R,C)$ contains, obviously, all trigonometric polynomials,

$$\sum_{j=1}^{n} a_j e^{i\lambda_j t}\ ,\ t\in R, \tag{2.22}$$

with $a_j \in C$ and $\lambda_j \in R,\ j \ge 1$. In case $r = 2$, one obtains the space $AP_2(R,C) = B^2(R,C)$, known as the Besicovitch space of almost periodic functions. See Corduneanu [6] for details in regard to spaces of almost periodic or, more general, oscillatory functions and their construction.

Fourth, the results in theorem 2.1 and its corollary show that the relatively new spaces of almost periodic functions, we deal with in this paper, are in line with the similar results in classical spaces of almost periodic functions.

3 A Second Order Neutral Equation

In this section we shall consider the neutral type functional differential equation

$$\frac{d}{dt}[\dot{x}(t)+(k*x)(t)] = f(t)\ ,\ t\in R, \tag{3.1}$$

as well as its nonlinear associate

$$\frac{d}{dt}[\dot{x}(t) + (k * x)(t)] = (fx)(t)\,,\ t \in R. \tag{3.2}$$

We intend to provide conditions on the data k, f or fx, such that the existence and uniqueness of a solution, $x \in AP_r(R,C)$, is assured. There is some similarity of the approach here, with the one adopted in Section 2 of the paper. Actually, we shall rely on the results we have obtained in preceding publications: Corduneanu [7], Mahdavi [10]. We shall transform the equation (3.1), by taking the integral of both sides:

$$\dot{x}(t) + (k * x)(t) = \int^{t} f(s)\,ds\,,\ t \in R, \tag{3.3}$$

an operation which is permitted/motivated by imposing the condition

$$|\lambda_j| \geq m > 0\,,\ j \geq 1, \tag{3.4}$$

to the set of exponential, involved in representing $f(t)$; see (2.5). Thus, we obtain an equation equivalent to (3.1), but of the first order. Treating (3.3) in the same manner as above, *i.e.*, looking for a series like in (2.5), representing the candidate solution, one obtains

$$\sum_{j=1}^{\infty} i x_j \lambda_j e^{i\lambda_j t} + \sum_{j=1}^{\infty} x_j \tilde{k}(-\lambda_j) e^{i\lambda_j t} = \sum_{j=1}^{\infty} f_j \lambda_j^{-1} e^{i\lambda_j t},$$

which leads to the equations

$$\sum_{j=1}^{\infty} [i\lambda_j + \tilde{k}(-\lambda_j)]\, x_j = f_j \lambda_j^{-1}\,,\ j \geq 1, \tag{3.5}$$

and further, to

$$x_j = f_j \lambda_j^{-1} [i\lambda_j + \tilde{k}(-\lambda_j)]^{-1}\,,\ j \geq 1, \tag{3.6}$$

provided the inverse, in the right hand side of (3.6), exists for any $j \geq 1$. A stronger condition than (3.6) is, obviously,

$$|is - \tilde{k}(s)| \geq \alpha > 0\,,\ s \in R, \tag{3.7}$$

playing the same role as condition (2.2) in section 2 of this paper. By the same argument as above, one finds out that (3.7) is equivalent to the (apparently) weaker condition

$$is - \tilde{k}(s) \neq 0\,,\ s \in R, \tag{3.8}$$

which is used only for $s = -\lambda_j$, $j \geq 1$. But, as noticed above, the assumption (3.8) makes the existence result valid for any $f \in AP_r(R,C)$, regardless of the distribution of its $\lambda_j' s$.

We can state now the basic existence, and uniqueness result, related to the equation (3.1), namely, the statement is:

Theorem 3.1. *Consider the equation (3.1), under the following assumptions:*

1) $f \in AP_r(R,C)$ *and* $k \in L^1(R,C)$*;*

2) *k satisfies the condition (3.8), equivalent to (3.7) frequency condition;*

3) *the Fourier exponents* λ_j, $j \geq 1$ *of* f *satisfy condition (3.4).*

Then, there exists a unique solution x *of the equation (3.1), such that* $x \in AP_r(R,C)$*. This solution is characterized by the generalized Fourier series*

$$x \simeq \sum_{j=1}^{\infty} f_j \lambda_j^{-1} [i\lambda_j + \tilde{k}(-\lambda_j)]^{-1} e^{i\lambda_j t}. \tag{3.9}$$

The *proof* of theorem 3.1 has been given before its statement, excepting the convergence in $AP_r(R,C)$ of the series (3.9). This can be easily accomplished, if one relies on hypotheses/conditions (3.4) and (3.7). Indeed, (3.4) implies

$$|\lambda_j^{-1}| \leq m^{-1}\ ,\ j \geq 1, \tag{3.10}$$

while (3.7) means

$$|i\lambda_j + \tilde{k}(-\lambda_j)| \leq \alpha^{-1}\ ,\ j \geq 1. \tag{3.11}$$

From (3.10), (3.9), (3.11) one derives

$$\left| f_j \lambda_j^{-1} [i\lambda_j + \tilde{k}(-\lambda_j)]^{-1} \right| \leq (m\alpha)^{-r} |f_j|^r\ ,\ j \geq 1. \tag{3.12}$$

The inequality (3.12) proves that $x(t)$, given by (3.9), is in $AP_r(R,C)$.

One more remark is, likely, necessary, when writing equation (3.3) as an equivalent of the equation (3.1), we did not add a constant in the right hand side of (3.3). This is explained by the fact that, in the left hand side, there is no term with zero exponential (the way we have chosen the $\lambda_j' s$ to satisfy (3.4) and also the candidate solution $x(t)$, given by (2.5)).

This ends the proof of theorem 3.1.

Concerning the nonlinear equation (3.2), one can treat it routinely if we require on the nonlinear operator $f : AP_r \to AP_r$, the Lipschitz type condition (2.20). We leave to the reader the task to prove existence and uniqueness for the equation (3.2), in the space $AP_r(R,C)$, when the Lipschitz constant L is sufficiently small (by the contraction principle of Banach).

In concluding this paper, we shall give an example of a nonlinear operator $f : AP_r \to AP_r$. This example is also discussed in the paper [8], by Corduneanu and Li, under somewhat different hypotheses. One can define a nonlinear operator acting on AP_r, of the form

$$\sum_{j=1}^{\infty} x_j e^{i\lambda_j t} \to \sum_{j=1}^{\infty} \phi(x_j) e^{i\lambda_j t},$$

with $\lambda_j \in R$, $j \geq 1$, and $\phi : C \to C$, such that

$$f x \simeq \sum_{j=1}^{\infty} \phi(x_j) e^{i\lambda_j t}, \tag{3.13}$$

with

$$\sum_{j=1}^{\infty} |\phi(x_j)|^r < \infty. \tag{3.14}$$

For instance, one can take $\phi : C \to C$, of the form $\phi(u) = Le^{-|u|}$, $u \in C$ or $\phi(u) = L|u|(|u|+1)^{-1}$ etc. The above choices for $\phi(u)$ are Lipschitzian on C, with constant L. Then, we obtain

$$\begin{aligned}
|fx - fy|_r &= \left(\sum_{j=1}^{\infty} |\phi(x_j) - \phi(y_j)|^r\right)^{\frac{1}{r}} \\
&\leq L\left(\sum_{j=1}^{\infty} |x_j - y_j|^r\right)^{\frac{1}{r}} \\
&= L|x-y|_r .
\end{aligned}$$

This shows that $f : AP_r \to AP_r$ is Lipschitz continuous on C, with constant L.

At this point in discussion, one can follow the argument in the proof of corollary 2.2, obtaining the existence and uniqueness of solution for the neutral equation (3.2).

The fixed point of the operator $x = Tu$, where x and u are connected by

$$\dot{x}(t) + (k * x)(t) = (fu)(t)\,,\ t \in R,$$

with $u, x \in AP_r(R,C)$, x being uniquely determined by u.

As mentioned above, the reader is invited to prove the existence and uniqueness result in $AP_r(R,C)$, for the equation (3.2).

Several *open* problems can be formulated on behalf of the considerations made in this paper. We shall look, in some detail, at one of this problems.

Let us notice that the convolution product $k*(k*x)$ makes sense even when $k \in L^1(R,C)$, and $x \in AP_r(R,C)$, $r \in [1,2]$. Moreover, one has $k*(k*x) \in AP_r(R,C)$. The repetition of the convolution, from the left only, $m > 2$ times, will also make sense and we "formally" can write

$$(k^m * x) = (k * \underbrace{(k*x)\cdots(k*x)}_{m-1\ times}) = \sum_{j=1}^{\infty} x_j \left(\int_R k(s)e^{-\lambda_j s}\,ds\right)^m e^{i\lambda_j t}. \tag{3.15}$$

We notice that in case $k \in L^1(R,C)$, which we assume, with $|k|_{L^1} < 1$ one obtains the absolute and uniform convergence in the right hand side of (3.15). Moreover, in case $|\lambda_j| \to \infty$ as $j \to \infty$, a situation occurring in many applications, the higher order of convoluting leads to an improvement of the convergence.

We challenge the reader to investigate the second order equation

$$\ddot{x}(t) + (k^2 * x)(t) = 0\ \ or\ f(t)\,,\ t \in R,$$

which has some resemblance with the classical equation of the harmonic oscillator: $\ddot{x} + \omega^2 x = 0$ or $f(t)$, when applied forces are involved. Series solution in $AP_r(R,C)$ must be sought, in accordance with the choice of the term $f(t)$.

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