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P W. D *
Department of Mathematics and Computer Science
The Lincoln University of the Commonwealth of Pennsylvania
1570 Old Baltimore Pike
Lincoln University, PA 19352, USA

Abstract

Solutions of the complex Monge-Ampère Equation are obtained in the Sobolev topology on complex manifolds and through the Delta-Delta-Bar Lemma, in case the manifold is compact Kähler, a simple proof is given of the Aubin-Calabi-Yau Theorem.

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1 Introduction

Whilst the real Monge-Ampère operator has been studied for a long time, the complex Monge-Ampère operator is of recent vintage. The pioneers in the study of the complex Monge-Ampère equation were Kerzman, Kohn and Nirenberg.

There are two approaches to the study of the complex Monge-Ampère equation-through pluripotential theory and through PDE. The PDE approach has been carried out mostly by Kerzman, Kohn, Nirenberg, Caffarelli, Spruck, Yau, et al. The pluripotential theory approach by Bedford, Taylor, Cegrell, Kolodziej, Demailly, et al.

In the work of the above mentioned people the complex Monge-Ampère equation was considered as a boundary value problem (except where the manifold was compact without boundary) and a unique solution was sought. In this paper we do not consider the complex Monge-Ampère equation as a boundary value problem, so we have an infinite number of solutions as in [1]. We start our estimates with results in bounded open subsets of $\mathbb{C}^n$ and then globalize to relatively compact subdomains of a complex manifold, or to compact complex manifolds without boundary. We then finish with the Aubin-Calabi-Yau Theorem, the approach being from PDE.

We consider the complex Monge-Ampère equation in the form

$$M_c(u) := det(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}) = f \tag{1.1}$$

*E-mail address: pdarko10@aol.com

where at least $f \geq 0$ in the domain in question.

2 Preliminaries

Let Ω be an open bounded subset of $\mathbb{C}^n$ with boundary of Lebesgue measure zero, a relatively compact subdomain of a complex manifold also with boundary of Lebesgue measure zero, or a compact complex manifold without boundary.

For s, p real numbers with $1 \leq s \leq \infty$, $1 \leq p \leq \infty$, $W_p^s(\Omega)$ are the usual Sobolev spaces on Ω (see [2]). Our main result is the following.

Theorem 2.1. *Let Ω be as above and let $f^{\frac{1}{n}} \in W_p^s(\Omega)$, $f \geq 0$, then there is $u \in W_p^{s+2}(\Omega)$ such that*

$$M_c(u) = f \quad \textit{and} \quad \|u\|_{W_p^{s+2}(\Omega)} \leq c\|f^{\frac{1}{n}}\|_{W_p^s(\Omega)} \tag{2.1}$$

where c is independent of f.

Corollary 2.2. *Let Ω be a relatively compact subdomain of a complex manifold with Lipschitz boundary, and let $f > 0$, $f \in C^\infty(\bar{\Omega})$, then there is $u \in C^\infty(\bar{\Omega})$, such that*

$$M_c(u) = f \quad \textit{on} \quad \Omega. \tag{2.2}$$

As an application of Corollary 2.2 we have

Theorem 2.3 (Aubin-Calabi-Yau)**.** *Let g_{jk}, $1 \leq j \leq n$, $1 \leq k \leq n$, f real be C^∞ function on a compact Kähler manifold Ω such that $det(g_{jk}) \geq 0$. Then there is a C^∞ function u such that, if the g_{jk} determine a d-closed (1,1)-form*

$$det(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} + g_{jk}) = e^f det(g_{jk}). \tag{2.3}$$

3 Local Estimates

In this section let Ω be a bounded open subset of $\mathbb{C}^n$ with boundary of Lebesgue measure zero and let $f \geq 0$, $f^{\frac{1}{n}} \in W_p^s(\Omega)$. Define f to be zero outside Ω and let e be a fundamental solution of the Laplacian Δ in $\mathbb{C}$, that is, $\Delta e = \delta$, where δ is the Dirac delta in $\mathbb{C}$. Define the distribution E_j in $\mathbb{C}^n$ by

$$E_j(\varphi) = e(\varphi(0,0,\cdots,j,\cdots,0,0)), \quad 1 \leq j \leq n \tag{3.1}$$

the action of e being in the jth coordinate. $\varphi \in D(\mathbb{C}^n)$-a test function. Define v by

$$v = \frac{1}{4}(E_1 + E_2 + \cdots + E_n) * f^{\frac{1}{n}} \tag{3.2}$$

where $*$ is convolution.

Then

$$M_c(v) = f \quad on \quad \mathbb{C}^n. \tag{3.3}$$

Let u be the restriction of v to Ω, then (2.1) holds.

4 Global Estimates

In this section Ω is a relatively compact subdomain of a complex manifold X, and Ω has a boundary with Lebesgue measure zero.

Let $\{U_j\}_{j=1}^N$ be an open covering of $\bar{\Omega}$ by coordinate neighbourhoods such that $\Omega \cap U_j$ has boundary of Lebesgue measure zero. Let $\theta_j : U_j \to C^n$ be the coordinate map in U_j. Let $\Omega_j = \theta_j(\Omega \cap U_j)$, and let ξ_j be a C^∞-partition of unity with each ξ_j supported in $\Omega \cap U_j$. Let $g_j = (\xi \cdot f^{\frac{1}{n}}) \circ \theta_j^{-1}$ in Ω_j, and let v_j on Ω_j be the solution from the construction in Section 3 of $M_c(u) = g_j^n$, so that

$$\frac{\partial^2 v_j}{\partial z_k \partial \bar{z}_k} = g_j \quad and \quad \frac{\partial^2 v_j}{\partial z_l \partial \bar{z}_k} = 0 \quad for \quad l \neq k \quad in \quad \Omega_j. \tag{4.1}$$

Now let $u = \sum_{j=1}^N v_j \circ \theta_j$ in Ω, with $v_j \circ \theta_j$ defined to be zero outside $\Omega \cap U_j$, then for $z_0 \in \Omega \cap U_j$, $\theta_j(z_0) \in \Omega_j$ and

$$\begin{aligned} \frac{\partial^2 u(z_0)}{\partial z_k \partial \bar{z}_k} &= \sum_{j=1}^N \frac{\partial^2}{\partial z_k \partial \bar{z}_k} v_j(\theta_j(z_0)) \\ &= \sum_{j=1}^N g_j(\theta_j(z_0)) \\ &= \sum_{j=1}^N (\xi_j f^{\frac{1}{n}}) \circ \theta_j^{-1}(\theta_j(z_0)) \\ &= \sum_{j=1}^N (\xi_j f^{\frac{1}{n}})(z_0) \\ &= f^{\frac{1}{n}}(z_0) \end{aligned} \tag{4.2}$$

and (2.1) holds.

The case of the compact complex manifold is similar.

5 The Aubin-Calabi-Yau Theorem

To prove Theorem 2.3, let Ω, f, g_{jk}, $1 \leq j \leq n$, $1 \leq k \leq n$ be as in that theorem and let $F = e^f det(g_{jk})$. Let the compact complex manifold Ω without boundary be covered by open sets $\{U_j\}_{j=1}^N$, where each U_j is biholomorphic to the unit polydisk. Let $\{\theta_j\}_{j=1}^N$ and $\{\xi_j\}$ be as in Section 4, so that $\theta_j : U_j \to \mathbb{C}^n$ is a coordinate map and $\Omega_j = \theta_j(U_j)$, and ξ_j is supported in U_j, where $\{\xi_j\}$ is a $\mathbb{C}^\infty$-partition of unity. Let $G_j = (\xi_j \cdot F^{\frac{1}{n}}) \circ \theta_j^{-1}$ in Ω_j, and let v_j be the solution of $M_c(u) = G_j^n$ in Ω_j constructed in Section 3.

From the local $\partial\bar{\partial}$-Lemma [3; Proposition 1.1 on page 85], there is H_j on each U_j such that

$$\frac{\partial H_j}{\partial z_l \partial \bar{z}_k} = g_{lk} \quad on \quad U_j. \tag{5.1}$$

Let $w_j = v_j \circ \theta_j - H_j$ on U_j and zero outside U_j, then

$$\frac{\partial w_j}{\partial z_l \partial \bar{z}_k} + g_{jk} = \frac{\partial v_j \circ \theta_j}{\partial z_l \partial \bar{z}_k} \quad in\ U_j. \tag{5.2}$$

Let $w = \sum_{j=1}^{N} w_j$ in Ω, then

$$\left(\frac{\partial w}{\partial z_l \partial \bar{z}_k} + g_{lk}\right)(z_0) = \sum_{j=1}^{N} \frac{\partial v_j \circ \theta_j(z_0)}{\partial z_l \partial \bar{z}_k} \quad for \quad z_0 \in \Omega. \tag{5.3}$$

Therefore from a result corresponding to (4.1) above

$$\left(\frac{\partial w}{\partial z_k \partial \bar{z}_k} + g_{kk}\right)(z_0) = F^{\frac{1}{n}}(z_0) \tag{5.4}$$

and

$$\left(\frac{\partial w}{\partial z_l \partial \bar{z}_k} + g_{lk}\right) = 0 \quad for\ l \neq k. \tag{5.5}$$

Therefore

$$det\left(\frac{\partial w}{\partial z_l \partial \bar{z}_k} + g_{lk}\right) = F = e^f\, det(g_{lk}). \tag{5.6}$$

Remark 5.1. Note that we did not mention the fact that Ω is Kähler in the above proof. We thus have a generalization.

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