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Abstract

In this paper, we characterize the solution of a nonlinear reflected Backward Stochastic Differential Equations (BSDE) as the unique solution of a Stochastic Variational Inequality (SVI). This approach leads to a priori estimate for the increment of the predictable component of the solution of the reflected BSDE.

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Introduction

Nonlinear backward stochastic differential equations have been introduced first by Pardoux and Peng [10]. They proved the existence and uniqueness of adapted solutions under suitable assumptions.

In [7], El-Karoui et *al.* have introduced the notion of continuous reflected BSDE. Actually, it is a backward equation, but one of the components of the solution is forced to stay above a given continuous boundary process. They proved that there exists a unique solution to this equation if the terminal condition ξ and the coefficient g satisfy smooth integrability assumptions and if $g(t,\omega,y,z)$ is Lipschitz in (y,z) uniformly in (t,ω).

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There are many works on reflected BSDE, among which Hamadène [9] studied the case when the barrier is a right continuous and left limited (r.c.l.l for short) process. In [13], Peng and Xu consider the case when the obstacle is only an L^2-process. In both these papers, it has been shown the existence and uniqueness of a solution for the reflected BSDE when the coefficient g is uniformly Lipschitz with respect to (y,z).

Many assumptions have been considered in order to relax the Lipschitz condition on the driver g. A large number of results devoted to this subject can be found in the literature review. In [16], Xu consider a driver g non-Lipschitz in z with monotonicity and general increasing conditions in y. He proves the existence and uniqueness of a solution for reflected BSDE with one continuous barrier. Bahlali and *al.* [1] proved the existence of a maximal solution and a minimal one for one barrier reflected backward doubly stochastic differential equations. Note that a continuous generator g has been considered in [1]. A recent result of Essaky and *al.* [8] shows the existence of a maximal solution for a generalized BSDE with two reflecting barriers when the terminal condition ξ is merely square integrable. In the latter paper, the authors have considered a generator g which is continuous, general growth with respect to y and stochastic quadratic growth with respect to z.

In this paper, we consider a nonlinear reflected BSDE with single L^2-obstacle. Using Peng's g-expectation [12], we formulate a SVI. We characterize the solution of the reflected BSDE as the unique solution of the SVI. The main idea of our method is the well known Girsanov transformation.

The paper is organized as follows: in Section 1, we present the basic assumptions and recall the notions of reflected BSDE and Snell envelope. In Section 2, we generalize some properties of the Snell envelope obtained in [4] to g-supermartingales. The nonlinear SVI is also introduced and studied in Section 2.

1 Reflected BSDE and Stopping Time Problem

On a given complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, let $B=(B_t,t\geq 0)$ be a standard d-dimensional Brownian motion defined on a finite time interval $[0,T]$. We denote by $\mathcal{F}=(\mathcal{F}_t)_{0\leq t\leq T}$ the augmentation of the natural filtration $\mathcal{F}^B=(\mathcal{F}_t^B)_{0\leq t\leq T}$ with $\mathcal{F}_t^B:=\sigma\{B_s,0\leq s\leq t\}$, generated by B. Throughout the work, we will assume that all stochastic processes to be considered are adapted to the given filtration.

Let $T\in[0,\infty)$, and introduce the following notations:
$|x|$ is the euclidean norm of an element $x\in\mathbb{R}^m$,
$\mathcal{P}$ is the σ-algebra of predictable sets in $[0,T]\times\Omega$,
(D) is the class of uniformly integrable processes,

$L^2(\mathcal{F}_T):=\Big\{\xi:\Omega\to\mathbb{R},\ \mathcal{F}_T\text{-measurable random variables s.t. } \mathbb{E}[|\xi|^2]<\infty\Big\}$,
$L^2_{\mathcal{F}}(0,T;\mathbb{R}^m):=\Big\{\varphi:\Omega\times[0,T]\to\mathbb{R}^m;\ \mathcal{F}\text{-predictable processes s.t. } \mathbb{E}\int_0^T|\varphi_s|^2ds<\infty\Big\}$,
$D^2_{\mathcal{F}}(0,T):=\Big\{\varphi:\Omega\times[0,T]\to\mathbb{R};\ \mathcal{F}\text{-progressively measurable; r.c.l.l processes s.t.}$

$$\mathbb{E}[\sup_{0\leq t\leq T}|\varphi_t|^2]<\infty\Big\}.$$

In this paper, $g:[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$, is a given $\mathcal{P}\times\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}^d)$-measurable function. It satisfies the following standard conditions (cf. [10]):

$$\mathbb{E}\int_0^T |g(t,0,0)|^2 dt < \infty, \tag{1.1}$$

$$|g(t,y_1,z_1)-g(t,y_2,z_2)| \le k(|y_1-y_2|+|z_1-z_2|), \tag{1.2}$$

for all $t\in[0,T],\ y_1,\ y_2\in\mathbb{R},\ z_1,\ z_2\in\mathbb{R}^d$, for some constant $k\ge 0$.

The following definition of g-supersolution (cf. [6], [11] and [13]) is a notion parallel to that in PDE theory (cf. [7]).

Definition 1.1. A triple $(Y,Z,A)\in D^2_{\mathcal{F}}(0,T)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)\times D^2_{\mathcal{F}}(0,T)$ is a g-supersolution if A is an increasing process in $D^2_{\mathcal{F}}(0,T)$ with $A_0=0$, and

$$Y_t = Y_T + \int_t^T g(s,Y_s,Z_s)ds + A_T - A_t - \int_t^T Z_s dB_s,\ 0\le t\le T. \tag{1.3}$$

We often simply call $Y := (Y_t)_{0\le t\le T}$ a g-supersolution. More particularly, when $A\equiv 0$, Y is called a g-solution on $[0,T]$.

We assume that

$$L\in L^2_{\mathcal{F}}(0,T;\mathbb{R}),\ \xi\in L^2(\mathcal{F}_T),\ \text{and}\ \mathbb{E}[ess\sup_{0\le t\le T}(L_t^+)^2]<\infty,\ L_T\le\xi,\ \text{(a.s.)}. \tag{1.4}$$

We give the generalized reflected BSDE with one L^2-obstacle as defined in [13].

Definition 1.2. Let ξ be a given random variable in $L^2(\mathcal{F}_T)$ and g the above function which satisfies (1.1) and (1.2). A triple $(Y,Z,A)\in D^2_{\mathcal{F}}(0,T)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)\times D^2_{\mathcal{F}}(0,T)$ is called a solution of the reflected BSDE with lower obstacle $L\in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ and terminal condition $\xi\in L^2(\mathcal{F}_T)$ if
(Y,Z,A) is a g-supersolution on $[0,T]$ with $Y_T=\xi$, i.e.

$$Y_t = \xi + \int_t^T g(s,Y_s,Z_s)ds + A_T - A_t - \int_t^T Z_s dB_s, \quad \text{with}\ \ Y_T=\xi, \tag{1.5}$$

Y dominates L, i.e.,

$$Y_t \ge L_t, \qquad \text{a.s., a.e.,} \tag{1.6}$$

and the following (generalized) Skorohod condition holds:

$$\int_0^T (Y_{s^-}-L^*_{s^-})dA_s = 0,\ \text{a.s.},\ \forall L^*\in D^2_{\mathcal{F}}(0,T)\ \text{s.t.}\ L_t\le L_t^*\le Y_t\ \text{a.s., a.e.} \tag{1.7}$$

The following existence and uniqueness result is given in [13] (see Theorem 2.1).

Theorem 1.3. *We assume that the lower obstacle $L\in L^2_{\mathcal{F}}(0,T)$ and $\xi\in L^2(\mathcal{F}_T)$ satisfy (1.4). Then there exists a unique solution (Y,Z,A) of the reflected BSDE with the lower obstacle L and terminal condition $Y_T=\xi$. Moreover, Y is the smallest g-supersolution that dominates L with terminal condition $Y_T=\xi$.*

Dealing with the notions of g-martingale and g-supermartingale, we can refer to [11] for interesting research works in this domain.

Definition 1.4. A g-martingale on $[0,T]$ is a g-solution on $[0,T]$.
An $\mathcal{F}_t$-progressively measurable real valued process $\tilde{Y}$ is called a g-supermartingale on $[0,T]$ in strong sense if, for each stopping time $\tau \leq T$ such that $\mathbb{E}[|\tilde{Y}_\tau|^2] < \infty$, and the g-solution Y on $[0,\tau]$ with terminal condition $Y_\tau = \tilde{Y}_\tau$, satisfies $Y_\sigma \leq \tilde{Y}_\sigma$ for all stopping time $\sigma \leq \tau$.

$\tilde{Y}$ is a g-supermartingale on $[0,T]$ in weak sense if and only if the stopping times $\sigma \leq \tau$ are changed to deterministic times $s \leq t$.

By comparison Theorem, we can prove that a g-supersolution on $[0,T]$ is also g-supermartingale in both strong and weak senses. Consequently, by Theorem 1.3, the solution Y of the reflected BSDE with lower obstacle L and terminal condition ξ is the smallest g-supermartingale that dominates L with $Y_T = \xi$.

1.1 The problem of Snell envelope

According to the fundamental Theorem due to Mertens (cf. [5], Appendix 1, Theorem 22), for any r.c.l.l process $L^* \in (D)$, there exists a smallest right continuous supermartingale Y bounding the given process L^* such that (a.s.):

$$Y_\sigma = ess \sup_{\sigma \leq \tau \leq T} \mathbb{E}(L^*_\tau / \mathcal{F}_\sigma), \quad \text{for any stopping time } \sigma.$$

This process Y is called the Snell envelope of the process L^*.

Proposition 1.5. *If the triple (Y,Z,A) is the solution of the reflected BSDE with lower obstacle L and terminal condition ξ, then for all $t \in [0,T]$*

$$Y_t = ess \sup_{t \leq \tau \leq T} \mathbb{E}\left[\int_t^\tau g(s,y_s,z_s)ds + L^*_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}} / \mathcal{F}_t\right]. \tag{1.8}$$

Proof
According to Theorem 1.3, let $(Y,Z,A) \in D^2_{\mathcal{F}}(0,T) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d) \times D^2_{\mathcal{F}}(0,T)$ be the solution of the reflected BSDE with lower obstacle L and terminal condition ξ.
Let $\tau \in [0,T]$ be a stopping time. We have:

$$g(.,y_s,z_s) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}), \ A \in D^2_{\mathcal{F}}(0,T) \ \text{ and } \ Y \in D^2_{\mathcal{F}}(0,T).$$

Therefore, we may pass to the conditional expectation in (1.5); hence

$$\begin{aligned} Y_t &= \mathbb{E}\left[\int_t^\tau g(s,Y_s,Z_s)ds + Y_\tau + A_\tau - A_t / \mathcal{F}_t\right] \\ &\geq \mathbb{E}\left[\int_t^\tau g(s,Y_s,Z_s)ds + L^*_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}} / \mathcal{F}_t\right]. \end{aligned}$$

Define the ϵ-optimal stopping time

$$\tau_t^\epsilon = \inf\{t \le s \le T : L_s^* \ge Y_s - \epsilon\} \wedge T,\ \forall t \in [0,T],\ \forall \epsilon > 0. \tag{1.9}$$

It follows that

$$Y_t = \mathbb{E}\left[\int_t^{\tau_t^\epsilon} g(s,Y_s,Z_s)ds + L^*_{\tau_t^\epsilon} 1_{\{\tau_t^\epsilon < T\}} + \xi 1_{\{\tau_t^\epsilon = T\}} + A_{\tau_t^\epsilon} - A_t / \mathcal{F}_t\right] \text{ (a.s.)}.$$

Let $s \in (t,\tau_t^\epsilon]$, then $Y_{s^-} - L^*_{s^-} \ge \epsilon > 0$.

Using the generalized Skorohod condition, we get

$$\epsilon \int_t^{\tau_t^\epsilon} dA_s \le \int_t^{\tau_t^\epsilon} (Y_{s^-} - L^*_{s^-}) dA_s \le \int_0^T (Y_{s^-} - L^*_{s^-}) dAs = 0.$$

Hence, we obtain

$$A_{\tau_t^\epsilon} - A_t = 0.$$

Therefore,

$$Y_t = \mathbb{E}\left[\int_t^{\tau_t^\epsilon} g(s,Y_s,Z_s)ds + L^*_{\tau_t^\epsilon} 1_{\{\tau_t^\epsilon < T\}} + \xi 1_{\{\tau_t^\epsilon = T\}} / \mathcal{F}_t\right]. \tag{1.10}$$

Consequently,

$$Y_t = ess \sup_{t \le \tau \le T} \mathbb{E}\left[\int_t^{\tau} g(s,Y_s,Z_s)ds + L^*_{\tau} 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} / \mathcal{F}_t\right].$$

Remark 1.6. If one takes $g \equiv 0$ in (1.8), by Theorem 1.3 we see that Y is the Snell envelope of L^*.

Let X be a semimartingale, a norm commonly used on the space of semimartingales is the H^p norm ($1 \le p < \infty$). One defines

$$\|X\|_{H^p} = \inf_{X=M+A} \left(\left\| [M]_T^{1/2} + \int_0^T |dA_s| \right\|_{L^p} \right) < \infty,$$

where the infinimum is taken over all decompositions of X into local martingale and process of bounded variation. Note that $\int_0^T |dA_s|$ is the total variation of the process A. We recall Emery's inequality (cf. [14], p. 246, Ch. V, Th. 3) for stochastic integrals of Itô type. It will play a crucial role in the proof of the main Theorem. It states that if Z is a semimartingale and H a r.c.l.l and adapted process such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}\ (1 \le p \le \infty,\ 1 \le q \le \infty)$$

then

$$\left\| \int_0^T H_s dZ_s \right\|_{H^r} \le \|H\|_{D^p_{\mathcal{F}}} \cdot \|Z\|_{H^q}.$$

In the sequel, for an arbitrary r.c.l.l process U we denote by

$$U_T := \sup_{0\leq t\leq T} |U_t|.$$

We known that if a r.c.l.l. supermartingale Y is of class (D), it has a Doob-Meyer decomposition

$$Y_t = M_t - B_t, \quad 0 \leq t \leq T, \tag{1.11}$$

where $M_t = \mathbb{E}(M_T/\mathcal{F}_t)$ is a uniformly integrable martingale and B is a predictable non-decreasing integrable process such that $B_0 = 0$.
We recall the result of proposition 1.3 in [4] for this supermartingale.

Proposition 1.7. *For the given r.c.l.l process L^* of the class (D) and its Snell envelope Y with the Doob-Meyer decomposition (1.11) the following estimates does hold*

$$\begin{aligned} ||Y_T||_{L^p} &\leq p(p-1)\left\|L_T^*\right\|_{L^p}, \qquad 1 < p \leq \infty,\\ ||B_T||_{L^p} &\leq 2p\left\|L_T^*\right\|_{L^p}, \qquad 1 \leq p \leq \infty,\\ ||M_T||_{L^p} &\leq (\frac{p}{p-1}+2p)\left\|L_T^*\right\|_{L^p}, \quad 1 < p < \infty, \end{aligned}$$

(where $\frac{p}{p-1}$ means 1 for $p = \infty$).

Consider the convex subset K of the space $D^2_{\mathcal{F}}(0,T)$ of the following form

$$K = \left\{V \in D^2_{\mathcal{F}}(0,T) : V_t \geq L_t^* \ \ dt\otimes dP \text{ a.e}, \ \ V_T = L_T^* \ \text{ a.s.}\right\}.$$

The problem of SVI associated to optimal stopping time was formulated in [4] by Danelia et *al.* as follows:

Find an element $U \in K \cap H^2$ such that for any $V \in K$ the following inequality should hold

$$\mathbb{E}\left(\int_{\tau_1}^{\tau_2} (U_{s^-} - V_{s^-})dU_s/\mathcal{F}_{\tau_1}\right) \geq 0 \ \text{ (a.s.)}, \tag{1.12}$$

for each pair $\tau_1,\ \tau_2$, $0 \leq \tau_1 \leq \tau_2 \leq T$, of stopping times.
They establish that the SVI (1.12) has one and only one solution which is given by the Snell envelope Y of the process L^*.

2 Variational Inequality and Nonlinear Expectation

In the following parts, our aim is to study the SVI given in (1.12) by using a nonlinear mathe-matical expectation. We are mainly concerned Peng's g-expectation defined by the solution of a BSDE. Therefore we will deal with nonlinear SVI to mark the fact of working with nonlinear mathematical expectation. When mentioning linear SVI, we refer to the one formulated under the classical mathematical expectation.

We first recall the notion of g-expectations defined in [12] and [11] from which the most basic material of this section is taken. According to [3], we introduce the next notion.

Definition 2.1. A (possibly nonlinear) expectation on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a map

$$\mathcal{E} : L^2(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathbb{R},$$

which satisfies the following properties:

$$\text{if } X_1 \geq X_2 \quad \text{(a.s.)}, \quad \mathcal{E}(X_1) \geq \mathcal{E}(X_2),$$

$$\text{if } X_1 \geq X_2 \quad \text{(a.s.)}, \quad \mathcal{E}(X_1) = \mathcal{E}(X_2) \;\Leftrightarrow\; X_1 = X_2 \quad \text{(a.s.)},$$

$$\mathcal{E}(c) = c, \quad \text{for each constant } c.$$

In particular, if $\mathcal{E}$ is linear, then it becomes a classic expectation under the probability measure defined by $\mathbb{P}_{\mathcal{E}}(A) = \mathcal{E}(1_A)$, $A \in \mathcal{F}$.

We will write $\mathcal{E}_g$ instead $\mathcal{E}$ to point out the dependence of the nonlinear map $\mathcal{E}$ to the generator g of the BSDE.

From now on, we make the following assumption on the generator g of the reflected BSDE with lower obstacle L and terminal condition ξ:

$$g(t,y,0) \equiv 0, \;\; \forall\, (t,y) \in [0,T] \times \mathbb{R}. \tag{2.1}$$

Definition 2.2. For $\xi \in L^2(\mathcal{F}_T)$, let Y be a g-solution on $[0,T]$ with terminal condition $Y_T = \xi$. The g-expectation of ξ is defined by $\mathcal{E}_g(\xi) := Y_0$.

Like classical expectation, g-expectation preserves almost all properties of the classical expectation except linearity. The interested reader can refer to [12].

Definition 2.3. Let Y be a g-solution on $[0,T]$ with terminal condition $Y_T = \xi$. For each $t \leq T$, we call Y_t the conditional g-expectation of ξ with respect to $\mathcal{F}_t$ and define it by $\mathcal{E}_g(\xi/\mathcal{F}_t) := Y_t$.

Remark 2.4. The g-expectation $\mathcal{E}_g(.)$ is the classical mathematical expectation $\mathbb{E}(.)$ if and only if g does not depend on y and is linear in z.

The following property provides another, and more familiar definition of g-supermartingales like the classical one (cf. [11]).

Proposition 2.5. *Under the assumption (2.1), a process Y satisfying $\mathbb{E}|Y_t|^2 < \infty$ is a g-martingale (resp. g-supermartingale) in weak sense if and only if*

$$\mathcal{E}_g(Y_t/\mathcal{F}_s) = Y_s, \;\; (resp.\ \mathcal{E}_g(Y_t/\mathcal{F}_s) \leq Y_s), \forall\ s \leq t \leq T. \tag{2.2}$$

They are g-martingales (resp. g-supermartingales) in strong sense if and only if, in the above relations, s and t are stopping times.

Now let us recall the nonlinear decomposition of Doob-Meyer's type (see [11]).

Corollary 2.6. *We suppose (1.1), (1.2) and $\xi \in L^2(\mathcal{F}_T)$. We assume furthermore that g is independent of y and that (2.1) holds. Let X be a g-supermartingale on $[0,T]$ in the strong sense satisfying $\mathbb{E}[\sup_{t\leq T}|X_t|^2] < \infty$. Then X has the following decomposition :*

$$X_t = M_t - A_t.$$

Here M is a g-martingale of the form $\mathcal{E}_g(\xi/\mathcal{F}_t)$ and A is r.c.l.l increasing process with $A_0 = 0$ and $\mathbb{E}[(A_T)^2] < \infty$. Moreover such decomposition is unique.

We note that the process A corresponds to the one in the classical supermartingales.

As mentioned in the introduction, we generalize some properties of the Snell envelope given in [4]. The objective is to have similar results for g-supermartingales. First, we have the following result.

Proposition 2.7. *Let Y be the solution of the reflected BSDE with lower obstacle L and terminal condition ξ, then*

$$\mathbb{E}(Y_t) = \sup_{t\leq\tau\leq T} \mathbb{E}\left[\int_t^\tau g(s,Y_s,Z_s)ds + L^*_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}}\right]. \tag{2.3}$$

Proof

Let $0 \leq \tau \leq T$, be a stopping. Then by (5) we have

$$Y_t = Y_\tau + \int_t^\tau g(s,Y_s,Z_s)ds + A_\tau - A_t - \int_t^\tau Z_s dB_s.$$

Hence,

$$\mathbb{E}(Y_t) \geq \mathbb{E}\left[\int_t^\tau g(s,Y_s,Z_s)ds + L^*_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}}\right].$$

Using the ϵ-optimal stopping time define in (1.9) we get

$$Y_t = \int_t^{\tau_t^\epsilon} g(s,Y_s,Z_s)ds + L^*_{\tau_t^\epsilon} 1_{\{\tau_t^\epsilon<T\}} + \xi 1_{\{\tau_t^\epsilon=T\}} + \int_t^{\tau^\epsilon} Z_s dB_s,$$

thus,

$$\mathbb{E}(Y_t) = \mathbb{E}\left[\int_t^{\tau_t^\epsilon} g(s,Y_s,Z_s)ds + L^*_{\tau_t^\epsilon} 1_{\{\tau_t^\epsilon<T\}} + \xi 1_{\{\tau_t^\epsilon=T\}}\right].$$

Hence we get the desired result.

The following Theorem is a generalization of a central result in the theory of the Snell envelopes (see [15], Theorem 1).

Theorem 2.8. *Let Y be the solution of the reflected BSDE with lower obstacle L and terminal condition ξ, then for all $\epsilon > 0$ and $t \in [0,T]$*

$$Y_t = \mathcal{E}_g(Y_{\tau_t^\epsilon}/\mathcal{F}_t), \quad where \quad \tau_t^\epsilon = \inf\{t \leq s \leq T : L^*_s \geq Y_s - \epsilon\} \wedge T. \tag{2.4}$$

Proof
Indeed, by the definition of τ_t^ϵ, we have $A_{\tau_t^\epsilon} - A_t = 0$. Then

$$Y_t = Y_{\tau_t^\epsilon} + \int_t^{\tau_t^\epsilon} g(s, Y_s, Z_s)ds + (A_{\tau_t^\epsilon} - A_t) - \int_t^{\tau_t^\epsilon} Z_s dB_s.$$

$$= Y_{\tau_t^\epsilon} + \int_t^{\tau_t^\epsilon} g(s, Y_s, Z_s)ds - \int_t^{\tau_t^\epsilon} Z_s dB_s = \mathcal{E}_g(Y_{\tau_t^\epsilon}/\mathcal{F}_t).$$

We generalize the proposition 1.2 in [4] to the class of g-supermartingales majorizing the process L^*. Indeed, Danelia et *al.* [4] show that the Skorohod condition is a characteristic property of the Snell envelope in the class of supermartingales majorizing a given r.c.l.l process L^*, and with the value L_T^* at final time T.

Theorem 2.9. *Let U be a g-supermartingale majorizing the process $L^* \in D^2_{\mathcal{F}}(0,T)$ with terminal condition $U_T = \xi$. Consider the nonlinear Doob-Meyer decomposition of U of the form*

$$U_t = M_t - A_t,\ 0 \le t \le T,$$

where $M_t = \mathcal{E}_g(\xi/\mathcal{F}_t)$ and A a non-decreasing r.c.l.l process with $A_0 = 0$ and $\mathbb{E}(A_T)^2 < +\infty$. Then U is equal to the solution of the reflected BSDE with lower obstacle L and terminal condition ξ if and only if the non-decreasing process A of U satisfies the following relation (a.s.)

$$\int_0^T (U_{s^-} - L_{s^-}^*)dA_s = 0. \tag{2.5}$$

We recall the following Lemma in [2] before the proof of the Theorem:

Lemma 2.10. *Let the function g satisfy the hypothesis (1.1), (1.2) and (2.1). Then*

$$\mathcal{E}_g(\xi + \eta/\mathcal{F}_t) = \mathcal{E}_g(\xi/\mathcal{F}_t) + \eta, \quad \forall \eta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$$

if and only if g does not depend on y.

Proof of Theorem 2.9
The necessity of the relation (2.5) is obvious, hence, the sufficiency needs to be proved. We have (a.s.)

$$\mathcal{E}_g(U_{\tau_t^\epsilon} - U_t/\mathcal{F}_t) = \mathcal{E}_g(M_{\tau_t^\epsilon} - A_{\tau_t^\epsilon} - M_t + A_t/\mathcal{F}_t).$$

Let $s \in (t, \tau_t^\epsilon]$, then from the definition of times τ_t^ϵ we get

$$L_{s^-}^* \ge U_{s^-} - \epsilon, \text{ that is } U_{s^-} - L_{s^-}^* \ge \epsilon > 0.$$

Thus we obtain

$$\epsilon \int_t^T I_{(t<s\le\tau_t^\epsilon)} dA_s \le \int_t^T (U_{s^-} - L_{s^-}^*)dA_s = 0.$$

Therefore,

$$A_{\tau_t^\epsilon} - A_t = 0.$$

When we use Lemma 2.10, we get:

$$\begin{aligned}\mathcal{E}_g(U_{\tau_t^\epsilon} - U_t/\mathcal{F}_t) &= \mathcal{E}_g(M_{\tau_t^\epsilon} - M_t/\mathcal{F}_t)\\ &= \mathcal{E}_g(M_{\tau_t^\epsilon}/\mathcal{F}_t) - M_t.\end{aligned}$$

Since

$$M_{\tau_t^\epsilon} = \mathcal{E}_g(\xi/\mathcal{F}_{\tau_t^\epsilon}),$$

then

$$\begin{aligned}\mathcal{E}_g(U_{\tau_t^\epsilon} - U_t/\mathcal{F}_t) &= \mathcal{E}_g[\mathcal{E}_g(\xi/\mathcal{F}_{\tau_t^\epsilon})/\mathcal{F}_t] - M_t\\ &= \mathcal{E}_g(\xi/\mathcal{F}_{\tau_t^\epsilon\wedge t}) - M_t\\ &= M_t - M_t = 0.\end{aligned}$$

Using once again Lemma 2.10, we get:

$$\mathcal{E}_g(U_{\tau_t^\epsilon}/\mathcal{F}_t) = U_t \quad \text{(a.s.)}.$$

Once again by the definition of τ_t^ϵ, the right continuity of the trajectories of the processes U, and L^* we have

$$U_{\tau_t^\epsilon} \le L^*_{\tau_t^\epsilon} + \epsilon \quad \text{(a.s.)},$$

then

$$U_t = \mathcal{E}_g(U_{\tau_t^\epsilon}/\mathcal{F}_t) \le \mathcal{E}_g(L^*_{\tau_t^\epsilon}/\mathcal{F}_t) + \epsilon \le \mathcal{E}_g(Y_{\tau_t^\epsilon}/\mathcal{F}_t) + \epsilon \le Y_t + \epsilon,$$

that is (a.s.)

$$U \le Y.$$

But Y is the smallest g-supermartingale that majorizes L^*, hence we obtain (a.s.)

$$U = Y.$$

2.1 Main result

Inspired by the work of Danelia et *al.* in [4], we formulate the nonlinear SVI as follows:

find an element $U \in K \cap H^1$ such that, for any element $V \in K$, the following inequality should hold

$$\mathcal{E}_g\left(\int_{\tau_1}^{\tau_2} (U_{s^-} - V_{s^-})dU_s/\mathcal{F}_{\tau_1}\right) \ge 0 \ \text{(a.s.)}, \tag{2.6}$$

for each pair $\tau_1, \tau_2,\ 0 \le \tau_1 \le \tau_2 \le T$, of stopping times (the stochastic integral is understood in Itô sense).

Our main result in this paper is:

Theorem 2.11. *If the function g does not depend on the variable y, then the SVI (2.6) has one and only one solution which is given by the solution Y of the reflected BSDE with lower obstacle L and terminal condition ξ.*

Proof
We will carry out the proof in several steps. We should check at first that the conditional g-expectation in (2.6) is well defined. We use Emery's inequality:

$$\left\|\int_{\tau_1}^{\tau_2}(U_{s^-}-V_{s^-})dU_s\right\|_{H^2} = \left\|\int_0^T I_{\tau_1<s\le\tau_2}(U_{s^-}-V_{s^-})dU_s\right\|_{H^2}$$
$$\le \|U-V\|_{D^2_{\mathcal{F}}}\cdot\|U\|_{H^1}<\infty,$$

hence the stochastic integral is a semimartingale belonging to the space H^2.
Since one can find a positive constant c_p such that

$$\left\|\int_0^T I_{\tau_1<s\le\tau_2}(U_{s^-}-V_{s^-})dU_s\right\|_{L^2}\le c_p\left\|\int_0^T I_{\tau_1<s\le\tau_2}(U_{s^-}-V_{s^-})dU_s\right\|_{H^2}<\infty,$$

we deduce that the the nonlinear conditional expectation in (2.6) is well defined.

Step 1: Assume that the generator g is equal to zero.
If $g\equiv 0$, then $\mathcal{E}_g$ is the classical mathematical expectation. By Theorem 1.3, the solution Y of the corresponding reflected BSDE is also the Snell envelope of the process L^*. From Theorem 2.1 in [4], Y is the unique solution of the linear SVI.

Step 2: Assume that the generator g is linear.
The function g does not depend on y and is linear with respect to z. In this step, we study a trivial example of the function g given by

$$g(t,z_t)=b_tz_t,$$

where (b_t) is a uniformly bounded and progressively measurable process. We have

$$Y_t=\xi+\int_t^T g(s,z_s)ds+A_T-A_t-\int_t^T Z_sdB_s.$$

Set

$$\tilde{Y}_t=\tilde{\xi}+\int_t^T b_sz_sds-\int_t^T Z_sdB_s \tag{2.7}$$
$$=\tilde{\xi}-\int_t^T Z_sdW_s,$$

where

$$W_t=B_t-\int_0^t b_sds,\ \text{ and }\ \tilde{\xi}=\xi+A_T.$$

Let

$$\Gamma_t:=\exp\left(\int_0^t b_s.dBs-\frac{1}{2}\int_0^t|b_s|^2\,ds\right),$$

it follows that

$$d\Gamma_t = \Gamma_t b_t dB_t.$$

Therefore, Γ is a local martingale. Since the process (b_t) is bounded, then there exists a constant $k > 0$ such that

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T |b_r|^2\, dr\right)\right) \leq k.$$

Consequently, Γ is a martingale. Thanks to Girsanov Theorem, W is a $\mathbb{Q}$-Brownian motion, where $\mathbb{Q}$ is the probability measure on $(\Omega, \mathcal{F}_T)$ which density with respect to $\mathbb{P}$ is Γ_T.

We have

$$\mathbb{E}|\Gamma_T|^2 = \mathbb{E}\left[\exp\left(\int_0^T 2b_s.dBs - \frac{1}{2}\int_0^T |2b_s|^2\, ds + \int_0^T b_r^2 ds\right)\right].$$

Since

$$\mathbb{E}\left[\exp\left(\int_0^T 2b_s.dBs - \frac{1}{2}\int_0^T |2b_s|^2\, ds\right)\right] = 1,$$

then

$$\mathbb{E}(|\Gamma_T|^2) \leq k.$$

Suppose that the constant k vary from line to line and denote by $\mathbb{E}^{\mathbb{Q}}$ the mathematical expectation under $\mathbb{Q}$, we get

$$\left(\mathbb{E}^{\mathbb{Q}}|\tilde{Y}_t|\right)^2 = \left(\mathbb{E}\left|\tilde{Y}_t\Gamma_T\right|\right)^2 \leq \mathbb{E}|\tilde{Y}_t|^2 k < \infty.$$

Therefore, $\tilde{Y}$ is $\mathbb{Q}$-integrable. Taking conditional expectation $\mathbb{E}^{\mathbb{Q}}(./\mathcal{F}_t)$ on both sides of the BSDE (2.7), we have

$$\tilde{Y}_t = \mathcal{E}_g(\tilde{\xi}/\mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}(\tilde{\xi}/\mathcal{F}_t).$$

Since $\tilde{Y}$ is a g-martingale, then

$$\mathbb{E}^{\mathbb{Q}}(\tilde{Y}_t/\mathcal{F}_s) = \mathcal{E}_g(\tilde{Y}_t/\mathcal{F}_s) = \tilde{Y}_s, \ \ \forall\, 0 \leq s \leq t.$$

Thus $\tilde{Y}$ is a $\mathbb{Q}$-martingale with respect to $(\mathcal{F}_t)$. As such, we write Y as

$$Y_t = M_t - A_t, \ \text{ with } \ M_t = \mathbb{E}^{\mathbb{Q}}(\xi/\mathcal{F}_t). \tag{2.8}$$

Hence Y is a $\mathbb{Q}$-supermartingale. Note that under $\mathbb{Q}$, relation (2.8) is the classical Doob-Meyer decomposition of Y and by Theorem 1.3, Y is the Snell envelope of the process L^*.

The problem of nonlinear SVI (2.6) becomes:
find an element $U \in K \cap H^1$ such that, for any element $V \in K$, the following inequality should hold

$$\mathcal{E}_g\left(\int_{\tau_1}^{\tau_2}(U_{s^-} - V_{s^-})dU_s/\mathcal{F}_{\tau_1}\right) = \mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2}(U_{s^-} - V_{s^-})dU_s/\mathcal{F}_{\tau_1}\right) \geq 0 \ \text{ (a.s.)},$$

for each pair $\tau_1, \ \tau_2, \ 0 \leq \tau_1 \leq \tau_2 \leq T$, of stopping times.

It is very important to note that, from now on, we restrict our analysis to the Banach space

$$S^2 := \left\{ \text{r.c.l.l processes } Y, \text{ such that } \mathbb{E}^{\mathbb{Q}}[\sup_{0\le t\le T} |Y_t|^2] < \infty \right\}.$$

Also, we can rewrite the convex space K as follows

$$K = \left\{ V \in S^2 \;\; V_t \ge L^*_t, \;\; 0 \le t \le T \quad V_T = L^*_T \;\text{ a.s.} \right\}.$$

We note that, since $\mathbb{Q}$ is equivalent to $\mathbb{P}$, then a $\mathbb{P}$-semimartingale is also a $\mathbb{Q}$-semimartingale and conversely. Therefore, if one has a decomposition of a semimartingale under one measure, there is no need to look for analogous decomposition. Moreover, if $[X]^{\mathbb{P}}$ denotes the quadratic variation of X considered as $\mathbb{P}$-martingale, we have

$$[X]^{\mathbb{P}} = [X]^{\mathbb{Q}} \quad \mathbb{Q}\text{-a.s.},$$

so we do not make any distinction.

We have that (a.s.) $\;Y \ge L^*$ and $\;Y_T = L^*_T$, then $Y \in K$.
Let us show that $\|Y\|_{H^1} < \infty$. We have

$$\begin{aligned} \|Y\|_{H^2} &\le \left\| [M]_T^{\frac{1}{2}} + \int_0^T |dA_s| \right\|_{L^2} \\ &\le \left\| [M]_T^{\frac{1}{2}} \right\|_{L^2} + \left\| \int_0^T |dA_s| \right\|_{L^2}. \end{aligned}$$

Using Burkholder-Davis-Gundy (BDG) inequality and proposition 1.7, we obtain

$$\begin{aligned} \|Y\|_{H^2} &\le c_p \|M_T\|_{L^2} + \left\| \int_0^T |dA_s| \right\|_{L^2} \\ &\le 6c_p \left\| L^*_T \right\|_{L^2} + \left\| \int_0^T |dA_s| \right\|_{L^2}. \end{aligned}$$

We have again

$$[Y]_T^{\frac{1}{2}} \le [M]_T^{\frac{1}{2}} + [A]_T^{\frac{1}{2}},$$

since the process A is of finite variation, then

$$[A]_T^{\frac{1}{2}} = \left(\sum_{0\le s\le t} (\Delta A_s)^2 \right)^{\frac{1}{2}} \le \sum_{0\le s\le t} |\Delta A_s|.$$

As

$$[Y]_T^{\frac{1}{2}} \le [M]_T^{\frac{1}{2}} + \int_0^T |dA_s|,$$

therefore,

$$\mathbb{E}^{\mathbb{Q}} \left| \int_0^T |dA_s| \right|^2 \le C_1 \mathbb{E}^{\mathbb{Q}}[M]_T + C_2 \mathbb{E}^{\mathbb{Q}}[Y]_T,$$

where C_1 and C_2 are constants. Using once again BDG inequality and proposition 1.7, we get

$$\left\|\int_0^T |dA_s|\right\|_{L^2} \le C_1 \left\|L_T^*|\right\|_{L^2} + 6C_2 \left\|L_T^*|\right\|_{L^2}.$$

Hence,

$$Y \in H^2 \subset H^1.$$

Therefore the process Y belongs to the space $K \cap H^1$.

Let $V \in K$ and consider the stochastic integral

$$\int_{\tau_1}^{\tau_2} (Y_{s^-} - V_{s^-}) dY_s.$$

Since $Y = M - A$, where M is a $\mathbb{Q}$-martingale, its martingale part vanishes after taking the conditional expectation under $\mathbb{Q}$. Thus, we obtain

$$\mathcal{E}_g\left(\int_{\tau_1}^{\tau_2} (Y_{t^-} - V_{t^-}) dY_t / \mathcal{F}_{\tau_1}\right)$$

$$= \mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} (Y_{t^-} - V_{t^-}) dY_t / \mathcal{F}_{\tau_1}\right)$$

$$= \mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} (Y_{t^-} - V_{t^-}) dM_t / \mathcal{F}_{\tau_1}\right)$$

$$+ \mathbb{E}^{\mathbb{Q}}\left(-\int_{\tau_1}^{\tau_2} (Y_{t^-} - V_{t^-}) dA_t / \mathcal{F}_{\tau_1}\right)$$

$$= \mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} (V_{t^-} - Y_{t^-}) dA_t / \mathcal{F}_{\tau_1}\right)$$

$$= \mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} (V_{s^-} - L^*_{s^-}) dA_t / \mathcal{F}_{\tau_1}\right)$$

$$- \mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} (Y_{s^-} - L^*_{s^-}) dA_t / \mathcal{F}_{\tau_1}\right) \ge 0 \text{ (a.s.)},$$

since $V_t \ge L_t^*$, $0 \le t \le T$, (a.s.) and $\int_{\tau_1}^{\tau_2} (Y_{s^-} - L^*_{s^-}) dA_t = 0$ (a.s.) by the generalized Skorokhod condition (1.7). Hence indeed the solution of the reflected BSDE is the one of the SVI (2.6). We need to show the uniqueness of solution. Suppose U^1 and U^2 are two arbitrary solutions of the problem

$$\mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} (U_{s^-} - V_{s^-}) dU_s / \mathcal{F}_{\tau_1}\right) \ge 0, \quad \text{(a.s.)}.$$

We have

$$\mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} (U^i_{s^-} - V_{s^-}) dU^i_s / \mathcal{F}_{\tau_1}\right) \ge 0, \quad \text{(a.s.)},$$

where V is the arbitrary element of K, and $i = 1,2$. Taking $V = U^2$ for $i = 1$ (resp. $V = U^1$ for $i = 2$), we get

$$\mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2}(U^1_{s^-} - U^2_{s^-})dU^1_s/\mathcal{F}_{\tau_1}\right) \geq 0, \text{ (a.s.)},$$

(resp.)

$$\mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2}(U^2_{s^-} - U^1_{s^-})dU^2_s/\mathcal{F}_{\tau_1}\right) \geq 0, \text{ (a.s.)}.$$

Adding the above terms together, one has

$$\mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2}(U^2_{s^-} - U^1_{s^-})d(U^2_s - U^1)/\mathcal{F}_{\tau_1}\right) \geq 0, \text{ (a.s.)}.$$

Introducing the notation $U_t = U^2_t - U^1_t$, we obtain

$$\mathbb{E}^{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} U_{s^-}dU_s/\mathcal{F}_{\tau_1}\right) \geq 0, \text{ (a.s.)}.$$

Taking $\tau_1 = t,\ \tau_2 = T$, and applying the Itô's formula for the function $f(x) = x^2$, we get

$$U^2_T = U^2_t + 2\int_t^T U_{s^-}dU_s + \int_t^T d[U]_s. \tag{2.9}$$

Since $U_T = 0$, then taking conditional expectation in both sides of equality (2.9) we obtain

$$-U^2_t = 2\mathbb{E}^{\mathbb{Q}}\left(\int_t^T U_{s^-}dU_s/\mathcal{F}_t\right) + \mathbb{E}^{\mathbb{Q}}([U]_T - [U]_t/\mathcal{F}_t).$$

Now,

$$[U]_T - [U]_t \geq 0 \text{ and } \mathbb{E}^{\mathbb{Q}}\left(\int_t^T U_{s^-}dU_s/\mathcal{F}_t\right) \geq 0 \text{ (a.s.)},$$

therefore,

$$U_t = 0 \quad \text{and} \quad U^1_t = U^2_t, \quad 0 \leq t \leq T, \text{ (a.s.)}.$$

Hence we get the uniqueness of the solution.

Step 3: Assume that the generator g is nonlinear.
Recall that Y can be rewritten as follows:

$$Y_t = \xi + \int_t^T g(s,z_s)ds - \int_t^T Z_s dB_s - A_t.$$

Let

$$M_t = \xi + \int_t^T g(s,z_s)ds - \int_t^T Z_s dB_s,$$

and

$$W_t = B_t - \int_0^t \frac{g(s,z_s)}{z_s}ds. \tag{2.10}$$

With the convention $(0/0 = 0)$ and thanks to (1.1) and (1.2), the relation (2.10) makes sense, and

$$M_t = \xi + \int_t^T Z_s dW_s.$$

Considering the process

$$D_t = \exp\left(\int_0^t \frac{g(s,z_s)}{z_s} \cdot dB_r - \frac{1}{2}\int_0^t \left|\frac{g(s,z_s)}{z_s}\right|^2 dr\right),$$

we have

$$dD_t = D_t \frac{g(t,z_t)}{z_t} \cdot dB_t.$$

Then D is a local martingale. Since the process $\frac{g(t,z_t)}{z_t}$ is bounded by the Lipschitz constant of the function g, then it satisfies the Novikov condition. Therefore D is a martingale.

Let $\mathbb{Q}$ be the probability whose density function, with respect to $\mathbb{P}$ is D_T. By Girsanov's Theorem, W is a $\mathbb{Q}$-Brownian motion. Set

$$b_t := \frac{g(t,z_t)}{z_t},$$

and use the arguments developed in Step 2 to complete the proof.

We give a priori estimate of the component A of the reflected BSDE. For the sake of simplicity, we omit the proof of this Theorem. Indeed, such a technique is presented in [4]. Considering an arbitrary stochastic interval $(\sigma_1, \sigma_2]$, we have the following Theorem.

Theorem 2.12. *Let Y be the solution of the reflected BSDE with lower obstacle L and terminal condition ξ. Then there exists a unique probability $\mathbb{Q}$ equivalent to $\mathbb{P}$, under which for the predictable component A of the process Y the following inequalities are valid*

$$\mathbb{E}^{\mathbb{Q}}(A_{\sigma_2} - A_{\sigma_1}/\mathcal{F}_{\sigma_1}) \leq \mathbb{E}^{\mathbb{Q}}(L^*_{\tau^\epsilon_{\sigma_1}\wedge\sigma_2} - L^*_{\sigma_2}/\mathcal{F}_{\sigma_1}) + \epsilon \tag{2.11}$$

for arbitrary ϵ,

$$\left\|A_{\sigma_2} - A_{\sigma_1}\right\|_{L^p} \leq p \left\| \sup_{\sigma_1 \leq u \leq \sigma_2} |L^*_u - L^*_{\sigma_2}| \right\|_{L^p}, \quad p \geq 1. \tag{2.12}$$

Conclusions

In this work, a nonlinear SVI is considered. The solution of the reflected BSDE with lower obstacle L and terminal condition ξ is characterized as being its unique solution. According to Theorem 4.1 in [13], there exists an equivalence between the solution of the reflected BSDE and the related smallest g-supemartingale that dominates the obstacle L. Hence, we deduce an equivalence between the above smallest g-supermartingale and the solution of our nonlinear SVI.

The valuation for American Contingent Claims (ACC) in general financial market model, the concept of the "certainty equivalent" in economic theory, as well as the study of stochastic geometry can be considered as applications.

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